Perloff and Salop with partial market coverage

Wejie Zhong; wz2269

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We consider a random utility model as in [Perloff and Salop, 1985] with outside option for consumers. The setup is as following.

1 Setup

- **Firms:** There are n firms supplying horizontally differentiated products at fixed marginal cost c. Each firm set prices simultaneously.
- **Consumers:** There are unit mass of consumers each with valuation for product i: $x_i^k \sim F$ with support $[\underline{x}, \bar{x}]$. x_i^k are i.i.d. across consumers and products. Consumers can choose outside option and earn zero surplus.

2 Equilibrium

We start from the consumers side. A consumer will buy firm i's product iff:

$$x_i - p_i \geq \max_{j \neq i} \{0, x_j - p_j\}$$

That is to say, the surplus from product i is no less than surplus from other products or outside option. Sum this up among all consumers, we can get the demand function for product i:

$$\begin{aligned} Q_i(p_i, p_{-i}) &= E\left[I_{\left(x_i^k - p_i \geq \max_{j \neq i}\left\{0, x_j^k - p_j\right\}\right)}\right] \\ &= E_{x_i}\left[Pr(x_i - p_i > \max_{j \neq i}\{0, x_j - p_j\})\right] \\ &= E_{x_i}\left[\Pi_{j \neq i} Pr(x_i - p_i > x_j - p_j) I_{(x_i > p_i)}\right] \\ &= \int_{\max\{p_i, \underline{x}\}}^{\bar{x}} \Pi_{j \neq i} F(x_i - p_i + p_j) dF(x_i) \end{aligned}$$

If we assume $\underline{x}$ to be sufficiently small (say less than c) that there will always be some counsumers willing to choose outside option (otherwise we go back to the classic full coverage case), then:

$$Q_i(p_i, p_{-i}) = \int_{p_i}^{\bar{x}} \Pi_{j \neq i} F(x_i - p_i + p_j) dF(x_i)$$

Assume we are in a symmetric equilibrium, then:

$$Q_i(p_i) = \int_{p_i}^{\bar{x}} F(x - p_i + p)^{n-1} dF(x_i)$$

$$F.O.C.: \ p - c = \frac{(1 - F(p)^n)/n}{\int_p^{\bar{x}} f(x) dF(x)^{n-1} + F(p)^{n-1} f(p)}$$

To get a cleaner form of equilibrium charactermization, we try to write the expression of $p-c$ in terms of order statistics. Fisrt we derive a statistic for the second largest value conditional on the largest value to be larger than a certain level:

$$P[x_{(n-1)} < x | x_{(n)>p}] = \begin{cases} \frac{n(\int_x^{\bar{x}} F^{n-1}(x)dF(\tau) + \int_p^x F^{n-1}(\tau)dF(\tau))}{1-F^n(p)} & \text{when } x > p \\ \frac{n(1-F(p)F^{n-1}(x))}{1-F^n(p)} & \text{when } x \leq p \end{cases}$$

$$f_{(n-1)}(x|x_{(n)>p}) = \begin{cases} \frac{n(n-1)f(x)F^{n-2}(x)(1-F(x))}{1-F^n(p)} & \text{if } x > p \\ \frac{n(n-1)(1-F(p))F^{n-2}(x)}{1-F^n(p)} & \text{if } x \leq p \end{cases} \quad (1)$$

$$\Rightarrow \ \frac{1}{p-c} = \frac{\int_p^{\bar{x}} f dF^{n-1} + f(p)F^{n-1}(p)}{(1-F^n(p))/n}$$

$$= \int_p^{\bar{x}} \frac{f(x)}{1-F(x)} dF_{(n-1)}(x|X_{(n)} > p) + F_{(n-1)}(p|X_{(n)} > p) \frac{f(p)}{1-F(p)}$$

$$\Rightarrow \ \frac{1}{p-c} = \int_{\underline{x}}^{\bar{x}} g(\max\{x,p\}) dF_{(n-1)}(x|X_{(n)} > p)$$

$$\text{where } g(x) = \frac{f(x)}{1-F(x)} \quad (2)$$

Before proceeding to analysis of equilibrium, we give the economic explanation for this expression (2).

