BEHAVIORAL MULTI-CRITERIA DECISION ANALYSIS: THE TODIM METHOD WITH CRITERIA INTERACTIONS

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Abstract

In this paper a multi-criteria decision aiding model is developed through the use of the Choquet integral. The proposed model is an extension of the TODIM method, which is based on nonlinear Cumulative Prospect Theory. The paper starts by reviewing the first steps of behavioral decision theory. A presentation of the TODIM method follows. The basic concepts of the Choquet integral as related to multi-criteria decision aiding are reviewed. It is also shown how the measures of dominance of the TODIM method can be rewritten through the application of the Choquet integral. From the ordering of decision criteria the fuzzy measures of criteria interactions are computed, which leads to the ranking of alternatives. A case study on the forecasting of property values for rent in a Brazilian city illustrates the proposed model. Results obtained from the use of the Choquet integral are then compared against a previously made usage of the TODIM method. It is concluded that significant advantages exist derived from the use of the Choquet integral. The paper closes with recommendations for future research.

Keywords: TODIM method, Multicriteria Decision Aid, Decision Analysis

1. Introduction

Many useful multi-criteria methods are available to practioners and researchers in decision aiding at present. These methods are based on different mathematical assumptions (Ehrgott, Figueira and Greco, 2010). A number of accomplishments have been achieved since the late 1970s and new developments and application fields are constantly emerging (Wallenius *et al*., 2008).

Behavioral decision theory is considered to have formally begun with Ward Edwards in Psychological Bulletin article (Edwards, 1954), although it went through some major advances in the 1970s and 1980s. Eisenführ, Weber, and Langer (2010) have established behavioral decision theory as "The approach of reflecting on axiomatic frameworks in the domain of descriptive theories (…) geared towards our goal of decision support". These three authors also point out that the Cumulative Prospect Theory by Tversky and Kahneman (Tversky and Kahneman, 1992) is "currently the most prominent descriptive decision theory under uncertainty". Again according to these three authors "the original Prospect Theory (OPT) from 1979 is only of historical importance today. However, to prevent possible misunderstandings, the cumulative version of Prospect Theory is commonly referred to as CPT" (Eisenführ, Weber and Langer, 2010).

OPT was developed by Daniel Kahneman and Amos Tversky and first published in 1979 (Kahneman and Tversky, 1979). The term prospect referred to a lottery in the original formulation of that theory. A prospect (x_1, p_1; …; x_n, p_n) is a contract that yields outcome x_i with probability p_i, where $p_1 + \ldots + p_n = 1$. With prospect theory Kahneman and Tversky aimed to describe how people choose between probabilistic alternatives and evaluate potential losses and gains defined with respect to a reference point or status quo. Consequently two domains are identified, the domains of gains and the domain of losses. A number of experiments have allowed researchers to conclude that humans tend to show risk-averse behavior in the domain of gains and a risk-seeking behavior in the domain of losses (Tversky and Kahneman, 1981; Kahneman and Tversky, 1982; Kahneman and Tversky, 1984; Tversky and Kahneman, 1986; Tversky and Kahneman, 1987; Quiggin, 1993).

The difference between cumulative prospect theory and OPT is that weighting is applied to the cumulative probability distribution function, as in rank-dependent expected utility theory, instead of being applied to the probabilities of individual outcomes. CPT is therefore a further development of prospect theory. CPT overcomes some clear limitations that OPT had and, due to its success as a descriptive theory of how people decide when facing risk, it is considered as having more accuracy than expected utility theory as a psychological theory of preferences under risk (Eisenführ, Weber and Langer, 2010).

The earliest attempt to apply prospect theory to multi-criteria decision making lays in the work of Korhonen, Moskowitz and Wallenius (1990). Those authors were investigating choice behavior in interactive multi-criteria decision making. The persistent patterns of intransitive choice behavior that were then revealed could be explained by using Tversky's additive utility difference model (Tversky, 1969) as well as Kahneman and Tversky's prospect theory (Kahneman and Tversky, 1979). Korhonen, Moskowitz and Wallenius 1990 therefore presented an explanatory use of prospect theory, not a prescriptive one. The validation of that theory made by Korhonen, Moskowitz and Wallenius (1990) made use of linear piecewise marginal value functions, which means that these three authors used a linear approximation to prospect theory. Korhonen, Moskowitz and Wallenius (1990) have shown that Tversky's (Tversky, 1969) additive difference model can indeed be regarded as a generalization of prospect theory to the multi-criteria context.

Salminen and Wallenius (1993) tested linear prospect theory in a deterministic multi-criteria decision-making environment. These two authors then concluded that prospect theory was a reasonable model of choice for many individuals within the framework of their research.

By making use of linear prospect theory Salminen (1991, 1992, 1994) proposed an interactive method for solving discrete deterministic multi-criteria decision problems and assumed prospect theoretical value functions for the decision makers. He then approximated the S-shaped value functions of prospect theory by piecewise linear marginal value functions. Salminen's proposed procedure was therefore valid only for convex preferences. This author pointed out that the major problem in making OPT operational was how to find an individual reference alternative. He then proposed as alternative possibilities the current option, the use of aspiration levels and the ideal point, but concluded that there was no unique solution to that problem (Salminen 1991).

The TODIM method was formulated in the early 1990s and has been the object of a number of publications since then (Gomes and Rangel, 2009; Gomes and Lima, 1992; Nobre, Trotta, and Gomes, 1999; Fa-dong et al, 2010; Moshkovich, Gomes and Mechitov, 2011). TODIM is the acronym for *Interactive and Multicriteria Decision Making* in Portuguese.

