# Optimal Dynamic Information Acquisition

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# Introduction

## Motivation

- Example of information acquisition:
  - A car manufacturer is deciding the design of new product:
    - with laser sensors v.s. without ?
  - Uncertainty: are autonomous technologies viable?

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  - Choose an R&D plan: what to learn & when to stop.
    - *Direction*: which specific technology to test.
    - *Precision*: the amount of data collected and analyzed.
    - *Frequency*: how intensively experiments are run.
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- Learning strategy is rich in multiple salient aspects
  - What is the optimal choice of "what to learn" and "when to stop"?

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  - Exogenous information.
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- My approach: flexible learning strategy.
  - Non-parametric information process.
    - Optimize in all aspects jointly.

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    - Repeated rational inattention: Hébert and Woodford 2016, Steiner, Stewart, and Matejka 2016.

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## Why a flexible model?

- Theoretically:
  - Provides intuitions in flexible benchmark
  - Identifies the endogenously important aspects of learning.

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- Theoretically:
  - Provides intuitions in flexible benchmark
  - Identifies the endogenously important aspects of learning.
- Practically:
  - Parametric models:
    - Can be misleading when wrong restriction is made.
    - Difficult to identify the restrictions.

## Preview of results

- Optimal learning strategy is a Poisson signal: induces Poisson belief process.

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- Optimal stopping strategy:
  - Immediately after signal arrival.

## Outline

1. Model Setup
   - Key assumptions
2. Main theorems:
   - Simplification: the HJB equation
   - Optimal strategy & proof: a concavification method
3. Discussion of key assumptions
4. Applications

# Model setup

## A flexible learning framework

- Decision problem:
  - Continuous time: $t \in [0, \infty)$.
  - One-shot choice of action: $e^{-\rho t} u(a, x)$.
    - $a \in A$, $x \in X$ both finite, $\rho > 0$, prior $\mu \in \Delta X$.

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- Strategy:
  - What to learn: stochastic belief $\langle \mu_t \rangle \in \mathbb{M}$
    - $\mathbb{M}$ contains all martingale processes.

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  - Choice of action: $F(\mu_t)$
    - $F(\mu_t) = \max_a E_{\mu_t}[u(a, x)]$

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$$V(\mu) = \sup_{\langle \mu_t \rangle \in \mathbb{M}, \tau} E\left[ \underbrace{e^{-\rho\tau} F(\mu_\tau)}_{\text{Stopping payoff}} - \int_0^\tau \underbrace{e^{-\rho t} C(I_t)}_{\text{flow control cost}} \, \mathrm{d}t \right] \tag{P}$$

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- Canonical models: $\mathbb{M}$ restricted to a parametric family.

# Model setup

## Key assumptions

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### Assumption 1

Let $H(\mu)$ be a concave function. $I_t = -E\left[ \frac{\mathrm{d}H(\mu_t)}{\mathrm{d}t} \middle| \mathcal{F}_t \right]$ (denoted by $-\mathcal{L}_t H(\mu_t)$).

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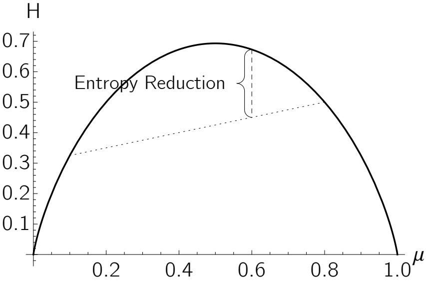
- $H(\mu)$ is an *uncertainty measure*
  - $E[H(\mu)]$ decreasing in MPS of $\mu$ distribution.
  - Example: Entropy function.



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- $H(\mu)$ is an *uncertainty measure*
- $I_t$ is the *uncertainty reduction speed* — *intensity* of learning
  - Assumption 1 makes the problem tractable.
  - Discrete-time foundation: posterior separability: Caplin, Dean, and Leahy 2017, Frankel and Kamenica 2018, Morris and Strack 2017.

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- Technical restriction on $\mathbb{M}$.
  - $\langle\mu_t\rangle$'s transition kernel is right-continuous in $t$ ($w-*$ topology).

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### Assumption 2

$C$ is weakly increasing, convex and continuous. $\lim_{I\to\infty} C'(I)=\infty$.

- Inada condition: strict incentive to smooth information.
  - Special case: $C$ is linear, optimal $\tau=0$. (Steiner, Stewart, and Matejka 2016)

# Simplification

## Dynamic programming and HJB equation

$$V(\mu) = \sup_{\langle \mu_t \rangle \in \mathbb{M}, \tau} E\left[ e^{-\rho\tau} F(\mu_\tau) - \int_0^\tau e^{-\rho t} C(-\mathcal{L}_t H(\mu_t)) \mathrm{d}t \right] \qquad (\mathrm{P})$$

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  - Generalized principle of DP:

$$\underbrace{\rho V(\mu_t)}_{\text{discount}} = \max\Big\{ \underbrace{\rho F(\mu_t)}_{\text{stopping value}}, \sup_{\mathrm{d}\mu_t} \Big\{ \underbrace{\mathcal{L}_t V(\mu_t)}_{\text{flow value}} - \underbrace{C(-\mathcal{L}_t H(\mu_t))}_{\text{flow control cost}} \Big\} \Big\} \tag{1}$$

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  - Need a verification theory for *equation* (P)$\Longleftrightarrow$*equation* (1).

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  - Need a verification theory for *equation* (P)$\Longleftrightarrow$*equation* (1).
  - Need a representation theory for $\mathcal{L}_t$.
  - Verification theory applies to different problems. Representation theory only shows existence. (Davis 1979,Boel and Kohlmann 1980,Striebel 1984)

# Simplification

Example: HJB for Gaussian learning

- General principle of DP

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  - Belief follows Gaussian process $\mathbb{M} = \{ \mu_t \,|\, \mathrm{d}\mu_t = \sigma_t \mathrm{d}\mu_t \}$. $\langle \sigma_t \rangle$ — signal *precision*.

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  - Moscarini and Smith 2001

# Simplification

## Verification and representation theorem

### Theorem 1

Assume $H$ is strictly concave and $C^{(2)}$, Assumption 1 and Assumption 2 are satisfied, then $V(\mu)\in C^{(1)}$ solves equation (P) if $V(\mu)$ is a solution of:

$$\rho V(\mu)=\max\Big\{\rho F(\mu), \sup_{p,\nu,\sigma} p(V(\nu)-V(\mu)-\nabla V(\mu)(\nu-\mu))+\frac{1}{2}\sigma^T \mathrm{H}V(\mu)\sigma$$
$$-C\left(p(H(\mu)-H(\nu)+\nabla H(\mu)(\nu-\mu))-\frac{1}{2}\sigma^T \mathrm{H}H(\mu)\sigma\right)\Big\} \tag{B}$$

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- Remark: optimal strategy represented by Markov jump-diffusion process.
  - Let $\mathbb{M}_{JD}=\Big\{\langle\mu_t\rangle \Big| d\mu_t=\underbrace{(\nu(\mu_t)-\mu_t)(dJ_t(p(\mu_t))-p(\mu_t)dt)}_{\text{compensated Poisson part}}+\underbrace{\sigma(\mu_t)dW_t}_{\text{Gaussian diffusion}}\Big\}$.

