Time preference and information acquisition

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Abstract. I consider the sequential implementation of a target information structure. I characterize the set of decision time distributions induced by all signal processes that satisfy a per-period learning capacity constraint. I find that all decision time distributions have the same expectation, and the maximal and minimal elements by mean-preserving spread order are deterministic distribution and exponential distribution. The result implies that when time preference is risk loving (e.g. standard or hyperbolic discounting), Poisson signal is optimal since it induces the most risky exponential decision time distribution. When time preference is risk neutral (e.g. constant delay cost), all signal processes are equally optimal.

Keywords: dynamic information acquisition, rational inattention, time preference
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1 Introduction

Consider a decision maker (DM) who is making a one-shot choice of action. The payoff of each action depends on an unknown state of the world. The DM can design a sequence of signal structures as her information source subject to a flow informativeness constraint. The informativeness of a signal structure is measured by a *posterior separable* measure. The DM is impatient and discounts future payoffs. Here I want to study a simple question: fix a target information structure, what is the optimal learning dynamics that implements this target information structure?

In Example 1, I solve this problem in a very simple setup. In the example, I consider three simple dynamic signal structures: pure accumulation of information before decision making, learning from a decisive signal arriving according to a Poisson process and learning from observing a Gaussian signal. The example suggests that different dynamic signal structures mainly differ in the induced decision time distribution. Since the form of discounting function prescribes the risk attitude on the time dimension, the discounting function (or time preference) should be a key factor determining the optimal dynamic signal structure.

Example 1. Consider a simple example with binary states $x = \{0, 1\}$. Prior belief is $\mu = 0.5$. Suppose the target information structure is full revelation (induces belief 0 or 1 each with 0.5 probability). I consider a model in continuous time. Let $H(\mu) = 1 - 4(\mu - 0.5)^2$, the flow information measure of belief process μ_t is $E[-\frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t) | \mathcal{F}_t]$ (the *uncertainty reduction speed*, introduced in Zhong (2017)). Assume flow cost constraint

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$c = 1$. DM has exponential discount function e^{-t}. I normalize expected decision utility from optimal action associated with full learning to be 1. The DM has three alternative learning strategies:

1. *Pure accumulation:* the DM uses up all resources pushing her posterior beliefs towards boundary. More precisely, at each prior μ, the strategy seeks posterior $\nu = 1 - \mu$ with arrival probability $p = \frac{1}{4(1-2\mu)^2}$[1]. The DM makes decision once her posterior arrives at 0 or 1.

   By standard property of Poisson process, the DM's posterior belief drifts towards 1 with speed $\frac{1}{4(2\mu-1)}$. Therefore, the dynamics of belief satisfies ODE:

$$\begin{cases} \dot{\mu} = \frac{1}{4(2\mu-1)} \\ \mu(0) = 0.5 \end{cases}$$

   It is easy to solve for $\mu(t) = \frac{1+\sqrt{t}}{2}$. As a result the DM's decision time is deterministic at $t = \mu^{-1}(1) = 1$. The expected utility from the pure accumulation strategy is $V_A = e^{-1} \approx 0.368$.

2. *Gaussian learning:* the DM observes a Gaussian signal, whose drift is the true state and variance is a control variable. By standard property of Gaussian learning, the DM's posterior belief follows a Brownian motion with zero drift. The flow variance of posterior belief process satisfies information cost constraint $E[-\frac{\mathrm{d}}{\mathrm{d}t}H(\mu_t)|\mathcal{F}_t] = -\frac{1}{2}\sigma^2 H''(\mu)\mathrm{d}t \leqslant c\mathrm{d}t$. Therefore, we can solve for $\sigma^2 = \frac{1}{4}$. Then value function is characterized by the HJB:

$$V(\mu) = \frac{1}{8}V''(\mu)$$

   with boundary condition $V(0) = V(1) = 1$. There is an analytical solution to the ODE:

$$V(\mu) = \frac{e^{2\sqrt{2}} + e^{4\sqrt{2}x}}{1 + e^{2\sqrt{2}}} e^{-2\sqrt{2}x}$$
$$\Longrightarrow V_G = V(0.5) \approx 0.459$$

3. *Poisson learning:* the DM learns states perfectly with a Poisson rate λ. If no information arrives, her belief stays at prior. By flow informativeness constraint $E[-\frac{\mathrm{d}}{\mathrm{d}t}H(\mu_t)|\mathcal{F}_t] = \lambda(H(\mu_t) - \frac{1}{2}H(1) - \frac{1}{2}H(0)) \leqslant c\mathrm{d}t \implies \lambda = 1$. The value function is characterized by the HJB:

$$\rho V_P = \lambda(1 - V_P)$$
$$\Longrightarrow V_P = 0.5$$

[1] This can be calculated using cost of Poisson signals $E[-\mathrm{d}H(\mu)] = (H(\mu) - H(\nu)p\mathrm{d}t + H'(\mu)(\nu-\mu))p\mathrm{d}t \leqslant c\mathrm{d}t$

Clearly:

$$V_P > V_G > V_A$$

Now we introduce the intuition why the values are ordered in this way. First, all of the three strategies induce the same expected decision time 1. This is due to the linearity of posterior separable information measure in compound experiments. The measure of a signal structure that fully reveals state at prior 0.5 is exactly 1, and it must equal the expected sum of all learning costs. Since in each continuing unit of time flow cost 1 is spent, expected learning time must be 1. Therefore, what determines expected decision utility is the dispersion of decision time distribution. Since discount function e^{-t} is a strictly convex function, a learning strategy that creates the most dispersed decision time attains the highest expected utility. Now let us study the decision time distribution induced by the three strategies:

1. Pure accumulation: $t = 1$ with probability 1. The decision time is deterministic.

2. Gaussian learning: With Gaussian learning, to characterize the decision time, it is equivalent to characterize the first passage time of a standard Brownian motion with two absorbing barriers:

$$\mathcal{T} = \min\left\{t \middle| \frac{1}{2} + \frac{1}{2}B_t = 0 \; or \; 1\right\}$$

Distribution of $\mathcal{T}$ can be solved by solving heat equation with two-sided boundary solution at $x = 0, 1$. There is no analytical solution (solution is only characterized by series). Here I numerically simulate this process:

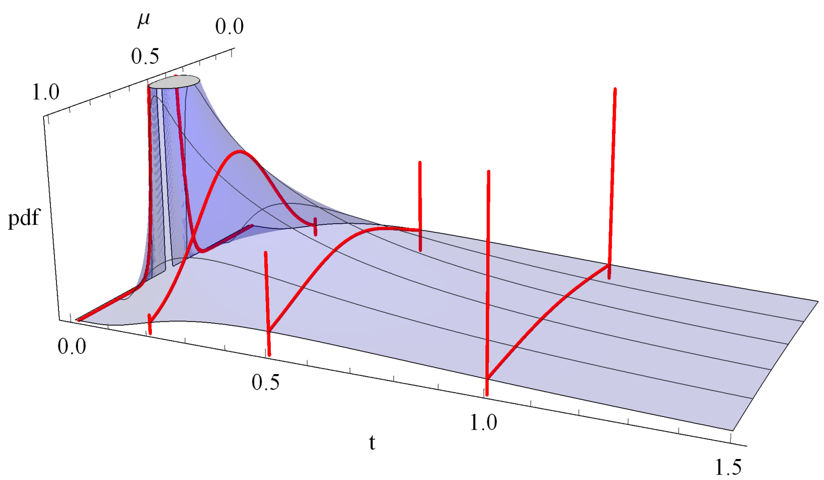

Figure 1: Belief distribution of Gaussian learning

Figure 1 depicts the evolution of distribution over posteriors over time. We can see that at any cross-section, the distribution over posteriors is a Normal distribution censored at two absorbing barriers [2]. The normal part is becoming flatter over time because learning leads to mean preserving spread of posterior beliefs.

[2] the distribution has point mass at 0, 1, represented by straight lines in figure. The relative height represents probability mass. But the point mass part and Normal part doesn't share same scale.

3. Poisson learning: As is calculated, the Poisson signal has a fixed arrival frequency $\lambda = 1$. The stopping time distribution can be calculated easily:

$$F(t) = 1 - e^{-t}$$

Evolution of posterior beliefs is shown in Figure 2:

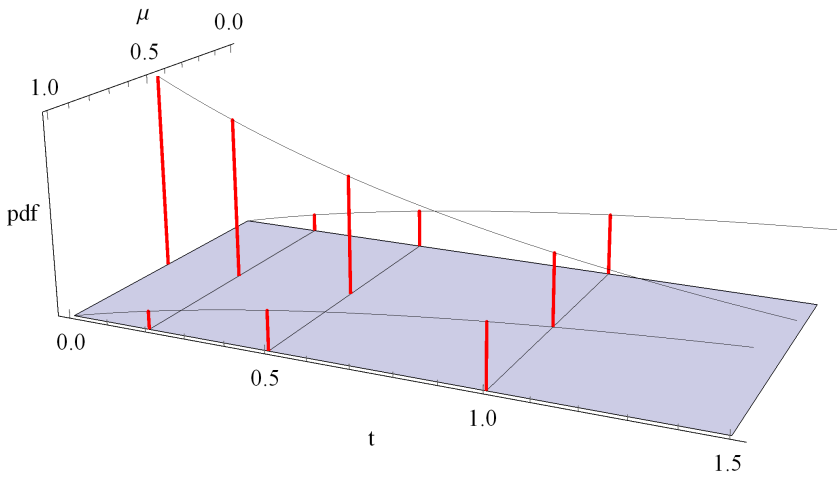

Figure 2: Belief distribution of Poisson learning

Figure 2 depicts the evolution of distribution over posteriors over time. At any cross-section, distribution over posterior has three mass points at prior and two target posteriors. The mass on prior is decreasing over time (following an exponential distribution) and the mass on posteriors is increasing over time.