The aim of this article is to show how criteria interactions can be determined in applications of the TODIM method. Korhonen and Wallenius' belief that decision aiders can do a much better job if their models are founded on a solid behavioral foundation (Korhonen and Wallenius, 1996) is followed here. The nonlinear cumulative prospect theory-based TODIM method is presented. Computations of criteria interactions as fuzzy measures by applying the Choquet integral to TODIM are then introduced. A numerical example is provided. Conclusions with suggestions for future research close the article.

2. Description of TODIM method

2.1 Basics

The TODIM method (an acronym in Portuguese of Interactive and Multicriteria Decision Making) is based on nonlinear CPT as the shape of its value function is the same as the gains/losses function of Cumulative Prospect Theory (Tversky and Kahneman, 1992). Here gains and losses are always established with respect to a reference point. In algorithmic form an application of TODIM would follow the steps below (all symbols are explained just after the mathematical formulae):

Step 1: From the evaluation matrix of size *m* (criteria) *versus n* (alternatives) and criteria weights, compute values of $\Phi_c(A_i, A_j)$ by using equation (2) and making θ vary in [1,10];

Step 2: Compute values of $\delta(A_i, A_j)$ with equation (1);

Step 3: Compute values of ξ_i with equation (3): those values lead to the ranking of alternatives.

Mathematical expressions (1), (2) and (3) constitute the modeling underlying the use of the TODIM method:

$$\delta(A_i, A_j) = \sum_{c=1}^{m} \Phi_c(A_i, A_j) \qquad i, j = 1, \ldots, n \tag{1}$$

$$\Phi_c(A_i, A_j) = \begin{cases} \sqrt{\dfrac{w_{rc}(P_{ic} - P_{jc})}{\sum_{c=1}^{m} w_{rc}}} & if\, (P_{ic} - P_{jc}) > 0 \\ 0 & if\, (P_{ic} - P_{jc}) = 0 \\ -\dfrac{1}{\theta}\sqrt{\dfrac{(\sum_{c=1}^{m} w_{rc})(P_{jc} - P_{ic})}{w_{rc}}} & if\, (P_{ic} - P_{jc}) < 0 \end{cases} \tag{2}$$

$$\xi_i = \frac{\sum_{j=1}^{n} \delta(A_i, A_j) - \min_i \sum_{j=1}^{n} \delta(A_i, A_j)}{\max_i \sum_{j=1}^{n} \delta(A_i, A_j) - \min_i \sum_{j=1}^{n} \delta(A_i, A_j)}, \ (i = 1,\ 2,\ 3,\ \ldots, n) \tag{3}$$

where:

$\delta(A_i, A_j)$ = measurement of dominance of alternative A_i over alternative A_j;

n = the total number of alternatives;

m = the total number of criteria;

c = a generic criterion;

w_{rc} = trade-off rate (or trade-off weighting factor) between the reference criterion r and any other, generic criterion c. The subscript r identifies a reference criterion for the decision maker. That can be, for example, the criterion that the decision maker considers as the most important one. It is easy to see that any criterion can be chosen as the reference criterion and this particular choice does not influence the final results from the computations.

P_{ic}, P_{jc} = evaluations of alternatives i and j with respect to criterion c;

θ = attenuation factor of the losses; different choices of θ lead to different shapes of the prospect theoretical value function in the negative quadrant;

$\Phi_c(A_i, A_j)$ = contribution of criterion c to function $\delta(A_i, A_j)$, when comparing alternatives A_i and A_j.

ξ_i = normalized global performance of alternative A_i, when compared against all other alternatives.

2.2 The CPT-based TODIM method

2.2.1. Need of a risk aversion parameter

The function Φ_c reproduces the value function of OPT and replicates the most relevant shape characteristics. That function fulfills the concavity for positive outcomes (convexity for negative outcomes) and second, it enlarges the perception of negative values for losses than positive values for gains, both value functions are steeper for negative outcomes than for positive ones.

First, each shape characteristic of the value function models psychological processes: the concavity for gains describes a risk aversion attitude, the convexity describes a risk seeking attitude; secondly, the assumption that losses carry more weight than gains is represented by a steeper negative function side.

Different kinds of decision makers can be understood in terms of their risk and loss attitude. Although the TODIM method does not deal with risk directly, the way the decision maker evaluates the outcomes of any decision can be expressed by their risk attitude: for instance, a cautious decision maker will undervalue a superior result more than a braver one. Apart from parameter θ, the attenuation factor of the losses, function Φ_c does not offer other parameters to delineate the behavior of diverse decision makers, therefore a generic formulation is proposed.

2.2.2. From trade-off weighting factor to CPT weighting function

Corresponding to TODIM method equation (2), trade off weighting factors w_{rc} are implemented differently to gains and losses domain when calculating function Φ_c. Let's assume, for example, two specific alternatives *i* and *j*, and two specific criteria *c* and *d*, suppose that alternative *i* performs equally bad compared to alternative *j* for both criteria *c* and *d*.

$$P_{ic} - P_{jc} = P_{id} - P_{jd} = -M \tag{4}$$

Consider also that criterion *c* is more important for the decision-maker than *d*.

$$w_c > w_d \tag{5}$$

It would be easy to prove that the contribution of criterion *c* is higher than that of criterion *d* to function $\delta(A_i, A_k)$ when comparing alternatives A_i and A_j.

$$-\frac{1}{\theta}\sqrt{\frac{(\sum_{c=1}^{m} w_{rc})M}{w_{rc}}} > -\frac{1}{\theta}\sqrt{\frac{(\sum_{d=1}^{m} w_{rd})M}{w_{rd}}} \qquad (6)$$

The contribution of criterion *c* should be lower than that of criterion *d*, because both performances are negative and criterion *c* is more important than *d*. Having the same weighting factor formulation structure for gains and losses will prevent this effect to happen.