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  - $\mathcal{L}V(\mu)\Big|_{\mathbb{M}_{JD}}=\underbrace{p(V(\nu)-V(\mu)-\nabla V(\mu)(\nu-\mu))}_{\text{flow value of Poisson jump \& drift}}+\underbrace{\frac{1}{2}\sigma^T \mathrm{H}V(\mu)\sigma}_{\text{flow value of diffusion}}$.

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- Trade-offs:
  1. Exploration – exploitation
  2. Gain from learning — cost of learning
  3. Poisson — Gaussian
  4. Precision — frequency

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- Trade-offs:
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- Proof methodology:
  - Discretize equation (P) and solve the discrete-time problem.
  - Characterize $V$ as the limit of discrete-time value function. ▸ Discrete-time analysis

# Optimal learning dynamics

## Existence and characterization of Solution

$$\rho V(\mu)=\max\Bigg\{\rho F(\mu),\ \sup_{p,\nu,\sigma}\ p(V(\nu)-V(\mu)-\nabla V(\mu)(\nu-\mu))+\frac{1}{2}\sigma^T \mathrm{H}V(\mu)\sigma \qquad \text{(B)}$$
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# Optimal learning dynamics

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(Binary states): $\|X\|=2$.

(Positive payoff): $F(\mu)>0$.

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### Theorem 2

$\exists$ quasi-convex value function $V \in C^{(1)}[0,1]$ solving equation (B). Let $E = \{\mu | V(\mu) > F(\mu)\}$ be *experimentation region*, then $\exists$ unique (a.e.) policy $(\nu, p)$ s.t. $\forall \mu \in E$:

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5. *Stopping time*: $\nu(\mu)\in E^C$.

# Optimal learning dynamics

## Example

- Decision problem:
  - $X=\{L,R\}$, $A=\{l,r\}$, $U(l,L)=U(r,R)=1$, $U(l,R)=U(r,L)=-1$.

# Optimal learning dynamics

## Example

- Decision problem:
  - $\mu_t \in [0,1]$, $F(\mu) = \max\{1-2\mu, 2\mu-1\}$.

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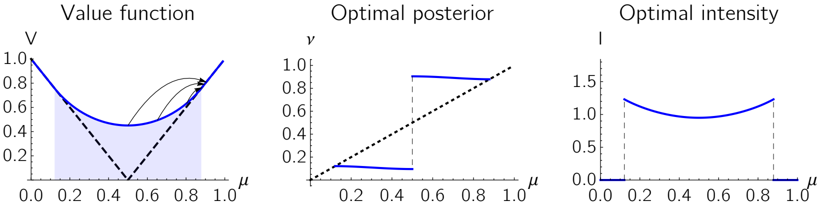

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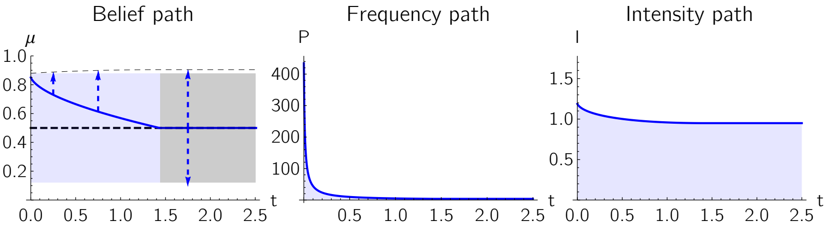
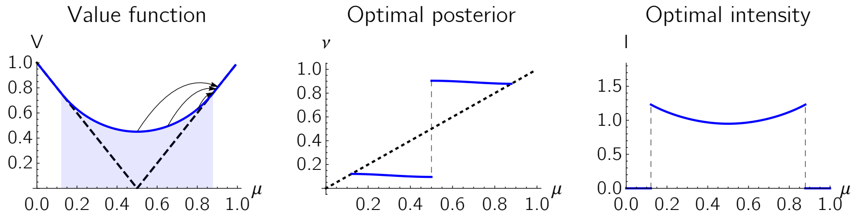


# Optimal learning dynamics

## Intuitions




- Key trade-off: precision-frequency trade-off.
  - Extreme belief → High continuation value → frequency > precision.
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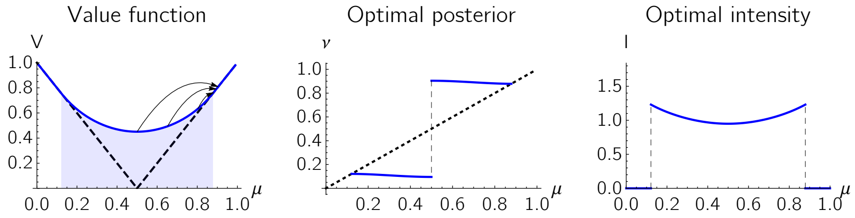


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- Poisson-Gaussian trade-off.
  - Gaussian signal: special Poisson signal — infinite frequency, low precision.
  - Gaussian signal dominated except for boundary of $E$.

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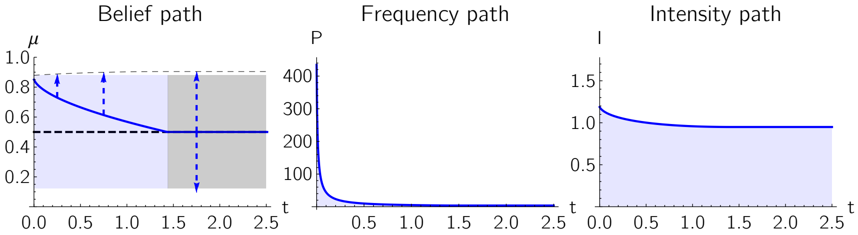


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- Confirmatory-contradictory trade-off.
  - Only confirmatory learning is consistent with the key trade-off.

