Indirect information measure and dynamic learning *

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Abstract. I study the robust predictions of optimal dynamic learning strategy when the measure of signal informativeness is an indirect measure from sequential cost minimization. I first show that an indirect information measure is supported by sequential cost minimization *iff* it satisfies: 1) monotonicity in Blackwell order, 2) sub-additivity in compound experiments and 3) linearity in mixing with no information. In a dynamic learning problem, if the cost of information depend on an indirect information measure and delay cost is fixed, then the optimal solution involves direct Poisson signals: arrival of signals directly suggest the optimal actions, and non-arrival of signal provides no information.

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1 Introduction

Information plays a central role in economic activities. It affects both strategic interaction in games and single agent decision making under uncertainty. Information is often endogenously acquired by decision maker, as opposed to being exogenously endowed. Therefore, it is important to understand how information is acquired. This boils down to a simple trade-off: the value of information and the cost of acquiring information. The value of information is often unambiguous in a single agent decision problem with uncertainty. It is measured by the increased expected utility from choosing optimal actions measurable to informative signal realizations(see Blackwell et al. (1951)). However, there has been less consensus on the proper form of information acquisition cost. One (probably most) popular measure of informativeness being used in studying information acquisition problems is the Entropy based mutual information and its generalizations. This approached was initiated by Sims (1998, 2003), and applied to a wide range of problems (Matejka and McKay (2014), Steiner et al. (2017), Yang (2015), Gentzkow and Kamenica (2014), etc.). Despite its great theoretical tractability, Entropy based model suffers from criticism on its unrealistic implications, including prior dependence, invariant likelihood ratio of action, etc.

Two approaches can be taken to build a solid foundation for studying information acquisition. One approach is to fully characterize behavior implications associated with mutual information and its generalizations. Then we will be able to test behavior

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validity of these models. Caplin et al. (2017) takes this approach and proposes testable axioms for the Shannon model of rational inattention and its generalizations. The other approach is to impose only minimal assumptions o then cost of information and study robust predictions from an information acquisition problem. In this paper I take the second robust approach and focus on dynamic information acquisition problem: a decision maker acquires information about payoff relevant state before choosing an action. She can choose an arbitrary random process as observed information, subject to cost on information and cost on waiting.

I accomplish two main goals. First, I characterize the "minimal assumptions" on a (static) information measure if a decision maker can choose from not only all information structures but also all combinations of them. I show that an indirect information measure is supported by expected learning cost minimization— given any general measure of information, and for any information structure (Blackwell experiment), the DM minimizes expected total measure of compound experiment which replicates original information structure— if and only it satisfies three simple conditions. 1. Monotonicity: Blackwell more informative experiment has higher measure. 2. Sub-additivity: expected total measure of replicating compound experiment is higher than measure of original experiment. 3. C-linearity: mixing uninformative experiment with a proportion of informative experiment has measure proportional to measure of the informative part.

Second, I solve a dynamic information acquisition problem with those assumptions imposed on the flow information measure. I show that solving the dynamic problem can be divided into two steps. The first step is to solve a static rational inattention problem for an optimal static information structure. The second step is to solve for the optimal dynamic implementation of the solution of the first step. The optimal information process involves direct Poisson signals: signal arrives according to a Poisson counting processes and the arrival of signal suggests the optimal action directly. When no signal arrives, posterior belief process stays at prior.

Related Literature

My paper is closed related to two sets of works that aim at understanding the measure of information. The first tries to characterize implications (testable or non-testable) of commonly used information measures. Basic mathematical implications and characterizations for Entropy and Entropy based mutual information was provided in standard information theory text books like Cover and Thomas (2012). Matejka and McKay (2014) and Dean (2013) study behavior implications of rational inattention model based on Mutual information and posterior separable information measure respectively. Caplin and Dean (2015) studied implications of rational inattention model based on general information measure. A full behavior characterization for mutual information, posterior separable information cost and their generalizations is provided in Caplin et al. (2017). Meanwhile, the second seeks to build dynamic foundation for common information measures. Morris and Strack (2017) shows that the posterior separable function can be represented as induced cost from random sampling. Hébert and Woodford (2016) justifies a class of information cost function (including mutual information) based on a continuous-time sequential information acquisition problem. My paper contributes to this literature by providing a new optimization foundation for posterior separability. Posterior separability is actually equivalent to additivity in expected measure of compound experiments. I show that sub-additivity is justified by expected information cost minimization.

My paper is also closed related to the dynamic information acquisition literature, in which the main goal is to characterize the learning dynamics. A common approach in this literature is to model information flow as a simple family of random process. The decision maker can control parameters which represents aspects of interest. Wald (1947) first studies stopping problem with exogenous information process. Moscarini and Smith (2001) and Che and Mierendorff (2016) go further by edogenizing information process into optimization problem in Brownian motion framework and Poisson bandits framework to study dynamics of learning intensity and direction respectively. Some recent papers edogenize the random process family as well and give decision maker full flexibility in designing information. Zhong (2017) studies flexible dynamic information acquisition with a posterior separable information measure and shows that confirmatory Poisson signal is optimal. Steiner et al. (2017) studies a repeated rational inattention problem with mutual information as cost. My paper contributes by relaxing the restriction on information cost to only minimal assumptions. I show that when impatience is measure by fixed delay cost, dynamic problem is closely related to static rational inattention problem, and Poisson learning is robustly optimal.

The rest of the paper is structured as follows. Section 2 introduces characterization of indirect information measure based on expected cost minimization. Section 3 setups a dynamic information acquisition problem and characterizes the solution.

2 Indirect information measure

2.1 Information structure and measure of informativeness

In this subsection, I make a formal definition for "information" and for a measure of informativeness in decision making problems. I extract key factors in any abstract "information" that matters in a decision making problem and characterize a well defined equivalence class that characterizes all information structures. On the other hand, I use an "indirect information measure" characterization to derive minimal assumptions that we should impose on an information measure.

Definition 1.

1. Bayesian plausible posteriors*: Let $\Delta X \in \mathcal{R}^{|X|}$ be belief space over X. Let $\Delta^2 X$ be the space of distributions over ΔX. $\Pi(\mu) = \left\{\pi \in \Delta^2 X \middle| \int \nu \mathrm{d}\pi(\nu) = \mu\right\}$ is the set of Bayesian plausible posterior distributions. Let $\Gamma = \left\{(\pi,\mu) \in \Delta^2 X \times \Delta X \middle| \pi \in \Pi(\mu)\right\}$*

2. Information structure*: Let S be an arbitrary set (set of signals). Let $p \in \Delta S \times X$ be a conditional distribution over S on $x \in X$. (S,p) is an information structure. (S,p) can be equivalently represented as $\mathcal{S}$, a random variable whose realization is determined by p.*

I would like to study the "set" of all information structures as a choice set for decision maker. However, since S is an arbitrary set, the "set" of all possible S is not even a well-defined object from the perspective of set theory. Instead, I use $\Pi(\mu)$ to equivalently characterize the "set" of all information structures. $\forall\ (S, p)$, $\forall s \in S$, the posterior belief from observing s can be calculated according to Bayes rule. The distribution of all such posterior forms a Bayesian plausible distribution as define in Definition 1. Since different signals inducing the same posterior belief affect neither the choice of action nor the expected utility, I claim that $\Pi(\mu)$ already summarizes all possible information structures (up to the equivalent of posterior beliefs). Γ is defined as the set of all pairs (π, μ) where π represents an information structure at μ.

Definition 2 (Information measure)**.** *An information measure is a mapping $I : \Gamma \to \overline{\mathbb{R}}^+$. $I(\pi, \mu)$ can be equivalently represented by $I(\mathcal{S}; \mathcal{X}|\mu)$ where μ is the distribution of $\mathcal{X}$ and $\mathcal{S}$ induces belief distribution π.*

Information measure I is defined as a mapping from prior-information structure pairs in Γ to extended non-negative real numbers. The only restriction I put on I is that different information structures that induce same distribution of posterior π at μ have the same measure. This restriction is actually without loss of generality because a DM only cares about induced distribution of posterior of an information structure. Suppose different information structures have different measure, then the DM is always able to choose an appropriate information structure with the lowest information measure.[1] Definition 2 is the same as *information cost function* defined in Caplin and Dean (2015). The only difference is that I explicitly modeled prior dependence of I: μ is an argument in I as opposed to in Caplin and Dean (2015) prior is chosen and fixed in the beginning so there is no need to explicitly specify information cost function for different priors.

From this point on, for simplicity I represent the choice set of DM with information structures $\mathcal{S}$. However, I don't differentiate two information structures that induces same distribution of posterior beliefs. By using notation $\cdot|\mathcal{S}$, I mean conditional on posterior beliefs induced by realization of $\mathcal{S}$. Next step is to impose some restrictive assumptions on I. The restriction I impose is about comparing measure of information structure when they satisfies some information order. So first let's formally define the information order.

Definition 3 (Information processing constraint)**.** *Given random variables $\mathcal{X}, \mathcal{S}, \mathcal{T}$ and their joint distribution $p(x, s, t)$. Let $p(t|s)$, $p(t|s, x)$ be the conditional distribution defined by Bayes rule: $p(t|s) = \frac{\int p(t,s,x)\mathrm{d}x}{\int p(t,s,x)\mathrm{d}x\mathrm{d}t}$ and $p(t|s, x) = \frac{p(t,s,x)}{\int p(t,s,x)\mathrm{d}s}$ and:*

$$p(t|s, x) = p(t|s)$$

for s, x with positive probability, then the triple $\mathcal{X}, \mathcal{S}, \mathcal{T}$ is defined to form a Markov chain:

$$\mathcal{X} \to \mathcal{S} \to \mathcal{T}$$

[1] Discussing this issue formally leads to the problem of choosing inf from all possible $\mathcal{S}$, which is not a well defined set. I avoid dealing with this problem by making this restriction explicitly.

The information processing constraint in Definition 3 defines a most natural constraint in the acquisition of information: When decision time $\mathcal{T}$ is chosen based on information $\mathcal{S}$, the choice should be purely a result of information. Therefore, conditional on knowing the information, choice should not be dependent to underlying state any more. This is the key constraint I'm going to impose in Section 3. The information processing constraint has several equivalent characterizations:

Proposition 1. *The following statements are equivalent:*

1. $\mathcal{X} \to \mathcal{S} \to \mathcal{T}$.
2. $\mathcal{X}$ *and* $\mathcal{T}$ *are independent conditional on* $\mathcal{S}$.
3. $\mathcal{S}$ *is a sufficient statistics for* $\mathcal{T}$ *w.r.t.* $\mathcal{X}$.
4. $\mathcal{S}$ *is Blackwell more informative than* $\mathcal{T}$ *about* $\mathcal{X}$.

Proposition 1 comes mostly from Blackwell et al. (1951) and links the information processing constraint to other well-known notions in probability theory and information theory. It is intuitive that these notions are equivalent. They essentially all characterize the fact that $\mathcal{S}$ carries more information about $\mathcal{X}$ than $\mathcal{T}$. From this point on, I use the four equivalent notions in an inter-changeable way.

Using Definition 3, I can define what I refer to as the minimal assumptions on the measure of information.

Assumption 1. $I(\mathcal{S};\mathcal{X}|\mu)$ *satisfies the following axioms:*

1. *(Monotonicity)* $\forall \mu$, *if* $\mathcal{X} \to \mathcal{S} \to \mathcal{T}$, *then:*
$$I(\mathcal{T};\mathcal{X}|\mu) \leqslant I(\mathcal{S};\mathcal{X}|\mu)$$
2. *(Sub-additivity)* $\forall \mu$, $\forall$ *information structure* $\mathcal{S}_1$ *and information structure* $\mathcal{S}_2|_{S_1}$ *whose distribution depends on the realization of* $\mathcal{S}_1$*:*
$$I((\mathcal{S}_1,\mathcal{S}_2);\mathcal{X}|\mu) \leqslant I(\mathcal{S}_1;\mathcal{X}|\mu) + E[I(\mathcal{S}_2;\mathcal{X}|\mathcal{S}_1,\mu)]$$
3. *(C-linearity)* $\forall \mu$, $\forall$ *information structure* $\mathcal{S} \sim (\mu_i, p_i)$. $\forall \lambda < 1$, *consider* $\mathcal{S}_\lambda \sim (\mu_i, \mu, \lambda p_i, 1-\lambda)$ [2], *then:*
$$I(\mathcal{S}_\lambda;\mathcal{X}|\mu) = \lambda I(\mathcal{S};\mathcal{S}|\mu)$$

Assumption 1 imposes three properties on information measure I. Monotonicity states that if an information structure $\mathcal{S}$ is Blackwell more informative than (statistically sufficient for) information structure $\mathcal{T}$, then the information measure of $\mathcal{S}$ is no lower than that of $\mathcal{T}$. Sub-additivity states that if one breaks a combined information structure into two components sequentially, then the information measure of the combined information structure is no higher than the expected total measure of the two components. C-linearity is a strengthen of sub-additivity in a special case: if a

[2] $\mathcal{S}_\lambda$ is defined that with $1-\lambda$ probability, the posterior is identical to the prior. With the remaining λ probability, the distribution of posteriors is identical to that of $\mathcal{S}$. That is to say, $\mathcal{S}_\lambda$ is obtained by mixing $\mathcal{S}$ with a constant signal by weight $(\lambda, 1-\lambda)$.

combined information structure can be decomposed into pure randomness and an information structure, then its information measure is exactly the expected total measure of these components.

