A Dynamic Network Model of Interbank Lending  
— Systemic Risk and Liquidity Provisioning  

Agostino Capponi\textsuperscript{1}, Xu Sun\textsuperscript{1}, and David D. Yao\textsuperscript{1}  

\textsuperscript{1}Industrial Engineering & Operations Research, Columbia University  
500 W120th St, New York, NY 10027.  

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Abstract  
We develop a dynamic model of interbank borrowing and lending activities in which banks are organized into clusters, and adjust their monetary reserve levels so as to meet prescribed capital requirements. Each bank has its own initial monetary reserve level and faces idiosyncratic risks characterized by an independent Brownian motion; whereas system wide, the banks form a hierarchical structure of clusters. We model the interbank transactional dynamics through a set of interacting measure-valued processes. Each individual process describes the intra-cluster borrowing/lending activities, and the interactions among the processes capture the inter-cluster financial transactions. We establish the weak limit of the interacting measure-valued processes as the number of banks in the system grows large. We then use the limiting results to develop asymptotic approximations on two proposed macro-measures, the liquidity stress index and the concentration index, both capturing the dynamics of systemic risk. Numerical examples are used to illustrate the applications of the asymptotics and related sensitivity analysis with respect to various indicators of financial activity.  

Keywords: dynamic interbanking networks, systemic risk, large networks asymptotics
1 Introduction

The interbank market plays a critical role in facilitating the provision of liquidity. Yet, this also subjects banks to risk exposures via a complex network of trading relations involving loans and derivatives transactions. Understanding the associated systemic phenomena and their dependence on the topological structure of the network is of critical importance for the design of policies aiming for financial stability.

Most studies in the literature on interbank networks have focused on static models, where all banks simultaneously clear their liabilities, which are exogenously specified. For instance, the seminal paper by Eisenberg and Noe (2001) develops what is essentially a fixed-point algorithm to derive the clearing vector and hence characterize how initial shocks spread through the financial network. Such static models provide a useful framework for quantifying the intensity of shocks and the sensitivity of contagion to structural parameters. They fail, however, to capture the often rapidly changing nature of financial networks in which borrowing and lending patterns adapt to the current economic environment and to the evolving idiosyncratic balance-sheet characteristics among the banks. Indeed, active balance sheet management by banks has been widely documented in empirical studies, see for instance Adrian and Shin (2010). In addition, a study by the European Central Bank (see Halaj and Kok (2013)), using balance sheet data from the banks involved, along with the geographical breakdown of their activities, indicates a pattern of connections via clusters: Most inter-banking transactions are among banks within the same country, hence forming a cluster; on the other hand, some of the largest domestic banks also actively transact with the largest banks in other countries; thus, there is also substantial inter-cluster connectivity.

Motivated by the above reality, we develop here a dynamic network model, where the financial system is partitioned into several clusters, each consisting of a group of banks actively managing their balance sheets to conform with a prescribed target leverage requirement; and we use a system of stochastic differential equations (SDE) to describe the interlinked dynamics of the monetary reserves of the banks in the network.

1.1 Contributions and Organization

A distinct feature of our model is the hierarchical structure of the network where clusters form the top layer; and a set of interacting measure-valued processes, in which each dimension captures the (empirical) reserve distribution of a specific cluster, model both the intra-cluster and the inter-cluster financial activities. We prove that the sequence of measure-valued processes converges
weakly, as the total number of banks in the system grows large, to a limiting process that can be explicitly characterized. This is different from most other studies in the related literature, where the system dynamics are represented by a one-dimensional measure-valued process, which lives in a measure space where the distance can be defined by the Levy-Prokhorov metric, a well-known topology for weak convergence. In contrast, our multi-dimensional process lives in $S$, the product of measure spaces; and for weak convergence, we need to construct a distance function that is a multidimensional extension of the Levy-Prokhorov metric. One might be tempted to use a one-dimensional measure-valued process to capture the empirical reserve distribution as in Bo and Capponi (2015). But in order to obtain meaningful large-network asymptotics, one has to impose some “homogeneity condition” (see eq. (4) and Assumption A1 in Bo and Capponi (2015)), which will add severe restrictions to the applicability of the model.

We demonstrate how the limiting results can be used to study both transient and steady-state performance measures of the network. In particular, we propose two macro-measures, the liquidity stress index (LSI) and the concentration index (CI), to characterize the systemic risk dynamics of the network. The LSI measures the proportion of banks in the system each with a reserve level falling below a threshold (a certain percentage of its target level). The CI measures the manner liquidity is distributed (more evenly or highly concentrated) throughout the system. ¹

Using numerical examples, we illustrate how our results lead to clear economic insights on the interplay between systemic risk and the network architecture. For instance, our results indicate that the transient response to a liquid shock may lead to “too-interconnected-to-fail” risk in a core-periphery network topology. In particular, suppose an initial shock occurred at a (small) subset of the banks in the network pushing down the value of their reserves below their target levels. If the size of the shock is moderate, its instantaneous amplification may be contained. Higher connectivity may thus serve to mitigate and eventually absorb the shock, and hence enhances robustness. If, however, the size of the shock is higher, connectivity becomes a mechanism that propagates and enhances the shock, leading to a high amplification and severe system-wide liquidity stress. By contrast, a liquidity stress takes longer time to propagate in a ring network, and thus in presence of regulatory intervention (e.g. cash injections by a lender of last resort) it may be possible to limit the contagion effect.

¹It is generally agreed upon that concentration threatens financial stability, primarily because of the government bailout of large financial institutions, see Acharya et al. (2014). Policy makers have designed policies limiting the market share of banking institutions. For instance, one of the prominent reforms after the global financial crisis, the Dodd–Frank Act, imposes concentration limits on the amount of debt that banks can hold.
The rest of the paper proceeds as follows. In what remains of this introductory section, we briefly review related theoretical and empirical studies of financial networks. In Section 2, we introduce our dynamic network model for interbank lending and the stochastic differential equations that govern its dynamics. In Section 3, we present our asymptotic analysis of the network model via a set of interacting measure-valued processes. In Section 4, we propose two systemic-risk indicators and develop useful approximations based on the limiting measure-valued process. Numerical examples are presented in Section 5, and concluding remarks summarized in Section 6. Proofs of the technical results are delegated to appendix.

1.2 Literature Review

As mentioned above, most studies in the literature concerning systemic risk in the banking system are motivated by the seminal work of Eisenberg and Noe (2001). Important extensions in this direction include the impact of bankruptcy losses as in Rogers and Veraart (2013); the quantification of contagion effects coming from direct counterparty exposures, and their relation to losses generated by inefficient asset liquidation as in Glasserman and Young (2015); the role played by the network topology in amplifying shocks as in the theoretical studies by Acemoglu et al. (2015) and Capponi et al. (2016). We refer the reader to Capponi (2016) and Glasserman and Young (2016) for excellent surveys on financial networks.

Our paper is related to a stream of literature studying network models with mean-field type interactions, in which banks mean-revert to the average monetary value of the system. Those studies include Fouque and Ichiba (2013), who propose a mean-field model where the monetary reserves of banks are modeled as a system of interacting Feller diffusion processes. They investigate how bank growth rates and lending preferences affect default probabilities and provide an interacting particle system algorithm to compute various performance measures of the network. In contrast to Fouque and Ichiba (2013), Bo and Capponi (2015) model the monetary reserves of banks as a system of interacting jump diffusion processes, where the jumps model inflows and outflows of customer deposits. A shortcoming of these models is that they are based on the assumption that the monetary reserves of banks eventually converge to the average monetary value of the system, regardless of the initial size of the bank. This stands in contrast with empirical evidence, suggesting that (I) larger banks have higher liabilities and hence more reserves (Adrian and Shin (2010)); (II) large banks are more actively engaged in the interbank lending market (Cocco et al. (2009)).

In our study, we take the banks’ target reserve levels as exogenous input parameters. Banks in
different clusters are allowed to have different speed of adjustments to their target reserve levels. Our framework can be specialized to mimic real-world scenarios, where large banks target higher reserve levels and are more actively engaged in the interbank lending market, providing higher financial intermediation to the system.

Our assumption that banks revert to their target reserve levels is strongly supported by empirical evidence from the last two decades. A study by Berger et al. (2008) reveals that banks in the U.S. hold far more equity than required by their regulatory authorities. They observe that banks, like non-financial firms, adjust their capital ratios to a predetermined target level, and set their capital targets significantly higher than regulatory minimum. A cross-section analysis done on a set of German banks by Memmel and Raupach (2010) reveals that a large portion of banks in the sample follows a target capital level. For these banks, adjusting the ratios via purchasing/selling of assets is less effective than by managing their liabilities. The empirical findings in Gropp and Heider (2010) mirror the findings by Berger et al. (2008) and Memmel and Raupach (2010). By analyzing a sample of large, publicly traded banks in sixteen countries, they conclude that banks have stable capital structures at levels that are specific to each individual bank. In addition, banks' target leverage/capital ratio is time invariant and bank specific.