Obviously, pure accumulation is always the worst since it has deterministic decision time. By comparing Figures 1 and 2, one can easily see the difference between Gaussian learning and Poisson learning: Gaussian learning accumulates some information that induces intermediate beliefs over time, while Poisson learning uses up all resources to draw conclusive signals. It seems the Poisson learning induces higher decision probability in the beginning and Gaussian learning induces higher decision probability later on (when prior is more dispersed). Therefore, Poisson learning has more dispersed decision time. We can verify this by plotting PDFs and integral of CDFs:

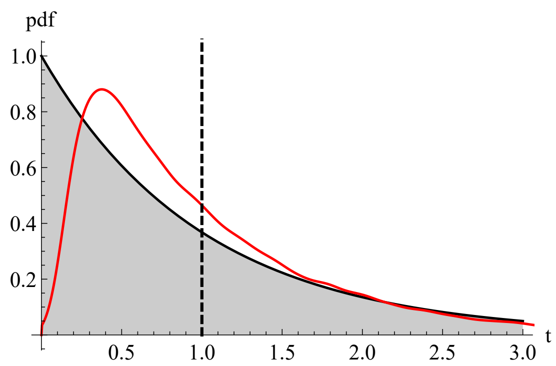

Figure 3: PDFs

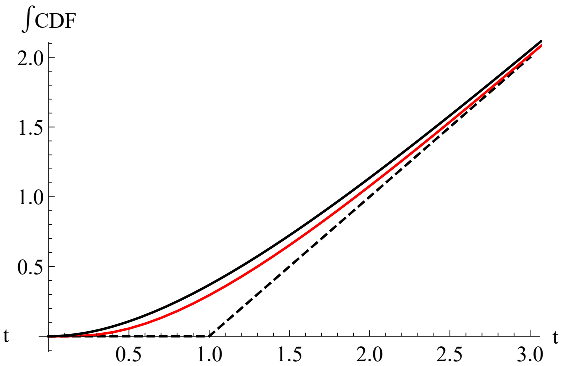

Figure 4: Integral of CDFs

In figure Figures 3 and 4, the black curves represent Poisson learning, the red curves represent Gaussian learning and the dashed lines represent pure accumulation. It is not hard to see from Figure 4 that decision time of Poisson learning is in fact a mean-preserving spread of that of Gaussian learning. So Poisson learning dominates Gaussian learning for not only exponential discounting, but also any convex discounting function.

In Example 1, I compare three kinds of dynamic learning strategies. These three strategies are chosen to be representative. First, these three strategies are simple heuristics that are very tractable. Second, these three strategies are also representative for three kinds of learning frameworks widely used in literature:

- Pure accumulation has a flavor of rational inattention models. Like in Matejka & McKay (2014), decision is made once and there is no dynamics. Even in dynamic rational inattention model like Steiner et al. (2017), information is acquired in one period, and there is no smooth of information. In this example, (decisive) learning has neither time dispersion nor cross-sectional dispersion with pure accumulation.
- Gaussian learning itself is well studied in literature, for example Moscarini & Smith (2001), Hébert & Woodford (2016). On the other hand, Gaussian learning is one kind of symmetric drift-diffusion model (Ratcliff & McKoon (2008)). Gaussian learning captures the idea of gradual learning both over time and over beliefs.
- Poisson learning relates to Wald's problem (Wald (1947), Che & Mierendorff (2016)). Poisson bandit is used as a building block for strategic experimentation models (see a survey by Hörner & Skrzypacz (2016)). My example considers a simplest stationary Poisson stopping strategy that stochastically reveals the true state. Poisson learning is only gradual over time, but is lump sum in belief space.

Example 1 suggests a key trade-off to be studied: gradual accumulation of information v.s. seeking decisive evidence. I want to learn how choice between gradual accumulation and decisive evidence seeking determines decision time distribution. In Section 2, I develop an information acquisition problem that imposes no restriction on the specific form of information a decision maker can acquire. The DM can choose an arbitrary random process as signals, and she observes signal realizations as her information. There are two constraints on the signal process. First, flow informativeness of the process is bounded. Second, the signal distribution conditional on stopping is fixed. If the DM chooses to learn gradually, then she is able to accumulate sufficient information before making any decision. After accumulating information, she can run the target experiment successfully with very high probability and achieves close to riskless decision time. On the contrary, if the DM chooses to only seek decisive signals, then they arrive only with low probabilities. So the corresponding decision time is more risky.

The main finding of this paper is that among all decision time distributions induced by feasible and exhaustive[3] learning strategies, the most dispersed decision time distribution is induced by decisive Poisson learning–only decisive signals arrive as Poisson process. Meanwhile, the least dispersed time distribution is induced by pure accumulation, as I already show in Example 1.

The paper is structured as follows. Section 2 setups a general discrete time information acquisition framework. Section 3 proves the main theorem. Section 4 extends the result to a continuous time model. Section 6 concludes.

2 Setup of model

The model is in discrete time. Consider a decision maker who has a discount function ρ_t decreasing and convex (both weakly) in time t. $\lim_{t\to\infty}\sum_{\tau=t}^{\infty}\rho_\tau = 0$. There is a finite state space X and action space A. Prior belief of the unknown payoff-relevant state is $\mu \in \Delta(X)$. The DM's goal is to implement a signal structure that induces distribution $\pi \in \Delta^2(X)$ over posterior beliefs[4]. By implementing a target signal structure, I mean conditional on stopping, the signal structure in current period must be a sufficient statistics for the target information structure. The informativeness of signal structure is measured by a posterior separable function $I(p_i, \nu_i|\mu) = \sum p_i(H(\mu) - H(\nu_i))$. In each period, the DM can acquire information for no more than c unit, i.e. $E\left[I(p_i^t, \nu_i^t|\mu^t)\right] \leqslant c$. The optimization problem is:

$$\sup_{\mathcal{S}_t,\mathcal{T}} E\left[\rho_{\mathcal{T}} u(\mathcal{A},\mathcal{X})\right] \tag{1}$$

$$\text{s.t.} \begin{cases} I\left(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t}\right) \leqslant c \\ \mathcal{X}\to\mathcal{S}_t\to\mathcal{A} \text{ conditional on } \mathcal{T}=t \\ \mathcal{X}\to\mathcal{S}_t\to\mathbf{1}_{\mathcal{T}\geqslant t} \end{cases}$$

where $\mathcal{T} \in \Delta\mathbb{N}, t \in \mathbb{N}$. $\mathcal{S}_{\mathbf{t-1}}$ is defined as summary of past information $(\mathcal{S}_1, \ldots, \mathcal{S}_{t-1})$. $\mathcal{S}_0 \equiv c_0$ is assumed to be degenerate. The objective function in Equation (1) is straight forward. The first constraint is the flow information cost constraint. The second constraint is the target information structure constraint. The remaining constraint is natural information process constraint.

Remark. This model is restrictive in design of information in the following sense: At any instant in time, conditional on stop, the information acquired must be sufficient for a time invariant $\mathcal{A}$. Other than this restriction, DM can freely choose her learning dynamics. This model does not necessarily cover Gaussian learning in general, but it does in symmetric cases (i.e. target posterior distribution and H are symmetric around prior μ).

I restrict learning dynamics in this way to abstract away from the fact that the optimal target information structure itself is changing over time, which creates time varying incentive for search direction, search precision and search intensity (highlighted in Zhong (2017)). In the current paper, I want to focus on the trade-off between gradual information accumulation and decision evidence seeking.

[3]A feasible strategy is exhaustive if it is not leaving any capacity unused or acquiring unrelated information.

[4]I equivalent represent state and signal realization as random variables $(\mathcal{X},\mathcal{A})$.

I assume that the DM follows suggestion of signal structure $\mathcal{A}$ in choosing action. This is WLOG since given any signal structure, the induced optimal action itself forms a Blackwell less informative signal structure. Therefore, the original learning strategy is still statistically sufficient for the direct signal structure. So if we take the optimization of $\mathcal{A}$ also into account, it is WLOG to assume $\mathcal{A}$ is a direct signal. Then the optimal implementation of $\mathcal{A}$ still follows solution to our problem. The optimization of $\mathcal{A}$ is studied in Section 5.1.

3 Solution

3.1 Relaxed problem

Let $\bar{I} = I(\mathcal{A};\mathcal{X})$ and $V^* = E[u(\mathcal{A};\mathcal{X})]$. Consider a relaxed problem which only tracks average accumulated information measure I rather than the whole signal process conditional on all histories:

$$\sup_{p_t} \sum_{t=1}^{\infty} \rho_t (1 - P_{t-1}) p_t V^* \tag{2}$$

$$\text{s.t.} \begin{cases} (\bar{I} - I_t) p_t + (I_{t+1} - I_t)(1 - p_t) \leqslant c \\ P_t = P_{t-1} + (1 - P_{t-1}) p_t \\ P_0 = 0, I_1 = 0 \end{cases}$$

where $p_t \in [0,1]$ and $I_t \geqslant 0$. $1 - P_{t-1}$ is the surviving probability at period t, p_t is the conditional stop probability. I_t is the information measure of the whole path of non-stopping signals up to period t.