With Assumption 1, my model nests some standard measure of information. Monotonicity directly states that my information measure is consistent with the Blackwell partial order of information (Blackwell et al. (1951)). My model includes the mutual information measure used in rational inattention models (Sims (2003), Matejka and McKay (2014) etc.) Mutual information is a special case where my sub-additivity assumption is replaced by additivity and an extra logarithm structure is imposed on the information measure. In Gentzkow and Kamenica (2014) and Zhong (2017), a *posterior separable* information measure, which is more general than mutual information is used to model the cost of information. Posterior separability is a special case of additivity (see discussion in Zhong (2017)), thus a special case of sub-additivity. Generally speaking, Assumption 1 nests most information measures used in the recent "information design" literature, where information is modeled in a non-parametric way. However, it still excludes many interesting settings. For example, it's hard to verify whether Assumption 1 is satisfied in a parametric model. It also fails prior independence (see Hébert and Woodford (2016)), which is a very natural assumption when we think of information as objective experimentations.

2.2 *Information cost minimization*

If a decision maker is allowed to flexibly choose any information structure to learn, then the cost of information is captured by a general measure of information as defined in Definition 2. Consider the information measure as the cost paid by the DM. Then, if the decision maker is further allowed to choose any (sequential) combinations of a set of information structures, then she might be able to replicate a single information structure using a combination of information structures with paying a lower cost on expectation. For each single information structure, I call the minimal expected sum of information measure of any sequential replication the *Indirect information measure*. In fact, if we consider the indirect information measure as the effective measure of informativeness of information structures, then Assumption 1 is without loss of any generality:

Proposition 2. *Information measure* $I^*(\mathcal{S};\mathcal{X}|\preceq)$ *satisfies Assumption 1* iff *there exists an information measure* $I(\mathcal{S};\mathcal{X}|\mu)$ *s.t.* $\forall \mu, \mathcal{S}$:

$$I^*(\mathcal{S};\mathcal{X}|\mu) = \inf_{(\mathcal{S}^i,N)} E\left[\sum_{i=1}^{N} I\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\dots,\mathcal{S}^{i-1}\right)\right]$$
$$\text{s.t. } \mathcal{X} \to \left(\mathcal{S}^1,\dots,\mathcal{S}^N\right) \to \mathcal{S}$$

Proposition 2 states that when a DM can choose from all sequential combinations of information structures that replicate a given information structure to minimize the expected total measure, then the effective measure for a piece of information satisfies Assumption 1. The intuition for Proposition 2 is quite simple. Consider the expected information measure as a cost for information. If a Blackwell less informative information structure has a higher measure, then it is never chosen because by choosing the more informative structure, a DM can still accomplish any decision rule feasible with the less informative structure and pays a lower cost. This implies both monotonicity and sub-additivity. C-linearity is in fact an implication of sub-additivity when adding irrelevant noise to information. On the one hand, combining white noise with a information structure $\mathcal{S}$, one can create $\mathcal{S}_\lambda$, implying inequality from one direction. On the other hand, by repeatedly acquiring $\mathcal{S}_\lambda$ conditional on observing only white noise, one can replicate $\mathcal{S}$. Therefore, additivity from both direction implies C-linearity.

In practice, there are many scenarios in which such minimization of expected information measure is present. If we consider information as a product provided in a competitive market, then the minimization problem in Proposition 2 is very natural. The price of information is the marginal cost of information. And cost minimization on sellers' side implies that the price of information satisfies Assumption 1. (In a monopolistic market there will be positive markups and varying information rents so pricing might be very different, as is discussed in Zhong (2016).) Modern computer programs are designed to balance work loads from independent processes onto nodes/threads. As a result what matters is the average informational bandwidth, (as opposed to peak bandwidth or other measures). If we consider information as data processed by a computer, then in each CPU tick time, an optimally designed algorithm will minimize expected bandwidth required to process information.

3 Dynamic decision problem

In this section, I consider an application of the indirect information measure in dynamic information acquisition problems. I consider a decision maker (DM) acquiring information about the payoffs of different alternatives before making a choice. She can choose any information structure which each period, contingent on history of signals. The cost of information acquired within a period depends on an indirect information measure, and the DM pays a constant cost of delay. I find that this model justifies learning by acquiring Poisson signals.

3.1 Model

Assume that the DM faces the following dynamic information acquisition problem:

- *Decision problem*: The time horizon $t = 0, 1, \dots, \infty$ is discrete. Length of each time interval is dt. The utility associated with action-state pair (a, x) is $u(a, x)$. The DM pays a constant cost m for delaying on period. If the DM takes action $a \in A$ at time t conditional on state being $x \in X$, then her utility gain is $u(a, x) - mt$. I assume that the utility gains from actions are bounded: $\sup_{a,x} u(a, x) < \infty$.

- *Uncertainty*: Not knowing the true state, the DM forms a prior belief $\mu \in \Delta X$ about the state. Her preference under uncertainty is expressed as von Neumann-Morgenstern expected utility. I am going to use two essentially equivalent formulations to express expected utility. 1) Given belief μ, the expected utility associated with each action $a \in A$ is $E_\mu[u(a, x)]$. 2) State and action are represented by random variables $\mathcal{X}, \mathcal{A}$. Expected utility is denoted by $E[u(\mathcal{A}, \mathcal{X})]$.

- *Information Cost*: Given an information measure I defined as in Definition 2, I define a time separable information cost structure. In each period, with prior belief μ, DM pays information cost $f(I(\mathcal{S},\mathcal{X}|\mu))$ which transforms informativeness of information structure acquired in the period into utility loss. $f:\mathbb{R}^+\to\overline{\mathbb{R}}^+$ is a non-decreasing convex function which maps to extended real values.

- *Dynamic Optimization*: The dynamic optimization problem of the DM is:

$$V(\mu)=\sup_{\mathcal{S}^t,\mathcal{A}^t,\mathcal{T}} E\left[u\left(\mathcal{A}^{\mathcal{T}},\mathcal{X}\right)-m\mathcal{T}-\sum_{t=0}^{\infty}f\left(I\left(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\leqslant t}\right)\right)\right] \tag{P}$$
$$\text{s.t.}\begin{cases}\mathcal{X}\to\mathcal{S}^{t-1}\to\mathbf{1}_{\mathcal{T}\leqslant t}\\ \mathcal{X}\to\mathcal{S}^{t-1}\to\mathcal{A}^t \text{ conditional on } \mathcal{T}=t\end{cases}$$

  where $\mathcal{T}\in\Delta\mathbb{N}$, $t\in\mathbb{N}$. $\mathcal{S}^{-1}$ is defined as an uninformative variable that induces belief same as prior belief μ of the DM (just for notation simplicity). $\mathcal{S}^{\mathbf{t-1}}$ is defined as summary of past information $\left(\mathcal{S}^1,\dots,\mathcal{S}^{t-1}\right)$. The DM chooses decision time $\mathcal{T}$, choice of action conditional on action time $\mathcal{A}^t$ and signal structure $\mathcal{S}^t$ subject to information cost, waiting cost and two natural constraints for information processing:

  1. Information received in last period is sufficient for stopping in current period.
  2. Information received in last period is sufficient for action in current period. [3]

In Equation (P), the DM is modeled as choosing information flow $\mathcal{S}^t$, decision time $\mathcal{T}$ and choice of action $\mathcal{A}^t$ simultaneously, to maximize utility gain from action profile net waiting cost and total information cost. Within each period, informativeness is measured by I and incurs cost $f(I)$. Across period, information costs are aggregated by expected sum of f. Since the information measure is defined on information structure-prior pairs. It's important to define clearly how prior is determined. In each period, information measure is evaluated conditional on realization of past signals and choice of stopping. This is a natural setup since past information plus whether action is taken in current period is exactly what the DM "knows" in current period. Therefore this is the finest filter on which she evaluates information cost.

I illustrate the cost structures of dynamic information acquisition with a simple two period model: $t\in\{0,1\}$ and DM has prior belief μ. Timing is as following: when $t=0$, DM first chooses whether to take an action and which action to take. Second she decides what information to acquire. when $t=1$, DM takes action based on information acquired in period 0. First let's consider deterministic continuation decision. In period 0 no information has been acquired yet so if DM want to make a choice, her expected utility will be calculated with prior μ: $E_\mu[u(a,\mathcal{X})]$ and there is no waiting or information cost. If DM wants to collect information before decision making, she can acquire information structure $\mathcal{S}$, now it's for sure $\mathcal{T}=1$ and $\mathcal{X}\to\mathcal{S}\to\mathcal{A}$. Therefore she gets expected utility $E[u(\mathcal{A},\mathcal{X})]$, pays waiting cost m and information cost $f(I(\mathcal{S};\mathcal{X}|\mu))$.

[3] Noticing that in every period, the information in current period has not been acquired yet. So decision can only be taken based on the information already acquired in the past. So the Markov chain property on information and action time/action will have information lagged by one period. This within-period timing can be defined in different ways and it doesn't affect main results.

The problem becomes less trivial when continuation is random: suppose DM chooses to continue with probability p (independent to states because she has no information yet about state). Only conditional on continuation, she acquires $\mathcal{S}$. Within my framework, total cost is $p \cdot f(I(\mathcal{S};\mathcal{X},\mu)) + (1-p)\cdot 0$ by calculating conditional cost on $\mathbf{1}_{\mathcal{T}\leqslant 0}$. One might think that just conditional on information but not continuation decision, the same information structure is essentially $\mathcal{S}_p$ and cost is $f\big(I(\mathcal{S}_p;\mathcal{X}|\mu)\big)$. However, this is saying that when DM is choosing information after decision making in period 0, she acquires a signal correlated to her previous choice of continuation. This piece of randomness (whether to continue) is already resolved. Since our DM can not revert time, this case is physically impossible.

3.2 Solution

In this section, I solve the dynamic information acquisition problem Equation (P) under Assumption 1 on the information measure. First, I characterize the optimal expected utility as a solution to a simple static information acquisition problem. Second, I provide a simple stationary strategy that implements the expected utility of any information and action strategy in the equivalent static problem.

Theorem 1. *If I satiesfies Assumption 1, $\forall \mu \in \Delta X$, suppose expected utility level $V(\mu)$ solves Equation (P), then:*

$$V(\mu) = \max\left\{\sup_{a\in A} E[u(a,\mathcal{X})],\ \sup_{I(\mathcal{A};\mathcal{X}|\mu)\geqslant\lambda} E[u(\mathcal{A},\mathcal{X})] - \left(\frac{m}{\lambda}+\frac{f(\lambda)}{\lambda}\right) I(\mathcal{A};\mathcal{X}|\mu)\right\} \tag{1}$$

The first superemum is taken over a, the second superemum is take over λ and $\mathcal{A}$.

Theorem 1 establishes that solving the optimal utility level in Equation (P) is essentially a static problem with Assumption 1. In the static problem, DM pays a fixed marginal cost $\left(\frac{m}{\lambda}+\frac{f(\lambda)}{\lambda}\right)$ on each unit of information measure $I(\mathcal{A};\mathcal{X}|\mu)$. Notice that the optimal parameter λ depends on only m, f when the constraint $I(\mathcal{A};\mathcal{X}|\mu) \geqslant \lambda$ doesn't bind. There is an explicit algorithm to solve Equation (1):

Proposition 3. *If I satisfies Assumption 1, $V(\mu)$ solves Equation (P) if and only if it solves the following problem: Let $\lambda^* = \sup\{\lambda \in \mathbb{R}^+ | m + f(\lambda) > \lambda \cdot \partial f(\lambda)\}$ and solve for*

$$V^0(\mu) = \sup_{a\in A} E[u(a,\mathcal{X})]$$

$$V^1(\mu) = \sup_{\mathcal{A}} E[u(\mathcal{A},\mathcal{X})] - \left(\frac{m}{\lambda^*}+\frac{f(\lambda^*)}{\lambda^*}\right) I(\mathcal{A};\mathcal{X}|\mu) \tag{2}$$

$$V^2(\mu) = \sup_{\mathcal{A}} E[u(\mathcal{A},\mathcal{X})] - m - f(I(\mathcal{A};\mathcal{X}|\mu)) \tag{3}$$

Let $\mathbb{A}$ be the set of solutions of Equation (2)[4]*, then*

$$V(\mu) = \begin{cases} \max\left\{V^0(\mu), V^1(\mu)\right\} & \text{if } \sup_{\mathcal{A}\in\mathbb{A}} I(\mathcal{A};\mathcal{X}|\mu) \geqslant \lambda^* \\ \max\left\{V^0(\mu), V^2(\mu)\right\} & \text{otherwise} \end{cases}$$

[4] If $\lambda^* = +\infty$, define $\mathbb{A} = \varnothing$. Here $\mathbb{A}$ includes both $\mathcal{A}$'s that exactly solve Equation (2) and sequences $\{\mathcal{A}^i\}$ that approach Equation (2). Given a sequence $\{\mathcal{A}^i\} \in \mathbb{A}$, $I(\mathcal{A};\mathcal{X}|\mu)$ is defined as $\limsup I(\mathcal{A}^i,\mathcal{X}|\mu)$

Proposition 3 states that value function in Equation (P) can be solved by solving three static problems. The first value V^0 is a no-information benchmark when value equals expected utility from choosing optimal action according to prior. The second problem Equation (2) is a standard rational inattention problem with marginal cost $\frac{m}{\lambda^*} + \frac{f(\lambda^*)}{\lambda^*}$ on information measure I. The interpretation is that under Assumption 1, the dynamic information acquisition problem is separable in two parts. First part is the dynamic allocation of information, keeping total information fixed. Marginal cost of increasing total information is reflected by $\frac{m}{\lambda^*} + \frac{f(\lambda^*)}{\lambda^*}$, which measures impatience and the smoothing incentive jointly. Second part is a static problem that optimizes total information. The third problem is a special case when there is under-smoothing. This happens only when waiting is so costly that it is optimal for decision maker to scale up information cost and wait for less than one period. Since fractional period length is not feasible, in this case decision maker solves a one-period problem.