A related line of research encompasses the study of trading relationships in the interbank lending market. The findings in Cocco et al. (2009) provide support for the notion that relationships play an important role in the process of liquidity provision in the interbank lending market. Afonso et al. (2013) find that interbank relationships are highly persistent over time and the majority of lending relationships are asymmetric, i.e. one party is providing liquidity while the other is always demanding it. Their analysis also supports the view that banks borrow funds when they lack liquidity and that when they are lending, they lend to banks that have dissimilar businesses. Earlier models proposed by Bo and Capponi (2015) and Fouque and Ichiba (2013) are unable to capture specific trading relationships or the network structure, because banks are assumed to have the same lending preferences. This excludes network topologies such as the core-periphery structure borne out by bilateral interbank data (see, for instance, Craig and Von Peter (2014) for the case of the German interbank market). In contrast to them, our model captures observed lending patterns and incorporates a wide range of topological structures including the core-periphery topology.

Finally, our study also connects to the measure-valued fluid model for a process-sharing queue; e.g., Gromoll et al. (2008) and Zhang et al. (2009). Applying scaling on certain system parameters similar to the scaling we do here, a family of measure-valued processes representing the dynamics
of the system are shown to converge to a fluid limit characterized as the solution of a time-changed functional differential equation.

1.3 Notation and Conventions

Here are the notation and conventions that will be extensively used throughout the paper. We denote by \( \mathbb{R} \) the set of real numbers. Let \( \mathbb{N} \) and \( \mathbb{Z}_+ \) denote the set of natural numbers and the set of positive integers respective. For a row vector \( a \) we use \( a^\top \) to denote its transpose. Let \( e_j \) denote the unit vector with the \( j \)th entry being one and the remaining entries being zero. \( 1_A(\cdot) \) denotes the indicator function of the event (set) \( A \). We use \( \overset{d}{=} \) to denote equality in distribution, and \( \Rightarrow \) to denote convergence in distribution. Let \( \mathcal{D} \) be the usual function space of right-continuous real-valued functions on the interval \([0, \infty)\) with left limits, as in Whitt (2002). The convergence \( f_n \to f \) in \( \mathcal{D} \) at the continuous points of \( f \) is equivalent to uniform convergence over bounded intervals. For a metric space \((E, d)\) with the distance function \( d \), we use \( \mathcal{C}(E) \) and \( \mathcal{C}_b(E) \) to denote, respectively, the space of continuous functions and bounded continuous functions on \( E \). Similarly, use \( \mathcal{C}_q^p(E) \) to denote the set of continuous functions on \( E \) that have bounded derivatives up to order \( q \). Lastly, for a set of metric spaces \( \{E_i\} \), we use \( \Pi_i E_i \) to denote the product space equipped with the usual product topology. A filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the filtration \( \mathcal{F}_t := (\mathcal{F}_t)_{t \geq 0} \) supports all stochastic processes defined below.

2 A Model of Interbank Lending

We consider an interbank network that has \( N \in \mathbb{N} \) clusters of banks interacting through borrowing and lending transactions. Each cluster \( j \) consists of \( K_j \) banks. Let \( \Xi \) be the collection of all pairs \((j, k), k = 1, \ldots, K_j, j = 1, \ldots, N\).

The state variable is \( \xi_{j,k}(t) \), representing the monetary reserve of bank \( k \) in cluster \( j \) at time \( t \). It is assumed to satisfy the following stochastic differential equation (SDE):

\[
\xi_{j,k}(t) = \xi_{j,k}(0) + \int_0^t \ell_j \theta_{j,k} - \xi_{j,k}(s) \, ds - \int_0^t \frac{\ell_j \pi_{jj}}{K_j} \sum_{k' \leq K_j} [\theta_{j,k'} - \xi_{j,k'}(s)] \, ds \\
- \int_0^t \sum_{h \neq j} \frac{\ell_h \pi_{h,j}}{K_j} \sum_{k' \leq K_h} [\theta_{h,k'} - \xi_{h,k'}(s)] \, ds + \sigma_{j,k} \int_0^t \sqrt{\xi_{j,k}(s)} \, dW_{j,k}(s). \tag{2.1}
\]

On the right hand side of (2.1), after the initial state \( \xi_{j,k}(0) \), the first integral accounts for the bank’s own (cumulative) input/output up to \( t \), where \( \theta_{j,k} \) is the bank’s required reserve level, and \( \ell_j > 0 \) is an intensity (or “pressure”) factor. Thus, if \( \theta_{j,k} > \xi_{j,k}(t) \) (resp. \( \theta_{j,k} < \xi_{j,k}(t) \)), then the
bank is more inclined—modulated by the rate $\ell_j$—to borrow (resp. to lend) and thereby increasing (resp. decreasing) its reserve. The second and the third integrals represent the bank’s transactions (borrowing and lending) with other banks ($k'$) within the same cluster ($j$) and those from other clusters ($h$), respectively. Note, as the vintage point switches to the other banks, the positive sign of the first integral becomes negative for the second and third integrals. We assume a certain symmetry among all the banks in the same cluster:

(i) any bank in cluster $j$ will have the same probability to transact with (a bank in) another cluster $h$; hence, this probability is denoted by $\pi_{j,h}$;

(ii) when an inter-cluster ($j, h$) transaction occurs, the originating bank in cluster $j$ chooses one of the $K_h$ banks in cluster $h$ with equal probability $(1/K_h)$.

The last integral on the right hand side of (2.1) captures the idiosyncrasy in the bank’s reserve (due, for instance, to the daily deposits and withdraws from retail customers), where $\sigma_{j,k}$ is a positive constant and $W \equiv (W_{j,k})$ the standard Brownian motion, with the latter being independent among the banks.

Let $\Pi \equiv [\pi_{j,h}]_{j,h=1}^{N}$ be the “routing” matrix of the transactions, also referred to as the transaction probability matrix throughout the paper. We assume $\Pi$ is a sub-stochastic matrix, i.e., with a spectral radius less than or equal to one. This implies that some rows of the $\Pi$ matrix may sum up to less than one. In that case, our network model is an open system, meaning that there exist some banks in the network that transact with banks outside of the system as well as those inside the system.

Denote $\bar{\theta}_j \equiv (1/K_j) \sum_{k \leq K_j} \theta_{j,k}, \bar{\xi}_j \equiv (1/K_j) \sum_{k \leq K_j} \xi_{j,k}$ for each $j \leq N$ and $\ell_{j,h} \equiv (K_j/K_h) \ell_j \pi_{j,h}$. We can rewrite the above equation (by combining the second and the third integrals) as follows:

$$
\xi_{j,k}(t) = \xi_{j,k}(0) + \int_0^t \ell_j [\theta_{j,k} - \xi_{j,k}(s)] \, ds - \int_0^t \sum_{h \leq N} \ell_{h,j} [\bar{\theta}_h - \bar{\xi}_h(s)] \, ds + \sigma_{j,k} \int_0^t \sqrt{\xi_{j,k}(s)} \, dW_{j,k}(s).
$$

(2.2)

The above formulation is designed to capture certain essential features of interbank lending activities as motivated in the Introduction (such as maintaining a target reserve level and operating in a clustered hierarchy), while modulated with simplifying assumptions (such as the linear “pressure” for borrowing/lending, and the symmetry among banks within the same cluster), so as to maintain tractability.
Throughout the paper, we impose the following Feller condition that guarantees the existence of a unique nonnegative solution.

**Condition 2.1** For each \((j, k) \in \Xi\),

\[
\ell_j (1 - \pi_{j,j}/K_j)\theta_{j,k} - (\ell_j \pi_{j,j}/K_j) \sum_{k' \neq k} \[\theta_{j,k'} - x_{j,k'}\] - \sum_{h \neq j} (\ell_j \pi_{h,j}/K_j) \sum_{k' \leq K_h} \theta_{h,k'} > \sigma_{j,k}^2/2 \quad (2.3)
\]

for \(\{x_{h,k'}\}_{(h,k') \neq (j,k)} \subset \mathbb{R}_+\).

A direct verification of the inequality (2.3) is not straightforward. Because the variables \(x_{h,k'}\) are nonnegative, however, the left-hand-side always admits the lower bound

\[
\ell_j (1 - \pi_{j,j}/K_j)\theta_{j,k} - (\ell_j \pi_{j,j}/K_j) \sum_{k' \neq k} \theta_{j,k'} - \sum_{h \neq j} (\ell_j \pi_{h,j}/K_j) \sum_{k' \leq K_h} \theta_{h,k'}.
\]

This bound is useful, because if one can verify that

\[
\ell_j (1 - \pi_{j,j}/K_j)\theta_{j,k} - (\ell_j \pi_{j,j}/K_j) \sum_{k' \neq k} \theta_{j,k'} - \sum_{h \neq j} (\ell_j \pi_{h,j}/K_j) \sum_{k' \leq K_h} \theta_{h,k'} > \sigma_{j,k}^2/2 \quad (2.4)
\]

for all \((j, k) \in \Xi\), Condition 2.1 necessarily holds.