# Proof methodology

Optimality condition and gross value function

- Consider a problem choosing optimal Poisson signal:

$$\sup_{p\geq 0,\nu} p\underbrace{(V(\nu)-V(\mu)-V'(\mu)(\nu-\mu))}_{\triangleq U(\mu,\nu)}-C(p\underbrace{(H(\mu)-H(\nu)+H'(\mu)(\nu-\mu))}_{\triangleq J(\mu,\nu)}) \tag{2}$$

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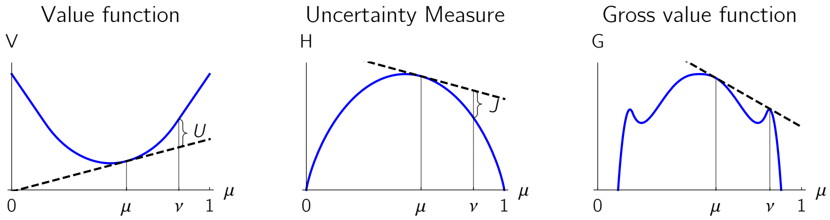
- Define $G(\mu)\triangleq V(\mu)+\lambda H(\mu)$, then $U-\lambda J=G(\nu)-G(\mu)-G'(\mu)(\nu-\mu)$:

$$\begin{cases}G(\nu)\leq G(\mu)+G'(\mu)(\nu-\mu) & \forall\nu\in[0,1]\\ G(\nu^*)=G(\mu)+G'(\mu)(\nu^*-\mu)\end{cases} \qquad (3)$$

# Proof methodology

## Geometric Characterization

$$\begin{cases} G(\nu) \leq G(\mu) + G'(\mu)(\nu - \mu) & \forall \nu \in [0,1] \\ G(\nu^*) = G(\mu) + G'(\mu)(\nu^* - \mu) \end{cases} \quad (3)$$



# Proof methodology

## Feasibility condition

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$$\begin{cases} G(\nu)\leq G(\mu)+G'(\mu)(\nu-\mu) & \forall \nu\in[0,1] \\ G(\nu^*)=G(\mu)+G'(\mu)(\nu^*-\mu) \end{cases} \qquad (3)$$

equations (3) and (4) pin down the whole solution.

# Proof methodology

Key trade-offs: utility-cost trade-off

- equation (4): $\rho V(\mu) = I^* \cdot C'(I^*) - C(I^*)$.
  - $\frac{\mathrm{d}}{\mathrm{d}I}(IC'(I) - C(I)) = IC''(I) > 0 \implies I^*$ is co-monotonic with continuation value $V(\mu)$.
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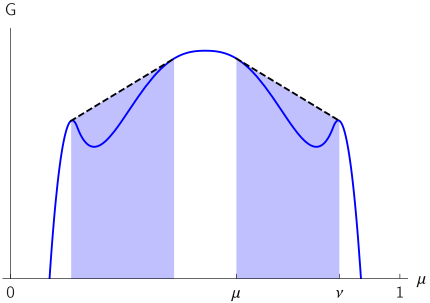
- A refinement of Moscarini and Smith 2001:
  - In a Gaussian learning model, $\sigma_t$ controls both precision and intensity.
  - The monotonicity is associated with intensity.

# Proof methodology

## Key trade-offs: precision-frequency trade-off

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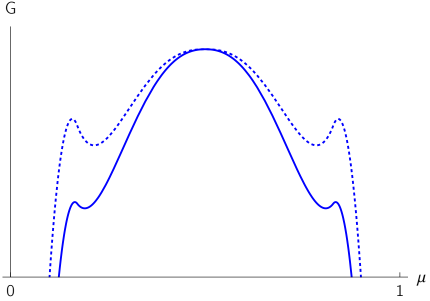


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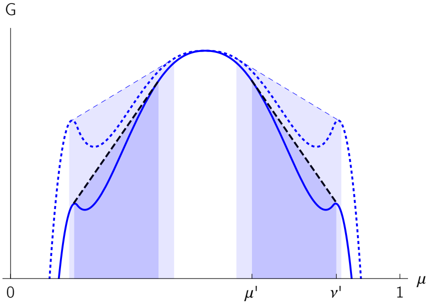


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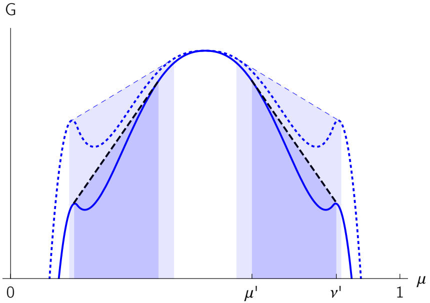


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  - $\mu$ and $\nu$ get closer — lower precision.
- Intuition:
  - Marginal rate of substitution of presision and frequency.
  - Higher continuation value $\implies$ lose more from waiting $\implies$ frequency is more prefered.

# Proof methodology

## Key trade-offs: Poisson-Gaussian trade-off

- Consider the optimal Gaussian signal:

$$\sup_{\sigma} \frac{1}{2}\sigma^2 V''(\mu) - C\left(-\frac{1}{2}\sigma^2 H''(\mu)\right)$$

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- If Gaussian signal is optimal, then:

$$\begin{cases} G(\nu) - G(\mu) - G'(\mu)(\nu - \mu) \leq 0 \\ G''(\mu) = 0 \end{cases} \quad \implies \nu^* \to \mu$$

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## Key trade-offs: Poisson-Gaussian trade-off

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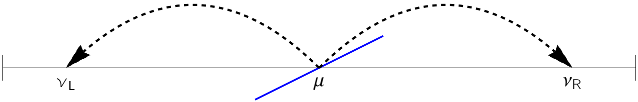
## Key trade-offs: confirmatory-contradictory trade-off

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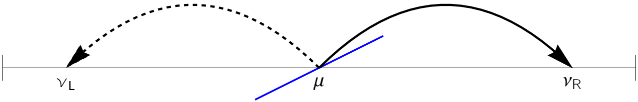


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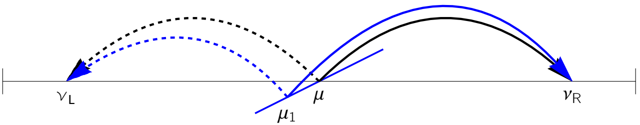


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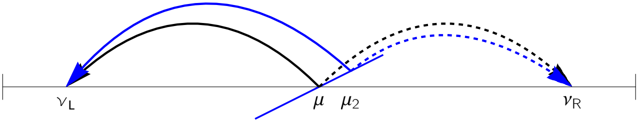


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# Proof methodology

## Proof of theorem 2

$$\rho V(\mu) = \max\Bigg\{\rho F(\mu),\ \sup_{p,\nu,\sigma}\ p(V(\nu)-V(\mu)-V'(\mu)(\nu-\mu))+\frac{1}{2}\sigma^2 V''(\mu) \\ -C\left(p(H(\mu)-H(\nu)+H'(\mu)(\nu-\mu))-\frac{1}{2}\sigma^2 H''(\mu)\right)\Bigg\} \qquad \text{(B)}$$

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  - Solve constrained problem: for $\mu \geq \mu^*$

Construction

$$\rho V(\mu)=\max_{\nu\geq\mu} I(\mu,\nu)\frac{F(\nu)-V(\mu)-V'(\mu)(\nu-\mu)}{H(\mu)-H(\nu)+H'(\mu)(\nu-\mu)}-C(I(\mu,\nu))$$

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- Step 2: verify that $V(\mu)$ also solves full problem equation (B).
  - Replace $F(\nu)$ with $V(\nu)$.
  - Remove constraint $\nu \geq \mu$.
  - Add Gaussian signals.