Once the static problems Equations (2) and (3) are solved, let $\mathcal{A}$ be an optimal information structure of the static problem, then $\mathcal{A}$ can be modified to construct an optimal dynamic information structure in Equation (P).

Proposition 4. *If I satisfies Assumption 1, $\forall \mu \in \Delta X$, $\mathcal{A} \in \Delta A \times X$ and $\lambda^* < I(\mathcal{A};\mathcal{X}|\mu)$, let $(\mathcal{S}^t, \mathcal{A}^t, \mathcal{T})$ be defined by*[5]*:*

1. $\mathcal{S}^{-1} = c_0$.
2. $$\mathcal{S}^t = \begin{cases} s_0 & \text{if } \mathcal{S}^{t-1} \in A \bigcup \{s_0\} \\ \mathcal{A} \text{ with probability } \frac{\lambda^*}{I(\mathcal{A};\mathcal{X}|\mu)} & \text{if } \mathcal{S}^{t-1} = c_0 \\ c_0 \text{ with probability } 1 - \frac{\lambda^*}{I(\mathcal{A};\mathcal{X}|\mu)} & \text{if } \mathcal{S}^{t-1} = c_0 \end{cases}$$
3. $$\begin{cases} \mathcal{T} = t \\ \mathcal{A}^t = \mathcal{S}^{t-1} \end{cases} \quad \text{if } \mathcal{S}^{t-1} \in A.$$

Then:

$$E[u(\mathcal{A},\mathcal{X})] - \left(\frac{m}{\lambda^*} + \frac{f(\lambda^*)}{\lambda^*}\right) I(\mathcal{A};\mathcal{X}|\mu) = E\left[u\left(\mathcal{A}^{\mathcal{T}},\mathcal{X}\right) - m\mathcal{T} - \sum_{t=0}^{\infty} f\left(I(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T}\leqslant t})\right)\right]$$

Proposition 4 complements Theorem 1 by showing that the optimal value from Equation (1) can be implemented using a simple stationary experimentation strategy feasible in Equation (P). The information structure $\mathcal{S}^t$ explicitly codes three kinds of signals: *Stop* s_0, *Wait* c_0 and *Action* in A. The first condition defines the initial information. The second condition defines the information structures in following periods by induction: If $\mathcal{S}^{t-1} = s_0$ or $\mathcal{A}$ it means that action is already taken and information acquisition stops from now on so $\mathcal{S}^t = s_0$ and so on so forth. If $\mathcal{S}^{t-1} = c_0$ it means that do nothing and delay all decision to the current period. Conditional on continuation, $\mathcal{S}^t$ realizes as $\mathcal{A}$ with $\frac{\lambda^*}{I(\mathcal{A};\mathcal{X}|\mu)}$ probability. And in the next period action is taken according to realization of $\mathcal{S}^t$. With $1 - \frac{\lambda^*}{I(\mathcal{A};\mathcal{X}|\mu)}$ probability c_0 realizes and the decision

[5] s_0 and c_0 are chosen to be distinguishable from any element in action set A.

is delayed to the next period. The Third condition explicitly defines $\mathcal{T}$: when action is taken in period t as indicated by $\mathcal{S}^{t-1}$, then $\mathcal{T} = t$. It's easy to verify the information processing constraints in Equation (P). First, conditional on $\mathcal{S}^{t-1}$, the distribution of $\mathbf{1}_{\mathcal{T}\leqslant t}$ is degenerate. When $\mathcal{S}^{t-1} = c_0$ it's 0 and 1 otherwise. So $\mathcal{S} \to \mathcal{S}^{t-1} \to \mathbf{1}_{\mathcal{T}} \leqslant t$. Second, conditional on $\mathcal{S}^{t-1}$ and knowing $\mathcal{T} = t$, $\mathcal{A}^t$ is also degenerate. It is exactly the realization of $\mathcal{S}^{t-1}$. Therefore $\mathcal{X} \to \mathcal{S}^{t-1} \to \mathcal{A}^t$.

Sketched proof.

Here I provide a simplified proof which illustrates the main intuition for Theorem 1 and Proposition 4. Since there is no discounting on the utility gain from actions, given an action profile $\mathcal{A}^{\mathcal{T}}$, the deterministic factors for expected utility is 1) the combined distribution of all actions $\mathcal{A}$. 2) the expected waiting time $E[\mathcal{T}]$. How actions are allocated over time doesn't affect the expected utility at all. Since actions are driven by information, this observation indicates that solving Equation (P) can be divide into three steps: Step 1 is to solve for the optimal distribution of information over time to minimize information cost given any combined information structure and expected waiting time. Step 2 is to solve for the optimal waiting time given any fixed combined information structure. Step 3 is to solve for the optimal combined information structure and associated action profile.

Step 1. Given any strategy $\left(\mathcal{S}^t, \mathcal{A}^t, \mathcal{T}\right)$, the DM can implement action distribution $\mathcal{A}^{\mathcal{T}}$ and expected waiting time $E[\mathcal{T}]$ with a better design of information. First, consider combining all information $\mathcal{S} = \left(\mathcal{S}^1, \dots, \mathcal{S}^t, \dots\right)$. By sub-additivity $I(\mathcal{S};\mathcal{X}|\mu) \leqslant \sum E\left[I\left(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right]$. Then consider averaging $I(\mathcal{S};\mathcal{X}|\mu)$ into $E[\mathcal{T}]$ periods:

$$
\begin{aligned}
\frac{I(\mathcal{S};\mathcal{X}|\mu)}{E[\mathcal{T}]} &\leqslant \frac{\sum E\left[I\left(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right]}{E[\mathcal{T}]}\\
\Longrightarrow f\left(\frac{I(\mathcal{S};\mathcal{X}|\mu)}{E[\mathcal{T}]}\right) &\leqslant \frac{\sum E\left[f\left(I\left(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right)\right]}{E[\mathcal{T}]}\\
\Longrightarrow E[\mathcal{T}]f\left(\frac{I(\mathcal{A};\mathcal{X}|\mu)}{E[\mathcal{T}]}\right) &\leqslant \sum E\left[f\left(I\left(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right)\right]
\end{aligned}
$$

Second inequality is first by monotonicity of f then by convexity of f. That's to say: there is incentive to combine small information (by sub-additivity of I) and smooth information over time (by convexity of f). Last inequality is from $I(\mathcal{S};\mathcal{X}|\mu) \leqslant I(\mathcal{A};\mathcal{X}|\mu)$ by Blackwell monotonicity of I. Then an ideal strategy is to spend $f\left(\frac{I(\mathcal{A};\mathcal{X}|\mu)}{E[\mathcal{T}]}\right)$ on information acquisition every period.

Then let's implement the aforementioned information cost using a strategy defined as in Proposition 4. By C-linearity, acquiring $\mathcal{A}$ with probability $\frac{1}{E[\mathcal{T}]}$ exactly has cost $f\left(\frac{I(\mathcal{A};\mathcal{X}|\mu)}{E[\mathcal{T}]}\right)$. On the other hand, taking action with probability $\frac{1}{E[\mathcal{T}]}$ in each period exactly implements combined action distribution $\mathcal{A}$ and expected waiting time $E[\mathcal{T}]$. Then it's WLOG to consider:

$$
\sup_{\mathcal{A},T} E[u(\mathcal{A},\mathcal{X})] - mT - Tf\left(\frac{I(\mathcal{A};\mathcal{X}|\mu)}{T}\right)
$$

Step 2. Maximizing over $E[\mathcal{T}]$ (or T in the simplified problem). This can be done easily by solving first order condition w.r.t. T: $-m - f\left(\frac{I}{T}\right) + \frac{I}{T}f'\left(\frac{I}{T}\right) = 0$. Replace

$\lambda = \frac{I}{T}$, we get the expression for λ: $m + f(\lambda) = \lambda f'(\lambda)$ and further simplified problem:

$$\sup_{\mathcal{A}} E[u(\mathcal{A}, \mathcal{X})] - \left(\frac{m}{\lambda} + \frac{f(\lambda)}{\lambda}\right) I(\mathcal{A}; \mathcal{X}|\mu)$$

The theorem deals with general case without smoothness assumption so f' is replaced with sub-differentials ∂f.

Step 3. I will refer to Wierestrass theorem to show existence of solution. See Proposition 5 for detailed discussion.

In the sketched proof I implicitly assumed f to be differentiable, first order condition has solution and optimal $T \geqslant 1$. The formal proof for more general cases is provided in Appendix B.1

3.3 Existence and uniqueness

In this section, I first show a general existence result for the solution of Equations (2) and (3). Then I established its uniqueness in different dimensions. By toggling inequality in defining monotonicity of I, concavity of g and sub-additivity of I to strict inequality, my model predicts unique belief profile, unique information cost allocation and unique strategy correspondingly.

Proposition 5. *If A, X are finite sets, I satisfies Assumption 1, then*

- *Existence: $\forall \varepsilon > 0, \nabla_\varepsilon = \{\mathcal{A}|\mathrm{P}[a|x] \geqslant \varepsilon\}$, then there exists a non-empty, convex and compact set of solution $\mathbb{A}_\varepsilon$ to Equation (1) subject to $\mathcal{A} \in \nabla_\varepsilon$.*
  - *If $\mathbb{A}_\varepsilon \not\subset \partial\nabla_\varepsilon$, then $\bigcup_{\varepsilon' \geqslant \varepsilon} \mathbb{A}_{\varepsilon'}$ is solution to Equation (1).*
  - *If $\mathbb{A}_\varepsilon \subset \partial\nabla_\varepsilon \ \forall \varepsilon > 0$, then any sequence in $\prod \nabla_\varepsilon$ approaches $V(\mu)$.*
- *Uniqueness:*
  - *If I satisfies strict-monotonicity, then posterior belief $\nu(a)$ associated with any action a is unique for all optimal $\mathcal{A}$.*
  - *If $f(\cdot)$ satisfies strict-convexity, then $\forall$ optimal strategy $(\mathcal{S}^t, \mathcal{A}^t, \mathcal{T})$ to Equation (P), $I(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} \leqslant t})$ is constant.*
  - *If I satisfies strict-sub-additivity, then solution $(\mathcal{S}^t, \mathcal{A}^t, \mathcal{T})$ to Equation (P) is unique.*

Proposition 5 establishes the existence of solution to Equation (1) and the uniqueness of different aspects of the solution. First, with Assumption 1, very mild extra assumptions (finite A and X) can guarantee existence of solution to Equation (1) (and solution to Equation (P) as well). Second, when strictly more informative information structure has strictly larger information measure, the belief inducing each action can be uniquely pinned down by optimization. Third, when information cost function f is strictly convex, then cost incurred in each experimentation period is constant over time. Finally, if combination of informative experiments has strictly larger measure than expected summation of components' measures, then whole dynamic strategy is uniquely pinned down.

The existence result is non-trivial in the sense that I don't impose any continuity assumption on I. However, I being an indirect information measure function actually guarantees it to be convex in an appropriate space. In Equation (1), the strategy space is all random variable $\mathcal{A}$. If we consider the space of all conditional distribution over A on X, then this is an Euclidean space and any indirect information measure I is a convex function on this space: if $\mathcal{S}$ is a linear combination of $\mathcal{S}_1$ and $\mathcal{S}_2$, then $\mathcal{S}$ can be implemented as randomly using $\mathcal{S}_1$ or $\mathcal{S}_2$ (and not knowing the choice of experiment). Therefore, monotonicity and sub-additivity guarantees $\mathcal{S}$ to have lower measure than linear combination of measures of $\mathcal{S}_1$, $\mathcal{S}_2$. Convexity of I implies both objective function to be continuous and choice set to be compact on any interior closed subset of the strategy space.

The incentive for inter-temporal smoothing of information is clearly illustrated in proof of Proposition 4 and Theorem 1: Convexity of information cost f implies incentive to smooth the cost over time. Sub-additivity of I implies incentive to smooth the choice of information structure over time. The incentive for choice of aggregate information structure is illustrated in the proof of existence: monotonicity and sub-additivity implies a concave objective function. Now if any of aforementioned incentives is strict, then the solution is uniquely pinned down in the corresponding aspect. First, consider the proof for convexity of I in the last part. Randomly using S_1 or S_2 (and knowing choice of experiment) carries strictly more information than $\mathcal{S}$ (which discards information about which experiment is used). Therefore, strict monotonicity implies that he objective function is strict concave (except when $\mathcal{S}_1$ and $\mathcal{S}_2$ have the same row vectors). Second, consider step 1 in the proof of Theorem 1. Suppose f is strictly convex, whenever information cost is not constant over time, the total cost is strictly dominated by a stationary strategy. Third, when there is strict sub-additivity, then any non-stationary experimentation strategy is dominated by the stationary one I constructed. Moreover, the objective function in Equation (1) is strictly concave w.r.t any $\mathcal{A}$. In this case, the whole solution is uniquely pinned down.

4 Conclusion

In this paper, I explore the robust predictions we can make when the measure of signal informativeness is an indirect measure from sequential cost minimization. I first show that an indirect information measure is supported by sequential cost minimization *iff* it satisfies: 1) monotonicity in Blackwell order, 2) sub-additivity in compound experiments and 3) linearity in mixing with no information. In a sequential learning problem, if the cost of information depend on an indirect information measure and delay cost is fixed, then the optimal solution involves direct Poisson signals: arrival of signal directly suggests the optimal action, and non-arrival of signal provides no information. I also characterize the existence and uniqueness of the optimal learning dynamics.