The relaxed problem Equation (2) captures a key feature of posterior separable information measure: I_t is accumulated linearly overtime and the information measure required to implement $\mathcal{S}$ is exactly the remaining information measure $\bar{I} - I_t$. It is more relaxed than Equation (1) in the following sense: in Equation (1), the flow informativeness constraint is imposed on all histories of $\mathcal{S}_{t-1}$ and $\mathbf{1}_{\mathcal{T} \leqslant t}$. However, in Equation (2), the first constraint is imposed only on average. p_t can be interpreted as the expected stopping probability and I_t's as the expected accumulated informativeness.

Lemma 1. *Value from solving Equation (1) is no larger than value from solving Equation (2)*

Lemma 1 verifies the previous intuition. Now I first focus on solving Equation (2), to provide some clue for solving the original problem Equation (1).

Theorem 1. $p_t \equiv \frac{c}{\bar{I}}$ *solves Equation (2).*

Theorem 1 states that the relaxed problem Equation (2) has a simple solution: no information should ever be accumulated. $I_t \equiv 0$ and the optimal value equals $\sum_{t=0}^{\infty} \rho_t \left(1 - \frac{c}{\bar{I}}\right)^{t-1} \frac{c}{\bar{I}} V^*$. I prove Theorem 1 by approximating the convex discount function ρ_t with finite summation of linear ones. For each linear discount function, I prove by backward induction that choosing $I_t \equiv 0$ is optimal.

3.2 Optimal learning dynamics

By Lemma 1 and Theorem 1, to solve Equation (1), it is sufficient to show that

$$\sum_{t=1}^{\infty} \rho_t \left(1 - \frac{c}{\bar{I}}\right)^{t-1} \frac{c}{\bar{I}} V^* \tag{3}$$

is attainable by feasible strategy in Equation (1). Consider the following experimentation strategy, $\mathcal{A}$ is observed with probability $\frac{c}{\bar{I}}$ in each period. If $\mathcal{A}$ is successfully observed, the corresponding action is taken. If not, go to the next period and follow the same strategy. Formally, $\mathcal{S}_t$ and $\mathcal{T}$ are defined as follows. Let $s_0, c_0 \notin A$ be two distinct degenerate signals.

$$\begin{aligned} \mathcal{S}_t =& \begin{cases} s_0 & \text{with probability } 1 \text{ if } \mathcal{S}_{t-1} \in A \bigcup \{s_0\} \\ \mathcal{A} & \text{with probability } \frac{c}{\bar{I}} \text{ if } \mathcal{S}_{t-1} = c_0 \\ c_0 & \text{with probability } 1 - \frac{c}{\bar{I}} \text{ if } \mathcal{S}_{t-1} = c_0 \end{cases} \\ \mathcal{T} =& t \ \text{ if } \mathcal{S}_t \in A \end{aligned} \tag{4}$$

Then it is not hard to verify that:

- *Objective function*:

$$\begin{aligned} & E[\rho_{\mathcal{T}} u(\mathcal{A}, \mathrm{X})] \\ =& \sum_{t=1}^{\infty} \rho_t \mathrm{P}(\mathcal{S}_t \in A) E[u(\mathcal{A}, \mathcal{X})] \\ =& \sum_{t=1}^{\infty} \rho_t \prod_{\tau=0}^{t-1} \mathrm{P}(\mathcal{S}_\tau = c_0 | \mathcal{S}_{\tau-1} = c_0) \mathrm{P}(\mathcal{S}_t \in A | \mathcal{S}_{t-1} = c_0) E[u(\mathcal{A}, \mathcal{X})] \\ =& \sum_{t=1}^{\infty} \rho_t \left(1 - \frac{c}{\bar{I}}\right)^t \frac{c}{\bar{I}} V^* \end{aligned}$$

- *Capacity constraint*:

$$\begin{aligned} & I(\mathcal{S}_t; \mathcal{X} | \boldsymbol{\mathcal{S}_{t-1}}, \mathbf{1}_{\mathcal{T} \geqslant t}) \\ =& \mathbf{1}_{\mathcal{S}_{t-1} = c_0} I(\mathcal{S}_t; \mathcal{X} | \{c_0\}, 1) + \mathbf{1}_{\mathcal{S}_{t-1} \neq c_0} I(s_0; \mathcal{X} | \boldsymbol{\mathcal{S}_{t-1}}, 0) \\ =& \mathbf{1}_{\mathcal{S}_{t-1} = c_0} \left(\mathrm{P}(\mathcal{S}_t \in A) I(\mathcal{A}; \mathcal{X}) + \mathrm{P}(\mathcal{S}_t = c_0) I(c_0; \mathcal{X})\right) + \mathbf{1}_{\mathcal{S}_{t-1} \neq c_0} \cdot 0 \\ =& \mathbf{1}_{\mathcal{S}_{t-1} = c_0} \cdot \frac{c}{\bar{I}} \cdot \bar{I} \leqslant c \end{aligned}$$

- *Decision time distribution*:

$$P_t = \mathrm{P}(\mathcal{T} \leqslant t) = 1 - \left(1 - \frac{c}{\bar{I}}\right)^t \tag{5}$$

I show that Equation (4) implements expected utility level Equation (3), hence solves Equation (1). It is easy to see that Equation (4) induces expected decision time $\frac{\bar{I}}{c}$. By

Lemma 2, $\frac{I}{c}$ is the lower bound of expected decision time for all feasible strategies. In fact, the proof of Lemma 2 suggests that $E[\mathcal{T}] > \frac{I}{c}$ only when there is some waste of information: either capacity constraint c is not fully used, or $\mathcal{S}_t$ contains strictly more information than $\mathcal{A}$ conditional on taking action.

Lemma 2. *Let $(\mathcal{S}_t, \mathcal{T})$ be information acquisition strategy satisfying constraints in Equation (1), then $E[\mathcal{T}] \geqslant \frac{I}{c}$.*

I call an information acquisition strategy *exhaustive* if corresponding $E[\mathcal{T}] = \frac{I}{c}$. The decision time distribution P_t induced by strategy Equation (4) is a exponential distribution with parameter $\frac{c}{I}$. Equation (4) being the optimal strategy, independent of choice of ρ_t implies that $\forall \rho_t$, $\forall$ information acquisition strategy $(\widetilde{\mathcal{S}}_t, \widetilde{\mathcal{T}})$:

$$\begin{aligned} & E[\rho_{\widetilde{\mathcal{T}}} u(\mathcal{A};\mathcal{X})] \leqslant E\left[\rho_{\mathcal{T}} u(\mathcal{A};\mathcal{X})\right] \\ \Longrightarrow \ & E[\rho_{\widetilde{\mathcal{T}}}] \leqslant E[\rho_{\mathcal{T}}] \end{aligned}$$

Since ρ_t ranges over all positive decreasing convex functions, P_t as distribution over time is second order stochastically dominated. Summarizing the analysis above, I get Theorem 2.

Theorem 2. *Equation (4) solves Equation (1). The decision time distribution of any feasible and exhaustive information acquisition strategy second order stochastically dominates P_t.*

3.3 Gradual learning v.s. decisive evidence

My analysis illustrates the gradual learning v.s. decisive evidence trade-off in the flexible learning environment. The trade-off is: the speed of future learning depends on how much information the DM has already possessed. Accumulating more information today speeds up future learning. So the DM is choosing between naively learning just for today or learning for the future. If all resources are invested in seeking decisive evidence, then signal arrives at a constant low probability, and the decision time distribution is dispersed. If some resources are invested in information accumulation, then learning will accelerate, at a cost of lower (or even zero) arrival rate of decisive signals in the early stage. As a result the decision time is less dispersed.

Surprisingly, when the decision maker has convex discounting function, decisive evidence seeking is optimal. The intuition behind this result is natural. Convex discounting function means that the decision maker is risk loving towards decision time. Seeking decisive evidence is the riskiest learning strategy one can take: it payoff quickly with high probability, but if it fails, learning is very slow in future. In practice, evidence seeking is a very natural learning strategies. A researcher tends to form a hypothesis, then seeks evidence either confirms or contradicts the hypothesis. Usually there is a clear target of what to prove (the hypothesis), and what kind of signals (data from experiments) proves/contradicts the hypothesis. Running the research protocol itself is usually more mechanical than the designing stage. What is common in natural science is that the principal investigator designs the whole research plan. Then all experiments, data collections and computations are run by doctoral students. PI usually diversifies over dozens of projects, so he can enjoy the expected payoff from this risky project design.

Two elements in my framework are key to this result. First is the flexibility in design of signal process. In contrast to my framework, if one consider a dynamic information acquisition problem with highly parametrized information process, then other kind of trade-offs tied to the parametrization constraints might have first order effects. For example, if one only allows Poisson learning or Gaussian learning, then the trade-off of gradual learning and decisive evidence is directly assumed away. As a result the choice among signal types (Che & Mierendorff (2016),Liang et al. (2017)) or the trade-off between intensity and information cost (Moscarini & Smith (2001)) becomes first order important. If one only allows DM to choose between to learn or not to learn in each period, then the trade-off between exploration and exploitation becomes first order. Meanwhile, in my framework, the DM can freely design the optimal signal type, and the corresponding decision time. So the aforementioned trade-offs actually do not exist, the trade-off between gradual learning and decisive evidence becomes central to the analysis.