An example (satisfying Condition 2.1) can be easily constructed. Assume \(\ell_j = \ell\) and \(K_j = K\) for all \(j\) and \(\sigma_{j,k} = \sigma\) for all \((j, k) \in \Xi\). Then, (2.4) reduces to

\[
\theta_j - \sum_h \pi_{h,j} \theta_h > \sigma^2/2, \quad \text{for all } j.
\]

In matrix form, the above equation may be rewritten as \((I - \Pi^\top)\theta > \sigma^2 I/2\) where \(>\) holds component-wise. Intuitively, so long as the spectral radius of the transaction probability matrix \(\Pi\) is sufficiently lower than 1 and each entry of the vector \(\theta\) is sufficiently larger than \(\sigma^2/2\), then the inequality holds.

### 3 Large Network Asymptotic Analysis

This section introduces the set of measure-valued processes which will be used in the large-network asymptotic analysis. Each process keeps track of the empirical distribution of the type (volatility and bank’s reserve level) of banks in a cluster, capturing the typical intra-cluster activities. The interaction between these processes captures the macroscopic behavior of the system’s activities.

Throughout the paper, we view \(\ell_j\) and \(\pi_{j,h}\) as fixed deterministic parameters, and treat the remaining parameters \(p_{j,k} \equiv (\xi_{j,k}(0), \theta_{j,k}, \sigma_{j,k})\), idiosyncratic to each bank \(k\) within the cluster \(j\), as
random variables taking values in $\Omega_p \equiv \mathbb{R}_+^3$. In addition, we will use the superscript $\eta$ to highlight the dependence of the relevant model quantities on a scaling parameter $\eta$. We make the following assumptions:

(A1) The number of banks in cluster $j$, for every $j$, is equal to $K^\eta_j = \eta \kappa_j$ where $\kappa_j > 0$ is a fixed parameter. We assume that $\sum_{j \leq N} \kappa_j = 1$, implying that the scaling parameter $\eta = \sum_{j \leq N} K_j^\eta$ is equal to the total number of banks in the network.

(A2) For every $j$ we assume that $\tilde{\phi}_j \equiv (1/K_j^\eta) \sum_{k \leq K_j^\eta} \delta(p_{j,k}) \Rightarrow \phi_j$ for some probability measure $\phi_j$ as $\eta \to \infty$. In addition, assume $p_{j,k} \leq C_p$ for all $(j,k)$, for some constant $C_p$.

**Remark 3.1** Recall that $\ell_{j,h} \equiv (K_h^\eta/K_j^\eta) \ell_j \pi_{j,h}$. Condition (A1) implies that $\ell_{j,h} = \left( \kappa_j / \kappa_h \right) \ell_j \pi_{j,h}$ being independent of the scaling coefficient $\eta$.

Let $\Omega \equiv \Omega_p \times \mathbb{R}$. We define a vector of interacting measure-valued processes

$$\nu^\eta(\cdot) \equiv (\nu_1^\eta(\cdot), \ldots, \nu_N^\eta(\cdot))^\top$$

where, for each cluster $j$ and time $t$, the empirical measure $\nu_j^\eta(t)$ is given by

$$\nu_j^\eta(t) = \frac{1}{K_j^\eta} \sum_{k=1}^{K_j^\eta} \delta(p_{j,k}, \xi_{j,k}(t)) \text{ for } t \geq 0. \quad (3.2)$$

Let $S = \prod_j P_j(\Omega)$, where $P_j(\Omega)$ represents the space of probability measures on the metric space $\Omega$. Then $\nu^\eta(\cdot)$ can be viewed as a $S$-valued stochastic process. Note that each component $\nu_j^\eta(\cdot)$ of the process $\nu^\eta(\cdot)$ is a standard measure-valued process. For notational brevity, set $\langle \mu, f \rangle \equiv \int_\Omega f \, d\mu$ for any $\mu \in \mathcal{P}(\Omega)$ and measurable function $f$. Hence, we obtain

$$\langle \nu_j^\eta, f \rangle_t \equiv \langle \nu_j^\eta(t), f \rangle = (1/K_j^\eta) \sum_{k=1}^{K_j^\eta} f(p_{j,k}, \xi_{j,k}(t)) \quad (3.3)$$

**Remark 3.2** For $(p, x) \in \Omega$, define the functions

$$\psi_1(p, x) = x \quad \text{and} \quad \Theta(p, x) = \theta. \quad (3.4)$$

Within each cluster, we can express the average bank’s monetary reserve and required reserve level using the representation (3.3):

$$\langle \nu_j^\eta, \psi_1 \rangle_t \equiv \xi_j(\cdot) \equiv \frac{1}{K_j^\eta} \sum_{k=1}^{K_j^\eta} \xi_{j,k}(\cdot) \text{ for } t \geq 0 \quad \text{and} \quad \langle \nu_j^\eta, \Theta \rangle_0 = \bar{\theta}_j \equiv \frac{1}{K_j^\eta} \sum_{k=1}^{K_j^\eta} \theta_{j,k} \quad (3.5)$$

We will make extensive use of the quantities in (3.5) throughout the paper.
3.1 The Topological Metric Space

We aim to show that the sequence of $S$-valued processes \( \{ \nu^n(\cdot) \equiv (\nu^1_\eta(\cdot), \ldots, \nu^N_\eta(\cdot)) \} \) indexed by $\eta$ converges weakly, as $\eta \to \infty$, to a limit $\nu(\cdot) \equiv (\nu_1(\cdot), \ldots, \nu_N(\cdot))$, with respect to an appropriate topology on $S$, which will be specified in the present section.

Single-Dimensional Case

We measure the distance between two distributions $\mu, \mu' \in \mathcal{P}(\mathcal{O})$ using the Prokhorov metric, i.e.,

\[
\rho(\mu, \mu') \equiv \inf \{ \epsilon > 0 : \mu(A) \leq \mu'(A^\epsilon) + \epsilon \},
\]

where $A$ is a Borel set, and $A^\epsilon \equiv \{ y \in \mathcal{O} : d(x, y) < \epsilon \text{ for some } x \in A \}$. It is known that the Prokhorov metric $\rho$ is topologically equivalent to

\[
\beta(\mu, \mu') \equiv \sup \left\{ \left\| \int f \, d(\mu - \mu') \right\|_{BL} : \|f\|_{BL} \leq 1 \right\},
\]

(3.6)

where $f$ is a bounded Lipschitz function and $\|f\|_{BL} \equiv \|f\|_L + \|f\|_\infty$ with $\|f\|_L \equiv \sup_{x \neq y} |f(x) - f(y)|/d(x, y)$ and $\|f\|_\infty \equiv \sup_x |f(x)|$, e.g. see Chapter 11 of Dudley (2002).

The finite-dimensional distribution of a diffusion process can be generally characterized as the unique solution of a martingale problem associated with a second-order elliptic differential operator. By analogy the finite-dimensional distribution of a rich class of measure-valued diffusion processes can be obtained as the solution to the martingale problem associated with a second-order differential operator of the form:

\[
\mathcal{L}\Phi(\mu) \equiv A(\mu)(\partial\Phi(\mu)/\partial\mu(x)) + B(\mu)(\partial^2\Phi(\mu)/\partial\mu(x)\partial\mu(y))/2
\]

(3.7)

for $\Phi(\cdot) \in \mathbb{D}(\mathcal{P}(\mathcal{O}))$ where $\partial\Phi(\mu)/\partial\mu(x)$ is the variational derivative given by $\partial\Phi(\mu)/\partial\mu(x) \equiv \lim_{\epsilon \to 0} [\Phi(\mu + \epsilon \delta_x) - \Phi(\mu)]/\epsilon$, $A(\cdot)$ and $B(\cdot)$ are linear and bilinear functionals defined on $\mathcal{C}(\mathcal{O})$ and $\mathcal{C}(\mathcal{O} \times \mathcal{O})$ respectively (see Dawson and Kurtz (1982)). The set $\mathbb{D}(\mathcal{P}(\mathcal{O})) \subset C_b(\mathcal{P}(\mathcal{O}))$ should be chosen so as to guarantee that both first and second variational derivatives exist and it must be sufficiently rich to generate $C_b(\mathcal{P}(\mathcal{O}))$ under the bounded pointwise convergence. This suggests the choice of the following family

\[
\Phi(\mu) = \phi(\langle \mu, f^1 \rangle, \ldots, \langle \mu, f^m \rangle)
\]

for some $m \in \mathbb{Z}_+$, $\phi \in C^2_b(\mathbb{R}^m)$ and $\{ f^i \}_{i=1}^m \subset C^2_b(\mathcal{O})$.

**Lemma 3.1** The function class $\mathbb{D}$ separates points in $\mathcal{P}(\mathcal{O})$ and is dense in $C_b(\mathcal{P}(\mathcal{O}))$. 
Multi-Dimensional Extension

To statistically correlate random measures, we propose a multivariate extension of the metric in (3.6). With a slight abuse of notation, we define

\[ \beta(\mu, \mu') \equiv \sup \left\{ \sum_{j=1}^{N} \left| \int f_j \, d(\mu_j - \mu'_j) \right| : \|f_j\|_{BL} \leq 1 \text{ for } j = 1, \ldots, N \right\} , \tag{3.8} \]

where both \( \mu \equiv (\mu_1, \ldots, \mu_N) \) and \( \mu' \equiv (\mu'_1, \ldots, \mu'_N) \) are elements in the space \( S \). We show that the function \( \beta \) is non-negative, indiscernible, symmetric, sub-additive and hence a metric on \( S \).