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Contents

A Proof in Section 2

A.1 Proof for Proposition 2

Proof. (Necessity) First suppose $I^*(\mathcal{S};\mathcal{X}|\mu)$ satisfies Assumption 1. Then choose I^* itself as I. $\forall\mu$ and $\mathcal{S}$. $\forall\mathcal{X} \to (\mathcal{S}^1,\ldots,\mathcal{S}^N) \to \mathcal{S}$:

$$
\begin{aligned}
E\left[\sum_{i=1}^{N} I^*\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] \geqslant & I^*\left(\left(\mathcal{S}^1,\ldots\mathcal{S}^N\right);\mathcal{X}|\mu\right) \\
\geqslant & I^*(\mathcal{S};\mathcal{X}|\mu) \\
\Longrightarrow \inf_{(\mathcal{S}^i,N)} E\left[\sum_{i=1}^{N} I^*\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] \geqslant & I^*(\mathcal{S};\mathcal{X}|\mu)
\end{aligned}
$$

First inequality is from sub-additivity. Second inequality is from monotonicity. On the other hand, let $\mathcal{S}^1 = \mathcal{S}$, $N = 1$, then

$$
\begin{aligned}
E\left[\sum_{i=1}^{N} I^*\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] = I^*(\mathcal{S};\mathcal{X}|\mu) \\
\Longrightarrow \inf_{(\mathcal{S}^i,N)} E\left[\sum_{i=1}^{N} I^*\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] \leqslant I^*(\mathcal{S};\mathcal{X}|\mu)
\end{aligned}
$$

Combining the two direction of inequality:

$$
\inf_{(\mathcal{S}^i,N)} E\left[\sum_{i=1}^{N} I^*\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] = I^*(\mathcal{S};\mathcal{X}|\mu)
$$

(Sufficiency) On the other hand, suppose given $I(\mathcal{S};\mathcal{X}|\mu)$,

$$
\begin{aligned}
I^*(\mathcal{S};\mathcal{X}|\mu) = & \inf_{(\mathcal{S}^i,N)} E\left[\sum_{i=1}^{N} I\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] \\
& \text{s.t. } \mathcal{X} \to \left(\mathcal{S}^1,\ldots,\mathcal{S}^N\right) \to \mathcal{S}
\end{aligned}
$$

Then

0. *Uninformative signal*: First it's not hard to observe that acquiring no informaiton is sufficient for an uninformative signal $\mathcal{S}$. Therefore if choose $N = 0$ we have, $0 \geqslant I^*(\mathcal{S};\mathcal{X}|\mu)$. Then:

$$
I^*(\mathcal{S};\mathcal{X}|\mu) = 0
$$

1. *Monotonicity*: $\forall(\mathcal{S}^i)$ s.t. $\mathcal{X} \to (\mathcal{S}^1,\ldots,\mathcal{S}^N) \to \mathcal{S}$. Since $\mathcal{X} \to \mathcal{S} \to \mathcal{T}$, we have $\mathcal{X} \to (\mathcal{S}^1,\ldots,\mathcal{S}^N) \to \mathcal{T}$. Therefore:

$$
\inf_{(\mathcal{T}^i,N)} E\left[\sum_{i=1}^{N} I\left(\mathcal{T}^i;\mathcal{X}|\mathcal{T}^1,\ldots,\mathcal{T}^{i-1}\right)\right]
$$

$$
\begin{aligned}
&\text{s.t. } \mathcal{X} \to \left(\mathcal{T}^1,\ldots,\mathcal{T}^N\right) \to \mathcal{T} \\
&\leqslant E\left[\sum_{i=1}^{N} I\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] \\
\Longrightarrow\ & \inf_{(\mathcal{T}^i,N)} E\left[\sum_{i=1}^{N} I\left(\mathcal{T}^i;\mathcal{T}|\mathcal{T}^1,\ldots,\mathcal{T}^{i-1}\right)\right] \\
&\text{s.t. } \mathcal{X} \to \left(\mathcal{T}^1,\ldots,\mathcal{T}^N\right) \to \mathcal{T} \\
&\leqslant \inf_{(\mathcal{S}^i,N)} E\left[\sum_{i=1}^{N} I\left(\mathcal{S}^i;\mathcal{X}|\mathcal{S}^1,\ldots,\mathcal{S}^{i-1}\right)\right] \\
&\text{s.t. } \mathcal{X} \to \left(\mathcal{S}^1,\ldots,\mathcal{S}^N\right) \to \mathcal{S} \\
\Longrightarrow\ & I^*(\mathcal{T};\mathcal{X}|\mu) \leqslant I^*(\mathcal{S};\mathcal{X}|\mu)
\end{aligned}
$$

First inequality comes from that factor that $(\mathcal{S}^i)$ serves as one feasible group of $(\mathcal{T}^i)$ in the minimization. Second inequality comes from taking inf on RHS. Final inequality comes from definition of I^*.

2. *Sub-additivity*: Suppose $\mathcal{S} = (\mathcal{S}_1,\mathcal{S}_2)$. $\forall\left(\mathcal{S}_1^1,\ldots,\mathcal{S}_1^{N_1}\right)$ s.t. $\mathcal{X} \to \left(\mathcal{S}_1^1,\ldots,\mathcal{S}_1^{N_1}\right) \to \mathcal{S}_1$. $\forall\left(\mathcal{S}_2^1,\ldots,\mathcal{S}_2^{N_2}\right)$ conditinal on $\mathcal{S}_1$ s.t. $\forall$ realization of $\mathcal{S}_1$, $\mathcal{X} \to \left(\mathcal{S}_2^1,\ldots,\mathcal{S}_2^{N_2}\right) \to \mathcal{S}_2$. Therefore:

$$
\begin{aligned}
&\mathcal{X} \to \left(\mathcal{S}_1^1,\ldots,\mathcal{S}_1^{N_1},\mathcal{S}_2^1,\ldots,\mathcal{S}_2^{N_2}\right) \to (\mathcal{S}_1,\mathcal{S}_2) \to \mathcal{S} \\
\Longrightarrow\ & I^*(\mathcal{S};\mathcal{X}|\mu) \leqslant E\left[\sum_{i=1}^{N_1} I(\mathcal{S}_1^i;\mathcal{S}|\mathcal{S}_1^1,\ldots,\mathcal{S}_1^{i-1})\right] + E\left[\sum_{i=1}^{N_2} I(\mathcal{S}_2^i;\mathcal{X}|\mathcal{S}_1,\mathcal{S}_2^1,\ldots,\mathcal{S}_2^{i-1})\right] \\
\Longrightarrow\ & I^*(\mathcal{S};\mathcal{X}|\mu) \leqslant \inf E\left[\sum_{i=1}^{N_1} I(\mathcal{S}_1^i;\mathcal{S}|\mathcal{S}_1^1,\ldots,\mathcal{S}_1^{i-1})\right] + \inf E\left[\sum_{i=1}^{N_2} I(\mathcal{S}_2^i;\mathcal{X}|\mathcal{S}_1,\mathcal{S}_2^1,\ldots,\mathcal{S}_2^{i-1})\right] \\
\Longrightarrow\ & I^*(\mathcal{S};\mathcal{X}|\mu) \leqslant I^*(\mathcal{S}_1;\mathcal{X}|\mu) + E[I^*(\mathcal{S}_2;\mathcal{X}|\mathcal{S}_1,\mu)]
\end{aligned}
$$

3. *C-linearity*: $\forall\mathcal{S}$, consider $\mathcal{S}^1 = (\{0,1\},\lambda,1-\lambda)$ being an uninformative binary signal. $\mathcal{S}^2 = \mathcal{S}$ when $\mathcal{S}^1 = 0$ and constant when $\mathcal{S}^1 = 1$. Therefore $\left(\mathcal{S}^1,\mathcal{S}^2\right) = \mathcal{S}_\lambda$. By sub-additivity:

$$
I^*(\mathcal{S}_\lambda;\mathcal{X}|\mu) \leqslant \lambda I^*(\mathcal{S};\mathcal{X}|\mu)
$$

On the other hand, consider $\mathcal{S}^1$ conditional on $\mathcal{S}_\lambda$. If $\mathcal{S}_\lambda$ induces $\nu \neq \mu$, then $\mathcal{S}^1$ is uninformative. Otherwise $\mathcal{S}^1 = \mathcal{S}$. Then $\left(\mathcal{S}_\lambda,\mathcal{S}^1\right) = \mathcal{S}$, by sub-additivity:

$$
\begin{aligned}
& I^*(\mathcal{S};\mathcal{X}|\mu) \leqslant I^*(\mathcal{S}_\lambda;\mathcal{X}|\mu) + (1-\lambda)I^*(\mathcal{S};\mathcal{X}|\mu) \\
\Longrightarrow\ & \lambda I^*(\mathcal{S};\mathcal{X}|\mu) \leqslant I^*(\mathcal{S}_\lambda;\mathcal{X}|\mu)
\end{aligned}
$$

To sum up, $\lambda I^*(\mathcal{S};\mathcal{X}|\mu) = I^*(\mathcal{S}_\lambda;\mathcal{X}|\mu)$.

Q.E.D.

B Proof in Section 3

B.1 Proof for Theorem 1

Proof. Let $V(\mu)$ be expected utility in Equation (P). Then by assumption $V(\mu) \geqslant 0$. Suppose $V(\mu) = 0$, then Theorem 1 is straight forward. $V(\mu)$ is achieved by choosing doing nothing and acquiring no information. From now on, we assume $V(\mu) > 0$. Pick any $\varepsilon < V(\mu)$, we want to show that there exists $\mathcal{A}, T$ s.t.:

$$V(\mu) - \varepsilon \leqslant E[u(\mathcal{A}, \mathcal{X})] - mT - Tf\left(\frac{I(\mathcal{A}; \mathcal{X}|\preceq)}{T}\right)$$

Suppose $(\mathcal{S}^t, \mathcal{A}^t, \mathcal{T})$ solves Equation (P) approches $V(\mu)$ up to $\frac{\varepsilon}{2}$:

$$V(\mu) - \frac{\varepsilon}{2} \leqslant E\left[u\left(\mathcal{A}^{\mathcal{T}}, \mathcal{X}\right) - m\mathcal{T} - \sum_{t=0}^{\infty} f\left(I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T}\leqslant t}\right)\right)\right]$$
$$\text{where } \begin{cases} \mathcal{X} \to \mathcal{S}^{t-1} \to \mathbf{1}_{\mathcal{T}\leqslant t} \\ \mathcal{X} \to \mathcal{S}^{t-1} \to \mathcal{A}^t \text{ conditional on } \mathcal{T} = t \end{cases}$$

Lemma B.1 (Lemma 7 in Zhong (2017)) shows that we can assume the signal structure WLOG takes the following form:

$$\mathcal{S}^t = \begin{cases} s_0 & \text{when } \mathcal{T} \leqslant t \\ \mathcal{A}^{t+1} & \text{when } \mathcal{T} = t+1 \end{cases}$$

Therefore, $\mathcal{A}^{t+1}$,$\mathbf{1}_{\mathcal{T}\leqslant t}$ and $\mathbf{1}_{\mathcal{T}=t+1}$ are all explicitly signal realizations included in $\mathcal{S}^t$. We discuss two cases separately:

Case 1. $E[\mathcal{T}] \geqslant 1$: Consider $\mathcal{S}^{\mathbf{T}} = (\mathcal{S}^0, \mathcal{S}^1, \cdots, \mathcal{S}^T)$ as a combined information structure of all signals in first T periods. By sub-additivity in Assumption 1:

$$\begin{aligned}
&E\left[\sum_{t=0}^{\infty} I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T}\leqslant t}\right)\right] \\
=&E\left[\sum_{t=0}^{\infty} I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right] \\
=&I\left(\mathcal{S}^0; \mathcal{X}|\mu\right) + E\left[\sum_{t=1}^{\infty} I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right] \\
=&I\left(\mathcal{S}^0; \mathcal{X}|\mu\right) + E\left[I\left(\mathcal{S}^1; \mathcal{X}|\mathcal{S}^0\right)\right] + E\left[\sum_{t=2}^{\infty} I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right] \\
\geqslant&I\left(\mathcal{S}^{\mathbf{1}}; \mathcal{X}|\mu\right) + E\left[\sum_{t=2}^{\infty} I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right] \\
\geqslant&\dots \\
\geqslant&I\left(\mathcal{S}^{\mathbf{T}}; \mathcal{X}|\mu\right) + E\left[\sum_{t=T+1}^{\infty} I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right]
\end{aligned}$$

$$\Longrightarrow \sum_{t=0}^{\infty} E\left[f\left(I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right)\right] \geqslant I\left(\mathcal{S}^{\mathbf{T}}; \mathcal{X}|\mu\right) \ \forall T \tag{4}$$

Now consider:

$$\begin{aligned}
&\sum_{t=0}^{\infty} E\left[I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)\right] \\
=&\sum_{t=0}^{\infty}\left(\text{Prob}(\mathcal{T} \leqslant t)E\left[I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)|\mathcal{T} \leqslant t\right] + \text{Prob}(\mathcal{T} > t)E\left[I\left(\mathcal{S}^t; \mathcal{X}|\mathcal{S}^{\mathbf{t-1}}\right)|\mathcal{T} > t\right]\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{\infty} \left(\text{Prob}(\mathcal{T} \leqslant t) E\left[I(s_0; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}) | \mathcal{T} \leqslant t \right] + \text{Prob}(\mathcal{T} > t) E\left[I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}) | \mathcal{T} > t \right] \right) \\
&= \sum_{t=0}^{\infty} \text{Prob}(\mathcal{T} > t) E\left[I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}) | \mathcal{T} > t \right] \qquad (5)
\end{aligned}
$$