# Discussion

## Assumptions

- Economic assumption:
  - $I_t =$ uncertainty reduction speed.
  - Exponential discounting $e^{-\rho\tau}$.
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- Restrictive assumption:
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- Technical assumption:
  - $F(\mu)>0$.
  - $H:\Delta X \mapsto \mathbb{R}^-$ is $C^{(2)}$ smooth.
  - $H''(\mu)$ is Lipschitz continuous and negative definite.
  - $C$ is $C^{(2)}$ smooth.

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Economic assumption: information measure

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  - $pJ(\mu,\nu)=\frac{1}{2}pJ''_{\nu\nu}(\mu,\mu)(\nu-\mu)^2+O(\nu-\mu)^3$.
  - Let $\sigma^2=p(\nu-\mu)^2$: flow variance, then $pJ(\mu,\nu)\sim\kappa(\mu,\sigma)$ when $\nu\to\mu$.

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Given Assumption 1 and Assumption 2, $V(\mu)$ solves equation (6) if and only if:

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  - Decision time distributions are ranked by MPS order.

# Discussion

## Economic assumption: discounting

- What is the role of discounting?
  - Discounting is the key factor determining precision-frequency trade-off.
  - What if cost of waiting is value-independent?

$$V(\mu) = \sup_{\langle \mu_t \rangle \in \mathbb{M}, \tau} E\left[ F(\mu_\tau) - m\tau - \int_0^\tau C(I_t) \mathrm{d}t \right] \quad (6)$$

**Theorem 4**

Given Assumption 1 and Assumption 2, $V(\mu)$ solves equation (6) if and only if:

$$V(\mu) = \sup_{P \in \Delta^2(X), \lambda > 0} E_P[F(\nu)] - \frac{m + C(\lambda)}{\lambda} E_P[H(\mu) - H(\nu)]$$

and any $\langle \mu_t \rangle$ s.t. $\mu_\infty \sim P^*$, $I_t = \lambda^*$ is optimal.

- Fixed waiting cost $\Longrightarrow$ indifference. (Hébert and Woodford 2016)
- More general analysis in Zhong 2018:
  - Decision time distributions are ranked by MPS order.
  - Poisson learning's decision time is MPS of any other strategies.

# Discussion

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### Theorem 5

Given Assumption 1 and $C(I) = \lambda I$, $V(\mu)$ solves equation (P) if and only if:

$$V(\mu) = \sup_{P \in \Delta^2(X)} E_P[F(\nu)] - \lambda E_P[H(\mu) - H(\nu)] \tag{7}$$

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**Theorem 5**

Given Assumption 1 and $C(I)=\lambda I$, $V(\mu)$ solves equation (P) if and only if:

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- Optimal strategy is to learn immediately (Steiner, Stewart, and Matejka 2016).

# Further discussion

## Convergence of policy

- equation (B) is proved by approximation using equation (B-dt).
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**Definition 1 (Lévy metric)**
Let $F, G: [0,1] \to [0,1]$ be two correspondences. Define the graph distance between them to be:

$$L(F,G) := d_H(\text{graph}(F), \text{graph}(G))$$

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**Theorem 6**

Let $N(\mu)=\{\mu\}\cup\nu(\mu)$. Let $N_{dt}(\mu)$ be support of optimal posteriors solving equation (B-dt). Then:

$$\lim_{dt\to 0} L(N,N_{dt})=0$$

# Further discussion

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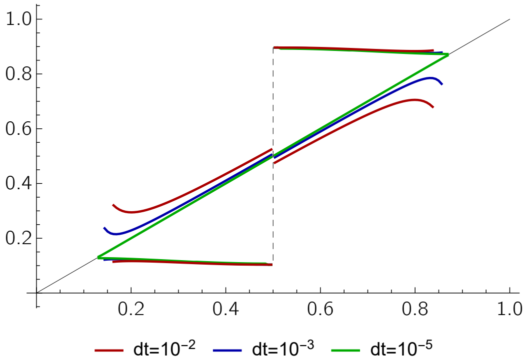
Figure: Convergence of policy w.r.t. $dt$



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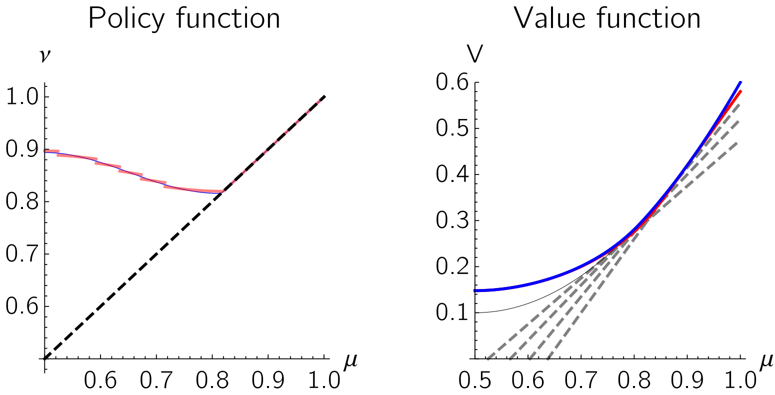


Figure: Approximation of policy function and value function.

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**Theorem 7 (Convergence of policy function)**

Let $\{F_n\}$ be a set of piecewise linear functions on [0,1] satisfying:

1. $\|F_n - F\|_\infty \to 0$;
2. $\forall \mu \in [0,1]$, $\lim F_n'(\mu) = F'(\mu)$.

Define $\mathcal{V}_{dt}(F_n)$ as the solution to equation (B-dt). Define functional $\mathcal{V}(F) = \lim_{dt\to 0} \mathcal{V}_{dt}(F)$. Then:

1. $\|\mathcal{V}(F) - \mathcal{V}(F_n)\| \to 0$.
2. $\mathcal{V}(F)$ solves equation (B).
3. $\forall \mu$ s.t. $V(\mu) > F(\mu)$, let $\nu_n$ be maximizer of $\mathcal{V}(F_n)$ s.t. $\nu = \lim_{n\to\infty} \nu_n$ exists, then $\nu$ achieves $\mathcal{V}(F)$ at $\mu$.