Second is the posterior separability assumption on information measure. Posterior separability is equivalent to the linear additivity of compound signal structures (see the discussion in Section 7.1 of Zhong (2017)). This assumption restricts the relative price between gradual learning and evidence seeking. Any amount of informativeness invested today to accumulate information transfers one-to-one to the amount of reduction of information cost tomorrow. Lemma 2 shows that the expected decision time is identical for all feasible and exhaustive learning strategy. As a result the trade-off between gradual learning and decisive learning translates to choice of dispersion of decision time distribution. If one assumes either sub-additivity or super-additivity in informativeness measure, then choosing different learning strategies might also change the expected decision time, which makes my key trade-off entangled with other effects.

4 Continuous time

In this section, I study a continuous time version of Equation (1). Let $\rho_t : \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing, convex and integrable discounting function. Let $F(\mu) = \sup_a E_\mu[u(a,x)]$. Consider stochastic control problem:

$$\sup_{\langle \mu_t \rangle, \mathcal{T}} E\left[\rho_\tau E_\pi[F(\mu)]\right] \tag{6}$$

$$\text{s.t.} \begin{cases} E\left[\mu_{t+s} | \mu_t\right] = \mu_t \\ -E\left[\frac{\mathrm{d}}{\mathrm{d}t} H(\mu_t) \middle| \mathcal{F}_t\right] \leqslant c\mathrm{d}t \\ \mu_0 = \mu, \ \mu_t\big|_{\tau=t} \sim \pi \end{cases}$$

where τ has continuous cdf and is measurable to μ_t. The objective function of Equation (6) is the same as that of Equation (1). In the stochastic control problem, the decision maker chooses the optimal posterior belief process $\langle \mu_t \rangle$ and stopping time τ, subject to 1) stopping time measurable to belief process. 2) belief process is a martingale. 3) flow increase in informativeness measure is bounded by c. 4) conditional on stopping time, μ_t has distribution π.

Equation (6) is a continuous time extension of Equation (1). However, I haven't not formally specify how a stochastic process of posterior beliefs can be induced by a stochastic information acquisition strategy. Equation (6) is constructed by taking analog of Equation (1). Let $V^* = E_\pi[F(\mu)]$. Then

Lemma 3. *Value from solving Equation (6) is no larger than value from solving Equation (7).*

$$V = \sup_{p_t} \int_0^\infty \rho_t \left(1 - P_t\right) p_t V^* \mathrm{d}t \tag{7}$$

$$\text{s.t. } \begin{cases} I_0 = 0,\ I_t \geqslant 0,\ \dot{I}_t \leqslant c - p_t(\bar{I} - I_t) \\ P_0 = 0,\ \dot{P}_t = (1 - P_t) p_t \end{cases}$$

where p_t is positive integrable function.

Theorem 3. $p_t \equiv \frac{c}{\bar{I}}$ *solves Equation (7).*

Lemma 3 and Theorem 3 are exactly the continuous time analog of Lemma 1 and Theorem 1. Lemma 3 states that Equation (7) is a relaxed problem of Equation (2). Theorem 3 characterizes the solution of Equation (7): no information should ever be accumulated. $I_t \equiv 0$ and the optimal value equals $\int_0^\infty \rho_t e^{-\frac{c}{\bar{I}}t} \frac{c}{\bar{I}} V^* \mathrm{d}t$. Theorem 3 is proved by discreizing the continuous time problem and invoking the result of Theorem 1.

4.1 Implementation

By Lemma 3 and Theorem 3, to solve Equation (6), it is sufficient to show that:

$$\int_0^\infty \rho_t e^{-\frac{c}{\bar{I}}t} \mathrm{d}t \frac{c}{\bar{I}} V^*$$

can be attained in Equation (6). Consider the following information acquisition strategy. Let ν be a random variable with distribution π and define:

$$\begin{cases} \mathrm{d}\mu_t = (\nu - \mu_t) \cdot \mathrm{d}N_t \\ \tau = t \text{ if } \mathrm{d}N_t{=}1 \end{cases} \tag{8}$$

where N_t a standard Poisson counting processes with parameter $\frac{c}{\bar{I}}$ and independent to ν. $\langle \mu_t \rangle$ is by definition a stationary compound Poisson process. The jump happens when the Poisson signal arrives and belief jumps to posteriors according to distribution π. Once the jump occurs, decision is made immediately. It is easy to verify:

- *Martingale property*: Each $\mathrm{d}N_t - \frac{c}{\bar{I}}\mathrm{d}t$ is martingale.

$$\begin{aligned} E[\mathrm{d}\mu_t|\mu_t] =& E[(\nu - \mu_t) \cdot \mathrm{d}N_t] \\ =& E_\pi\left[E[(\nu - \mu) \cdot \mathrm{d}N_t|\nu]\right] \\ =& E_\pi\left[(\nu - \mu) \cdot E\left[\mathrm{d}N_t - \frac{c}{\bar{I}}\mathrm{d}t\right]\right] + E_\pi\left[(\nu - \mu) \cdot \frac{c}{\bar{I}}\mathrm{d}t\right] \\ =& 0 \end{aligned}$$

  therefore, μ_t is a martingale. Second equality is law of iterated expectation. Third equality is by martingale assumption of ν and $\mathrm{d}N_t - \frac{c}{\bar{I}}\mathrm{d}t$ being martingale.

- *Capacity constraint*: If $\mathrm{d}N_t \geqslant 1$, then $E[-\mathrm{d}H(\mu_t)|\mu_t] = 0 \leqslant c$. If $\mathrm{d}N_t < 1$, then by Ito formula for jump process:

$$\begin{aligned}
\mathrm{d}H(\mu_t) =&(H(\nu) - H(\mu)) \cdot \mathrm{d}N_t \\
\implies \quad E[-\mathrm{d}H(\mu_t)|\mu_t] =&E_\pi \left[E\left[(H(\mu_t) - H(\nu)) \cdot \mathrm{d}N_t|\nu\right]\right] \\
=&E_\pi \left[\frac{c}{\overline{I}}(H(\mu_t) - H(\nu))\mathrm{d}t\right] \\
=&c\mathrm{d}t
\end{aligned}$$

  Second equality is law of iterated expectation. Third equality is martingale property of $\mathrm{d}N_t - \frac{c}{\overline{I}}\mathrm{d}t$.

- *Decision time distribution*:

$$P_t = 1 - e^{-\frac{c}{\overline{I}}t}$$

  Therefore, Equation (8) implements utility level $\int_0^\infty \rho_t \frac{c}{\overline{I}} e^{-\frac{c}{\overline{I}}t} V^* \mathrm{d}t$.

Lemma 4. *Let $(\mu_t, \mathcal{T})$ be information acquisition strategy satisfying constraints in Equation (6), then $E[\mathcal{T}] \geqslant \frac{\overline{I}}{c}$.*

As in the discrete case, I call an information acquisition strategy *exhaustive* if the corresponding $E[\mathcal{T}] = \frac{\overline{I}}{c}$. Since Equation (8) is optimal independent of choice of convex ρ_t, previous analysis implies Theorem 4.

Theorem 4. *Equation (8) solves Equation (6). The decision time distribution of any feasible and exhaustive information acquisition strategy second order stochastically dominates P_t.*

5 Discussion

5.1 Optimal target signal structure

In this section, I solve for the optimal target signal structure in decision problem Equation (6). Assume that ρ_t is differentiable. By Theorem 4, the optimization problem can be written as:

$$\begin{aligned}
\sup_{\pi \in \Delta^2(X)} & \int_0^\infty \rho_t e^{-\frac{c}{H(\mu) - E_\pi[H(\nu)]}t} \mathrm{d}t \cdot \frac{c \cdot E_\pi[F(\nu)]}{H(\mu) - E_\pi[H(\nu)]} \\
\text{s.t. } & E_\pi[\nu] = \mu
\end{aligned} \tag{9}$$

Define $f(V^1, V^2) = \int_0^\infty \rho_t e^{-\frac{c}{H(\mu) - V^1}t} \mathrm{d}t \cdot \frac{c \cdot V^2}{H(\mu) - V^1}$. Then it is not hard to verify that $f(V^1, V^2)$ is differentiable[5] in V^1, V^2. Apply Theorem 2 of Zhong (2018), a necessary condition for π^* solving Equation (9) is:

$$\pi^* \in \arg \max_{\substack{\pi \in \Delta^2(X) \\ E_\pi[\nu] = \mu}} E_\pi \left[F(\nu) + \frac{\int_0^\infty (-\dot{\rho}_t)\, e^{-\frac{c}{H(\mu) - E_{\pi^*}[H(\nu)]}t} \frac{E_{\pi^*}[F(\nu)]}{H(\mu) - E_{\pi^*}[H(\nu)]} t \mathrm{d}t}{\int_0^\infty \rho_t e^{-\frac{c}{H(\mu) - E_{\pi^*}[H(\nu)]}t} \mathrm{d}t} \cdot H(\nu) \right]$$

[5]Differentiabiliy can be shown by definition, noticing that $e^{-\frac{c}{H(\mu) - V^1}t} \cdot t$ is absolutely integrable.