Since $V(\mu) - \varepsilon > 0$, then:

$$
\begin{aligned}
& m \cdot E[\mathcal{T}] \leqslant \max v \\
\Longrightarrow\ & \text{Prob}(\mathcal{T} > T) \cdot T \cdot m \leqslant \max v \\
\Longrightarrow\ & \text{Prob}(\mathcal{T} > T) \leqslant \frac{\max v}{mT} \\
\Longrightarrow\ & \text{Prob}(\mathcal{T} \leqslant T) E\left[u\left(\mathcal{A}^{\mathcal{T}}, \mathcal{X} \right) | \mathcal{T} \leqslant t \right] \geqslant E\left[u\left(\mathcal{A}^{\mathcal{T}}, \mathcal{X} \right) \right] - \text{Prob}(\mathcal{T} > T) \cdot \max v \\
\Longrightarrow\ & \text{Prob}(\mathcal{T} \leqslant T) E\left[u\left(\mathcal{A}^{\mathcal{T}}, \mathcal{X} \right) | \mathcal{T} \leqslant t \right] \geqslant E\left[u\left(\mathcal{A}^{\mathcal{T}}, \mathcal{X} \right) \right] - \frac{\max v^2}{mT}
\end{aligned}
$$

Choose $T + 1 > \frac{m\varepsilon}{\max v^2}$. Now combine Equations (4) and (5), and $\sum_{t=0}^{\infty} \text{Prob}(\mathcal{T} > t) = E[\mathcal{T}]$, then we have:

$$
\begin{aligned}
& I(\mathcal{S}^{\mathbf{T}}; \mathcal{X} | \mu) \leqslant \sum_{t=0}^{\infty} \text{Prob}(\mathcal{T} > t) E[I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}) | \mathcal{T} > t] \\
\Longrightarrow\ & \frac{I(\mathcal{S}^{\mathbf{T}}; \mathcal{X} | \mu)}{E[\mathcal{T}]} \leqslant \sum_{t=0}^{\infty} \frac{\text{Prob}(\mathcal{T} > t)}{\sum \text{Prob}(\mathcal{T} > \tau)} E[I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}) | \mathcal{T} > t] \\
\Longrightarrow\ & f\left(\frac{I(\mathcal{S}^{\mathbf{T}}; \mathcal{X} | \mu)}{E[\mathcal{T}]} \right) \leqslant \sum_{t=0}^{\infty} \frac{\text{Prob}(\mathcal{T} \geqslant t)}{\sum \text{Prob}(\mathcal{T} > \tau)} f(E[I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}) | \mathcal{T} > t]) \\
\Longrightarrow\ & E[\mathcal{T}] f\left(\frac{I(\mathcal{S}^{\mathbf{T}}; \mathcal{X} | \mu)}{E[\mathcal{T}]} \right) \leqslant \sum_{t=0}^{\infty} \text{Prob}(\mathcal{T} > t) E\left[f(I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} > t})) \right] \\
\Longrightarrow\ & E[\mathcal{T}] f\left(\frac{I(\mathcal{S}^{\mathbf{T}}; \mathcal{X} | \mu)}{E[\mathcal{T}]} \right) \leqslant E\left[\sum_{t=0}^{\infty} f(I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} \leqslant t})) \right]
\end{aligned}
$$

Consider $\mathcal{A}^{\mathbf{T+1}} = \left(\mathcal{A}^0, \mathcal{A}^2, \cdots, \mathcal{A}^{T+1} \right)$ as a random variable which summarizes realizations of all $\mathcal{A}^t$. Since $\mathcal{A}^{t+1}$ are directly included in $\mathcal{S}^t$, we have $\mathcal{X} \to \mathcal{S}^{\mathbf{T}} \to \mathcal{A}^{\mathbf{T+1}}$. Therefore, by Assumption 1:

$$
\begin{aligned}
& I\left(\mathcal{A}^{\mathbf{T}+1}; \mathcal{X} | \mu \right) \leqslant I(\mathcal{S}^{\mathbf{T}}; \mathcal{X} | \mu) \\
\Longrightarrow\ & E[\mathcal{T}] f\left(\frac{I(\mathcal{A}^{\mathbf{T+1}}; \mathcal{X} | \mu)}{E[\mathcal{T}]} \right) \leqslant E\left[\sum_{t=0}^{\infty} f(I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}})) \right]
\end{aligned}
$$

That's to say, if we can implement $\mathcal{A}^{\mathbf{T+1}}$ with expected waiting time $E[\mathcal{T}]$ and information cost $E[\mathcal{T}] f\left(\frac{I(\mathcal{A}^{\mathbf{T+1}}; \mathcal{X} | \mu)}{E[\mathcal{T}]} \right)$, then utility level will be weakly higher than V. We define the new strategy as follows:

1. In each period, acquire a combined information structure by mixing $\mathcal{A}^{\mathbf{T+1}}$ with probability $\frac{1}{E[\mathcal{T}]}$ and uninformative signal structure with probability $1 - \frac{1}{E[\mathcal{T}]}$.
2. Following arrival of signal $\mathcal{A}^{\mathbf{T+1}}$, choosing the corresponding action.
3. If no informative signal arrive, do nothing and go to next period.

It's not hard to see that in this strategy, action and signal are identical thus the three information processing constraint are naturally satisfied. In each period, the probabiltiy of decision making is $\frac{1}{E[\mathcal{T}]}$ and the distribution of actions is $\mathcal{A}^{\mathbf{T+1}}$. Therefore, totally utility gain is:

$$\sum_{t=0}^{\infty}\left(1-\frac{1}{E[\mathcal{T}]}\right)^{t}\frac{1}{E[\mathcal{T}]}E[u(\mathcal{A}^{\mathbf{T+1}},\mathcal{X})]=E[u(\mathcal{A}^{\mathbf{T+1}},\mathcal{X})]$$

Expected waiting time is:

$$\sum_{t=0}^{\infty}\left(1-\frac{1}{E[\mathcal{T}]}\right)^{t}\frac{1}{E[\mathcal{T}]}\cdot t=E[\mathcal{T}]$$

Expected experimentation cost is:

$$\sum_{t=0}^{\infty}\left(1-\frac{1}{E[\mathcal{T}]}\right)^{t}f\left(\frac{I(\mathcal{A}^{\mathbf{T+1}};\mathcal{X}|\mu)}{E[\mathcal{T}]}\right)=E[\mathcal{T}]f\left(\frac{I(\mathcal{A}^{\mathbf{T+1}};\mathcal{X}|\mu)}{E[\mathcal{T}]}\right)$$

Therefore, we find a strategy which is no worse than original strategy than $\frac{\varepsilon}{2}$. Then:

$$\begin{aligned}
V(\mu)\leqslant & E[u(\mathcal{A}^{\mathbf{T+1}},\mathcal{X})]-mE[\mathcal{T}]-E[\mathcal{T}]f\left(\frac{I(\mathcal{A}^{\mathbf{T+1}};\mathcal{X}|\mu)}{E[\mathcal{T}]}\right)+\varepsilon\\
\leqslant & \sup_{\mathcal{A},T}E[u(\mathcal{A},\mathcal{X})]-mT-Tf\left(\frac{I(\mathcal{A};\mathcal{X}|\preceq)}{T}\right)+\varepsilon\ \forall\varepsilon\\
\implies V(\mu)\leqslant & \sup_{\mathcal{A},T}E[u(\mathcal{A},\mathcal{X})]-mT-Tf\left(\frac{I(\mathcal{A};\mathcal{X}|\preceq)}{T}\right)
\end{aligned}$$

Therefore, we proved Theorem 1 when $E[\mathcal{T}]\geqslant 1$.
Case 2. $E[\mathcal{T}]<1$: Since $\mathcal{T}\in\mathbb{N}$, $E[\mathcal{T}]<1$ means $P(\mathcal{T}=0)>0$. When $\mathcal{T}=0$, no informatin is acquired yet and decision making is based on prior. Therefore:

$$\begin{aligned}
&E\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=0}^{\mathcal{T}}f\big(I(\mathcal{S}^{t};\mathcal{X}|\mathcal{S}^{\mathbf{t-1}})\big)\right]\\
=&\text{Prob}(\mathcal{T}=0)E\big[u(\mathcal{A}^{0},\mathcal{X})|\mathcal{T}=0\big]+\text{Prob}(\mathcal{T}\geqslant 1)E\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=1}^{\mathcal{T}}f\big(I(\mathcal{S}^{t};\mathcal{X}|\mathcal{S}^{\mathbf{t-1}})\big)\middle|\mathcal{T}\geqslant 1\right]\\
\leqslant&\text{Prob}(\mathcal{T}=0)\max_{a}E_{\mu}[u(a,\mathcal{X})]+\text{Prob}(\mathcal{T}\geqslant 1)E\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=1}^{\mathcal{T}}f\big(I(\mathcal{S}^{t};\mathcal{X}|\mathcal{S}^{\mathbf{t-1}})\big)\middle|\mathcal{T}\geqslant 1\right]
\end{aligned}$$

First equality is from law of iterated expectation. Inequality is from when $\mathcal{T}=0$, choice of $\mathcal{A}^{0}$ is not necessarily optimal. Suppose:

$$\begin{aligned}
&\max_{a}E_{\mu}[u(a,\mathcal{X})]\geqslant E\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=1}^{\mathcal{T}}f\Big(I(\mathcal{S}^{t};\mathcal{X}|\mathcal{S}^{t-1})\Big)\middle|\mathcal{T}\geqslant 1\right]\\
\implies &E\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=0}^{\mathcal{T}}f\Big(I(\mathcal{S}^{t};\mathcal{X}|\mathcal{S}^{t-1})\Big)\right]\leqslant\max_{a}E_{\mu}[u(a,\mathcal{X})]
\end{aligned}$$

Then strategy $(\mathcal{S}^{t},\mathcal{A}^{t},\mathcal{T})$ is dominated by acquiring no information and this already proves Theorem 1. Suppose on the other hand:

$$\max_{a}E_{\mu}[u(a,\mathcal{X})]<E\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=1}^{\mathcal{T}}f\Big(I(\mathcal{S}^{t};\mathcal{X}|\mathcal{S}^{t-1})\Big)\middle|\mathcal{T}\geqslant 1\right]$$

$$\begin{aligned}\Longrightarrow E&\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=0}^{\mathcal{T}}f\Big(I(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{t-1})\Big)\right]\\<E&\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=1}^{\mathcal{T}}f\Big(I(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{t-1})\Big)\middle|\mathcal{T}\geqslant 1\right]\end{aligned}$$

Then we define strategy $\mathcal{S}_1^t,\mathcal{A}_1^t,\mathcal{T}_1$ where: $(\mathcal{S}_1^t,\mathcal{A}_1^t,\mathcal{T}_1)=(\mathcal{S}^t,\mathcal{A}^t,\mathcal{T}\mid\mathcal{T}\geqslant 1)$. Then it's straight foward that:

$$\begin{aligned}&E\left[u(\mathcal{A}_1^{\mathcal{T}_1},\mathcal{X})-m\mathcal{T}_1-\sum_{t=0}^{\mathcal{T}_1}f\Big(I(\mathcal{S}_1^t;\mathcal{X}|\mathcal{S}_1^{t-1})\Big)\right]\\=&E\left[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})-m\mathcal{T}-\sum_{t=1}^{\mathcal{T}}f\Big(I(\mathcal{S}^t;\mathcal{X}|\mathcal{S}^{t-1})\Big)\middle|\mathcal{T}\geqslant 1\right]\end{aligned}$$

We only need to verify the information processing constraints.

- When $\mathcal{T}_1\leqslant t$: $\mathcal{S}_1^t=s_0$
- When $\mathcal{T}_1=t+1$: $\mathcal{S}_1^t=\mathcal{S}^t=\mathcal{A}^{t+1}=\mathcal{A}_1^{t+1}$.
- $\mathcal{T}_1=0$ happen with zero probability.

However, in this case $E[\mathcal{T}\geqslant 1]$. Therefore this goes back to case one. To sum up, we showed that:

$$V(\mu)\leqslant\max\left\{\sup_{\mathcal{A}}E[u(\mathcal{A},\mathcal{X})],\ \sup_{\mathcal{A},T\geqslant 1}E[u(\mathcal{A},\mathcal{X})]-mT-Tf\left(\frac{I(\mathcal{A};\mathcal{X}|\preceq)}{T}\right)\right\}$$

On the other hand the inequality of the other hand is straight foward, any strategy achieve the RHS can be achieved in origianl problem Equation (P). Therefore:

$$V(\mu)=\max\left\{\sup_{\mathcal{A}}E[u(\mathcal{A},\mathcal{X})],\ \sup_{\mathcal{A},T\geqslant 1}E[u(\mathcal{A},\mathcal{X})]-mT-Tf\left(\frac{I(\mathcal{A};\mathcal{X}|\preceq)}{T}\right)\right\}\tag{6}$$

Finally, we consider solving optimal T in Equation (1). Fix I, consider:

$$\inf_{T\geqslant 1}\left(mT+Tf\left(\frac{I}{T}\right)\right)$$

I first show that the objective function is quasi-convex. mT is already linear, so it's sufficient to show quasi-convexity of $T\left(\frac{I}{T}\right)$. By transforming argument, it's not hard to see that it's equivalent to show quasi-convexity of $\frac{f(I)}{I}$ w.r.t. I. Now consider $I_1<I_2$ and $\lambda\in(0,1)$. Suppose by contradiction:

$$\begin{aligned}&\frac{f(I_1)}{I_1}=\frac{f(I_2)}{I_2}<\frac{f(\lambda I_1(1-\lambda)I_2)}{\lambda I_1+(1-\lambda)I_2}\\\Longrightarrow&\frac{\lambda f(I_1)+(1-\lambda)f(I_2)}{\lambda I_1+(1-\lambda)I_2}<\frac{f(\lambda I_1(1-\lambda)I_2)}{\lambda I_1+(1-\lambda)I_2}\end{aligned}$$

contradicting convexity of $f(I)$. Therefore, $mT+T\left(\frac{I}{T}\right)$ is quasi-convex in T. Since f is convex, it always has one-side derivatives well defined. So an necessary condition for T solving the problem will be:

$$m+f\left(\frac{I}{T}\right)-\frac{I}{T}f_+'\left(\frac{I}{T}\right)\leqslant 0\leqslant m+f\left(\frac{I}{T}\right)-\frac{I}{T}f_-'\left(\frac{I}{T}\right)$$

$$\overset{\lambda=\frac{I}{T}}{\iff} m + f(\lambda) - \lambda f'_+(\lambda) \leqslant 0 \leqslant m + f(\lambda) - \lambda f'_-(\lambda)$$
$$\iff \frac{m + f(\lambda)}{\lambda} \in \partial f(\lambda)$$

What's more, since f is convex, the correspondence $f(\lambda) - \lambda f'(\lambda)$ is increasing (in set order). Therefore, the set of λ such that $\frac{m+f(\lambda)}{\lambda} \in \partial f(\lambda)$ must be an connected interval. Therefore, $\frac{m+f(\lambda)}{\lambda} \in \partial f(\lambda)$ is actually also sufficient for minimizing $mT + Tf(\frac{I}{T})$.