# Further discussion

Other extensions: larger state space

$$\rho V(\mu) = \max\left\{\rho F(\mu), \max_{\nu,p,\Sigma} p(V(\nu) - V(\mu) - \nabla V(\mu)\cdot(\nu - \mu)) + \sigma^T \mathrm{H} V(\mu)\sigma\right\}$$
$$s.t.\ -p(H(\nu) - H(\mu) - \nabla H(\mu)\cdot(\nu - \mu)) - \sigma^T \mathrm{H} H(\mu)\sigma \leq c \tag{8}$$

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## Theorem 8

Let $E = \{\mu \in \Delta(X) | V(\mu) > F(\mu)\}$ be experimentation region. Suppose there exists $C^{(2)}$ smooth $V(\mu)$ on $E$ solving equation (8), then $\exists$ policy function $\nu : E \mapsto \Delta(X)$ s.t.

$$\rho V(\mu) = -c\frac{F(\nu(\mu)) - V(\mu) - \nabla V(\mu)(\nu(\mu) - \mu)}{H(\nu(\mu)) - H(\mu) - \nabla H(\mu)(\nu(\mu) - \mu)}$$

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where $\nu$ satisfies the following properties:

1. *Poisson learning:* $\rho V(\mu)\geq\sup_\sigma -c\frac{\sigma^T \mathrm{H}V(\mu)\sigma}{\sigma^T \mathrm{H}H(\mu)\sigma}$.
2. *Direction:* $D_{\nu-\mu}V(\mu)\geq 0$ and $F(\nu)>V(\mu)$.
3. *Precision:* $D_{\mu-\nu}\nu(\mu)\cdot\mathrm{H}H(\nu)(\nu-\mu)\leq 0$.
4. *Stopping time:* $\nu(\mu)\in E^C$.

There exists a nowhere dense set $K$ s.t. strict inequality holds on $E\backslash K$ in property 1,3,4.

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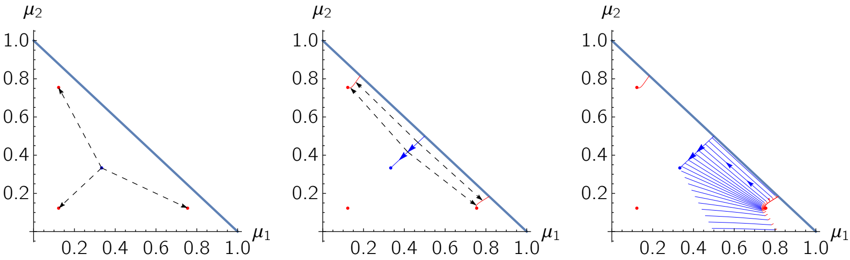


Figure: Optimal Policy of 3X3 problem.

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# Applications

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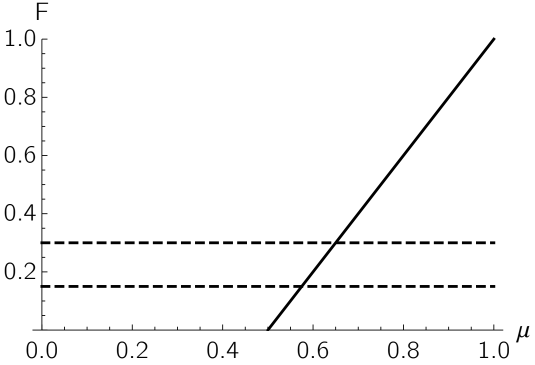

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**Proposition 1**

$\exists \mu_c$ s.t. $\forall \mu \in E$, firm $H$ innovates more radically *iff* $\mu > \mu_c$.
Moreover, $E \bigcap (0, \mu_c) \neq \emptyset$ and $E \bigcap (\mu_c, 1) \neq \emptyset$.

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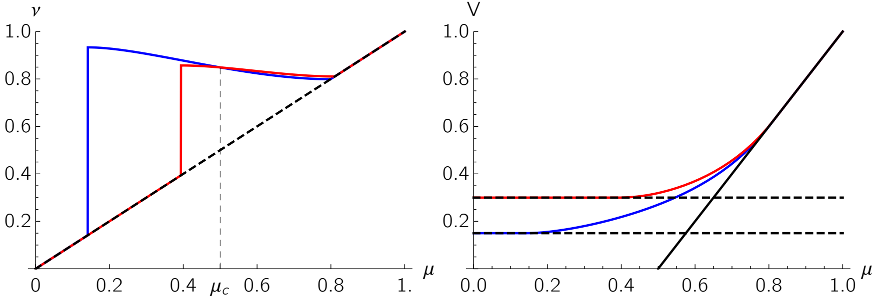

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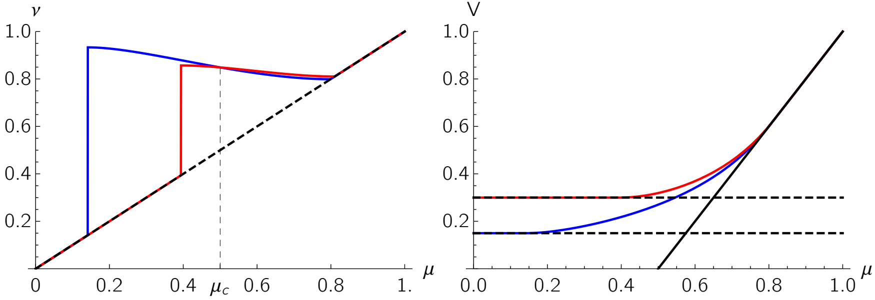


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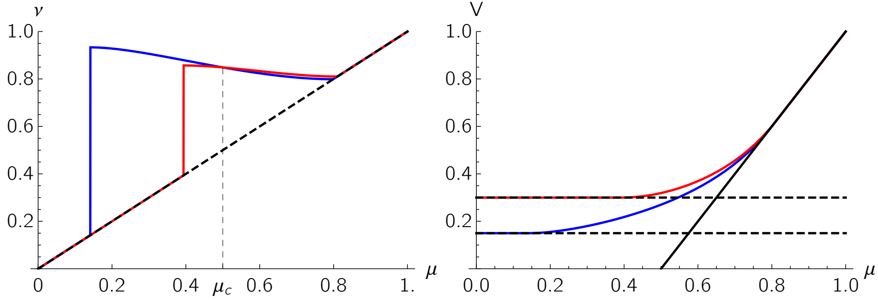


  - The first effect decreases with $\mu$, as value functions get closer.
  - The second effect increases with $\mu$, as adopting the risky options gets more likely.