2.2.3. An analogy to CPT

CPT is a model for descriptive decisions under risk. As OPT, CPT treats gains and losses separately. Basically CPT considers: (i) the evaluation of possible outcomes relative to a certain reference point (often the *status quo*); (ii) different risk attitudes towards gains (i.e., outcomes above the reference point) and losses (i.e., outcomes below the reference point) and care generally more about potential losses than potential gains (loss aversion); and (iii) a tendency to overweight extreme, but unlikely events, but underweight "average" events.
Suppose a gamble is composed of *a* + *b* + 1 monetary outcomes, $x_{-a} < \ldots < x_0 < \ldots < x_b$, which occur with probabilities $p_{-a}, \ldots, p_b$, respectively. Recall that outcomes are defined relative to a reference point, which serves as the zero point of the value scale. Hence, v measures the value of deviations from that reference point, i.e., gains and losses (Neilson and Stowe, 2002). The corresponding gamble can be denoted by the pair (x; p), where $x = (x_{-a}, \ldots, x_b)$ and $p = (p_{-a}, \ldots, p_b)$. The preference value of the gamble (x; p) is given by (7).

$$V(x; p) = V^{+}(x; p) + V^{-}(x; p) \qquad (7)$$

In equation (7) V^{+} measures the contribution of gains and V^{-} measures the contribution of losses. The two parts of the sum in (7) can be rewritten as in (8) and (9).

$$V^{+}(x;p) = \sum_{i=0}^{b} \pi^{+}.v(x_i) = g^{+}(p_b)v(x_b) + \sum_{k=1}^{b}\left[g^{+}\left(\sum_{j=0}^{k} p_{b-j}\right) - g^{+}\left(\sum_{j=0}^{k-1} p_{b-j}\right)\right]v(x_{b-k}) \qquad (8)$$

$$V^{-}(x;p) = \sum_{j=-a}^{0} \pi^{-}.v(x_j) = g^{-}(p_{-a})v(x_{-a}) + \sum_{k=1}^{b}\left[g^{-}\left(\sum_{j=0}^{k} p_{-(a-j)}\right) - g^{-}\left(\sum_{j=0}^{k-1} p_{-(a-j)}\right)\right]v(x_{-(a-j)}) \qquad (9)$$

where π^{+} and π^{-} are decision weights associated to positive and negative outcomes respectively. The function g (*p*) is a probability weighting function assumed to be increasing with g (0) = 0 and g (1) = 1. As such it can be computed as in (10) and (11).

$$g^{+}(p) = \frac{p^{\gamma}}{\left(p^{\gamma} + (1-p)^{\gamma}\right)^{\gamma}} \tag{10}$$

$$g^{-}(p) = \frac{p^{\delta}}{\left(p^{\delta} + (1-p)^{\delta}\right)^{\delta}} \tag{11}$$

v (*x*) is a utility (or value) function assumed to be increasing with u(0) = 0 and it is formulated as in (12).

$$v(x) = \begin{cases} x^{\alpha} & if \quad x \geq 0 \\ (-\lambda)(-x)^{\beta} & if \quad x < 0 \end{cases} \tag{12}$$

where:

x = economic outcome relative to a reference point; it can also be understood as an evaluation of an alternative relative to another one;

α = curvature of the subjective value function for gains;

β = curvature of the subjective value function for losses;

λ = loss-aversion coefficient.

According to CPT, *v* (*x*) is a utility (or value) function assumed to be increasing with u(0) = 0 and it is formulated as in (12).

In regards to multi-criteria models based on CPT, an association can be made between the economic outcomes of the gambling and the evaluation of alternatives with respect to several criteria. Such association can also be made between outcome probabilities and the trade-off rate weighting factors. Here x = ($x_{-a}, \ldots, x_b$) are the economic outcomes relative to a reference point and can also be understood as an evaluation of an alternative relative to another one. p = ($p_{-a}, \ldots, p_b$) are the probabilities of each economic outcome and can be interpreted as the trade-off importance. For instance, let's suppose that alternative *i* is evaluated with respect to alternative *j*, and that there are *a*+*b*+1 different criteria to evaluate such alternatives. Table 1 displays the CPT elements associated with the TODIM elements.

Table 1: Analogy between CPT and TODIM elements

CPT elements				Analogous TODIM elements			
Outcome	Result	Outcome	Probability	Criteria	Result	Evaluation	Weighting

Index			of outcome	Index		alternative *i* compared to *j*	factor
a	Neg.	x_{-a}	p_{-a}	*c*	Neg.	P_{ci} -P_{cj}	$w_{rc}/(\sum w_{rc})$
b	Pos.	X_b	P_b	*d*	Pos.	P_{di} -P_{dj}	$w_{rd}/(\sum w_{rd})$

As the CPT outcome vector x = (x_{-a},…, x_b), TODIM relative evaluation score differences can be sorted low to high. For some criteria *c*, alternative *j* will perform better than *i* and the evaluation difference will be negative, i.e., $P_{ci} – P_{cj} < 0$. For some other criteria *d,* alternative *i* will show better evaluations than *j* and the evaluation differences will be positive, i.e., $P_{di} – P_{dj} > 0$. The performance evaluation differences among a set of criteria can be sorted and written as x = (P_{ci}-P_{cj},…, P_{di}-P_{dj}).