Lemma 3.2 The function \( \beta : \prod_j P_j(O) \times \prod_j P_j(O) \to [0, \infty] \) given above is a metric.

Lemma 3.2 leads to the conclusion that \( S \) is a Polish space. A topology is generated in the usual way for the Skorokhod space \( D_S[0, \infty) \).

Convergence in \( S \equiv \prod_j P_j(O) \) can be characterized through the following lemma, whose proof follows by a straightforward extension of that in the unidimensional case \( (N = 1) \), e.g., see Chapter 11 of Dudley (2002).

Lemma 3.3 Let \((U, d)\) be a separable metric space. For any \( \mu^\alpha \equiv (\mu^\alpha_1, \ldots, \mu^\alpha_N) \) and \( \mu \) in \( S \), the following statements are equivalent:

(a) \( \beta(\mu^\alpha, \mu) \to 0 \);

(b) \( (\langle \mu^\alpha_1, f_1 \rangle, \ldots, \langle \mu^\alpha_N, f_N \rangle) \to (\langle \mu_1, f_1 \rangle, \ldots, \langle \mu_N, f_N \rangle) \) for all \( \{f_j\}_{j=1}^{N} \subset BL(O) \), where \( BL \) denotes the collection of bounded Lipschitz functions;

(c) \( (\langle \mu^\alpha_1, f_1 \rangle, \ldots, \langle \mu^\alpha_N, f_N \rangle) \to (\langle \mu_1, f_1 \rangle, \ldots, \langle \mu_N, f_N \rangle) \) for all \( \{f_j\}_{j=1}^{N} \subset C^q_b(O), q \in \mathbb{Z}_+ \).

Remark 3.3 Lemma 3.3 is especially useful in analyzing the sample path properties of processes living in the Skorokhod space \( D_S[0, \infty) \). In particular, we will use the equivalent characterization (c) to verify the modulus of continuity condition for the set of interacting processes describing the interbanking activities.

With a slight abuse of notation, let \( \mathcal{D} \) be the collection of functions of the form

\[ \Phi(\mu) = \phi(\langle \mu, f^1 \rangle, \ldots, \langle \mu, f^m \rangle) \quad \text{for } \mu = (\mu_1, \ldots, \mu_N) \in S \tag{3.9} \]

where

\[ f^i(\cdot) \equiv (f^i_1(\cdot), \ldots, f^i_N(\cdot))^\top \quad \text{and} \quad \langle \mu, f^i \rangle \equiv (\langle \mu_1, f^i_1 \rangle, \ldots, \langle \mu_N, f^i_N \rangle)^\top \]
for $m \in \mathbb{Z}_+$, $f_j^i \in \mathcal{C}_b$ and a test function $\phi \in \mathcal{C}_b^2(\mathbb{R}^{N \times m})$. Paralleling the proof of Lemma 3.1 in the appendix, one can easily argue that the function class $\mathcal{D}$ separates $S$ and is dense in $\mathcal{C}_b(S)$.

### 3.2 Weak Convergence in $S$

We will show that the sequence of stochastic processes $\{\nu^n(\cdot)\}$ indexed by the scaling factor $\eta$ converges weakly to a limit $\nu(\cdot)$ in the Skorokhod space $D_S[0, \infty)$. The main result is formally stated in Theorem 3.1, and is essentially a functional law of large numbers (FLLN) for a family of interacting measure-valued processes.

For each $j \leq N$, let $p_j \equiv (x_j, \theta_j, \sigma_j)$ be a sample from the limiting distribution $\phi_j$, specified in Assumption (A2). For such a $p_j \in \mathcal{O}$, define a mean-reverting square-root process $X_j(p_j; \cdot)$ with time-varying coefficients:

$$X_j(p_j; t) = x_j + \int_0^t \ell_j (\theta_j - X_j(p_j; s)) \, ds - \int_0^t \sum_{h \leq N} \ell_{h,j} (V_h - Q_h(s)) \, ds + \int_0^t \sigma_j \sqrt{X_j(p_j; s)} \, dW_j(s).$$

(3.10)

Notice that the above notation stresses the dependence of the underlying state process $X_j(p_j; \cdot)$ on the realized parameter set $p_j$. For each $j \in \{1, \ldots, N\}$, $V_j$ is a constant that satisfies

$$V_j = \langle \nu_j, \Theta \rangle_0 \equiv \int_{\mathcal{O}} \Theta(p, x) \phi_j(dp_j) = \int_{\mathcal{O}_p} \theta_j \phi_j(dp_j)$$

(3.11)

where we recall that $\Theta(p, x) = \theta$. The time-varying vector-valued function $Q(\cdot) \equiv (Q_1(\cdot), \ldots, Q_N(\cdot))^\top$ satisfies a set of integral equations:

$$Q_j(t) = \int_{\mathcal{O}_p} e^{-\ell_j t} \left[ x_j + \int_0^t \left( \ell_j \theta_j - \sum_{h \leq N} \ell_{h,j} (V_h - Q_h(s)) \right) e^{\ell_j s} \, ds \right] \phi_j(dp_j)$$

$$= \int_{\mathcal{O}_p} \left[ e^{-\ell_j t} x_j + (1 - e^{-\ell_j t}) \left( \theta_j - \sum_{h \leq N} \ell_{h,j} V_h/\ell_j \right) + \sum_{h \leq N} \int_0^t e^{-\ell_j (t-s)} Q_h(s) \, ds \right] \phi_j(dp_j)$$

(3.12)

for $j = 1, \ldots, N$.

Using the state process $X_j(p_j; t)$ defined by (3.10)-(3.12), we characterize the limiting measure-valued process of the sequence $\{\nu^n(\cdot)\}$. For each $j \in \{1, \ldots, N\}$, define a measure-valued process $\nu_j(\cdot)$ via

$$\langle \nu_j, 1_A \times B \rangle_t \equiv \langle \nu_j(t), 1_A \times B \rangle \equiv \int_{\mathcal{O}} 1_A(p_j) \mathbb{P}(X_j(p_j; t) \in B) \phi_j(dp_j) \quad \text{for} \quad t \geq 0,$$

(3.13)

where $A \in \mathcal{B}(\mathcal{O}_p)$ and $B \in \mathcal{B}(\mathbb{R})$. Let $\nu(\cdot) \equiv (\nu_1(\cdot), \ldots, \nu_N(\cdot))$. The following lemma plays an important role in the proof of the main theorem and as well as in the development of useful approximations for the systemic-risk indicators studied in §5.
Lemma 3.4 The time-varying vector-valued function \( Q(t) \equiv (Q_1(t), \ldots, Q_N(t))^\top \) given through the set of integral equations (3.12) equals the limiting measure-valued process \( \nu_t \) acting on the identity function, i.e.,
\[
Q(t) = (\langle \nu_1, \psi_1 \rangle_t, \ldots, \langle \nu_N, \psi_1 \rangle_t)^\top \quad \text{for} \quad t \geq 0,
\]
where \( \nu \) is specified by (3.13) and \( \psi_1 \) given by (3.4).

Corollary 3.1 If, for each \( j \leq N \), the limiting measure \( \phi_j \) in Assumption (A2) is a Dirac measure, i.e., \( \phi_j \equiv \delta_{p^*_j} \) and \( p^*_j \equiv (x^*_j, \theta^*_j, \sigma^*_j) \) as \( \eta \to \infty \), then the vector-valued function \( Q(t) \) is the solution of the following linear system:
\[
dQ(t) = (I - \Lambda^{-1} \Pi^\top \Lambda)(\theta^* - Q(t))dt \quad \text{and} \quad Q(0) = x^* \equiv (x^*_1, \ldots, x^*_N)
\]
where \( I \) is the identity matrix, \( \Pi \) the transaction probability matrix, \( L \) a diagonal matrix whose diagonal elements are \( \ell_j \), \( \Lambda \) a diagonal matrix whose diagonal elements are \( \kappa_j \) and \( \theta^* \equiv (\theta^*_1, \ldots, \theta^*_N)^\top \).

Let \( R \equiv (I - \Lambda^{-1} \Pi^\top \Lambda)L \). Then the equation (3.15) admits a closed-form solution:
\[
Q(t) = e^{-Rt}x^* + (I - e^{-Rt})\theta^*,
\]
where \( e^M \) is understood to be exponential of the matrix \( M \).