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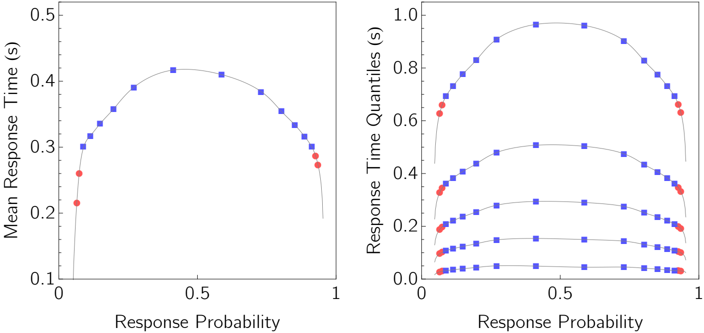

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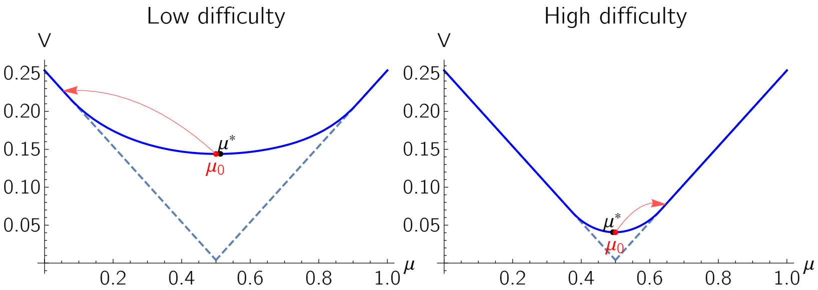

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## Response time

### Proposition 2

Suppose $|A|=2$, $H_0(\mu)$ and $F(\mu)$ are symmetric around $\mu_0=0.5$ and satisfy Assumption 3. $\forall$ partition of $\mathbb{R}^+$: $\{0, c_1 \ldots, c_K, \infty\}$, there exists $H(\mu)$ satisfying Assumption 3 such that then sign of $\mu^* - \mu_0$ alternates on each partition.

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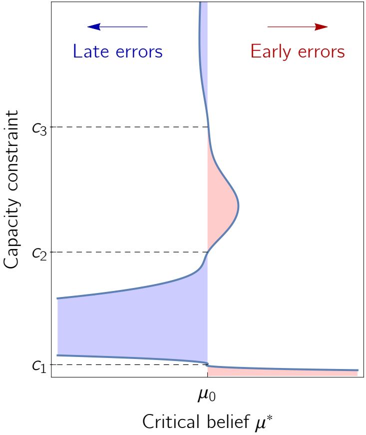

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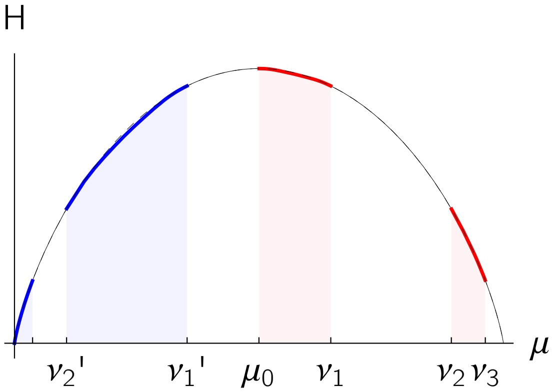

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  - Response time and decision accuracy.

# Reference I


Arrow, K. J., Blackwell, D., & Girshick, M. A. (1949). Bayes and minimax solutions of sequential decision problems. *Econometrica, Journal of the Econometric Society*, 213–244.

Boel, R., & Kohlmann, M. (1980). Semimartingale models of stochastic optimal control, with applications to double martingales. *SIAM Journal on Control and Optimization*, *18*(5), 511–533.

Caplin, A., & Dean, M. (2015). Revealed preference, rational inattention, and costly information acquisition. *The American Economic Review*, *105*(7), 2183–2203.

Caplin, A., Dean, M., & Leahy, J. (2017). *Rationally inattentive behavior: Characterizing and generalizing shannon entropy*. (Working Paper No. 23652). National Bureau of Economic Research. doi:10.3386/w23652

Che, Y.-K., & Mierendorff, K. (2016). Optimal sequential decision with limited attention.

Davis, M. H. (1979). Martingale methods in stochastic control. In *Stochastic control theory and stochastic differential systems* (pp. 85–117). Springer.

Frankel, A., & Kamenica, E. (2018). *Quantifying information and uncertainty*. Working paper.

Hébert, B., & Woodford, M. (2016). Rational inattention with sequential information sampling.

Kamenica, E., & Gentzkow, M. (2009). *Bayesian persuasion*. National Bureau of Economic Research.

Liang, A., Mu, X., & Syrgkanis, V. (2017). Optimal learning from multiple information sources.

Luce, R. D. et al. (1986). *Response times: Their role in inferring elementary mental organization*. Oxford University Press on Demand.

# Reference II


Matejka, F., & McKay, A. (2014). Rational inattention to discrete choices: A new foundation for the multinomial logit model. *The American Economic Review*, *105*(1), 272–298.

Morris, S., & Strack, P. (2017). *The wald problem and the equivalence of sequential sampling and static information costs*. Retrieved from https://ssrn.com/abstract=2991567

Moscarini, G., & Smith, L. (2001). The optimal level of experimentation. *Econometrica*, *69*(6), 1629–1644.

Ratcliff, R., Smith, P. L., Brown, S. D., & McKoon, G. (2016). Diffusion decision model: Current issues and history. *Trends in cognitive sciences*, *20*(4), 260–281.

Steiner, J., Stewart, C., & Matejka, F. (2016). *Rational inattention dynamics: Inertia and delay in decision-making*. Wiley Online Library.

Striebel, C. (1984). Martingale conditions for the optimal control of continuous time stochastic systems. *Stochastic processes and their applications*, *18*(2), 329–347.

Wald, A. (1947). Foundations of a general theory of sequential decision functions. *Econometrica, Journal of the Econometric Society*, 279–313.

Zhong, W. (2018). Time preference and information acquisition. *ArXiv e-prints*. arXiv: 1809.05120

# Discrete time foundation

## Discretization of equation (1)

- Given Assumption 1, $\forall \langle \mu_t \rangle$, $\tau$ admissible,
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  - $\langle \widehat{\mu}_i \rangle$ is discrete-time martingale and $\iota$ is stopping time.
  - Define Riemann sum:

$$W_{dt}(\widehat{\mu}_i, \iota) = E\left[ e^{-\rho \iota dt} F(\widehat{\mu}_\iota) - \sum_{i=0}^{\iota} e^{-\rho i dt} C\left( E\left[ \frac{H(\widehat{\mu}_{idt}) - H(\widehat{\mu}_{(i+1)dt})}{dt} \right] \right) dt \right]$$

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  - By definition, $V(\mu){=}\sup_{\langle\mu_t\rangle,\tau}\lim_{dt\to 0}W_{dt}(\widehat{\mu}_i,\iota)$.
- Consider a discrete-time stochastic control problem:
  - Define $W^*_{dt}(\mu){=}\sup_{\langle\widehat{\mu}_t\rangle,\iota}W_{dt}(\widehat{\mu}_i,\iota)$.
  - Obviously, $V(\mu){\leq}\lim_{dt\to 0}W^*_{dt}(\mu)$.

**Lemma 1**

$V(\mu){=}\lim_{dt\to 0}W^*_{dt}(\mu)$.