Besides the linkage between CPT outcomes and TODIM evaluation differences, we can assume that, as in CPT, for high positive and low negative prospects people tend to overweight the probabilities and for mid-level outcomes they tend to underweight probabilities. One can extend this assumption by considering that, in multicriteria decision making, for the highest and lowest evaluation differences the decision maker will overweight in a second stage a low criteria weighting factor (and he will underweight a similar evaluation between alternatives). In other words, the trade-off weighting factors of TODIM can be understood as probabilities that allow implementing the same formulation.

To sum up, the CPT formulation to calculate the value of a gamble V(x; p), x = (x_{-a},…, x_b) being relative to a reference point and p = (p_{-a},…, p_b) as probabilities of the outcomes, can be adopted in association to the use of the TODIM method to calculate the value of an alternative *i* relative to a second alternative *j* V_{ij}(x; p) whose performances according to a set of criteria are x = (P_{ci}-P_{cj},…, P_{di}-P_{dj}) and the trade-off weighting factors are p = ($w_{rc}/(\sum w_{rc})$,…, $w_{rd}/(\sum w_{rd})$).

Equation (2) of TODIM can be rewritten as in (13).

$$\delta\,(A_i, A_j) = V_{ij}(x; p) \qquad (13)$$

such that x and p can be written as in (14).

$$x = (P_{ci}\text{-}P_{cj},\ldots, P_{di}\text{-}P_{dj}) \text{ and } p = (w_{rc}/(\textstyle\sum w_{rc}),\ldots, w_{rd}/(\textstyle\sum w_{rd})) \qquad (14)$$

The parametric formulation of the TODIM method is now completed with the following: α quantifies the curvature of the subjective value function for gains, β does for losses, and λ quantifies the loss aversion. For α, β < 1, the value function exhibits risk aversion over gains and risk seeking over losses. Furthermore, if λ, the loss-aversion coefficient, is greater than one, individuals are more sensitive to losses than gains. By using non-linear regression the values of these three parameters of CPT were estimated as $\alpha=\beta=0.88$ and $\lambda=2.25$. The values of γ and δ were estimated as 0.61 and 0.69 respectively (Neilson and Stowe, 2002). Other estimations from experimental data are present in the literature (Camerer and Ho, 1994; Gonzalez and Wu, 1999).

In essence, we just showed that the ratio $w_{rc}/(\sum w_{rc})$ can be interpreted as a probability. This allows us to make full use of CPT, including its decision weights (Tversky and Kahneman, 1992). We can therefore say that the formulation of TODIM in terms of CPT is indeed a formulation in terms of the concept of capacity (Choquet, 1953; Grabisch and Labreuche, 2010). Here a capacity is a non-additive set function that generalizes the standard notion of probability. Capacities are also known under the name of fuzzy measures (Sugeno, 1974). It is shown in the coming section that, by relying on CPT, the TODIM method is indeed a multi-criteria formulation in terms of the concept of capacity (Choquet, 1953).

Besides the linkage between CPT outcomes and TODIM evaluation differences, we can assume that, as in CPT, for high positive and low negative prospects people tend to overweight the probabilities and for mid-level outcomes they tend to underweight probabilities. In analogy, in multicriteria decision making for the highest and lowest evaluation differences the decision maker will overweight in a second stage a low criteria weighting factor (and he will underweight a similar evaluation between alternatives). In other words, the trade-off weighting factors of TODIM can be understood as probabilities that allow implementing the same formulation.

An important mathematical model that has been used for modeling interactions between criteria is the Choquet integral (Choquet, 1953). In decision theory, the Choquet integral is a way to measure the expected utility of an uncertain event (Gilboa & Schmeidler, 1992). The Choquet integral is indeed a generalization of the weighted arithmetic mean and has been extensively used since the last decade in Multiple Criteria Decision Aiding in modeling interactions between criteria (Grabisch, 1996; Grabisch, 2006; Grabisch & Labreuche, 2005; 2010). A critical analysis of the use of the Choquet integral for modeling interactions between criteria was presented by Roy (2009) though. This author has pointed out that the generalization of the Choquet integral known as bipolar model (or model with bi-capacities) should be utilized in order to capture some particular aspects of interactions between criteria.

In the coming sections the basic notions of the Choquet integral are presented. It is then shown how interactions between criteria can be determined by applying the Choquet integral to the TODIM method.

3. The Choquet integral

3.1 The Choquet integral

Given a finite and not empty set $S = \{1,2...K\}$ and considering a family of subsets 2^S of S, a fuzzy measure is a function $\mu : 2^S \to [0,1]$ such that:

i. $\mu(\phi) = 0$ and $\mu(S) = 1$; when $\mu(S) = 1$, this fuzzy measure is said to be normalized;

ii. $\mu(A) \le \mu(B),\ for\ all\ A \subset B, A, B \in S$, given a function a : S →R , the Choquet integral in relation to the fuzzy measure μ is given by (15):

$$I_\mu(a) = \sum_{s=1}^{K} [\mu(\{s,...K\}) - \mu(\{s+1,...K\})] a_s + \mu(K) a_K \text{ ,} \tag{15}$$

where $a_s = a(s)$ and $a_1 < a_2 < ... < a_K$.