Notice that the objective function is the expectation of the linear combination of two belief dependent functions. If we define:

$$g(x) = \frac{\int_0^\infty (-\dot{\rho}_t) e^{-\frac{c}{H(\mu)-x}t} \frac{t}{H(\mu)-x} dt}{\int_0^\infty \rho_t e^{-\frac{c}{H(\mu)-x}t} dt}$$

Then by the standard argument in Bayesian persuasion, π^* can by characterized by concavifying the *gross value function* $F + (g(E_{\pi^*}[H(\nu)]) \cdot E_{\pi^*}[F(\nu)]) H$. Moreover, by Theorem 1 of Zhong (2018), there exists π^* with support size $2|X|$ solving Equation (9). So I get the following characterization:

Proposition 1. *There exists π^* solving Equation (9) and $|\text{supp}(\pi^*)| \leqslant 2|X|$. Let $\lambda = g(E_{\pi^*}[H(\nu)]) \cdot E_{\pi^*}[F(\nu)]$, any such π^* satisfies:*

$$\pi^* \in \arg \max_{\substack{\pi \in \Delta^2(X) \\ E_\pi[\nu]=\mu}} E_\pi[F(\nu) + \lambda \cdot H(\nu)]$$

Suppose the discounting function is a standard exponential function: $\rho_t = e^{-\rho d}$, then $g(x) = \frac{\rho}{c+\rho(H(\mu)-x)}$. Notice that objective function:

$$V(\mu) = \int_0^\infty e^{-\left(\rho + \frac{c}{H(\mu)-E_{\pi^*}[H(\nu)]}\right)t} \frac{c \cdot E_{\pi^*}[F(\nu)]}{H(\mu) - E_{\pi^*}[H(\nu)]} dt = \frac{c \cdot E_{\pi^*}[F(\nu)]}{c + \rho(H(\mu) - E_{\pi^*}[H(\nu)])}$$

Therefore, optimality condition becomes:

$$\pi^* \in \arg \max_{\substack{\pi \in \Delta^2(X) \\ E_\pi[\nu]=\mu}} E_\pi \left[F(\nu) + \frac{\rho}{c} V(\mu) H(\nu) \right] \tag{10}$$

Equation (10) is very similar to optimality condition derived in Zhong (2017), where optimal posterior is solved from concavifying $V(\cdot) + \frac{\rho}{c} V(\mu) H(\cdot)$. The problem solved in Zhong (2017) is the continuous time limit of Equation (1) without constraint on constant target signal structure and exponential discounting. In both problems, $\frac{\rho}{c} V(\mu)$ is adjusting the concavity of gross value function. Therefore, higher continuation value corresponds to more concave gross value function and less informative signal structure. This suggests that the monotonicity in precision-frequency trade-off is extended to our model as well. In Zhong (2017), the trade-off is illustrated as decrease in precision of target information structure at each decision time. In the current paper, target information structure is forced to be constant over time. However, if I endogenize target information structure, then at more extreme prior beliefs associated with higher decision value, less informative target information structure is optimal (and corresponding expected waiting time is shorter).

6 Conclusion

In this paper, I characterize the decision time distributions that can be induced by a dynamic information acquisition strategy, and study how time preference determines the optimal form of learning dynamics. No restriction is placed on the form of information acquisition strategy, except for a fixed target signal structure and a flow informativeness constraint.I find that all decision time distributions have the same expectation, and the maximal and minimal elements by mean-preserving spread order are deterministic distribution and exponential distribution. The result implies that when time preference is risk loving (e.g. standard or hyperbolic discounting), Poisson signal is optimal since it induces the most risky exponential decision time distribution. When time preference is risk neutral (e.g. constant delay cost), all signal processes are equally optimal.

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A Omitted proofs

A.1 Proof for Lemma 1

Proof.

Step 1. Value from solving Equation (1) is no larger than value from solving Equation (11):

$$\sup_{\mathcal{S}_t,\mathcal{T}} E\left[\rho_{\mathcal{T}} u(\mathcal{A},\mathcal{X})\right] \tag{11}$$

$$\text{s.t.} \begin{cases} E\left[I\left(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t}\right)|\mathcal{T}\geqslant t\right]\leqslant c \\ \mathcal{X}\to\mathcal{S}_t\to\mathcal{A} \text{ conditional on } \mathcal{T}=t \\ \mathcal{X}\to\mathcal{S}_t\to\mathbf{1}_{\mathcal{T}\geqslant t} \end{cases}$$

Equation (11) is more relaxed than Equation (1) in the first constraint. In Equation (1), the flow cost constraint is imposed on each prior induced by previous information and decision choice. Equation (11) only requires the average cost conditional on not having stopped yet being bounded by c:

$$\begin{aligned} & I(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t})\leqslant c \\ \Longrightarrow\ & E\left[I\left(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t}\right)|\mathcal{T}\geqslant t\right]\leqslant E\left[c|\mathcal{T}\geqslant t\right]=c \end{aligned}$$

Therefore, any feasible strategy for Equation (1) is feasible for Equation (11). So Equation (11) is a more relaxed problem than Equation (1).

Step 2. Value from solving Equation (11) is no larger than value from solving Equation (2). $\forall\ (\mathcal{S}_t,\mathcal{T})$ satisfying constraints in Equation (11), define:

$$\begin{cases} I_t = I\left(\mathcal{S}_{\mathbf{t-1}};\mathcal{X}|\mathcal{T}\geqslant t\right) \\ p_t = \mathrm{P}(\mathcal{T}=t|\mathcal{T}\geqslant t) \\ P_t = \mathrm{P}(\mathcal{T}\leqslant t) \end{cases}$$

Want to show that (I_t,p_t) is feasible and implements same utility in Equation (2) as $(\mathcal{S}_t,\mathcal{T})$ in Equation (11). First, consider the objective function:

$$\begin{aligned} & E\left[\rho_{\mathcal{T}} u(\mathcal{A};\mathcal{X})\right] \\ =& \sum_{t=0}^{\infty}\mathrm{P}(\mathcal{T}=t)\rho_t E\left[u(\mathcal{A};\mathcal{X})|\mathcal{T}=t\right] \\ =& \sum_{t=0}^{\infty}\mathrm{P}(\mathcal{T}=t|\mathcal{T}\geqslant t)\mathrm{P}(\mathcal{T}\geqslant t)\rho_t V^* \\ =& \sum_{t=0}^{\infty}\rho_t(1-P_{t-1})p_t V^* \end{aligned}$$

Second, consider feasibility constraint:

$$\begin{aligned} c \geqslant & E\left[I\left(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t}\right)|\mathcal{T}\geqslant t\right] \\ =&\mathrm{P}(\mathcal{T}=t|\mathcal{T}\geqslant t)E\left[I(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t})|\mathcal{T}=t\right] \\ &+\mathrm{P}(\mathcal{T}>t|\mathcal{T}\geqslant t)E\left[I(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t})|\mathcal{T}>t)\right] \\ =&p_t\left(I(\mathcal{S}_{\mathbf{t}},\mathbf{1}_{\mathcal{T}\geqslant t};\mathcal{X}|\mathcal{T}=t)-I(\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t};\mathcal{X}|\mathcal{T}=t)\right) \\ &+(1-p_t)\left(I(\mathcal{S}_{\mathbf{t}},\mathbf{1}_{\mathcal{T}\geqslant t};\mathcal{X}|\mathcal{T}>t)-I(\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t};\mathcal{X}|\mathcal{T}>t)\right) \end{aligned}$$

$$
\begin{aligned}
&= p_t I(\mathcal{S}_\mathbf{t};\mathcal{X}|\mathcal{T}=t) + (1-p_t) I(\mathcal{S}_\mathbf{t};\mathcal{X}|\mathcal{T}>t)\\
&- (p_t I(\mathcal{S}_\mathbf{t-1},\mathbf{1}_{\mathcal{T}\geqslant t};\mathcal{X}|\mathcal{T}=t) + (1-p_t) I(\mathcal{S}_\mathbf{t-1},\mathbf{1}_{\mathcal{T}\geqslant t};\mathcal{X}|\mathcal{T}>t))\\
&\geqslant p_t I(\mathcal{A};\mathcal{X}) + (1-p_t) I_{t+1} - I(\mathcal{S}_\mathbf{t-1},\mathbf{1}_{\mathcal{T}\geqslant t};\mathcal{X}|\mathcal{T}\geqslant t)\\
&= p_t \bar{I} + (1-p_t) I_{t+1} - I_t
\end{aligned}
$$

First inequality is feasibility constraint. Fist equality is law of iterated expectation. Second equality is chain rule from posterior separability. Third equality is rewriting terms. Noticing that condition on $\mathcal{T} = t+1$, $\mathbf{1}_{\mathcal{T}\leqslant t}$ is degenerate. Second inequality is from information processing inequality and applying chain rule again. Last equality is by definition. It is easy to verify by law of total probability that:

$$
\begin{aligned}
P_t =& \mathrm{P}(\mathcal{T}\leqslant t) = \mathrm{P}(\mathcal{T}=t) + \mathrm{P}(\mathcal{T}\leqslant t-1)\\
=& \mathrm{P}(\mathcal{T}\geqslant t)\mathrm{P}(\mathcal{T}=t|\mathcal{T}\geqslant t) + \mathrm{P}(\mathcal{T}\leqslant t-1)\\
=& (1-P_{t-1})p_t + P_{t-1}
\end{aligned}
$$

Then we verify initial conditions:

$$
\begin{cases}
I_1 = I(\mathcal{S}_0;\mathcal{X}|\mathcal{T}\geqslant 1) = 0\\
P_0 = \mathrm{P}(\mathcal{T}\leqslant 0) = 0
\end{cases}
$$

Q.E.D.