Case 1.: $\{\lambda | m + f(\lambda) \in \lambda \partial f(\lambda)\} \neq \varnothing$: Since f is convex, ∂f is a continuous correspondence, therefore the set is closed. Pick the smallest λ:

$$\begin{aligned} mT + Tf\left(\frac{I}{T}\right) &= m\frac{I}{\lambda} + \frac{I}{\lambda} f(\lambda) \\ &= \left(\frac{m}{\lambda} + \frac{f(\lambda)}{\lambda}\right) I \end{aligned}$$

Therefore, the total cost paid can be summarized by:

$$\left(\frac{m}{\lambda} + \frac{f(\lambda)}{\lambda}\right) I(\mathcal{A}; \mathcal{X} | \mu)$$

Finally, the constraint $T \geqslant 1$ can be replaced by:

$$\frac{I(\mathcal{A}; \mathcal{X}|\mu)}{\lambda} \geqslant 1$$
$$\iff I(\mathcal{A}; \mathcal{X}|\mu) \geqslant \lambda$$

Theorem 1 is proved.

Case 2.: $m + f(\lambda) - \lambda \partial f(\lambda) > 0 \; \forall \lambda$. That is to say:

$$mT + Tf\left(\frac{I}{T}\right)$$

is strictly increasing in $T \; \forall I$. Therefore, independent of choice I, choosing smaller T will yield higher utility. T will eventually be smaller than 1. So we can rule out this case.

Case 3.: $m + f(\lambda) - \lambda \partial f(\lambda) < 0 \; \forall \lambda$. That is to say:

$$mT + Tf\left(\frac{I}{T}\right)$$

is strictly decreasing in $T \; \forall I$. However this is not possible since:

$$\lim_{T \to \infty} mT + Tf\left(\frac{I}{T}\right) = +\infty$$

To sum up, if $\{\lambda | m + f(\lambda) \in \lambda \partial f(\lambda)\} = \varnothing$, then we define $\lambda = \infty$. Then the constraint for second term in Equation (6) can never be satisfied and $V(\mu) = \sup_a E[u(a, \mathcal{X})]$. *Q.E.D.*

Lemma B.1 (Reduction of redundency). *$(\mathcal{S}^t, \mathcal{T}, \mathcal{A}^{\mathcal{T}})$ solves Equation (P) if and only if there exists $\left(\tilde{\mathcal{S}}^T, \mathcal{T}, \mathcal{A}^{\mathcal{T}}\right)$ solving :*

$$\sup_{\mathcal{S}^t, \mathcal{T}, \mathcal{A}^{\mathcal{T}}} \sum_{t=0}^{\infty} \Big(e^{-\rho dt} \mathbf{P}[\mathcal{T} = t] \big(E\big[u(\mathcal{A}^t, \mathcal{X}) | \mathcal{T} = t\big]\big) - \mathbf{P}[\mathcal{T} > t] E\Big[f\Big(I\Big(\tilde{\mathcal{S}}^t; \mathcal{X} | \tilde{\mathcal{S}}^{\mathbf{t-1}}\Big)\Big) | \mathcal{T} > t\Big]\Big) \tag{7}$$

$$\text{s.t. } \tilde{\mathcal{S}}^t = \begin{cases} s_0 & \textit{when } \mathcal{T} < t+1 \\ \mathcal{A}^{t+1} & \textit{when } \mathcal{T} = t+1 \\ \mathcal{S}^t & \textit{when } \mathcal{T} > t+1 \end{cases}$$

What's more, the optimal utility level is same in Equation (P) and Equation (7).

Proof. Suppose $(\mathcal{S}^t, \mathcal{T}, \mathcal{A}^t)$ is a feasible strategy to Equation (P). Let first show that it's WLOG that the DM can discard all information after taking an action: take given $\mathcal{T}$ and $\mathcal{A}^t$, take s_0 as a given degenerate signal, define $\hat{\mathcal{S}}^t$ as:

$$\hat{\mathcal{S}}^t = \begin{cases} \mathcal{S}^t & \text{when } \mathcal{T} \geqslant t+1 \\ s_0 & \text{when } \mathcal{T} \leqslant t \end{cases}$$

By definition, $\hat{\mathcal{S}}^{\mathbf{t}} = \mathcal{S}^{\mathbf{t}}$ conditional on $\mathcal{T} \geqslant t+1$. Therefore:

$$I\left(\hat{\mathcal{S}}^t; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} \leqslant t}\right) = \begin{cases} I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} \leqslant t}) & \text{when } \mathcal{T} \leqslant t \\ 0 & \text{when } \mathcal{T} \geqslant t+1 \end{cases}$$

$$\mathcal{X} \rightarrow \hat{\mathcal{S}}^t \rightarrow \mathcal{A}^{t+1} \text{ conditional on } \mathcal{T} = t$$

To show that the first information processing constraint is satisfied, we discuss the case $\hat{\mathcal{S}} = s_0$ and $\hat{\mathcal{S}} \neq s_0$ separately:

- When $\hat{\mathcal{S}}^{t-1} = s_0$, $\mathcal{T} \leqslant t-1$. Therefore:

$$\text{Prob}\left(\mathcal{T} > t | \hat{\mathcal{S}}^{t-1} = s_0, \mathcal{X}\right) = 0$$
$$\text{Prob}\left(\mathcal{T} \leqslant t | \hat{\mathcal{S}}^{t-1} = s_0, \mathcal{X}\right) = 1$$

  which is independent of realization of $\mathcal{X}$.
- When $\hat{\mathcal{S}}^{t-1} \neq s_0$, $\mathcal{T} \geqslant t$. Then by law of total probability:

$$\begin{aligned}
&\text{Prob}\left(\mathcal{T} > t | \mathcal{S}^{t-1}\right) \\
=&\text{Prob}\left(\mathcal{T} > t | \mathcal{S}^{t-1}, \mathcal{X}\right) \\
=&\text{Prob}\left(\mathcal{T} > t | \mathcal{S}^{t-1}, \mathcal{X}, \mathcal{T} \geqslant t\right) \text{Prob}(\mathcal{T} \geqslant t | \mathcal{S}^{t-1}, \mathcal{X}) \\
&+ \text{Prob}\left(\mathcal{T} > t | \mathcal{S}^{t-1}, \mathcal{X}, \mathcal{T} < t\right) \text{Prob}(\mathcal{T} < t | \mathcal{S}^{t-1}, \mathcal{X}) \\
=&\text{Prob}\left(\mathcal{T} > t | \mathcal{S}^{t-1}, \mathcal{X}, \mathcal{T} \geqslant t\right) \text{Prob}(\mathcal{T} \geqslant t | \mathcal{S}^{t-1}, \mathcal{X}) \\
\Longrightarrow\ & \text{Prob}\left(\mathcal{T} > t | \hat{\mathcal{S}}^{t-1}, \mathcal{X}\right) \\
=&\frac{\text{Prob}\left(\mathcal{T} > t | \hat{\mathcal{S}}^{t-1}\right)}{\text{Prob}\left(\mathcal{T} \geqslant t | \hat{\mathcal{S}}^{t-1}, \mathcal{X}\right)} \\
=&\text{Prob}\left(\mathcal{T} > t | \hat{\mathcal{S}}^{t-1}\right)
\end{aligned}$$

  which is independent of realization of $\mathcal{X}$.

Therefore, we proved that:

$$\mathcal{X} \to \hat{\mathcal{S}}^{t-1} \to \mathbf{1}_{\mathcal{T}\leqslant t}$$

Therefore $\left(\hat{\mathcal{S}}^t, \mathcal{A}^t, \mathcal{T}\right)$ is feasible and :

$$\begin{aligned}
&E\left[\sum_{t=0}^{\infty} e^{-\rho dt\cdot t} f\left(I\left(\hat{\mathcal{S}}^t; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T}\leqslant t}\right)\right)\right] \\
=&E\left[\sum_{t=0}^{\mathcal{T}-1} e^{-\rho dt\cdot t} f\left(I\left(\hat{\mathcal{S}}^t; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T}\leqslant t}\right)\right)\right] \\
=&E\left[\sum_{t=0}^{\mathcal{T}-1} e^{-\rho dt\cdot t} f\left(I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T}\leqslant t})\right)\right] \\
\leqslant&E\left[\sum_{t=0}^{\infty} e^{-\rho dt\cdot t} f\left(I(\mathcal{S}^t; \mathcal{X} | \mathcal{S}^{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T}\leqslant t})\right)\right]
\end{aligned}$$

Therefore, $\left(\hat{\mathcal{S}}^t, \mathcal{A}^t, \mathcal{T}\right)$ is a feasible strategy dominating $(\mathcal{S}^t, \mathcal{A}^t, \mathcal{T})$. Now we define $\tilde{\mathcal{S}}^t$:

$$\tilde{\mathcal{S}}^t = \begin{cases} s_0 & \text{when } \mathcal{T} < t+1 \\ \mathcal{A}^{t+1} & \text{when } \mathcal{T} = t+1 \\ \hat{\mathcal{S}}^t & \text{when } \mathcal{T} > t+1 \end{cases}$$

Initial information $\tilde{\mathcal{S}}^{-1}$ is defined as a degenerate(uninformative) signal and induced belief is the prior. Verify the properties of $\tilde{\mathcal{S}}^t$:

1. When $\tilde{\mathcal{S}}^{t-1} \in \{s_0\} \cup A$, it's for sure that $\mathcal{T} \leqslant t$. Otherwise, $\mathcal{T} > t$. Therefore $\mathbf{1}_{\mathcal{T}\leqslant t}$ is a direct garbling of $\tilde{\mathcal{S}}^{t-1}$. So we must have $\mathcal{X} \to \tilde{\mathcal{S}}^{t-1} \to \mathbf{1}_{\mathcal{T}\leqslant t}$.
2. When $\mathcal{T} = t$, $\mathcal{A}^t = \tilde{\mathcal{S}}^{t-1}$. Therefore $\mathcal{X} \to \tilde{\mathcal{S}}^{t-1} \to \mathcal{A}^t$ conditional on $\mathcal{T} = t$.
3. Information measure associated with $\left(\tilde{\mathcal{S}}^t, \mathcal{A}^t, \mathcal{T}\right)$ when $\mathcal{T} > t$:

$$\begin{aligned}
&I\left(\tilde{\mathcal{S}}^t; \mathcal{X} | \tilde{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} > t\right) \\
=&\mathbf{1}_{\mathcal{T}=t+1} I\left(\mathcal{A}^{t+1}; \mathcal{X} | \tilde{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} = t+1\right) \\
&+ \mathbf{1}_{\mathcal{T}>t+1} I\left(\tilde{\mathcal{S}}^t; \mathcal{X} | \tilde{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} > t+1\right) \\
=&\mathbf{1}_{\mathcal{T}=t+1} I\left(\mathcal{A}^{t+1}; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} = t+1\right) \\
&+ \mathbf{1}_{\mathcal{T}>t+1} I\left(\hat{\mathcal{S}}^t; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} > t+1\right) \\
\leqslant&\mathbf{1}_{\mathcal{T}=t+1} I\left(\hat{\mathcal{S}}^t; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} = t+1\right) \\
&+ \mathbf{1}_{\mathcal{T}>t+1} I\left(\hat{\mathcal{S}}^t; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} > t+1\right) \\
=&I\left(\hat{\mathcal{S}}^t; \mathcal{X} | \hat{\mathcal{S}}^{\mathbf{t-1}}, \mathcal{T} > 1\right)
\end{aligned}$$