If μ is additive, the Choquet integral is the expected value (or the weighted average), i.e., as in (16) and (17) (Dubois and Prade, 1986; 1989).

$$[\mu(\{s,...K\}) - \mu(\{s+1,...K\})] = \mu(s) \text{ ,} \tag{16}$$

$$I_\mu(a) \sum_{s=1}^{K} [\mu(s)] a_s \text{ .} \tag{17}$$

Consider $S = \{1,2,3\}$, μ a fuzzy measure $\mu : 2^{\{1,2,3\}} \to [0,1]$ and a function $b = (2,3,1)$; in order to calculate the Choquet integral of b, we have (18) and (19):

$$I_\mu(b) = \sum_{s=1}^{3} [\mu(\{s,...K\}) - \mu\{s+1,...K\}] b_s + \mu(K) b_K \tag{18}$$

$$I_\mu(b) = [\mu(\{1,2,3\}) - \mu(\{2,3\})] b_1 + [(\mu(\{2,3\} - \mu(\{3\})] b_2 + \mu(3) b_3 \tag{19}$$

$b_1 < b_2 < b_3$.

As $b_3 < b_2 < b_1$, to calculate the Choquet integral we have to take a permutation $n:\{1,2,3\} \longrightarrow \{1,2,3\}$ so that n(1) = n_1=1; n_2 = 2 and n_3 = 3 where $b_{n_1} \le b_{n_2} \le b_{n_3}$; therefore $I_\mu(b) = [\mu\{n_1, n_2, n_3\} - \mu\{n_2, n_3\}] \times 1 + [\mu(\{n_2, n_3\}) - \mu(\{n_3\})] \times 2 + \mu(\{n_3\} \times 3$.

In general, given a function b: S → R we can always consider a permutation n: S → S so that $b_{n_1} \le b_{n_2} \le b_{nk}$, and we can write (20):

$$I_\mu(a) = \sum_{s=1}^{K} [\mu(\{n_s, ...n_K\}) - \mu\{n_{s+1}, ...n_K\}] a_{n_s} + \mu(\{n_K\}) a_{n_K} \tag{20}$$

$\mu(1) = \mu(2) = 0.3$.

Consider $a, b \in R^2$, so that $a_1 = 2$, $a_2 = 3$, $b_1 = 3$, and $b_2 = 1$; we have c = a + b = (5.4). The Choquet integral for the function a is:

$$I_\mu(a) = [\mu(\{1,2\}) - \mu(\{2\})] a_1 + \mu(2) a_2 = [1 - \mu(2)] a_1 + \mu(2) a_2 =$$
$$[1 - \mu(2)] a_1 + \mu(2) a_2 = [1 - 0.3] \times 5 + 0.3 \times 3 = 0.7 \times 5 + 0.9 = 4.4$$

The Choquet integral for the function b is:

$$I_\mu(b) = [\mu(\{1,2\}) - \mu(\{2\})] b_1 + \mu(2) b_2 = [1 - \mu(2)] b_1 + \mu(2) b_2 =$$

$$[1 - 0.3] \times 5 + 0.3 \times 1 = 0.7 \times 5 + 0.3 = 3.8$$

Summing these Integrals (the sum of the Integrals of functions *a* and *b*) we have:

$$I_\mu(a) + I_\mu(b) = 4.4 + 3.8 = 8.2$$

The Choquet integral for the function c (an additive function) is given by:

$$I_\mu(c) = [\mu(\{1,2\}) - \mu(\{2\})] c_1 + \mu(2) c_2 = [1 - \mu(2)] c_1 + \mu(2) c_2 =$$
$$[1 - 0.3] \times 5 + 0.3 \times 4 = 0.7 \times 5 + 1.2 = 4.7$$

Therefore, we have: $I_\mu(a) + I_\mu(b) \ne I_\mu(a + b)$.

4. The Choquet – extended TODIM Method

From the classical formulation of TODIM we can compute the measure of relative dominance of each alternative A_i over another alternative A_j as show as equation (1):

$$\delta(A_i, A_j) = \sum_{c=1}^{m} \Phi_c(A_i, A_j), \qquad \forall (A_i, A_j) \tag{1}$$

Through considering the fuzzy measures μ of interactions between criteria we can obtain the overall value of each alternative with no need of normalization. This is accomplished by rewriting the equation above as equation (21):

$$\delta(A_i, A_j) = I_\mu(a) \Phi_c(A_i, A_j) \tag{21}$$

where a : S →R , and I is the Choquet integral in relation to the fuzzy measure μ. In order to illustrate, assume the following evaluation matrix as showed in Table 2.

Table 2: Evaluation matrix

Criteria	Alternatives			
	A_1	A_2	. . .	A_n
C_1	$\Phi(A_1,C_1)$	$\Phi(A_2,C_1)$	. . .	$\Phi(A_n,C_1)$
C_2	$\Phi(A_1,C_2)$	$\Phi(A_2,C_2)$	. . .	$\Phi(A_n,C_2)$
. . .	. . .	. . .	. . .	. . .
C_m	$\Phi(A_1,C_m)$	$\Phi(A_2,C_m)$	. . .	$\Phi(A_n,C_m)$

Suppose that criteria are ordered as follow: $C_1 > C_2 > ... > C_m$. We can now determine the fuzzy measures (i.e., the criteria interactions)
as follows as equation (22):

$$\mu_1 = k_1;\ \mu_{12} = k_2\mu_1; .....\mu_{j-1,j} = k_j\mu_{j-1}, \sum_{j=1}^{m} k_j\mu_j = 1 \qquad (22)$$

where k_j are constants.
The evaluation matrix can now be rewritten as follows in Table 3:

Table 3: Evaluation matrix with fuzzy measures

Criteria	Alternatives			
	A_1	A_2	. . .	A_n
C_1	$\mu_1\Phi(A_1,C_1)$	$\mu_1\Phi(A_2,C_1)$	. . .	$\mu_1\Phi(A_n,C_1)$
C_2	$\mu_{12}\Phi(A_1,C_2)$	$\mu_{12}\Phi(A_2,C_2)$	. . .	$\mu_{12}\Phi(A_n,C_2)$
. . .	. . .	. . .	. . .	. . .
C_m	$\mu_{m-1,m}\Phi(A_1,C_m)$	$\mu_{m-1,m-1}\Phi(A_1,C_m)$	. . .	$\mu_{m-1,m-1}\Phi(A_n,C_m)$

5. An application case study

5.1. Problem definition

The case study is a valuation of residential properties carried out by real estate agents in the city of Volta Redonda, Brazil. Fifteen properties in different neighborhoods were analyzed as alternatives and a total of eight evaluation criteria were identified. A detailed description of the alternatives and criteria can be found in Gomes and Rangel (2009).
The initial weights assigned to the criteria used to evaluate the properties were defined by decision makers (i.e., the real estate agents), assigning a number between 1 and 5 to each criterion where 1 would mean ‘least important’ and 5 would mean ‘most important’.

The alternatives are presented.

A_1 – A house in an average location, with 290 m^2 of constructed area, a high standard of finishing, in a good state of conservation, with one garage space, 6 rooms, a swimming pool, barbecue and other attractions, without a security system.

A_2 – A house in a good location, with 180 m^2 of constructed area, an average standard of finishing, in an average state of conservation, with one garage space, 4 rooms, a backyard and terrace without a security system.

A_3 – A house in an average location, with 347 m^2 of constructed area, a low standard of finishing, in an average state of conservation, two garage spaces, 5 rooms, a large backyard, without a security system.

A_4 – A house in an average location, with 124 m^2 of constructed area, an average standard of finishing, in a good state of conservation, two garage spaces, 5 rooms, a fruit orchard, a swimming pool and barbecue, without security system.

A_5 – A house in an excellent location, with 360 m^2 of constructed area, a high standard of finishing, in a very good state of conservation, four garage spaces, 9 rooms, a backyard and manned security boxes in the neighborhood streets.

A_6 – A house located between the periphery and the city center (periphery/average location) with 89 m^2 of constructed area, an average standard of finishing, in a good state of conservation, with one garage space, 5 rooms, a backyard, without a security system.

A_7 – An apartment located in the periphery, with 85 m^2 of constructed area, a low standard of finishing, in a bad state of conservation, one garage space, 4 rooms, a manned entrance hall with security.

A_8 – An apartment in an excellent location, with 80 m^2 of constructed area, average standard of finishing, good state of conservation, with one garage space, 6 rooms, manned entrance hall with security.

A_9 – An apartment located between the periphery and the city center (periphery/average location), with 121 m^2 of constructed area, an average standard of finishing, in a good state of conservation, no garage space, 6 rooms, without a security system.

A_{10} – A house located between the periphery and the city center (periphery/average location), with 120 m^2 of constructed area, a low standard of finishing, in a good state of conservation, with one garage space, 5 rooms, a large backyard, without a security system.

A_{11} – A house in a good location, with 280 m^2 of constructed area, an average standard of finishing, in an average state of conservation, with two garage spaces, 7 rooms, with an additional security system.

A_{12} – An apartment located in the periphery, with 90 m^2 of constructed area, a low standard of finishing, in a bad state of conservation, one garage space, 5 rooms, without additional security.

A_{13} – An apartment located in the periphery in an average location, with 160 m^2 of constructed area, a high standard of finishing, in a good state of conservation, two garage spaces, 6 rooms, with additional security features.

A_{14} – An apartment in a good location, with 320 m^2 of constructed area, high standard of finishing, in a good state of conservation, 2 garage spaces, 8 rooms, with in addition a security system.

A_{15} – A house in a good location, with 180 m^2 of constructed area, an average standard of finishing, in a very good state of conservation, one garage space, 6 rooms, with in addition a security system.

Table 4 shows a list and a description of criteria, with their assigned and normalized weights. Table 5 is the evaluation matrix.

Table 4: Criteria weights

Criterion	Description	Assigned weights	Criteria weights
C_1	Localization	5	0.25
C_2	Construction area	3	0.15
C_3	Quality of construction	2	0.1
C_4	State of conservation	4	0.2
C_5	Number of garage spaces	1	0.05
C_6	Number of rooms	2	0.1
C_7	Attractions	1	0.05
C_8	Security	2	0.1

Table 5: Evaluation matrix

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	3	290	3	3	1	6	4	0
A_2	4	180	2	2	1	4	2	0
A_3	3	347	1	2	2	5	1	0
A_4	3	124	2	3	2	5	4	0
A_5	5	360	3	4	4	9	1	1
A_6	2	89	2	3	1	5	1	0
A_7	1	85	1	1	1	4	0	1
A_8	5	80	2	3	1	6	0	1
A_9	2	121	2	3	0	6	0	0
A_{10}	2	120	1	3	1	5	1	0
A_{11}	4	280	2	2	2	7	3	1

A_{12}	1	90	1	1	1	5	2	0
A_{13}	2	160	3	3	2	6	1	1
A_{14}	3	320	3	3	2	8	2	1
A_{15}	4	180	2	4	1	6	1	1

Computations are performed in 4 steps:

Step 1: Fuzzification of the scales of criteria in order to become non dimensional. In this presentation fuzzy triangular membership functions with null amplitude and mode equal to the original scale are used. Those fuzzy triangular membership functions are written as equation (23) below:

$$f(x,b,c,d) = \max(\min(\frac{x-b}{c-b};\frac{d-x}{d-c}),0) \tag{23}$$

where *b, c, d* are parameters. Parameters *b* and *c* locate the base of the triangle and parameter *d* locates the vertex.