Remark 3.4 The structure of Eq. (3.16) highlights the idiosyncratic effect and, more importantly, the systemic impact of a shock to the initial monetary reserves of a cluster. Recall that each component of \( Q(t) \) represents the large-network approximation for the average reserve level of a cluster. Suppose that an initial shock occurred at the \( j \)th cluster of the network, pushing down the value of its average reserve level below the average target level by \( \Delta x \). Using (3.16), the transient (negative) impact at time \( t \) is
\[
\Delta Q(t) \equiv e^{-Rt}(\Delta xe_j) \approx \Delta x(I - Lt)e_j + \Delta x(\Lambda^{-1} \Pi^\top \Lambda L)e_j,
\]
where we have used the Taylor approximation to highlight the short-term systemic effects of this shock. The first term on the right side is the idiosyncratic component of the shock. It indicates that if banks in cluster \( j \) have a high propensity to transact and adjust to the target level, then they will recover quickly from the shock. The second term captures the impact of the network structure through the dependence on the transaction probability matrix \( \Pi \). If cluster \( j \) has a high propensity to transact (large \( \ell_j \)) and distributes its transactions uniformly over the network (\( \pi_{j,k} \approx \frac{1}{N-1} \)), then
the short-term impact will be high on all clusters and may result in a systemic breakdown when 
\( \Delta x \) is sufficiently large. On the other hand, if cluster \( j \) concentrates its transactions among a few
clusters \( \pi_{j,k} \gg \frac{1}{N-1} \) for some \( k \)'s, and \( \pi_{j,k} = 0 \) for other values of \( k \),
then the shock will take a longer time to propagate to those components of the network that have weak connections to cluster
\( j \). Note, however, that in the long run, a more heterogenous transaction probability matrix always
leads to faster recovery. This can be directly seen from Eq. (3.16). The term \( (I - e^{-R})e_j \) will be
low if the matrix \( R = I - \Pi \) has large eigenvalues (i.e. if \( \Pi \) has small eigenvalues). For example,
in a fully diversified network, the eigenvalues of \( \Pi \) will be close to \( \frac{1}{N} \), whereas in a ring structure
they will be closer to one.

We are now in a position to state the main result which is formalized as the following theorem.

**Theorem 3.1** Under Assumptions (A1) - (A2), the sequence of interacting measure-valued pro-
cesses \( \{\nu^\eta(\cdot) \equiv (\nu^\eta_1(\cdot), \ldots, \nu^\eta_N(\cdot))\} \) indexed by \( \eta \) converges weakly to the limit \( \nu(\cdot) \), i.e.,

\[
\nu^\eta(\cdot) \Rightarrow \nu(\cdot) \equiv (\nu_1(\cdot), \ldots, \nu_N(\cdot)) \quad \text{in} \quad D_S[0, \infty), \quad \text{as} \quad \eta \to \infty,
\]
where each entry of the vector \( \nu(\cdot) \) is a measure-valued process and satisfies (3.13).

Theorem 3.1 characterizes the weak limit as a vector of deterministic (hence statistically inde-
pendent) measure-valued processes, where each dimension describes the transient behavior of the
empirical distribution of bank reserves within a cluster.

The proof of the theorem consists of several steps. First, we establish the existence of at least one
limit point. This is achieved by proving that the sequence of measures \( \{\nu^\eta(\cdot)\} \) is tight. We then
identify the limit via the martingale approach as described, for instance, in Stroock and Varadhan
(1972). Convergence of finite-dimensional distributions (via the martingale approach) along with
a uniqueness lemma for the martingale problem concludes the proof of the main theorem. By
uniqueness, we mean that any two solutions to the martingale problem share the same distribution
law.

4 Systemic-Risk Indicators

The objective of this section is to compute asymptotic approximation formulas for systemic risk
metrics. The limiting measure-valued process \( \nu \) can be used to construct law-of-large number
approximations for a wide variety of performance measures. We choose to focus on two types
of systemic-risk indicators, namely, the liquidity stress index and the concentration index. These
measures provide not only an overall risk outlook of the network, but also capture excess correlation and volatility in the network.

**Liquidity Stress Index**

A bank is said to be experiencing liquidity stress if its reserve level falls short of a certain percent of the target, i.e., $\xi < \alpha \theta$ for $\alpha < 1$. The following quantity, which we call the “liquidity stress index”, is simply the fraction of banks experiencing liquidity stress at time $t$:

$$L_j^n(t) \equiv \frac{1}{K_j^n} \sum_{k=1}^{K_j^n} 1_{\{\xi_{j,k} < \alpha \theta_{j,k}\}},$$

where we recall that $\eta$ is the scaling parameter denoting the total number of banks in the network.

A larger value for $L_j^n(t)$ corresponds to a situation when normal banking intermediation at time $t$ is severely disrupted and the credit supply is reduced with potentially adverse consequences on the real economy. Let $A \equiv \{x | x < \alpha \theta\}$. We can then write

$$L^n(t) \equiv (L_1^n(t), \ldots, L_N^n(t)) = (\langle \nu_1^n, 1_A \rangle_t, \ldots, \langle \nu_N^n, 1_A \rangle_t).$$

For each $j$ and a fixed $t \geq 0$, $A$ is a continuity set for $\nu_j(t)$, e.g., see (3.13) and (3.10). Thus

$$\langle \nu_j^n, 1_A \rangle_t \Rightarrow (\nu_j, 1_A)_t \text{ in } \mathbb{R} \text{ as } \eta \to \infty.$$  

We can use the converging-together lemma (see, e.g., Theorem 11.4.3 of Whitt (2002)) to establish the joint convergence

$$\left(\langle \nu_1^n, 1_A \rangle_t, \ldots, \langle \nu_N^n, 1_A \rangle_t\right) \Rightarrow \left(\nu_1, 1_A \right)_t, \ldots, \langle \nu_N, 1_A \rangle_t) \text{ in } \mathbb{R}^N \text{ as } \eta \to \infty,$$

where

$$\langle \nu_j, 1_A \rangle_t = \int_{O} \mathbb{P}(X_j(p_j; t) < \alpha \theta_j) \phi_j(dp_j) \quad (4.2)$$

and $X_j(p_j; t)$ follows the dynamics given by (3.10).

**Concentration Index**

The “concentration-fragility” view holds that concentrated systems lead to excessive risk-taking, because of moral hazard stemming from the implicit government bail out of too-big-to-fail institutions (O’Hara and Shaw (1990), Acharya et al. (2014)), or the complex and opaque structures that are often associated with large institutions (Cetorelli et al. (2014)). Our analysis of the concentration level of the financial network serves to highlight how the interplay of shocks, volatilities, and
inter-dependencies of financial activities can lead to a rise in the concentration of banks’ monetary reserves.

We measure concentration using the Herfindahl index. For a vector of non-negative real numbers \( a \equiv (a_1, \ldots, a_n) \), the Herfindahl index of \( a \) is defined to be \( H(a) \equiv \frac{\sum_{k=1}^{n} a_i^2}{(\sum_{k=1}^{n} a_1)^2} \), i.e., the sum of the squares normalized by the square of the sum. It is easy to verify that \( H \) attains its maximum when all \( a_i \)'s are equal; and \( H \) attains its minimum when all \( a_i \)'s, except for one, are zero. This notion can be easily generalized to vector-valued functions.

**Definition 4.1** The concentration index of the interbank network is the sum of the squares of banks’ monetary reserves normalized by the squared aggregate amount of monetary reserves, i.e.

\[
H^\eta(t) = \frac{\sum_{(j,k) \in \Xi} (\xi_{j,k}(t))^2}{\left(\sum_{(j,k) \in \Xi} \xi_{j,k}(t)\right)^2}. \tag{4.3}
\]

The time series \( \{H^\eta(t)\} \) defined by (4.3) is a stochastic process adapted to the natural filtration. We scale the process \( H^\eta \) in a way that the sequence of scaled processes converges weakly to a proper limit. Let

\[
\overline{H}^\eta(t) \equiv H^\eta(t)/\eta, \quad \text{for} \quad t \geq 0. \tag{4.4}
\]

Using the definition of \( H^\eta \), we can write

\[
\overline{H}^\eta(t) = \frac{\sum_{j=1}^{N} (K^\eta_j/\eta) \cdot \langle \nu_j^\eta, \psi_2 \rangle_t}{\left[\sum_{j=1}^{N} (K^\eta_j/\eta) \cdot \langle \nu_j^\eta, \psi_1 \rangle_t\right]^2} \tag{4.5}
\]

where we have defined \( \psi_2(p,x) \equiv x^2 \), and we recall that \( \psi_1(p,x) = x \) is the identify function defined in Eq. (3.4). Using Theorem 3.1 and the moment conditions established by Lemma A.3, we can deduce for each \( j \)

\[
\langle \nu_j^\eta, \psi_1 \rangle_t \Rightarrow \langle \nu_j, \psi_1 \rangle_t \quad \text{and} \quad \langle \nu_j^\eta, \psi_2 \rangle_t \Rightarrow \langle \nu_j, \psi_2 \rangle_t \quad \text{as} \quad \eta \to \infty. \tag{4.6}
\]

Because all the limits are deterministic, the above convergence can be strengthened to joint convergence by the converging-together lemma (see, e.g., Theorem 11.4.3 of Whitt (2002)), i.e.,

\[
(\langle \nu_1^\eta, \psi_1 \rangle, \ldots, \langle \nu_N^\eta, \psi_1 \rangle) \Rightarrow (\langle \nu_1, \psi_1 \rangle, \ldots, \langle \nu_N, \psi_1 \rangle) \tag{4.7}
\]

and

\[
(\langle \nu_1^\eta, \psi_2 \rangle, \ldots, \langle \nu_N^\eta, \psi_2 \rangle) \Rightarrow (\langle \nu_1, \psi_2 \rangle, \ldots, \langle \nu_N, \psi_2 \rangle) \tag{4.8}
\]

16
as \( \eta \to \infty \). We can then use the continuous mapping theorem (CMT) with continuity of additive functions and the converging-together lemma to obtain

\[
\left( \sum_{j=1}^{N} \left( K_{j}^{\eta}/\eta \right) \langle \nu_{j}^{\eta}, \psi_{2} \rangle, \sum_{j=1}^{N} \left( K_{j}^{\eta}/\eta \right) \langle \nu_{j}^{\eta}, \psi_{1} \rangle \right) \Rightarrow \left( \sum_{j=1}^{N} \kappa_{j} \langle \nu_{j}, \psi_{2} \rangle, \sum_{j=1}^{N} \kappa_{j} \langle \nu_{j}, \psi_{1} \rangle \right)
\]

as \( \eta \to \infty \). Using the CMT with division operator yields the result below.