# Discrete time foundation

## Bellman equation

### Lemma 2 (Bellman equation)

$W^*_{dt} = V_{dt}$, where $V_{dt}$ solves Bellman equation:

$$V_{\mathrm{d}t}(\mu) = \max\left\{ F(\mu), \sup_{p,\nu,c} e^{-\rho \mathrm{d}t} \sum p_i V_{\mathrm{d}t}(\nu_i) - C_{\mathrm{d}t}\Big( H(\mu) - \sum p_i H(\nu_i) \Big) \right\} \tag{B-dt}$$
$$\text{s.t. } \sum p_i \nu_i = \mu$$

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### Lemma 3

Assume $H$ is strictly concave and $C^{(2)}$, Assumption 1 and Assumption 2 are satisfied, then if $V(\mu)$ solves HJB equation (B) and $V_{dt}$ solves equation (B-dt): $V_{dt} \xrightarrow[L_\infty]{dt \to 0} V$.

# Discrete time foundation

## Proof of theorem 1

### Theorem 1

Assume $H$ is strictly concave and $C^{(2)}$, Assumption 1 and Assumption 2 are satisfied, then $V(\mu)\in C^{(1)}$ solves equation (P) if $V(\mu)$ is a solution of:

$$\rho V(\mu)=\max\Bigg\{\rho F(\mu),\ \sup_{p,\nu,\sigma}\ p(V(\nu)-V(\mu)-\nabla V(\mu)(\nu-\mu))+\frac{1}{2}\sigma^T \mathrm{H}V(\mu)\sigma \\ -C\left(p(H(\mu)-H(\nu)+\nabla H(\mu)(\nu-\mu))-\frac{1}{2}\sigma^T \mathrm{H}H(\mu)\sigma\right)\Bigg\} \tag{B}$$

# Discrete time foundation

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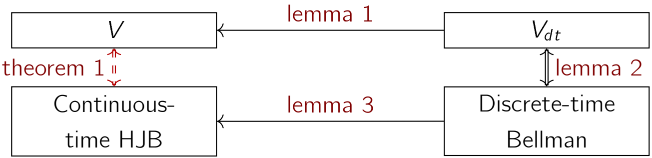
- Indirect method:
  - lemma 2: discrete-time Bellman $\Longleftrightarrow$ discrete-time value function.
  - lemma 1: discrete-time value function $\rightarrow$ continuous-time value function.
  - lemma 3: discrete-time Bellman $\rightarrow$ continuous-time HJB.



# Proof of lemma 3

## Convergence

**Proof.**

- Fix any $\mathrm{d}t$, $V_{\mathrm{d}t/2^k}(\mu)$ converges monotonically to $\bar{V}_{\mathrm{d}t}$. Because with period length $\mathrm{d}t/2^k$, any policy with $k' < k$ can be replicated by decomposition.

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- For any $\mathrm{d}t, \mathrm{d}t'$, $V_{\mathrm{d}t}$ should be less than $V_{\mathrm{d}t'}$. By slicing $dt'/2^k$, any policy with $\mathrm{d}t$ can be approximated arbitrarily well.

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- So there exists a unique $\overline{V} = \overline{V}_{\mathrm{d}t}$. Convergence speed is $O(\mathrm{d}t)$.

□

# Proof of lemma 3

## Equivalence

**Proof.**

- $\bar{V}$ is unimprovable. Suppose $c\frac{\bar{V}(\mu')-\bar{V}(\mu)-D\bar{V}(\mu,\mu')(\mu'-\mu)}{H(\mu)-H(\mu')+H'(\mu)(\mu'-\mu)}\geq\rho V(\mu)+\varepsilon$.

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Equivalence

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  - Pick $\mu_1$ close enough to $\mu$:
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## Equivalence

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  - This can be replicated in a $\mathrm{d}t_n=\frac{I(\mu_1,\mu'|\mu)}{c2^n}$ problem. Then $V_{\mathrm{d}t_n}$ will be improvable.

    Therefore $\bar{V}\geq V$.

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## Equivalence

**Proof.**

- $\overline{V}$ is unimprovable. Suppose $c\frac{\overline{V}(\mu')-\overline{V}(\mu)-D\overline{V}(\mu,\mu')(\mu'-\mu)}{H(\mu)-H(\mu')+H'(\mu)(\mu'-\mu)}\geq\rho V(\mu)+\varepsilon$.
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  - This can be replicated in a $\mathrm{d}t_n=\frac{I(\mu_1,\mu'|\mu)}{c2^n}$ problem. Then $V_{\mathrm{d}t_n}$ will be improvable. Therefore $\overline{V}\geq V$.
- Suppose $\overline{V}(\mu)>V(\mu)$, then $\forall\mathrm{d}t>0$ small enough, $V_{\mathrm{d}t}(\mu)\geq V(\mu)+\varepsilon$. Then $V$ will be improvable.

□

# Proof of theorem 2

## Construction of $V(\mu)$

- Step 1: Construct $\mu^*$.

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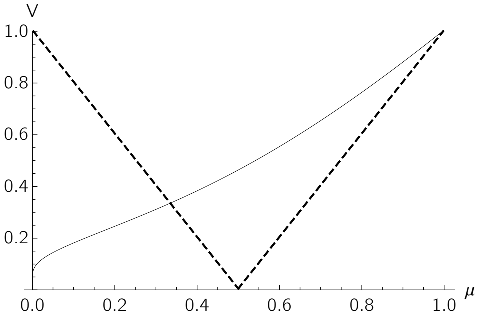

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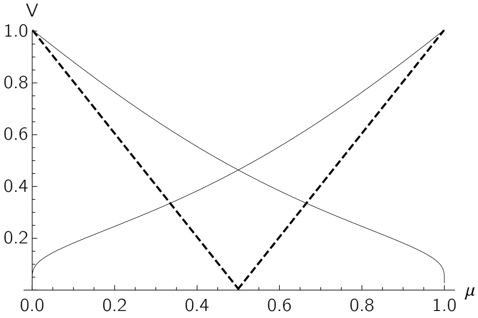
  - Calculate utility from searching $\nu>\mu^*$ (while assuming $V'=0$).
  - Calculate utility from searching $\nu<\mu^*$.



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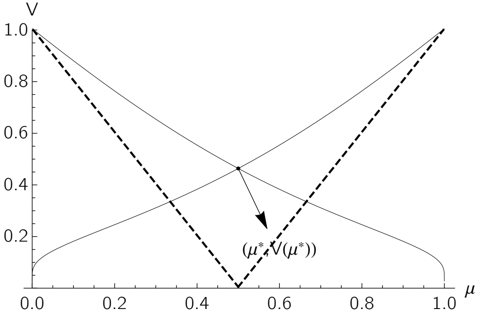
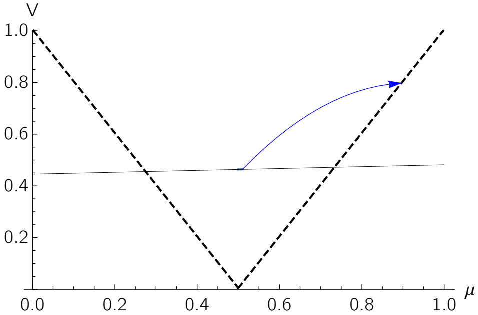
  - Calculate utility from searching $\nu>\mu^*$ (while assuming $V'=0$).
  - Calculate utility from searching $\nu<\mu^*$.
  - Unique intersection determines $\mu^*$ and $V(\mu^*)$.