After a number of studies on the joint transformation of all scales in non dimensional values it was decided that a triangular fuzzy membership function should be used. Therefore for all criteria the value of 0.067 was chosen as the highest value of the original scale. That number is equal to the inverse of the number of alternatives which is 15. For example, for C_1 (Localization) the top value is equal to 5 and this value is associated to 0.067. All other values for that particular criterion are proportional to 0.067 in order to maintain the relative importance of readings in the original scale. The second highest value is 4 as it represents 80% of 5, i.e., 0.053. The third highest value is 3, corresponding to 60% of that top value of 5, i.e., 0.040. By following this procedure one obtains 0.027 for the fourth value and 0.013 for the fifth value for criterion $C_{1.}$ Table 6 shows the fuzzification of $C_{1.}$ Exactly the same procedure was followed for all other criteria.

Table 6 – Fuzification of the scale of criterion C_1

Original scale	Fuzzified scale
5	0.067
4	0.053
3	0.040
2	0.027
1	0.013

Table 7 shows the evaluation matrix obtained after fuzzification.

Table 7: The evaluation matrix can now be rewritten after accomplishing the fuzzification

Alternative	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8
A_1	0.040	0.053	0.040	0.053	0.027	0.027	0.067	0.013

A_2	0.053	0.040	0.040	0.040	0.027	0.007	0.040	0.013
A_3	0.040	0.067	0.016	0.040	0.040	0.013	0.027	0.013
A_4	0.040	0.027	0.040	0.053	0.040	0.013	0.067	0.013
A_5	0.067	0.067	0.053	0.067	0.067	0.067	0.027	0.067
A_6	0.027	0.013	0.040	0.053	0.027	0.013	0.027	0.013
A_7	0.013	0.013	0.016	0.027	0.027	0.007	0.013	0.067
A_8	0.067	0.013	0.040	0.053	0.027	0.027	0.013	0.067
A_9	0.027	0.027	0.040	0.053	0.013	0.027	0.013	0.013
A_{10}	0.027	0.027	0.016	0.053	0.027	0.013	0.027	0.013
A_{11}	0.053	0.053	0.040	0.040	0.040	0.040	0.053	0.067
A_{12}	0.013	0.013	0.016	0.027	0.027	0.013	0.040	0.013
A_{13}	0.027	0.040	0.040	0.053	0.040	0.027	0.027	0.067
A_{14}	0.040	0.067	0.053	0.053	0.040	0.053	0.040	0.067
A_{15}	0.053	0.040	0.053	0.067	0.027	0.027	0.027	0.067

Step 2 - Determination of fuzzy measures

Considering the order of criteria:

$C_1 > C_4 > C_2 > C_3 = C_6 = C_8 > C_5 = C_7$

We have the fuzzy measures to calculate the Choquet integral as:

$$\mu_1 = 0.25; \quad \mu_{14} = 0.84\mu_1; \quad \mu_{42} = 0.49\mu_{42}; \quad \mu_{23} = 0.9\mu_{42}$$

$$\mu_{36} = \mu_{68} = \mu_{23}; \quad \mu_{85} = 0.5\mu_{68}; \quad \mu_{57} = \mu_{85}; \quad \mu_{36} = \mu_{68} = \mu_{23}$$

where μ_{ij} are fuzzy measures which are the weights for the different criteria group. We have taken the highest value for μ_1 because criterion 1 is the most important one. The other values are proportional or equal following the criteria order. This weighting is performed in a way such that the sum of all measures is equal to 1.0.

Step 3 - Computation of the Choquet integral

Table 8 presents the computed values of the Choquet integral.

Table 8: Computation of the Choquet integral

Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	Choquet integral
A_1	0.010	0.006	0.004	0.011	0.001	0.003	0.003	0.001	0.041
A_2	0.013	0.005	0.004	0.008	0.001	0.001	0.002	0.001	0.036
A_3	0.010	0.008	0.002	0.008	0.002	0.001	0.001	0.001	0.034
A_4	0.010	0.003	0.004	0.011	0.002	0.001	0.003	0.001	0.037
A_5	0.017	0.008	0.006	0.013	0.003	0.007	0.001	0.007	0.063
A_6	0.007	0.002	0.004	0.011	0.001	0.001	0.001	0.001	0.029
A_7	0.003	0.002	0.002	0.005	0.001	0.001	0.001	0.007	0.022
A_8	0.017	0.002	0.004	0.011	0.001	0.003	0.001	0.007	0.046
A_9	0.007	0.003	0.004	0.011	0.001	0.003	0.001	0.001	0.031
A_{10}	0.007	0.003	0.002	0.011	0.001	0.001	0.001	0.001	0.028
A_{11}	0.013	0.006	0.004	0.008	0.002	0.004	0.003	0.007	0.049
A_{12}	0.003	0.002	0.002	0.005	0.001	0.001	0.002	0.001	0.018
A_{13}	0.007	0.005	0.004	0.011	0.002	0.003	0.001	0.007	0.040
A_{14}	0.010	0.008	0.006	0.011	0.002	0.006	0.002	0.007	0.052
A_{15}	0.010	0.000	0.010	0.010	0.000	0.000	0.000	0.010	0.050

Some of the computed values of the Choquet integral are shown in Table 9 as an example.