First we calculate the *market coverage*. Assume we are in a symmetric equilibrium, a consumer will be active in the market iff:

$$\max_i \left\{x_i^k - p\right\} \geq 0$$

Thus market coverage will be:

$$\begin{aligned} & E\left[I_{\max_i\{x_i^k - p\} \geq 0}\right] \\ =& Pr\left(\max_i \{x_i\} \geq p\right) \\ =& 1 - F^n(p) \end{aligned}$$

Then we calculate the density of *marginal consumers*, which are the consumers on the margin of buying good from highest valuation firm. From the assuption of i.i.d. valuation distribution, we know that the market coverage is quivalent to participation probability of a single agent, and the desity of marginal consumers is equivalent to a single agent's expectation to be on margin.

For a single agent, conditional on his highest valuation to be lower than p, it's impossible for him to be on margin (he isn't even in the market). Conditional on his highest valuation to be no less than p:

We can derive his CDF for the highest valuation conditional on second highest valuation to be x:

$$\begin{cases} \frac{F(t)-F(p)}{1-F(p)} \text{ if } (x < p) \\ \frac{F(t)-F(x)}{1-F(x)} \text{ if } (x \geq p) \end{cases}$$

- If his second highest valuation is $x < p$, then he will be on margin iff his highest valuation is p. Thus the density will be $\frac{f(p)}{1-F(p)}$
- If his second highest valuation is $x \geq p$, then he will be on margin iff his highest valuation is x. Thus the density will be $\frac{f(x)}{1-F(x)}$

To sum up, the desity of marginal consumers is:

$$\begin{aligned}
&\int_{X_{(n)} \geq p} g(\max\{x,p\}) dF_{n-1}(x) \\
&= \left(\int_{\underline{x}}^{\bar{x}} g(\max\{x,p\}) dF_{(n-1)}(x|X_{(n)} \geq p) \right) Pr\left[X_{(n)\geq p}\right] \\
&= \left(\int_{\underline{x}}^{\bar{x}} \underbrace{g(\max\{x,p\})}_{\substack{\text{Density of margianl consumer} \\ \text{conditional on } X_{(n-1)}}} \underbrace{dF_{(n-1)}(x|X_{(n)} \geq p)}_{\substack{\text{Density of second} \\ \text{largest valuation}}} \right) \underbrace{(1-F^n(p))}_{\substack{\text{Probability that there} \\ \text{exist marginal consumers}}}
\end{aligned}$$

Then we can re-write (2) as:

$$\begin{aligned}
\underbrace{p-c}_{\text{Unit profit}} \times \underbrace{\int_{\underline{x}}^{\bar{x}} g(\max\{x,p\}) dF_{(n-1)}(x|X_{(n)} \geq p)(1-F^n(p))/n}_{\text{Marginal consumers of a single firm}} \\
= \underbrace{(1-F^n(p))/n}_{\text{Market coverage of a single firm}} \qquad (3)
\end{aligned}$$

Finally we can interprete (2) as (3).

3 Analysis of Equilibrium

First we prove a lemma about the property of the second largest order statistics we constructed, which will be useful later on:

Lemma 3.1. *$X_{(n-1)}(x|X_{(n)\geq p})$ is increaseing in both n and p in the sense of first order stochastic dominance, i.e. $F_{(n-1)}(x|X_{(n)\geq p})$ is decreasing in both n and p.*

Proof. We make two claims:

1. $X_{(n)}|X_{(n)} > p$ is increasing in n and p in the sense of FOSD.

$$P[X_{(n)<x}|x_{(n)>p}] = F_{(n)}(x|p) = \frac{F(x)^n - F(p)^n}{1 - F(p)^n} \text{ (For all } x > p)$$

   It's easy to see that CDF is decreasing in p. To prove $F_{(n)}(x|p)$ is decreasing with n, it's equivalent to prove $\frac{a^n-1}{b^n-1}$, $(1 < a < b)$ is decreasing with n

$$\begin{aligned} &\frac{a^n - 1}{b^n - 1} > \frac{a^{n+1} - 1}{b^{n+1} - 1} \\ \Leftrightarrow &\frac{1 + \cdots + a^{n-1}}{1 + \cdots + b^{n-1}} > \frac{1 + \cdots + a^n}{1 + \cdots + b^n} \\ \Leftrightarrow &\frac{1 + \cdots + a^{n-1}}{1 + \cdots + b^{n-1}} > \frac{a^n}{b^n} \end{aligned}$$

   The last inequality is not hard to verify by observing $1 < a < b$ and using induction.