A.2 *Proof of Theorem 1*

Proof. First, assume $\rho_t = \max\left\{0, \frac{T-t}{T}\right\}$. We show that the statement in Theorem 1 is correct with assumed ρ_t. Since $\rho_t = 0$ when $t \geqslant T$, Equation (2) is finite horizon. So we can apply backward induction. Define:

$$
\begin{aligned}
V_t(I) = &\sup_{p_\tau} \sum_{\tau=t}^{T} \frac{T-\tau}{T}(1-P_{\tau-1})p_\tau V^*\\
&\text{s.t. } \begin{cases}
(\bar{I}-I_\tau)p_\tau + (I_{\tau+1}-I_\tau)(1-p_\tau) \leqslant c\\
P_\tau = P_{\tau-1} + (1-P_{\tau-1})p_\tau\\
P_{t-1} = 0, I_t = I
\end{cases}
\end{aligned}
$$

Then V_t solves functional equation:

$$
\begin{aligned}
V_t(I) = &\sup_p \frac{T-t}{T} p V^* + (1-p)V_{t+1}(I') \\
&\text{s.t. } (\bar{I}-I)p + (I'-I)(1-p) \leqslant c
\end{aligned}
\tag{12}
$$

I conjecture that for $I \geqslant 0$:

$$
\widetilde{V}_t(I) = \begin{cases}
\frac{T-t}{T}\frac{c+I}{\bar{I}}V^* + \left(1-\frac{c+I}{\bar{I}}\right)\sum_{\tau=t+1}^{T}\frac{T-\tau}{T}V^*\frac{c}{\bar{I}}\left(1-\frac{c}{\bar{I}}\right)^{\tau-t-1} & \text{when } \frac{c+I}{\bar{I}} < 1\\
\frac{T-t}{T}V^* & \text{when } \frac{c+I}{\bar{I}} \geqslant 1
\end{cases}
\tag{13}
$$

solves Equation (12). This is clearly true for $t = T-1$. Since when $t = T-1$, $V_{t+1} \equiv 0$ so there is no utility gain from accumulating I. Now we prove the conjecture by backward induction on t. Suppose the conjecture is true for t. Consider solving V_{t-1} from Equation (12).

- *Case 1*: $\bar{I} \leqslant c + I$. Then choosing $p = 1$ gives utility $\frac{T-\tau}{T}V^*$ immediately, thus optimal and $V_t(I) = \frac{T-t}{T}V^* = \widetilde{V}_t(I)$.

- *Case 1*: $\bar{I} > c + I$. Consider the one-step optimization problem choosing I':

$$V_t(I) = \sup_{I' \geqslant 0} \frac{T-t}{T}\frac{c+I-I'}{\bar{I}-I'}V^* + \frac{\bar{I}-I-c}{\bar{I}-I'}\widetilde{V}_{t+1}(I')$$

When $I' \leqslant \bar{I} - c$, the objective function is:

$$\frac{T-t}{T}\frac{c+I-I'}{\bar{I}-I'}V^* + \frac{\bar{I}-I-c}{\bar{I}-I'}\frac{T-t-1}{T}V^*$$
$$\implies \text{FOC}: \ -\frac{1}{T}\frac{\bar{I}-I-c}{(\bar{I}-I')^2}V^* \leqslant 0$$

When $I' > \bar{I} - c$, the objective function is:

$$\frac{T-t}{T}\frac{c+I-I'}{\bar{I}-I'}V^* + \frac{\bar{I}-I-c}{\bar{I}-I'}\left(\frac{T-t-1}{T}\frac{c+I'}{\bar{I}}V^* + \left(1-\frac{c+I'}{\bar{I}}\right)\sum_{\tau=t+2}^{T}\frac{T-\tau}{T}V^*\frac{c}{\bar{I}}\left(1-\frac{c}{\bar{I}}\right)^{\tau-t-2}\right)$$
$$\implies \text{FOC}: \ -\left(1-\frac{c}{\bar{I}}\right)^{T-t-1}\frac{\bar{I}-I-c}{T(\bar{I}-I')^2}V^* < 0$$

To sum up, decreasing I' is always utility improving. So optimal $I' = 0$ and optimal solution of Equation (12) is

$$\begin{aligned}V_t(I) =& \frac{T-t}{T}\frac{c+I}{\bar{I}}V^* + \left(1-\frac{c+I}{\bar{I}}\right)\left(\frac{T-t-1}{T}\frac{c}{\bar{I}}V^* + \left(1-\frac{c}{\bar{I}}\right)\sum_{\tau=t+2}^{T}\frac{T-\tau}{T}V^*\frac{c}{\bar{I}}\left(1-\frac{c}{\bar{I}}\right)^{\tau-t-2}\right)\\ =& \frac{T-t}{T}\frac{c+I}{\bar{I}}V^* + \left(1-\frac{c+I}{\bar{I}}\right)\sum_{\tau=t+1}^{T}\frac{T-\tau}{T}V^*\frac{c}{\bar{I}}\left(1-\frac{c}{\bar{I}}\right)^{\tau-t-1}\\ =& \widetilde{V}_t(I)\end{aligned}$$

Therefore, $\widetilde{V}_t(I)$ solves Equation (12). So with ρ_t defined by $\max\left\{0, \frac{T-t}{T}\right\}$, Equation (2) is solved by strategy $I_t \equiv 0$ (i.e. $p_t = \frac{c}{\bar{I}}$) and optimal utility is $\widetilde{V}_1(0)$.

Now, consider a general convex ρ_t. We want to show that $p_t = \frac{c}{\bar{I}}$ is still optimal strategy for Equation (2). By definition $\lim_{t\to\infty}\sum_{\tau\geqslant t}\rho_\tau = 0$, so $\forall\varepsilon$ there exists T s.t $\sum_{t\geqslant T}\rho_t < \varepsilon$. Pick T to be an even number. Now define ρ_τ^t recursively:

- $\rho_\tau^T = \max\{\rho_T + (\tau - T)(\rho_T - \rho_{T-1}), 0\}$. Define $\widehat{\rho}_\tau^T = \rho_\tau - \rho_\tau^T$ when $\tau \leqslant T$ and $\widehat{\rho}_\tau^T = 0$ otherwise. It is not hard to verify that $\widehat{\rho}_\tau^T$ is convex in τ and $\widehat{\rho}_\tau^T = 0\ \forall\tau \geqslant T-1$.

- $\rho_\tau^{T-2} = \max\left\{\widehat{\rho}_{T-2}^T + (\tau - T + 2)(\widehat{\rho}_{T-2}^T - \widehat{\rho}_{T-3}^T), 0\right\}$. Define $\widehat{\rho}_\tau^{T-2} = \widehat{\rho}_\tau^T - \rho_\tau^{T-2}$. It is not hard to verify that $\widehat{\rho}_\tau^{T-2}$ is convex in τ and $\widehat{\rho}_\tau^{T-2} = 0\ \forall\tau \geqslant T-3$.

- $\cdots$

- $\rho_\tau^{T-2k} = \max\left\{\widehat{\rho}_{T-2k}^{T-2k+2} + (\tau - T + 2k)\left(\widehat{\rho}_{T-2k}^{T-2k+2} - \widehat{\rho}_{T-2k-1}^{T-2k+2}\right), 0\right\}$. Define $\widehat{\rho}_\tau^{T-2k} = \widehat{\rho}_\tau^{T-2k+2} - \rho_\tau^{T-2k}$.

$\forall p'_t$ satisfying constraints in Equation (2) and corresponding P'_t:

$$
\begin{aligned}
&\sum_{t=1}^{\infty} \rho_t (1 - P'_{t-1}) p'_t V^* \\
&= \sum_{t=1}^{\infty} \left(\sum_{k=1}^{[T/2]} \rho_t^{T-2k} + \left(\rho_t - \sum_{k=1}^{[T/2]} \rho_t^{T-2k} \right) \right) (1 - P'_{t-1}) p'_t V^* \\
&< \sum_{k=1}^{[T/2]} \sum_{t=1}^{\infty} \rho_t^{T-2k} (1 - P'_{t-1}) p'_t V^* + \varepsilon V^* \\
&\leqslant \sum_{k=1}^{[T/2]} \sum_{t=1}^{\infty} \rho_t^{T-2k} (1 - P_{t-1}) p_t V^* + \varepsilon V^* \\
&\leqslant \sum_{t=1}^{\infty} \rho_t (1 - P_{t-1}) p_t V^* + \varepsilon V^*
\end{aligned}
$$

First inequality is from $\sum_{t \geqslant T} \rho_t < \varepsilon$. Second inequality is from optimality of p_t in last part. Last inequality is from $\sum_{t \geqslant T} \rho_t \geqslant 0$. Therefore, by taking $\varepsilon \to 0$, we showed that:

$$\sum_{t=1}^{\infty} \rho_t (1 - P'_{t-1}) p'_t V^* \leqslant \sum_{t=1}^{\infty} \rho_t (1 - P_{t-1}) p_t V^*$$

Q.E.D.

A.3 Proof of Lemma 2

Proof. First of all, redefine $\widetilde{\mathcal{S}}_t$ s.t.

$$\widetilde{\mathcal{S}}_t = \begin{cases} \mathcal{S}_t & \text{conditional on } \mathcal{T} \geqslant t \\ s_0 & \text{conditional on } \mathcal{T} < t \end{cases}$$

where equality is defined as signal distribution conditional on $\mathcal{X}$ and $\mathcal{T}$ being identical. It is not hard to verify that $\widetilde{\mathcal{S}}_t, \mathcal{T}$ still satisfies constraints in Equation (1):

- If $\mathcal{T} < t$, $I(\widetilde{\mathcal{S}}_t; \mathcal{X} | \widetilde{\mathcal{S}}_{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} \geqslant t}) = 0$ since $\widetilde{\mathcal{S}}_t$ is degenerate. If $\mathcal{T} \geqslant t$, then $\mathcal{T} \geqslant t - 1$ so $I(\widetilde{\mathcal{S}}_t; \mathcal{X} | \widetilde{\mathcal{S}}_{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} \geqslant t}) = I(\mathcal{S}_t; \mathcal{X} | \mathcal{S}_{\mathbf{t-1}}, \mathbf{1}_{\mathcal{T} \geqslant t}) \leqslant c$.
- Conditional on $\mathcal{T} = t$, $\widetilde{\mathcal{S}}_t = \mathcal{S}_t$ so $\mathcal{X} \to \widetilde{\mathcal{S}}_t \to \mathcal{A}$.
- If $\widetilde{\mathcal{S}}_t = s_0$, then $\mathcal{T} < t$ for sure, so $\mathbf{1}_{\mathcal{T} \geqslant t}$ is independent to $\mathcal{X}$. If $\widetilde{\mathcal{S}}_t \neq s_0$, then $\mathcal{T} \geqslant t$ for sure, so $\mathbf{1}_{\mathcal{T} \geqslant t}$ is independent to $\mathcal{X}$.