First inequality is simply rewriting two possible cases of $\mathcal{T}$. Second equality is from definition of $\tilde{\mathcal{S}}^t$ when $\mathcal{T} > t+1$. First inequality is from $\mathcal{X} \to \hat{\mathcal{S}}^t \to \mathcal{A}^{t+1}$ conditional on $\mathcal{T} = t+1$. Therefore, $\left(\tilde{\mathcal{S}}^t, \mathcal{T}, \mathcal{A}^t\right)$ dominates the original solution in Equation (*P*) by achieving same action profile but lower costs. $\left(\tilde{\mathcal{S}}^t, \mathcal{T}, \mathcal{A}^t\right)$ is a feasible solution to Equation (7). Therefore solving Equation (7) yields a weakly higher utility than Equation (*P*). What remains to be proved is that any $\left(\tilde{\mathcal{S}}^t, \mathcal{T}, \mathcal{A}^t\right)$ feasible in Equation (7) can be dominated by some strategy feasible in Equation (*P*). It's not hard to see that it's feasible in Equation (*P*). Finally we show that the two formulation gives same utility:

$$
\begin{aligned}
&E\left[e^{-\rho dt\cdot\mathcal{T}}E[u(\mathcal{A}^{\mathcal{T}},\mathcal{X})] - \sum_{t=0}^{\infty} e^{-\rho dt\cdot t} f\left(I\left(\tilde{\mathcal{S}}^t;\mathcal{X}|\tilde{\mathcal{S}}^{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\leqslant t}\right)\right)\right]\\
=&\sum_{t=0}^{\infty}\left(e^{-\rho dt}\mathbf{P}[\mathcal{T}=t]\left(E[u(\mathcal{A}^t,\mathcal{X})|\mathcal{T}=t]\right) - E\left[f\left(I\left(\tilde{\mathcal{S}}^t;\mathcal{X}|\tilde{\mathcal{S}}^{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\leqslant t}\right)\right)\right]\right)\\
=&\sum_{t=0}^{\infty}\left(e^{-\rho dt}\mathbf{P}[\mathcal{T}=t]\left(E\left[u(\mathcal{A}^t,\mathcal{X})|\mathcal{T}=t\right]\right)\right.\\
&\left.-\mathbf{P}[\mathcal{T}>t]E\left[f\left(I\left(\tilde{\mathcal{S}}^t;\mathcal{X}|\tilde{\mathcal{S}}^{\mathbf{t-1}}\right)\right)\middle|\mathcal{T}>t\right]\right)
\end{aligned}
$$

Therefore, Equation (*P*) is equivalent to Equation (7). *Q.E.D.*

B.2 Proof for Proposition 3

Proof. The outer maximization of Equation (1) is trivial. We focus on solving:

$$\overline{V}(\mu) = \sup_{I(\mathcal{A},\mathcal{X}|\mu)\geqslant\lambda} E[u(\mathcal{A},\mathcal{X})] - \left(\frac{m}{\lambda}+\frac{f(\lambda)}{\lambda}\right)I(\mathcal{A};\mathcal{X}|\mu) \tag{8}$$

Case 1. $\lambda^* < \infty$. By definition of λ^*, we know that

$$\lambda^* = \inf\arg\min_{\lambda}\left(\frac{m}{\lambda}+\frac{f(\lambda)}{\lambda}\right)$$

Let

$$g(I) = \left(\frac{m+f(\min\{I,\lambda^*\})}{\min\{I,\lambda^*\}}\right)I$$

Then $\frac{m+f(\lambda^*)}{\lambda^*}I \leqslant g(I) \leqslant m+f(I)$ and $g(I)$ is a convex function on $[0,\infty)$.Equation (8) can be rewritten as:

$$\overline{V}(\mu) = \sup_{\mathcal{A}} E[u(\mathcal{A},\mathcal{X})] - g(I(\mathcal{A};\mathcal{X}|\mu)) \tag{9}$$

Therefore by definition:

$$V^2(\mu) \leqslant \overline{V}(\mu) \leqslant V^1(\mu)$$

Now it is sufficient to show that if $\sup_{\mathcal{A}\in\mathbb{A}} I(\mathcal{A};\mathcal{X}|\mu) \geqslant \lambda^*$ then $\overline{V}(\mu) \geqslant V^1(\mu)$, otherwise $\overline{V}(\mu) \leqslant V^2(\mu)$. First of all, suppose $\sup_{\mathcal{A}\in\mathbb{A}} I(\mathcal{A};\mathcal{X}|\mu) \geqslant \lambda^*$, then by definition of $\sup_{\mathcal{A}\in\mathbb{A}} I(\mathcal{A};\mathcal{X}|\mu)$ there exists $\left\{\mathcal{A}_j^i\right\}$ s.t:

$$
\begin{aligned}
&E[u(\mathcal{A}_j^i,\mathcal{X})] - \left(\frac{m}{\lambda^*}+\frac{f(\lambda^*)}{\lambda^*}\right)I\left(\mathcal{A}_j^i,\mathcal{X}|\mu\right) \geqslant V^1(\mu) - \frac{1}{i}\\
&I(\mathcal{A}_j^i;\mathcal{X}|\mu) \geqslant \lambda^* - \frac{1}{j} - \frac{1}{i}
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \begin{cases} E[u(\mathcal{A}_i^i, \mathcal{X})] - \frac{m + f(\lambda^*)}{\lambda^*} I\left(\mathcal{A}_i^i; \mathcal{X}|\mu\right) \to V^1(\mu) \\ I(\mathcal{A}_i^i; \mathcal{X}|\mu) \to \lambda^* \end{cases} \\
&\Longrightarrow V^2(\mu) \geqslant E[u(\mathcal{A}_i^i, \mathcal{X})] - m - f\left(I\left(\mathcal{A}_i^i; \mathcal{X}\middle|\mu\right)\right) \\
&= E[u(\mathcal{A}_i^i, \mathcal{X})] - \frac{m + f(\lambda^*)}{\lambda^*} I\left(\mathcal{A}_i^i; \mathcal{X}|\mu\right) + \left(\frac{m + f\left(I\left(\mathcal{A}_i^i; \mathcal{X}\middle|\mu\right)\right)}{I\left(\mathcal{A}_i^i; \mathcal{X}\middle|\mu\right)} - \frac{m + f(\lambda^*)}{\lambda^*}\right) I\left(\mathcal{A}_i^i; \mathcal{X}\middle|\mu\right) \\
&\to V^1(\mu) \\
&\Longrightarrow V^2(\mu) = \overline{V}(\mu)
\end{aligned}
$$

Now suppose $\sup_{\mathcal{A} \in \mathbb{A}} I(\mathcal{A}; \mathcal{X}|\mu) < \lambda^*$. Assume by contradiction that $\overline{V}(\mu) > V^2(\mu)$. Then I first claim that $\forall \mathcal{A}^i$ solving Equation (9), $\limsup I(\mathcal{A}^i; \mathcal{X}|\mu) \leqslant \lambda^*$. If this claim is true, then there is immediately a contradiction:

$$
\begin{aligned}
&\begin{cases} \lim I(\mathcal{A}^i; \mathcal{X}|\mu) = \lambda^* \\ \lim E[u(\mathcal{A}^i, \mathcal{X})] - g\left(I\left(\mathcal{A}^i; \mathcal{X}|\mu\right)\right) = \overline{V}(\mu) \end{cases} \\
&\Longrightarrow \lim E[u(\mathcal{A}^i, \mathcal{X})] - g(\lambda^*) = \overline{V}(\mu) \\
&\Longrightarrow \lim E[u(\mathcal{A}^i, \mathcal{X})] - f\left(I(\mathcal{A}^i; \mathcal{X}|\mu)\right) = \overline{V}(\mu) > V^2(\mu)
\end{aligned}
$$

Suppose the claim is not true, then $\overline{V}(\mu) < V^1(\mu)$ and there exists:

$$
\begin{cases} \lim I(\mathcal{A}_1^i; \mathcal{X}|\mu) = \lambda' > \lambda^* \\ \lim E[u(\mathcal{A}_1^i, \mathcal{X})] - \frac{m + f(\lambda^*)}{\lambda^*} I\left(\mathcal{A}_1^i; \mathcal{X}|\mu\right) = \overline{V}(\mu) \end{cases}
$$

$$
\begin{cases} \lim I(\mathcal{A}_2^i; \mathcal{X}|\mu) = \lambda'' < \lambda^* \\ \lim E[u(\mathcal{A}_2^i, \mathcal{X})] - \frac{m + f(\lambda^*)}{\lambda^*} I\left(\mathcal{A}_2^i; \mathcal{X}|\mu\right) = V^1(\mu) \end{cases}
$$

$\forall \alpha \in [0, 1]$ consider compound experiment: $\mathcal{S}^0$ is an unrelated random draw with outcome 1 with probability $1 - \alpha$ and 2 with α. Conditional on 1, do experiment $\mathcal{A}_1^i$ and follow recommendation. Otherwise do $\mathcal{A}_2^i$ and follow recommendation. Call this information structure $\mathcal{A}_\alpha^i$. Then Assumption 1 implies:

$$
I(\mathcal{A}_\alpha^i; \mathcal{X}|\mu) \leqslant (1 - \alpha) I(\mathcal{A}_1^i; \mathcal{X}|\mu) + \alpha I(\mathcal{A}_2^i; \mathcal{X}|\mu)
$$

Since $l' > \lambda^* > \lambda''$, WLOG we can assume $I(\mathcal{A}_j^i; \mathcal{X}|\mu)$ is bounded within λ', λ'' by ε and $2\varepsilon < \lambda' - \lambda^*$. Now consider the utlity of strategy $\mathcal{A}_\alpha^i$ in Equation (9). Suppose $I(\mathcal{A}_\alpha^i; \mathcal{X}|\mu) < \lambda^*$ for all $\alpha > 0$, then:

$$
\begin{aligned}
&\lim_{\alpha \to 0} E\left[u(\mathcal{A}_\alpha^i; \mathcal{X})\right] - g\left(I(\mathcal{A}_\alpha^i; \mathcal{X}|\mu)\right) \\
\geqslant& E\left[u\left(\mathcal{A}_1^i\right); \mathcal{X}\right] - g(\lambda^*) \\
\geqslant& \overline{V}(\mu) + \left(g(\lambda' - \varepsilon) - g(\lambda^*)\right) - \frac{1}{i}
\end{aligned}
$$

Since g is a strictly increasing function with $\lambda > \lambda^*$, given any $\delta < g(\lambda' - \varepsilon) - g(\lambda^*)$, there exists α^i s.t.

$$
E\left[u(\mathcal{A}_{\alpha^i}^i; \mathcal{X})\right] - g\left(I(\mathcal{A}_{\alpha^i}^i; \mathcal{X}|\mu)\right) \geqslant \overline{V}(\mu) - \frac{1}{i} + \delta
$$

Suppose there exists α^i s.t. $I(\mathcal{A}^i_{\alpha^i};\mathcal{X}|\mu)=\lambda^*$, then:

$$
\begin{aligned}
&E\left[u(\mathcal{A}^i_{\alpha^i};\mathcal{X})\right]-g\left(I(\mathcal{A}^i_{\alpha^i};\mathcal{X}|\mu)\right)\\
=&E\left[u(\mathcal{A}^i_{\alpha^i};\mathcal{X})\right]-\frac{m+f(\lambda^*)}{\lambda^*}I(\mathcal{A}^i_{\alpha^i};\mathcal{X}|\mu)\\
\geqslant&\overline{V}(\mu)+\alpha(V^1(\mu)-\overline{V}(\mu))-\frac{1}{i}\\
&+\frac{m+f(\lambda^*)}{\lambda^*}\left((1-\alpha)I(\mathcal{A}^i_1;\mathcal{X}|\mu)+\alpha I(\mathcal{A}^i_2;\mathcal{X}|\mu)-I(\mathcal{A}^i_{\alpha^i};\mathcal{X}|\mu)\right)\\
\geqslant&\max\begin{cases}\overline{V}(\mu)+\alpha(V^1(\mu)-\overline{V}(\mu))-\frac{1}{i}\\ \overline{V}(\mu)-\frac{1}{i}+\frac{m+f(\lambda^*)}{\lambda^*}((1-\alpha)(\lambda'-\varepsilon)+\alpha(\lambda''-\varepsilon)-\lambda^*)\end{cases}\\
=&\overline{V}(\mu)-\frac{1}{i}+\max\left\{\alpha(V^1(\mu)-\overline{V}(\mu)),\frac{m+f(\lambda^*)}{\lambda^*}(\lambda'-\alpha(\lambda'-\lambda'')-\varepsilon-\lambda^*)\right\}
\end{aligned}
$$

The maximum is independent to i and strictly positive for any α. Therefore:

$$
\lim_{i\to\infty}E\left[u(\mathcal{A}^i_{\alpha^i};\mathcal{X})\right]-g\left(I(\mathcal{A}^i_{\alpha^i};\mathcal{X}|\mu)\right)>\overline{V}(\mu)
$$

Contradicting optimality of $\overline{V}(\mu)$. To sum up, I show that when $\sup_{\mathcal{A}\in\mathbb{A}}I(\mathcal{A};\mathcal{X}|\mu)<\lambda^*$, $\overline{V}(\mu)=V^2(\mu)$. Therefore:

$$
\overline{V}(\mu)=\begin{cases}V^1(\mu) & \text{if }\sup\limits_{\mathcal{A}\in\mathbb{A}}I(\mathcal{A};\mathcal{X}|\mu)\geqslant\lambda^*\\ V^2(\mu) & \text{if }\sup\limits_{\mathcal{A}\in\mathbb{A}}I(\mathcal{A};\mathcal{X}|\mu)<\lambda^*\end{cases}
$$

Case 2. $\lambda^*=+\infty$. By definition of λ^*, $\left(\frac{m}{\lambda}+\frac{f(\lambda)}{\lambda}\right)$ is strictly decreasing in λ. $\forall\mathcal{A},\lambda$ being feasible in Equation (8), it can be improved by replacing λ with $I(\mathcal{A};\mathcal{X}|\mu)$ (feasibility is still satisfied). Therefore, it is without loss of optimality to assume constraint binding and Equation (8) becomes:

$$
\sup_{\mathcal{A}}E[u(\mathcal{A},\mathcal{X})]-m-f(I(\mathcal{A};\mathcal{X}|\mu))
$$

which is exactly Equation (3). *Q.E.D.*

B.3 Proof for Proposition 5

Proof. *Existence:* Equations (2) and (3) can be solved prior by prior. Therefore, I sometimes don't explicitly include prior any more in this proof. It's not hard see that it's sufficient to prove existence of solution to:

$$
\sup_{\mathcal{A}}E[u(\mathcal{A},\mathcal{X})-f(I(\mathcal{A};\mathcal{X}|\mu))] \tag{10}
$$

where $\mathcal{A}\in\Delta A\times X$ and f is convex. Equation (10) can be modified to accommodate Equation (2) by set f to be a linear function. This can be WLOG directly modeled in changing information measure I. Equation (10) is different from Equation (3) by only a constant. Therefore, it is sufficient to show existence of solution to Equation (10) under Assumption 1.