**Proposition 4.1** Let \( K = (K_{1}^{\eta}, \ldots, K_{N}^{\eta}) \), if \( K_{j}^{\eta}/\eta \to \kappa_{j} \) for all \( j \) and \( \sum_{j=1}^{N} \kappa_{j} \langle \nu_{j}, \psi_{1} \rangle > 0 \) for all \( t \), then

\[
\mathcal{N}_{t}^{\eta} \Rightarrow \frac{\sum_{j=1}^{N} \kappa_{j} \langle \nu_{j}, \psi_{2} \rangle}{\left[ \sum_{j=1}^{N} \kappa_{j} \langle \nu_{j}, \psi_{1} \rangle \right]^{2}},
\]

as \( \eta \to \infty \), where \( \mathcal{N}_{t}^{\eta} \) is given by (4.4) and the measure-valued process \( \nu \equiv (\nu_{1}, \ldots, \nu_{N}) \) is defined through (3.12) - (3.17).

### 5 Network Topology and Systemic Risk Dynamics

We now use the asymptotic formulas for the network performance measures derived in the previous section to analyze the interplay between network topology and systemic risk. Furthermore, our dynamic interbanking model allows investigating how the network topology affects both the transient and the steady-state behavior of the liquidity stress and Herfindahl indices.

We work under the assumption that, for each \( j \leq N \), the limiting measure \( \phi_{j} \) in Assumption (A2) is a Dirac measure, i.e., \( \phi_{j} \equiv \delta_{p_{j}^{*}} \) and \( p_{j}^{*} \equiv (x_{j}^{*}, \theta_{j}^{*}, \sigma_{j}^{*}) \) as \( \eta \to \infty \).

Using Corollary 3.1 and Lemma 3.4, the mean-reserve processes \((\langle \nu_{1}, \psi_{1} \rangle, \ldots, \langle \nu_{N}, \psi_{1} \rangle)\) are approximated by the solution of the linear system (3.15), i.e.,

\[
Q(t) = e^{-Rt}x^{*} + (I - e^{-Rt})\theta^{*},
\]

where we recall that \( R \equiv (I - \Lambda^{-1}\Pi^T \Lambda)L \). Using this explicit representation, we can compute the denominator in (4.9). To compute the numerator, however, we need to develop a computational scheme for the time-varying functions \((\langle \nu_{1}, \psi_{2} \rangle, \ldots, \langle \nu_{N}, \psi_{2} \rangle)\). Using (A.8) with \( f \) replaced by \( \psi_{2} \), we get

\[
\langle \nu_{j}, \psi_{2} \rangle_{t} = \int_{\mathcal{O}_{p}} \mathbb{E} \left[ X_{j}(p_{j};t)^{2} \right] \phi_{j}(dp_{j}) = \mathbb{E} \left[ X_{j}(p_{j}^{*};t)^{2} \right] \equiv \epsilon_{j}(t).
\]

Application of Ito’s formula, along with (3.10), gives immediately the following dynamics

\[
d\epsilon_{j}(t) = -2\ell_{j}\epsilon_{j}(t)dt + \left( 2\ell_{j}\theta_{j}^{*} + (\sigma_{j}^{*})^{2} - 2 \sum_{h \leq N} \ell_{h,j} \left[ \theta_{h}^{*} - Q_{h}(t) \right] \right) Q_{j}(t)dt \equiv -2\ell_{j}\epsilon_{j}(t)dt + a(t)dt.
\]
The above is a first order, linear nonhomogenous differential equation, whose solution can be obtained explicitly. In particular,
\[ \dot{\varepsilon}_j(t) = e^{-2\ell_j t}(x^*_j)^2 + \int_0^t e^{-2\ell_j (t-s)}a_j(s)\,ds, \]
where the function \( a_j(\cdot) \) is given by
\[ a_j(\cdot) \equiv \left( 2\ell_j \theta^*_j + (\sigma^*_j)^2 - 2 \sum_{h \leq N} \ell_{h,j} [\theta^*_h - Q_h(\cdot)] \right) Q_j(\cdot). \]

Differently from the Herfindahl index, the computation of the liquidity stress index requires knowledge of the entire distribution of \( X_j(p^*_j; t) \) at any time \( t \). Using (4.2), and choosing \( \phi_j \equiv \delta_{p^*_j} \), the approximation formula reduces to
\[ \int \mathbb{P}(X_j(p_j; t) < \alpha \theta^*_j) \phi_j(dp_j) = \mathbb{P}(X_j(p^*_j; t) < \alpha \theta^*_j). \]

The calculation of these probabilities can be achieved by inverting the Laplace transform which admits closed-form expression. Recall that the dynamics of the underlying state process \( X_j(p^*_j; \cdot) \) is given by (3.10) which admits the general form
\[ dX(t) = \ell(\theta - X(t))dt + q(t)dt + \sigma X(t)^{1/2}dW(t), \]
where \( q(\cdot) \) is a deterministic time-varying function. Let
\[ \psi(t, u) \equiv \mathbb{E}[e^{uX(t)}] = \exp[\alpha(t, u) + \beta(t, u)X(0)] \quad (5.1) \]
be the Laplace transform.

**Proposition 5.1** The Laplace transform \( \psi(t, u) \) given in (5.1) admits an explicit expression as provided below
\[ \psi(t, u) = \exp[\alpha(t, u) + \beta(t, u)X(0)], \]
where
\[ \alpha(t, u) = -\frac{2\ell \theta}{\sigma^2} \log \left( 1 - \frac{\sigma^2}{2\ell} u (1 - e^{-\ell t}) \right) + \int_0^t \frac{ue^{-\ell s}q(t-s)}{1 - \frac{\sigma^2}{2\ell} u (1 - e^{-\ell s})}\,ds, \]
\[ \beta(t, u) = \frac{ue^{-\ell t}}{1 - \frac{\sigma^2}{2\ell} u (1 - e^{-\ell t})} \quad \text{and} \]
\[ \alpha(t, u) = -\frac{2\ell \theta}{\sigma^2} \log \left( 1 - \frac{\sigma^2}{2\ell} u (1 - e^{-\ell t}) \right) + \int_0^t \frac{ue^{-\ell s}q(t-s)}{1 - \frac{\sigma^2}{2\ell} u (1 - e^{-\ell s})}\,ds. \]

Notice that if \( q(t) \equiv 0 \), then \( \psi(t, u) \) is simply the Laplace transform of a non-central \( \chi^2 \)-distribution.
5.1 Interplay of Network Topology and Systemic Risk

We consider a network consisting of four clusters. Each cluster consists of a dozen banks with identical lending/borrowing preferences. Banks within the same cluster have the same target reserve level and are initially endowed with the same amount of monetary reserves. The two considered network configurations, namely the core-periphery structure and the ring structure, are reported in Figure 1.

![Diagram](image)

Figure 1: The transaction probability matrix $\Pi^1$ of the core-periphery network is specified as follows: $\pi_{1,1} = \pi_{2,2} = \pi_{3,3} = \pi_{4,4} = 0, \pi_{1,2} = \pi_{1,3} = \pi_{1,4} = 0.3, \pi_{2,1} = \pi_{3,1} = \pi_{4,1} = 0.60$ and $\pi_{23} = \pi_{24}, = \pi_{32}, = \pi_{34}, = \pi_{42} = \pi_{43} = 0.15$. The transaction probabilities between core and periphery banks are higher than the transaction probabilities between periphery banks, reflecting empirically observed patterns according to which core banks are primarily intermediaries, while periphery banks are retailer or small commercial banks. The transaction probability matrix $\Pi^2$ of the ring network is given as follows: $\pi_{1,1} = \pi_{2,2} = \pi_{3,3} = \pi_{4,4} = 0.3$ and $\pi_{1,2} = \pi_{2,3} = \pi_{3,4} = \pi_{4,1} = 0.6$; the remaining entries are zero.