2. $X_{(n-1)}|X_{(n)} = p$ is increasing in both n and p in the sense of FOSD.

$$P[X_{(n-1)} < x|x_{(n)} = p] = \frac{nF(x)^{n-1}f(p)}{nF(p)^{n-1}f(p)} = (\frac{F(x)}{F(p)})^{n-1}$$

   Easy to see that it's decreasing in n and p.
   When $x > p$, the CDF is 1, then FOSD is trivial.

Then

$$\begin{aligned} P[X_{(n-1)} < x|X_{(n)} > p] &= \int_0^x \int_p^{\bar{x}} P[X_{(n-1)} = s|X_{(n)=q}]P[X_{(n)} = q|X_{(n)} > p]dqds \\ &= \int_p^{\bar{x}} P[X_{(n-1)} < x|X_{(n)} = q]f_{(n)}(q|p)dq \\ &> \int_p^{\bar{x}} P[X_{(n)} < x|X_{n+1} = q]f_{(n)}(q|p)dq \\ &> \int_p^{\bar{x}} P[X_{(n)} < x|X_{n+1} = q]f_{(n+1)}(q|p)dq \\ &= P[X_{(n)} < x|X_{(n+1)} > p] \end{aligned}$$

First inequality is directly from FOSD of $X_{(n-1)}|X_{(n)} = p$.Second inequality is from the throrem: $\forall$ increasing function, the expacted value of this function on FOSD distribution is higher, combining with the fact that $P[X_{(n)} < x|X_{n+1} = q]$ decreases with q.

For $p' > p$

$$\begin{aligned} P[X_{(n-1)} < x|X_{(n)} > p] &= \int_p^{\bar{x}} P[X_{(n-1)} < x|X_{(n)} = q]f_{(n)}(q|p)dq \\ &> \int_p^{\bar{x}} P[X_{(n-1)} < x|X_{(n)} = q]f_{(n)}(q|p')dq \\ &= \int_{p'}^{\bar{x}} P[X_{(n-1)} < x|X_{(n)} = q]f_{(n)}(q|p')dq \\ &= P[X_{(n-1)} < x|X_{(n)} > p'] \end{aligned}$$

The inequality is from FOSD of $X_{(n)}|X_{(n)} > p$. The second equality is from $f_{(n)}(q|p') = 0 \ \forall q < p'$. □

We make an assumption of the density function of valuation:

Assumption 3.2. *The CDF F is log-concave in x*

With this assumption we will have the following proposition fully characterizing symmetric equilibrium:

Proposition 1. *There exists a unique symmetric equilibrium characterized by the equilibrium price:*

$$\frac{1}{p-c} = \int_{\underline{x}}^{\bar{x}} g(\max\{x,p\}) dF_{(n-1)}(x|X_{(n)} > p)$$
$$\textit{where } g(x) = \frac{f(x)}{1-F(x)}$$

With this construction, existence and uniqueness of equilibrium and $p \in (c, p_M)$ are all very straight forward results.

1. *Existence and Uniqueness:* LHS covers $(0, \infty)$ and decreases with p. By log-concavity, $g(x)$ is increasing function, thus since $F_{(n-1)}(x|X(n) > p)$ increases with p in the sense of FOSD, RHS is always positive and finite and increases with p.

2. *Range of p:* $g(x) \geq \frac{f(p)}{1-F(p)}$ (equal only when $p = \bar{x}$), Thus RHS $\geq \frac{f(p)}{1-F(p)}$. Thus $p \leq p_M$. And since $p_M < \bar{x}$, we have $p < p_M$, that is, equlibrium price is strictly less than monopoly price.

We can also perform comparative statics with respect to the number of firms:

Proposition 2. *The equilibrium price p will strictly decrease when n increase.*

Proof.
By log-concavity of f, we know $d\ln(1-F)$ to be monotonicly decreasing. By construction of the order statistic, we know that when n increase, it decreases and dominates this statistic for a smaller n in the sense of FOSD. Thus the RHS of (2) increases unambiguously when n increase.

Now let's study the change in equlibrium price p. If p increases, the function being taken expectation becomes larger due to log-concavity. Also the distribution becomes larger in the sense of FOSD when p increases). Thus the equation will have smaller LHS and larger RHS, which will always fail.
$\Rightarrow$ p decreases when n increases.

□

References

[Perloff and Salop, 1985] Perloff, J. M. and Salop, S. C. (1985). Equilibrium with product differentiation. *The Review of Economic Studies*, 52(1):107–120.