Next let's explicitly model the set of all feasible $\mathcal{A}$'s: $\Delta A\times X\in\mathbb{R}^{(|A|-1)\times|X|}$. Let's call this set Λ and any conditional distribution $p(a|x)\in\Lambda$. We define $\widetilde{I}:\Lambda\to\mathbb{R}^+$:

$$
\widetilde{I}(p(\cdot|\cdot))=I(\gamma)
$$

where $\gamma = (\pi, \mu) \in \Gamma$ and π is defined by distribution of posteriors induced by p:

$$\begin{cases} \mu_s(x) = \frac{p(s|x)\mu(\mu)}{\sum_y p(s|y)\mu(\mu)} \\ \pi(\mu_s) = \sum_y p(s|y)\mu(y) \end{cases}$$

Our original problem Equation (10) can be written as:

$$\sup_{p \in \Lambda} \sum_{a,x} p(a|x)\mu(x)u(a,x) - f\left(\widetilde{I}(p)\right)$$

To show existence of solution, it will be sufficient to show convexity of $\widetilde{I}$. If $\widetilde{I}$ is convex, the objective function is continuous in p and the space Λ is compact (a closed and bounded set in Eclidean space). Now let's study convexity of $\widetilde{I}$. Consider $\forall\, p_1, p_2 \in \Lambda$. Let $p = \lambda p_1 + (1-\lambda)p_2$. It's not hard to verify that $p \in \Lambda$ as well. Want to show:

$$\widetilde{I}(p) \leqslant \lambda \widetilde{I}(p_1) + (1-\lambda)\widetilde{I}(p_2)$$

Now define p' on $A \times \{1, 2\} = \{a_1, a_2, \ldots\}$ with twice number of signals than A. Let $\lambda_1 = \lambda, \lambda_2 = 1 - \lambda, \forall a, x$

$$p'(a_i|x) = \lambda_i p_i(a|x)$$

Then p' will be Blackwell more informative than p:

$$\begin{bmatrix} 1\,1 & 0\,0 & \cdots & 0\,0 \\ 0\,0 & 1\,1 & \cdots & 0\,0 \\ \vdots & \vdots & \ddots & \vdots \\ 0\,0 & 0\,0 & \cdots & 1\,1 \end{bmatrix} \cdot p' = p \tag{11}$$

On the other hand, p' can be written as combination of p_1 and p_2. Let $\mathcal{S}_0$ be randomly $\{1, 2\}$ with probability λ_1, λ_2. Let $(\mathcal{S}_1|1, \mu) \sim p_1$ and $(\mathcal{S}_1|2, \mu) \sim p_2$. Then it's easy to see that $(\mathcal{S}_0, \mathcal{S}_1, \mu) \sim p'$. Therefore:

$$\begin{aligned} \widetilde{I}(p) \leqslant& \widetilde{I}(p') \\ =& I(\mathcal{S}_0, \mathcal{S}_1; \mathcal{X}|\mu) \\ \leqslant& I(\mathcal{S}_0; \mathcal{X}|\mu) + \lambda_1 I(\mathcal{S}_1|1; \mathcal{X}|\mu) + \lambda_2 I(\mathcal{S}_1|2; \mathcal{X}|\mu) \\ =& \lambda \widetilde{I}(p_1) + (1-\lambda)\widetilde{I}(p_2) \end{aligned}$$

First inequality is from monotonicity, second inequality is from sub-additivity. Therefore $\widetilde{I}$ is a convex (and continuous) function. It's easy to see that Λ is a compact set. So we can apply Wierestrass theorem to conclude existence of solution.

Now suppose p_1,p_2 are two distinct maximizer. Consider $p = \alpha p_1 + (1-\alpha)p_2$. By convexity of $\widetilde{I}$ and f:

$$\begin{aligned} E_\mu[u(a,x)p(a,x)] =& \alpha E_\mu[u(a,x)p_1(a,x)] + (1-\alpha)E_\mu[u(a,x)p_2(a,x)] \\ f\left(\widetilde{I}(p)\right) \leqslant& \alpha f\left(\widetilde{I}(p_1)\right) + (1-\alpha) f\left(\widetilde{I}(p_2)\right) \end{aligned}$$

Therefore p weakly dominates p_1 and p_2 and $p \in \mathbb{A}$. $\mathbb{A}$ is convex.

Uniqueness: Now suppose I also satisfies strict-monotonicity. Then consider proof in last section. First, let $p_1 \neq p_2$. Suppose equality $\widetilde{I}(p) = \widetilde{I}(p')$ holds, then strict-monotonicity implies that p is Blackwell sufficient for p':

$$M \cdot p = p'$$

Where M is a stochastic matrix. Consider the following operation: If $p'_1 \sim p'_2$, then proof is done. Otherwise, first remove replication of p' (when two rows of p' are mutiplications of each other, then add them up) and get $\widetilde{p}'$. Since $p'_1 \nsim p'_2$, we can assume $\widetilde{p}'_1 = p'_1, \widetilde{p}'_2 = p'_2$. Define $\widehat{p}_1 = p'_1 + p'_2$ and $\widehat{p}_i = \widetilde{p}'_{i+1}$. By definition $\widetilde{p}'$ Blackwell dominates $\widehat{p}$. On the other hand, $\widehat{p}$ Blackwell dominates p, so dominates p', and $\widetilde{p}'$. By Lemma B.2, $\widetilde{p}'$ and $\widehat{p}$ are identical up to permutation. Then p'_1 must equal to some $\widehat{p}_i$.

- *Case 1.* If $i = 1$, then $p'_1 + p'_2$ is a multiplication of $\widetilde{p}'^1_1$, which is a multiplication of $\widetilde{p}'_1$. This means p'_1 and p'_2 are replication, contradiction.
- *Case 2.* If $i > 1$, then $\widetilde{p}'_1$ is a multiplication of $\widehat{p}_i$, which is a multiplication of $\widetilde{p}'_{i+1}$. Contradicting definition of $\widetilde{p}'$.

Therefore, p'_1 and p'_2 are replications. Now permute p' and apply the same analysis on all p'_{2i-1}, p'_{2i}. We can conclude that any row of p_1 is a replication of that of p_2. To sum up, a necessary condition for $\widetilde{I}(p) = \alpha\widetilde{I}(p_1) + (1-\alpha)\widetilde{I}(p_2)$ is that each row in p_1 and p_2 induces same posterior belief ν.

Now consider $\mathbb{A}$ being set of solutions to Equation (1). Suppose by contradiction there exists $\mathcal{A}_1$ and $\mathcal{A}_1$ and a such that they induces different posterior with realization a. Let p_1, p_2 be corresponding stochastic matrices, consider any $\mathcal{A} \sim \alpha p_1 + (1-\alpha)p_2$. By previous proof, $I(\mathcal{A}; \mathcal{X}|\mu) < \alpha\widetilde{I}(p_1) + (1-\alpha)\widetilde{I}(p_2)$. In first part, we show that $\mathbb{A}$ is convex, so $\mathcal{A}$ is feasible. This contradicts unimprovability.

To sum up, solutions to Equation (10) always have the same support. Of course if $\mathcal{A}$ is uninformative, then it induces prior μ. In both case, support of posteriors is uniquely determined.

Q.E.D.

Lemma B.2 (Blackwell equivalence). *Let P and P' be two stochastic matrices. P has no replication of rows. Suppose there exists stachatic matrices $M_{PP'}$ and $M_{P'P}$ s.t.:*

$$P' = M_{P'P} \cdot P$$
$$P = M_{PP'} \cdot P'$$

Then $M_{PP'}$ and $M_{P'P}$ are permutation matrices.

Proof. Let $P_i = (p_{i1}, p_{i2}, \ldots)$ be ith row of P. Suppose P_i can not be represented as positive combination of P_{-i}'s. Then by construction $P_i = M_{PP'i} \cdot M_{P'P} \cdot P$, we have:

$$M_{PP'i} \cdot M_{P'P} = (\underbrace{0, \cdots, 0}_{i-1}, 1, 0, \cdots, 0)$$

Then by non-negativity of stochastic matrices, suppose $M_{PP'ij} > 0$, then $M_{P'Pj}$ are all 0 except $M_{P'Pji}$. Then for all such rows j, we have $M_{PP'j}$ be a vector with only ith column being non-zero. However this suggests they are replicated rows. So the only possibility is that j s.t. $M_{PP'ij} > 0$ is unique. And

$$M_{PP'ij} \times M_{P'Pji} = 1$$

Since stochastic matrices have elements no larger than 1, it must be $M_{PP'ij} = M_{P'Pji} = 1$. This is equivalently saying $P'_j = P_i$. Since permutation of rows of P' doesn't affect our statement, let's assume $P'_i = P_i$ afterwards for simplicity.

So far we showed that if P_i is not a positive combinations of P_{-i}'s, then $P'_i = P_i$. We do the folloing transformation: $\widetilde{P}, \widetilde{P}'$ are P, P' removing ith row. $\widetilde{M}_{PP'}, \widetilde{M}_{P'P}$ are $M_{PP'}, M_{P'P}$ removing ith row and column. It's easy to verify that we still have:

$$\widetilde{P}' = \widetilde{M}_{P'P} \cdot \widetilde{P}$$
$$\widetilde{P} = \widetilde{M}_{PP'} \cdot \widetilde{P}'$$

and $\widetilde{M}_{PP'}, \widetilde{M}_{P'P}$ still being stochastic matrices since previous argument shows $M_{PP'ii}$ and $M_{P'Pii}$ being the only non-zero element in their rows. Since they are both 1, they must also be only non-zero element in their columns. So removing them doesn't affect the matrices being stochastic matrices.

Now we can repeat this process iteratively until any row $\widetilde{P}_i$ will be a positive combination of $\widetilde{P}_{-i}$. If $\widetilde{P}$ has one unique row, then the proof is done. We essentially showed that $P = P'$ (up to permutation of rows). Therefore we only need to exclude the possibility of $\widetilde{P}$ having more than one rows.

Suppose $\widetilde{P}$ has n rows. Then $\widetilde{P}_1$ is a posistive combination of $\widetilde{P}_{-i}$'s:

$$\widetilde{P}_1 = \sum_{i=2}^{n} a_i^1 \widetilde{P}_i$$

and $\widetilde{P}_2$ is a positive combination of $\widetilde{P}_{-i}$'s:

$$\begin{aligned}
\widetilde{P}_2 &= \sum_{i\neq 2}^{n} a_i^2 \widetilde{P}_i \\
&= a_1^2 \widetilde{P}_1 + \sum_{i>2}^{n} a_i^2 \widetilde{P}_i \\
&= a_1^2 a_2^1 \widetilde{P}_2 + \sum_{i>2}^{n} \left(a_i^2 + a_1^2 a_i^1 \right) \widetilde{P}_i
\end{aligned}$$

Since all rows in $\widetilde{P}$ are non-negative (and strictly positive in some elements). This is possible only in two cases:

- *Case 1.* $a_1^2 a_2^1 = 1$ and $\sum_{i>2} \left(a_i^2 + a_1^2 a_i^1 \right) = 0$. This implies $\widetilde{P}_1 = a_2^1 \widetilde{P}_2$. Contradicting non-replication.
- *Case 2.* $a_1^2 a_2^1 < 1$. Then $\widetilde{P}_2$ is a positive combination of $\widetilde{P}_{i>2}$. Of course $\widetilde{P}_1$ is also a positive combination of $\widetilde{P}_{i>2}$.

Now by induction suppose $\widetilde{P}_1, \ldots, \widetilde{P}_i$ are positive combinations of $\widetilde{P}_{j>i}$. Then:

$$\begin{aligned}
\widetilde{P}_{i+1} &= \sum_{j=1}^{i} a_j^{i+1} \widetilde{P}_j + \sum_{j=i+1}^{n} a_j^{i+2} \widetilde{P}_j \\
&= \sum_{k=i}^{n} \left(\sum_{j=1}^{i} a_j^{i+1} a_k^j \right) \widetilde{P}_k + \sum_{j=i+2}^{n} \widetilde{P}_j
\end{aligned}$$

$$= \sum_{j=1}^{i} a_j^{i+1} a_{i+1}^{j} \widetilde{P}_{i+1} + \sum_{k=i+2}^{n} \left(\sum_{j=1}^{i} a_j^{i+1} a_k^{j} + a_j^{i+1} \right) \widetilde{P}_j$$

Similar to previous analysis, non-replication implies $\sum_{j=1}^{i} a_j^{i+1} < 1$ and $\widetilde{P}_{i+1}$ is a positive combination of $\widetilde{P}_{j>i+1}$. Then by replacing $\widetilde{P}_{i+1}$ in combination of all $\widetilde{P}_{j\leqslant i}$, we can conclude that $\widetilde{P}_1, \ldots, \widetilde{P}_{i+1}$ are all positive combinations of $\widetilde{P}_{j>i+1}$. Finally, by induction we have all $\widetilde{P}_{i<n}$ being positive combination of $\widetilde{P}_n$. However, this contradicts non-replication. To sum up, we proved by contradiction that $\widetilde{P}$ has one unique row. Therefore, P must be indentical to P' up to permutations. *Q.E.D.*