Table 9: Example_computation of the Choquet integral

Alternative	Criteria	
	C_1 - Localization	C_2 – Constructed Area
A_1	0.25 * 0.04 = 0.010	0.12 * 0.05 = 0.006
A_2	0.25 * 0.053 = 0.013	0.12 * 0.040 =0.005

In other words, $0.25 * 0.04 = 0.010$ is the product of the fuzzy measure of criteria 1(0.25) by the fuzzified value of the utility for alternative A_1 in relation of criteria C_1. Similarly, $0.25 \cdot 0.053 = 0.013$ is the product of the fuzzy measure of criteria C_1 (0.25) by the fuzzified value of the utility for alternative A_2 in relation of criteria C_1.

The calculations of the Choquet integral are the sum of all the values obtained for each column of the matrix.

For the alternative A_1, we have:

$$0.010 + 0.006 + 0.004 + 0.011 + 0.001 + 0.003 + 0.003 + 0.001 = 0.041$$

For the alternative A_2, we have:

$$0.013 + 0.005 + 0.004 + 0.008 + 0.001 + 0.001 + 0.002 + 0.001 = 0.036$$

and so on.

Thus we obtain Table 10, with values of the Choquet integral for alternatives A_1 and A_2.

Table 10: Values of the Choquet integral for alternatives A_1 and A_2

Criteria	A_1	A_2
C_1 - Localization	0.010	0.013
C_2 - Constructed Area	0.006	0.005
C_3 - Quality of Construction	0.004	0.004
C_4 - State of Conservation	0.011	0.008
C_5 - Number of garage spaces	0.001	0.001
C_6 - Number of rooms	0.003	0.001
C_7 - Attractions	0.003	0.002
C_8 - Security	0.001	0.001
Values of the Choquet integral	0.041	0.036

Step 4 – Ranking of the alternatives

With the values of the Choquet integral we obtain the ranking of the alternatives. This ranking is performed by ordering the obtained values of the Choquet integral. The ranking of the alternatives ordering is shown in Table 11.

Table 11: Ranking of Alternatives and values of the Choquet integral

Alternative	Values of the Choquet integral	Ranking
A_1	0.041	6
A_2	0.036	9
A_3	0.034	10
A_4	0.037	8
A_5	0.063	1
A_6	0.029	12
A_7	0.022	14
A_8	0.046	5
A_9	0.031	11
A_{10}	0.028	13
A_{11}	0.049	4
A_{12}	0.018	15
A_{13}	0.040	7
A_{14}	0.052	2
A_{15}	0.050	3

A comparative analysis of the results is performed by comparing the ranking deisplayed in Table 8 with these obtained by using the original TODIM method as in Gomes and Rangel (2009). Table 12 displays the two rankings. The Spearman coefficient of correlation between the two ranks was found equal to 0.9142. This indicates that these two ranks are indeed quite close.

Table 12: Rankings from using Choquet and the original TODIM method

Alternatives	Choquet ranking	TODIM ranking	Comparison
A_1	6	5	—
A_2	9	10	—
A_3	10	9	—
A_4	8	7	—
A_5	1	1	Same
A_6	12	11	—
A_7	14	15	—
A_8	5	8	—
A_9	11	14	—
A_{10}	13	12	—
A_{11}	4	3	—
A_{12}	15	13	—
A_{13}	7	4	—
A_{14}	2	2	Same
A_{15}	3	6	—

Sensitivity Analysis

The sensitivity analysis was performed by modifying the fuzzy measures by increasing and decreasing their values, and recalculating the Choquet integral. The fuzzy measures used in the sensitivity analysis for the Choquet integral were:

$$\mu_1 = 0.21;\ \mu_{14} = 0.693\mu_1;\ \mu_{42} = 0.93\mu_{14};\ \mu_{23} = 0.835\mu_{42}$$

$$\mu_{36} = \mu_{68} = \mu_{23};\ \mu_{85} = 0.75\mu_{68};\ \mu_{57} = \mu_{85}$$

The results from the sensitivity analysis are presented in Table 13.

Table 13: Comparing Choquet ranking and Todim ranking

Alternatives	Choquet's initial ranking	Choquet's ranking after sensitivity analysis
A_1	6	6
A_2	9	8
A_3	10	9
A_4	8	7
A_5	1	1
A_6	12	11
A_7	14	13
A_8	5	5
A_9	11	10
A_{10}	13	12
A_{11}	4	3
A_{12}	15	14
A_{13}	7	6
A_{14}	2	2
A_{15}	3	4

6. Conclusions and recommendations for future research

The key conclusions from this case study are listed below:

i. The use of the Choquet integral minimizes the calculations of the TODIM method since it is unnecessary to normalize the raw data;

ii. Not only crisp values can be used but also interval data; this second situation would lead to using a fuzzy triangular number;

iii. By using the Choquet integral more complex additive models can be used that allow for taking dependencies between criteria into consideration.

Suggestions for future research follow:

i. Tackling situations where input data on preferences are either entirely unavailable or only partially available and the decision analyst still wants to use TODIM for providing a framework on which an analysis can be based. This case can then be treated as in inverse problem and therefore approached by Monte Carlo simulation. This will lead to a SMAA-P method following Lahdelma and Salminen (2009);

ii. Extending the TODIM method to situations when input data are not only crisp, but also liable to be described by interval or by fuzzy numbers;

iii. Using more complex additive models that allow for taking dependencies between criteria into consideration;

iv. Making use of both Mamdani and Sugeno's fuzzy inferential systems (Oliveira *et al*., 2007) in order to compare the obtained results against these computed by the Choquet-extended TODIM method.

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