The core-periphery network architecture has been identified as the most accurate description of interbanking activities. Craig and Von Peter (2014) performed an empirical analysis using bilateral interbank data from German banks from 1999 to 2007 and found that the matrix of interbank liabilities follows a core-periphery structure. These findings are in line with the analysis by Fricke and Lux (2015), who employed a dataset of overnight interbank transactions in the Italian market from 1999 to 2010, and found that a core-periphery structure provides the best fit for these interbank data, with high degree of persistence over time. In the core-periphery model, each core bank transacts with any other core bank in the network, but peripheral banks do not directly interact with each other. In our numerical examples, the model parameters are chosen to match empirical
evidence suggesting that core banks are significantly larger and more active than peripheral banks (Craig and Von Peter (2014), Fricke and Lux (2015)). We choose the ring network in representation of sparsely connected network architectures, to contrast their capacity of absorbing and propagating shocks with the more densely connected core-periphery topology. Such a choice is quite standard in the literature; see, for instance Acemoglu et al. (2015).

We first test the asymptotic accuracy of the systemic measures, by comparing the values obtained using the large network approximation with the corresponding Monte-Carlo estimates. Intuitively, we expect that as the size of the network increases, the asymptotic approximation gets closer to the Monte-Carlo estimates. This statement is visually confirmed from Figure 2, which reports selected sample paths of the LSI for different network sizes \( \eta \), together with the large-network approximation given by (4.2).

5.2 Transient and Steady-State Network Performance

We analyze the transient and steady-state performance of the network, measured in terms of liquidity stress and Herfindahl indices. At time zero, we apply an exogenous shock to all banks in cluster 1, which leads to a downward deviation from the target reserve level for each bank in the cluster.

We consider two shock regimes to highlight the qualitatively different behavior of the core-periphery and ring architectures in amplifying an initial shock through the network.

In Figure 3, the shock yields a downward deviation from the target level by 10 units for the banks in the first cluster. While the liquidity stress index ramps up instantaneously in the core-periphery topology, it propagates at a slower speed in the ring network. Noticeably, the core-periphery network recovers more rapidly from the shock relative to the more sparsely connected ring network. This suggests that connectivity improves the ability for a banking network to absorb shocks over a long term, in line with the existing literature of one-period models of network contagion (e.g. Acemoglu et al. (2015)). However, our analysis highlights an important effect which is absent in static models: the instantaneous response to a shock is higher in a more densely connected network. While a shock of moderate size may not lead to a systemic distress of the network (e.g. the LSI in clusters 3 and 4 does not peak to very high values), a shock of larger size may have more serious consequences. For instance, Figure 4 considers a similar setup, but applies a larger shock to the monetary reserves of banks in the first cluster, leading to a downward deviation of 20 units. The systemic consequences are stronger: the shock wipes out 16.67% of the total reserves in cluster
Figure 2: Liquidity-stress-indices for different network sizes and the large network approximation.

We choose \( \alpha = 0.95 \). We choose \((K_1, K_2, K_3, K_4) = \eta \times (2/11, 3/11, 3/11, 3/11)\), the speed of adjustment \( \ell_1 = 2\ell_2 = 2\ell_3 = 2\ell_4 = 2 \), and the loading factor \( \sigma_{j,k} = 0.25 \) identically equal for all banks. The initial reserves of each bank in cluster \( j \) equals \( x_j^* \) plus a small perturbation, where \((x_1^*, x_2^*, x_3^*, x_4^*) = (100, 24, 30, 27)\); similarly, the target reserve level of each bank in cluster \( j \) equals \( \theta_j^* \) plus a small random perturbation where \((\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*) = (120, 24, 30, 27)\). The transaction probability matrix \( \Pi^2 \) is specified in Figure 1.

1. The core-periphery network experiences a system-wide liquidity stress almost immediately after the shock, which leaves over 90% of the banks in the network under liquidity stress. In contrast, the shock propagates at a much lower speed in the ring network. For instance, in the ring network, the LSI of cluster 2 reaches its maximum at \( t_2 = 0.1 \), while the LSIs of clusters 3 and 4 reach their peak at a later time, respectively \( t_3 = 2.5 \) and \( t_4 = 4.2 \). Even though the core-periphery network always recovers better in the long run, the transient behavior of the network in response to a large shock raises serious concerns for financial stability. It is unlikely that any form of government intervention would be able to mitigate the severe shortage of liquidity arising in a densely connected core-periphery network. In contrast, the effects of liquidity stress take more time to propagate in the ring network and thus allowing the possibility of restoring financial stability through say, liquidity
Figure 3: Asymptotic approximations for the liquidity-stress-index at level $\alpha = 0.95$. We choose $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = \left(\frac{2}{11}, \frac{3}{11}, \frac{3}{11}, \frac{3}{11}\right)$, $(x_1^*, x_2^*, x_3^*, x_4^*) = (110, 24, 30, 27)$, $(\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*) = (120, 24, 30, 27)$, $\ell_1 = 2\ell_2 = 2\ell_3 = 2\ell_4 = 2$, and $\sigma_{j,k} = 0.25$ for all banks. The transaction probability matrices $\Pi^1$ and $\Pi^2$ are specified in Figure 1.

Figure 5 shows that higher idiosyncratic risk (higher $\sigma$) leads to a higher concentration in monetary reserves. This is intuitively expected because higher volatility increases the variability of the sample paths of the inter-banking network, and thus the probability of observing higher heterogeneity in the distribution of monetary reserves in the network. In line with intuition, the Herfindahl index is generally higher in the ring network, because a more sparsely connected network architecture reduces the amount of risk sharing in the network. A larger shock to the initial monetary reserves of a cluster leads to a higher concentration index both for the ring and core-periphery network topology.
Figure 4: Asymptotic approximations for the liquidity-stress-index at level $\alpha = 0.95$. We choose $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (2/11, 3/11, 3/11, 3/11), (x^*_1, x^*_2, x^*_3, x^*_4) = (100, 24, 30, 27), (\theta^*_1, \theta^*_2, \theta^*_3, \theta^*_4) = (120, 24, 30, 27), \ell_1 = 2\ell_2 = 2\ell_3 = 2\ell_4 = 2$, and $\sigma_{j,k} = 0.25$ for all banks. The transaction probability matrices $\Pi^1$ and $\Pi^2$ are specified in Figure 1.

6 Concluding Remarks

In this paper, we have developed a dynamic network model driven by empirically observed banking behavior. Banks manage their trading activities to maintain a desired target level of reserve capital, and the network structure is organized according to a hierarchical structure. We have modeled the intra-cluster and inter-cluster transactional activities using a vector of interacting measure-valued processes, each component of which captures the trading characteristics of the banks in a specific cluster. We have established the weak limit of the interacting system of measure-valued processes as the number of banks in the system grows large. We have provided an explicit characterization of the limit vector process in which each component tracks the typical behavior of a bank in the cluster. The explicit analytical form of the limit allows us to obtain tractable representations for statistical measures of systemic performance. We have analyzed in detail two important indicators
of systemic distress—the liquidity stress index and the concentration index. Through illustrative numerical examples, we have analyzed the sensitivity of these systemic risk indicators with respect to network parameters, including banks’ volatilities and target leverages.

The approach developed in this paper may extend to other stochastic systems. As further studies, we plan to investigate the applicability of techniques developed here to queueing networks with heterogeneous pools of non-exponential servers. We expect this system to be fully characterized by a vector of interacting measure-valued processes, in which each component tracks the evolution of the empirical distribution of the elapsed/residual service times within a pool. This analysis may result in an extension of studies done by Kaspi et al. (2011) (see also Kaspi et al. (2013) for a second-order refinement via martingale measures), from a one-dimensional to a high-dimensional setting, with multiple heterogeneous server pools and appropriate routing rules.

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References


Appendix

A Technical Proofs

Proof of Theorem 3.1 The proof consists of the tightness proof and characterization step showing the convergence of finite-dimensional distributions (via the martingale approach).

Tightness of the Sequence of Measure-Valued Processes. The proof of tightness for the sequence of measure-valued processes \( \nu^\eta(\cdot) \) is implied by (i) the compact containment condition (CCC) and (ii) the modulus of continuity condition (MCC). The CCC holds if and only if for each \( \epsilon > 0 \) there exists a compact set \( K \) of \( S \) such that

\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^\eta(t) \in K \text{ for all } t \in [0, T]) > 1 - \epsilon \tag{A.1}
\]

The CCC is often difficult to verify. However, a weaker condition which we will refer to as pointwise containment condition (PCC) can often be used in conjunction with the MCC to establish the CCC. The PCC holds if for all \( \epsilon > 0 \) and \( t \geq 0 \), there exists a compact set \( K(\epsilon, t) \) that depends on both \( \epsilon \) and \( t \) such that

\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^\eta(t) \in K(\epsilon, t)) > 1 - \epsilon. \tag{A.2}
\]

We have shown that \((S, \beta)\) is a complete metric space. By Theorem 17 of Ledger (2016), if the family of \( S \)-valued processes \( \nu^\eta \) satisfy both the MCC and the PCC, then the CCC holds. The forthcoming lemma verifies the PCC.

Lemma A.1 For each \( \epsilon \) and \( t \geq 0 \), there exists a compact subset \( K^* \) of \( S \) such that

\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^\eta(t) \in K^*(\epsilon, t)) > 1 - \epsilon.
\]

The PCC will be strengthened to CCC if MCC holds. Set \( E_t[\cdot] \equiv E[\cdot | \mathcal{F}_t] \) for \( t \geq 0 \). The following proposition uses Lemma 3.3 to verify the MCC.