So replacing $\mathcal{S}$ with $\widetilde{\mathcal{S}}$ we still get a feasible strategy and induced decision time distribution $\mathcal{T}$ is unchanged. From now on, we assume WLOG that $\mathcal{S}_t \equiv s_0$ when $\mathcal{T} < t$. I only discuss the case $E[\mathcal{T}] < \infty$. If $E[\mathcal{T}] = \infty$ then Lemma 2 is automatically true.

$$
\begin{aligned}
E[\mathcal{T}] &= \sum_{t=1}^{\infty} \mathrm{P}(\mathcal{T} = t) \cdot t = \sum_{t=1}^{\infty} \mathrm{P}(\mathcal{T} = t) \sum_{\tau=1}^{t} 1 = \sum_{\tau=1}^{\infty} \sum_{t=\tau}^{\infty} \mathrm{P}(\mathcal{T} = t) \\
&= \sum_{t=1}^{\infty} \mathrm{P}(\mathcal{T} \geqslant t) = \frac{1}{c} \sum_{t=0}^{\infty} \mathrm{P}(\mathcal{T} \geqslant t) \cdot c
\end{aligned}
$$

$$
\begin{aligned}
&\geqslant \frac{1}{c}\sum_{t=1}^{\infty} \mathrm{P}(\mathcal{T} \geqslant t) E\left[I(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t})|\mathcal{T} \geqslant t\right] \\
&= \frac{1}{c}\sum_{t=1}^{\infty} \left(\mathrm{P}(\mathcal{T} \geqslant t) E\left[I(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t})|\mathcal{T} \geqslant t\right] + \mathrm{P}(\mathcal{T} < t) E\left[I(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t})|\mathcal{T} < t\right]\right) \\
&= \frac{1}{c}\sum_{t=1}^{\infty} E\left[I(\mathcal{S}_t;\mathcal{X}|\mathcal{S}_{\mathbf{t-1}},\mathbf{1}_{\mathcal{T}\geqslant t})\right] = \frac{1}{c}\sum_{t=1}^{\infty}\left(I(\mathcal{S}_{\mathbf{t}};\mathcal{X}) - I(\mathcal{S}_{\mathbf{t-1}};\mathcal{X})\right) \\
&= \frac{1}{c}\left(\lim_{t\to\infty} I(\mathcal{S}_{\mathbf{t}};\mathcal{X}) + \lim_{\tau\to\infty}\sum_{t=\tau}^{\infty} I(\mathcal{S}_{\mathbf{t}};\mathcal{X}) - I(\mathcal{S}_{\mathbf{t-1}};\mathcal{X})\right) \\
&\geqslant \frac{1}{c}\lim_{\tau\to\infty}\sum_{t=1}^{\tau} \mathrm{P}(\mathcal{T} = t) E\left[I(\mathcal{S}_{\mathbf{t}};\mathcal{X}|\mathcal{T} = t)|\mathcal{T} = t\right] \\
&\geqslant \frac{1}{c}\lim_{\tau\to\infty}\sum_{t=1}^{\tau} \mathrm{P}(\mathcal{T} = t) I(\mathcal{A};\mathcal{X}) \\
&= \frac{I(\mathcal{A};\mathcal{X})}{c}
\end{aligned}
$$

Third line is from flow informativeness constraint. Forth line is from $\mathcal{S}_t|_{\mathcal{T}<t} \equiv s_0$. Fifth and sixth line is from chain rule of posterior separable information measure. Seventh line is from information process inequality and law of interated expectation. Second last line is from information processing constraint. *Q.E.D.*

A.4 Proof of Lemma 3

Proof. Take any strategy $(\mu_t, \mathcal{T})$ feasible in Equation (6). Define

$$
\begin{cases} P_t = \mathrm{P}(\mathcal{T} \leqslant t) \\ I_t = E\left[H(\mu) - H(\mu_t)|\mathcal{T} > t\right] \end{cases} \tag{14}
$$

Now we prove that Equation (14) is a feasible strategy in Equation (7) and implements same value. First, since H is concave, then $I_t \geqslant 0$. Since $\mu_0 = \mu$, $I_0 = 0$. Since $\mu_t|_{\mathcal{T}=t} = \pi$ and $\mu_0 = \mu$, then $P_0 = 0$. Now we verify $\dot{I}_t \leqslant c - p_t(\bar{I} - I_t)$

$$
\begin{aligned}
E[H(\mu_{t+dt})|\mathcal{T} > t] =& -H(\mu) - E[H(\mu_{t+dt})|\mathcal{T} > t+dt]\mathrm{P}(\mathcal{T} > t+dt|\mathcal{T} > t) \\
& - E[H(\mu_{t+dt})|\mathcal{T} \in (t,t+dt]]\mathrm{P}(\mathcal{T} \in (t+dt]|\mathcal{T} > t) \\
\implies I_{t+dt} =& H(\mu) - \frac{1-P_t}{1-P_{t+dt}}\left(E[H(\mu_{t+dt})|\mathcal{T} > t] - \frac{P_{t+dt}-P_t}{1-P_t}E[H(\mu_{t+dt})|\mathcal{T} \in (t,t+dt]]\right) \\
\implies I_{t+dt} - I_t =& E[H(\mu_t)|\mathcal{T} > t] - \frac{1-P_t}{1-P_{t+dt}}\left(E[H(\mu_{t+dt})|\mathcal{T} > t] - \frac{P_{t+dt}-P_t}{1-P_t}E[H(\mu_{t+dt})|\mathcal{T} \in (t,t+dt]]\right) \\
=& \frac{1-P_{t+dt}}{1-P_t}\left(E\left[H(\mu_t) - H(\mu_{t+dt})|\mathcal{T} > t\right]\right) \\
& - \frac{1-P_{t+dt}}{1-P_t}\frac{P_{t+dt}-P_t}{1-P_t}\left((E[H(\mu) - H(\mu_{t+dt})|\mathcal{T} \in (t,t+dt]]) - E[H(\mu) - H(\mu_t)|\mathcal{T} > t]\right) \\
& - \left(\frac{P_{t+dt}-P_t}{1-P_t}\right)^2 E[H(\mu_t)|\mathcal{T} > t] \\
\implies \mathrm{d}I_t =& E\left[-\mathrm{d}H(\mu_t)|\mathcal{T} > t\right] - \frac{\mathrm{d}P_t}{1-P_t}\left(E_\pi\left[H(\mu) - H(\nu)\right] - I_t\right) \\
\implies \dot{I}_t \leqslant& c - \frac{\dot{P}_t}{1-P_t}(\bar{I} - I_t)
\end{aligned}
$$

First equality is law of iterated expectation. Second, third and forth equalities are rearranging terms. Fifth equality is from taking $dt \to 0$. Inequality is from $E[\mathrm{d}H(\mu_t)|\mu_t] \leqslant d\mathrm{d}t$.

Finally, deifne $p_t = \frac{\dot{P}_t}{1-P_t}$. Then

$$E[\rho_T] = \int_0^\infty \rho_t \mathrm{d}P_t = \int_0^\infty \rho_t (1-P_t) p_t \mathrm{d}t$$

To sum up, for any feasible strategy in Equation (6), there exists an feasible strategy in Equation (7) attaining same value. So the statement in Lemma 3 is true. *Q.E.D.*

A.5 proof of Theorem 3

Proof. It is easy to verify that $p_t \equiv \frac{c}{\bar{I}}$ is feasible in Equation (7) and the objective function is exactly $\int_0^\infty \rho_t e^{-\frac{c}{\bar{I}}t}\frac{c}{\bar{I}}\mathrm{d}t$. Therefore, it is sufficient to show that $V \leqslant \int_0^\infty \rho_t e^{-\frac{c}{\bar{I}}t}\frac{c}{\bar{I}}\mathrm{d}t$. Pick any p_t satisfying constraints in Equation (7). Now since p_t and ρ_t are integrable, $\forall \varepsilon > 0$, there exists T s.t.

$$\int_0^\infty \rho_t(1-P_t)p_t \mathrm{d}t \leqslant \int_0^T \rho_t(1-P_t)p_t \mathrm{d}t + \varepsilon$$