# Proof of theorem 2

## Construction of $V(\mu)$

- Step 2: Construct $V(\mu)$.
  - Take $\mu^*$,$V(\mu^*)$,$V'(\mu^*)$=0 as starting point.
  - At $\mu^*+\mathrm{d}\mu$, take $V'=0$ and maximize $V$.
  - At $\mu^*+2\mathrm{d}\mu$, take $V'=\frac{V(\mu^*+\mathrm{d}\mu)-V(\mu^*)}{\mathrm{d}\mu}$ and maximize $V$.
  - Continue this process (with $\mathrm{d}\mu\to 0$). $V$ determined by ODE.



# Proof of theorem 2

## Multiple actions

- Step 3: Update value function by adding more actions.

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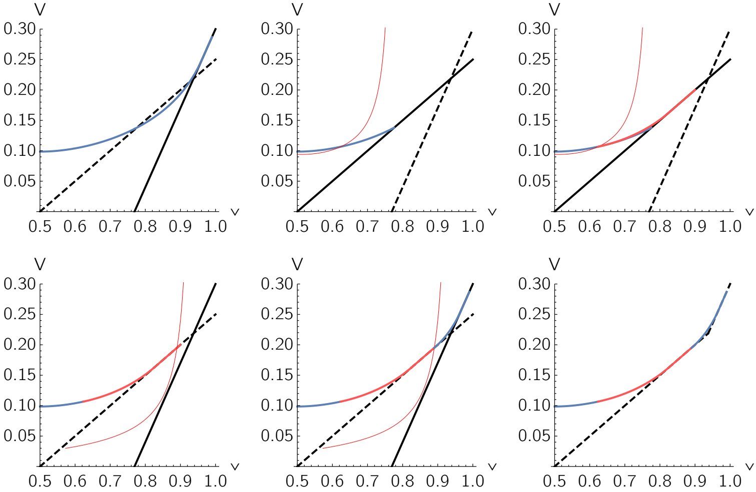


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# General cost structure

## Construction of the special cost

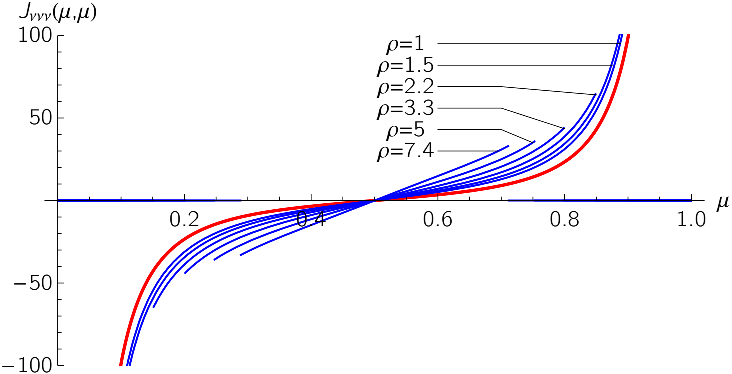
- Primitives:
  - $F(\mu)=\max\{1-2\mu,2\mu-1\}$; $\rho=\bar{l}=1$.
  - $H(\mu)=-\mu\log(\mu)-(1-\mu)\log(1-\mu)$—Entropy function.
- Suppose Gaussian learning is optimal:

$$V(\mu)=\frac{V''(\mu)}{J''_{\nu\nu}(\mu,\mu)}=-\frac{V''(\mu)}{H''(\mu)}$$

$$\iff V(\mu)=C_1 G^{2,0}_{2,2}\left(\begin{matrix}1+(-1)^{\frac{2}{3}},1-(-1)^{\frac{2}{3}}\\ 0,1\end{matrix}\middle|\mu\right)-C_2\mu_2F_1\left(1-(-1)^{\frac{1}{3}},1+(-1)^{\frac{2}{3}};2;\mu\right)$$

  - Apply smooth pasting to pin down $C_1,C_2$.
  - Optimality of Gaussian learning implies: $V(\mu)\geq\frac{V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)}{J(\mu,\nu)}$, $\forall\nu$.
- Define $J_0(\mu,\nu)=\frac{V(\nu)-V(\mu)-V'(\mu)(\nu-\mu)}{V(\mu)}$.
  1. $J_{0\nu\nu}(\mu,\mu)=-H''(\mu)$. $J_0$ satisfies Assumption 3.
  2. If $J_0$ is the cost function, then all strategies are equally optimal.

- Compare $J(\mu,\nu)$ and $J_0(\mu,\nu)$.
  - Gaussian learning supported by $J$ only if: $J(\mu,\nu) - J_0(\mu,\nu) = o((\nu-\mu)^3)$.



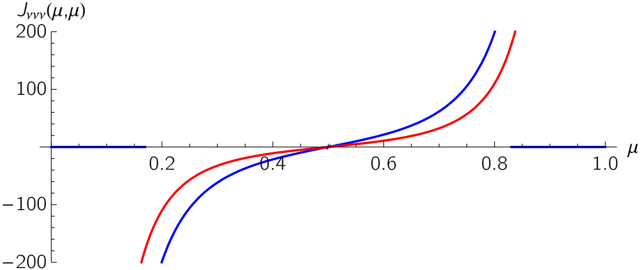
# General cost structure

## Construction of the special cost

- Suppose the cost depends only on the information structure:

$$\mathbf{P}=\begin{array}{c|cc} & l & r \\ \hline s_1 & 1-\lambda\sqrt{\beta}\,\mathrm{d}t & 1-\frac{\lambda}{\sqrt{\beta}}\mathrm{d}t \\ s_2 & \lambda\sqrt{\beta}\,\mathrm{d}t & \frac{\lambda}{\sqrt{\beta}}\mathrm{d}t \end{array}$$

  - Let cost be $\lambda\cdot(\log(\beta))^2$.
  - Then $J(\mu,\nu)=\frac{\left(\log\left(\frac{1-\mu}{\mu}\right)-\log\left(\frac{1-\nu}{\nu}\right)\right)^2}{\mu\sqrt{\frac{1-\mu}{1-\nu}\frac{\nu}{\mu}}+(1-\mu)\sqrt{\frac{1-\nu}{1-\mu}\frac{\mu}{\nu}}}$. Construct $J_0(\mu,\nu)$ from $J_{\nu\nu}(\mu,\mu)=\frac{2}{\mu^2(1-\mu)^2}$.

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# Deterministic decision time

- Learning stratgy with:
  - Deterministic decision time
  - Constant flow cost

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