Then there exists $dt > 0$ s.t.:

$$\begin{aligned}
\int_0^T \rho_t(1-P_t)p_t \mathrm{d}t &\leqslant \sum_{k=1}^{[T/dt]} \rho_{kdt} \int_{kdt}^{(k+1)dt} (1-P_\tau)p_\tau \mathrm{d}\tau + \varepsilon \\
&= \sum_{k=1}^{[T/dt]} \rho_{kdt} \int_{kdt}^{(k+1)dt} e^{-\int_0^\tau p_s \mathrm{d}s} p_\tau \mathrm{d}\tau + \varepsilon \\
&= \sum_{k=1}^{[T/dt]} \rho_{kdt} \left(e^{-\int_0^{kdt} p_\tau \mathrm{d}\tau} - e^{-\int_0^{(k+1)dt} p_\tau \mathrm{d}\tau} \right) + \varepsilon \\
&= \sum_{k=1}^{[T/dt]} \rho_{kdt} e^{-\int_0^{kdt} p_\tau \mathrm{d}\tau} \left(1 - e^{-\int_{kdt}^{(k+1)dt} p_\tau \mathrm{d}\tau} \right) + \varepsilon \\
&= \sum_{k=1}^{[T/dt]} \rho_{kdt} P_{kdt} \left(1 - e^{-\int_{kdt}^{(k+1)dt} p_\tau \mathrm{d}\tau} \right) + \varepsilon
\end{aligned}$$

Now consider the following sequence:

$$\begin{cases}
\hat{\rho}_k = \rho_{k\cdot dt} \\
\hat{p}_k = 1 - e^{-\int_{kdt}^{(k+1)dt} p_\tau \mathrm{d}\tau} \\
\hat{P}_{k-1} = P_{kdt} \\
\hat{I}_k = I_{kdt} \\
\hat{c} = cdt
\end{cases}$$

We verify that:

$$\begin{cases}
(\bar{I} - \hat{I}_k)\hat{p}_k + (\hat{I}_{k+1} - \hat{I}_k)(1-\hat{p}_k) \leqslant \hat{c} \\
\hat{P}_k = \hat{P}_{k-1} + (1-\hat{P}_{k-1})\hat{p}_k
\end{cases}$$

- Solve ODE defining P_t, we get $P_t = 1 - e^{-\int_0^t p_\tau \mathrm{d}\tau}$. Apply this to calculate $\hat{P}_k - \hat{P}_{k-1} = P_{(k+1)dt} - P_{kdt} = (1-P_{kdt})\left(1 - e^{-\int_{kdt}^{(k+1)dt} p_\tau \mathrm{d}\tau}\right) = (1-\hat{P}_{k-1})\hat{p}_k$.

- Solve ODE defining I_t, we get:

$$
\begin{aligned}
I_t =& \int_0^t e^{\int_\tau^t p_s \mathrm{d}s}(c - \bar{I}p_\tau)\mathrm{d}\tau \\
\implies \quad I_{(k+1)dt} - I_{kdt} =& \int_0^{(k+1)dt} e^{\int_\tau^{(k+1)dt} p_s \mathrm{d}s}(c - \bar{I}p_\tau)\mathrm{d}\tau - \int_0^{kdt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}(c - \bar{I}p_\tau)\mathrm{d}\tau \\
=& \int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}(c - \bar{I}p_\tau)\mathrm{d}\tau + \left(e^{\int_{kdt}^{(k+1)dt} p_s \mathrm{d}s} - 1\right) \int_0^{(k+1)dt} e^{\int_\tau^{t} p_s \mathrm{d}s}(c - \bar{I}p_\tau)\mathrm{d}\tau \\
=& \left(e^{\int_{kdt}^{(k+1)dt} p_s \mathrm{d}s} - 1\right) I_{kdt} + e^{\int_{kdt}^{(k+1)dt} p_s \mathrm{d}s} \int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}(c - \bar{I}p_\tau)\mathrm{d}\tau \\
\implies \quad \Delta\widehat{I}_k(1 - \widehat{p}_k) =& \left(1 - e^{-\int_{kdt}^{(k+1)dt} p_s \mathrm{d}s}\right) \widehat{I}_k + c\int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}\mathrm{d}\tau - \bar{I}\int_{kdt}^{(k+1)dt} e^{\int_\tau^{t} p_s \mathrm{d}s}p_\tau \mathrm{d}\tau \\
=& \widehat{p}_k\left(\widehat{I}_k - \bar{I}\right) + c\int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}\mathrm{d}\tau - \bar{I}\left(\int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}p_\tau \mathrm{d}\tau - \widehat{p}_k\right)
\end{aligned}
$$

First, since when $\tau \in [kdt, (k+1)dt]$, $\int_\tau^{kdt} p_s \mathrm{d}s \leqslant 0$, $\int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}\mathrm{d}\tau \leqslant dt$. Then we consider

$$
\begin{aligned}
&\int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}p_\tau \mathrm{d}\tau - \widehat{p}_k \\
=& \int_{kdt}^{(k+1)dt} e^{\int_\tau^{kdt} p_s \mathrm{d}s}p_\tau \mathrm{d}\tau - 1 + e^{-\int_{kdt}^{(k+1)dt} p_s \mathrm{d}s}
\end{aligned}
$$

Let

$$
\begin{aligned}
H(t, t') =& \int_t^{t'} e^{\int_\tau^{t} p_s \mathrm{d}s}p_\tau \mathrm{d}\tau - 1 + e^{-\int_t^{t'} p_s \mathrm{d}s} \\
\implies \quad \frac{\partial H(t,t')}{\partial t'} =& 0 \& H(t,t) = 0 \\
\implies \quad H(t,t') \equiv& 0
\end{aligned}
$$

Therefore, to sum up:

$$
\Delta\widehat{I}_k(1 - \widehat{p}_k) + \widehat{p}_k(\bar{I} - \widehat{I}_k) \leqslant \widehat{c}
$$

We have checked that $\widehat{p}_k, \widehat{P}_k, \widehat{I}_k$ is feasible in problem Equation (2) with parameter $\widehat{\rho}_k$ and $\widehat{c}$. Then by Theorem 1:

$$
\begin{aligned}
\sum_{k=1}^{[T/dt]} \widehat{\rho}_k\left(1 - \widehat{P}_{k-1}\right)\widehat{p}_k \leqslant& \sum_{k=1}^{\infty} \widehat{\rho}_t \left(\frac{\bar{I} - \widehat{c}}{\bar{I}}\right)^{k-1} \frac{\widehat{c}}{\bar{I}} \\
\implies \int_0^\infty \rho_t(1 - P_t)p_t \mathrm{d}t \leqslant& \sum_{k=1}^{\infty} \widehat{\rho}_t \left(\frac{\bar{I} - \widehat{c}}{\bar{I}}\right)^{k-1} \frac{\widehat{c}}{\bar{I}} + 2\varepsilon \\
=& \sum_{k=1}^{\infty} \rho_{kdt}\left(1 - \frac{c}{\bar{I}}dt\right)^k \frac{c}{\bar{I}}dt + \varepsilon
\end{aligned}
$$

Since $\log(1-x) \leqslant -x$, $\left(1 - \frac{cdt}{\bar{I}}\right)^k \leqslant e^{-\frac{c}{\bar{I}}kdt}$ Then:

$$
\int_0^\infty \rho_t(1 - P_t)p_t \mathrm{d}t \leqslant \sum_{k=1}^{\infty} \rho_{kdt}e^{-\frac{c}{\bar{I}}kdt}\frac{c}{\bar{I}}dt + 2\varepsilon
$$

On the other hand, since $\rho_t e^{-\frac{c}{\bar{I}}t}$ is integrable, there exists dt sufficiently small that:

$$\sum_{k=1}^{\infty} \rho_{kdt} e^{-\frac{c}{\bar{I}}kdt} \frac{c}{\bar{I}} dt \leqslant \int_{t=0}^{\infty} \rho_t e^{-\frac{c}{\bar{I}}t} \mathrm{d}t + \varepsilon$$
$$\Longrightarrow \int_0^{\infty} \rho_t (1-P_t) p_t \mathrm{d}t \leqslant \int_0^{\infty} \rho_t e^{-\frac{c}{\bar{I}}t} \mathrm{d}t + 3\varepsilon$$

Let $\varepsilon \to 0$, then we showed that:

$$V \leqslant \int_0^{\infty} \rho_t e^{-\frac{c}{\bar{I}}t} \mathrm{d}t$$

Q.E.D.

A.6 Proof of Lemma 4

Proof. Similar to discussion in proof of Lemma 2, I only prove for $E[\mathcal{T}] < \infty$. Let:

$$\begin{cases} P_t = \mathrm{P}(\mathcal{T} \leqslant t) \\ I_t = E\left[H(\mu) - H(\mu_t) | \mathcal{T} > t\right] \end{cases}$$

Then be proof of Lemma 3:

$$\mathrm{d}I_t = E[-\mathrm{d}H(\mu_t)|\mathcal{T} > t] - \frac{\mathrm{d}P_t}{1-P_t}\left(\bar{I} - I_t\right) \tag{15}$$

Consider $E[\mathcal{T}]$:

$$\begin{aligned} E[\mathcal{T}] &= \frac{1}{c}\int_0^{\infty}(1-P_t)c\mathrm{d}t \\ &\geqslant \frac{1}{c}\int_0^{\infty}(1-P_t)E[-\mathrm{d}H(\mu_t)|\mathcal{T} > t] \\ &= \frac{1}{c}\left(\int_0^{\infty}(1-P_t)\mathrm{d}I_t + \int_0^{\infty}(\bar{I}-I_t)\mathrm{d}P_t\right) \\ &= \frac{\bar{I}}{c} + \int_0^{\infty}((1-P_t)\mathrm{d}I_t + I_t\mathrm{d}(1-P_t)) \\ &= \frac{\bar{I}}{c} + I_t(1-P_t)\big|_0^{\infty} \\ &= \frac{\bar{I}}{c} \end{aligned}$$

Inequality is flow informativeness constraint. Second equality is by Equation (15). Forth equality is by intergral by part. *Q.E.D.*