Proposition A.1 Let \( g(x, y) = \|x - y\|_2 \wedge 1 \) for any \( x, y \in \mathbb{R}^N \). Then there exists a positive random variable \( a^\eta(\gamma) \) with \( \lim_{\gamma \to 0} \sup_{\eta} E[a^\eta(\gamma)] = 0 \) such that for all \( 0 \leq t \leq T, 0 \leq u \leq \gamma \) and \( 0 \leq v \leq \gamma \wedge 1 \), it holds that

\[
E_t \left[ g^2(\langle \nu^\eta, f \rangle_t + u, \langle \nu^\eta, f \rangle_t) g^2(\langle \nu^\eta, f \rangle_t, \langle \nu^\eta, f \rangle_t - v) \right] \leq E_t[a^\eta(\gamma)]
\]

where \( f \equiv (f_1, \ldots, f_N) \) with \( f_j \in C_b^2(O) \) and \( \langle \nu^\eta, f \rangle_t \equiv (\langle \nu^\eta, f_1 \rangle_t, \ldots, \langle \nu^\eta, f_N \rangle_t)^\top \).
Proof of Proposition A.1 In view of the equation (A.15), we have

\[ \langle \nu^v_j, f_j \rangle_t = \langle \nu^\eta_j, f_j \rangle_0 + A_j(t) + B_j(t) + E_j(t) + F_j(t) \]  \hspace{1cm} (A.3)

where we have defined

\[ A_j(t) \equiv \ell_j \int_0^t \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} \frac{\partial f_j(\xi_j,k(s))}{\partial x} (\theta_{j,k} - \xi_j,k(s)) \, ds, \]

\[ B_j(t) \equiv - \sum_{h \leq N} \ell_{h,j} \int_0^t \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} \frac{\partial f_j(\xi_j,k(s))}{\partial x} (\theta_h - \xi_h(s)) \, ds, \]

\[ E_j(t) \equiv \int_0^t \frac{1}{2K_j^\eta} \sum_{k \leq K_j^\eta} \sigma_{j,k}^2 \xi_j,k(s) \frac{\partial^2 f_j(\xi_j,k(s))}{\partial x^2} \, ds, \]  \hspace{1cm} (A.4)

\[ F_j(t) \equiv \int_0^t \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} \sigma_{j,k} \frac{\partial f_j(\xi_j,k(s))}{\partial x} (\xi_j,k(s))^{1/2} dW_{j,k}(s). \]

From the definition of \( g \) and (A.4), it follows that

\[ g^2 (\langle \nu^\eta, f \rangle_{t+u}, \langle \nu^\eta, f \rangle_t) \leq \sum_{j=1}^N \left| \langle \nu^\eta_j, f_j \rangle_{t+u} - \langle \nu^\eta_j, f_j \rangle_t \right|^2 \leq 4 \sum_{j=1}^N \left( |A_j(t+u) - A_j(t)|^2 + |B_j(t+u) - B_j(t)|^2 + |E_j(t+u) - E_j(t)|^2 + |F_j(t+u) - F_j(t)|^2 \right). \]  \hspace{1cm} (A.5)

To proceed, we will be making extensive use of the inequality (6.1) in Giesecke et al. (2013). We start with \( A_j \) and \( B_j \):

\[ |A_j(t+u) - A_j(t)| \leq C_p \left\| \frac{\partial f}{\partial x} \right\| \int_t^{t+u} \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} |\xi_j,k(s)| \, ds + C_p^2 \left\| \frac{\partial f}{\partial x} \right\| u \]

\[ \leq C_p \left\| \frac{\partial f}{\partial x} \right\| \gamma^{1/4} \left( 1 + \int_0^T \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} |\xi_j,k(s)|^2 \, ds \right) + C_p^2 \left\| \frac{\partial f}{\partial x} \right\| \gamma \equiv a_{j,1}^\eta(\gamma) \]

and

\[ |B_j(t+u) - B_j(t)| \leq C_p \left\| \frac{\partial f}{\partial x} \right\| \int_t^{t+u} \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} \left\| \xi_{h,k'}(s) \right\| \, ds + NC_p^2 \left\| \frac{\partial f}{\partial x} \right\| u \]

\[ \leq C_p \left\| \frac{\partial f}{\partial x} \right\| \gamma^{1/4} \left( N + \int_0^T \sum_{h \leq N} \frac{1}{K_h^\eta} \sum_{k' \leq K_h^\eta} |\xi_{h,k'}(s)|^2 \, ds \right) + NC_p^2 \left\| \frac{\partial f}{\partial x} \right\| \gamma \equiv a_{j,2}^\eta(\gamma), \]

for \( 0 \leq u \leq \gamma \). Similar to the proof of of Lemma 3.5 in Bo and Capponi (2015), we can show that

\[ |E_j(t+u) - E_j(t)| \leq \frac{C_p^2}{4} \left\| \frac{\partial^2 f}{\partial x^2} \right\| \gamma^{1/4} \left( 1 + \int_0^T \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} |\xi_j,k(s)|^2 \, ds \right) \equiv a_{j,3}^\eta(\gamma) \]

and

\[ |F_j(t+u) - F_j(t)| \leq C_p \left\| \frac{\partial^2 f}{\partial x^2} \right\| \gamma^{1/4} \left( 1 + \int_0^T \frac{1}{K_j^\eta} \sum_{k \leq K_j^\eta} |\xi_j,k(s)|^2 \, ds \right) \equiv a_{j,4}^\eta(\gamma). \]
\( \mathbb{E}_t \left[ |F_j(t+u) - F_j(t)|^2 \right] \leq \mathbb{E}_t \left[ \frac{C_p}{2} \left\| \frac{\partial f}{\partial x} \right\|_{2}^{2} \gamma^{1/4} \left( 1 + \int_{0}^{T} \frac{1}{K_j} \sum_{k=1}^{K_j} |\zeta_{j,k}(s)|^2 \, ds \right) \right] \equiv \mathbb{E}_t \left[ a_{\gamma,4}^\eta(\gamma) \right], \)

for \( 0 \leq u \leq \gamma. \) Note that \( g^2 ((\nu^n, f)_t, (\nu^n, f)_{t-v}) \leq 1 \) and let

\[
a_{\gamma}^\eta(\gamma) \equiv \sum_{j=1}^{N} \left( a_{\gamma,1}^j(\gamma) \right)^2 + \left( a_{\gamma,2}^j(\gamma) \right)^2 + \left( a_{\gamma,3}^j(\gamma) \right)^2 + a_{\gamma,4}^j(\gamma).
\]

It follows from Lemma A.3 in the appendix that \( \lim_{\gamma \to 0} \sum_{\eta} \mathbb{E}[a_{\gamma}^\eta(\gamma)] = 0. \)

**Identification of the Limit.** We formulate and solve the martingale problem that pins down the limiting measure-valued process; see Lemma A.2. This requires to identify the generator of the limiting process; see (A.7).

We start by introducing the following operators on the space \( C_b(O) \):

\[
\mathcal{T}_o^{dr} f = \partial f / \partial x, \quad \mathcal{T}_1^{dr} f = x(\partial f / \partial x), \quad \mathcal{T}_2^{dr} f = \theta(\partial f / \partial x) \quad \text{and} \quad \mathcal{T}^v f = \sigma^2 x (\partial^2 f / \partial x^2) / 2. \quad (A.6)
\]

Next define the operator \( \mathcal{A} \) acting on \( \Phi(\cdot) \) to be

\[
\mathcal{A} \Phi(\mu) \equiv \sum_{n=1}^{m} \sum_{j=1}^{N} \frac{\partial \phi}{\partial x_j,n} \left[ \ell_j(\mu_j, \mathcal{T}_2^{dr} f_j^n) - \ell_j(\mu_j, \mathcal{T}_1^{dr} f_j^n) + \langle \mu_j, \mathcal{T}^v f_j^n \rangle \right] - \sum_{h \leq N} \ell_{h,j}(\mu_j, \mathcal{T}_o^{dr} f_j^n) \langle \mu_h, \Theta - \psi_1 \rangle \quad \text{for} \quad \mu \equiv (\mu_1, \ldots, \mu_N) \in S \quad (A.7)
\]

**Proposition A.2** The operator \( \mathcal{A} \) is the generator of our limit martingale problem in the sense of

\[
\lim_{\eta \to \infty} \mathbb{E} \left[ \Phi(\nu^\eta(t_{r+1})) - \Phi(\nu^\eta(t_r)) - \int_{t_r}^{t_{r+1}} \mathcal{A} \Phi(\nu^\eta(u)) \, du \right] \prod_{i=1}^{r} \Psi_i(\nu^\eta(t_j)) = 0,
\]

where \( 0 \leq t_1 < \ldots < t_{r+1} < +\infty \) with \( r \in \mathbb{N} \), and \( \Psi_i \in B(S) \) (the set of all bounded measurable functions on \( S \)) with \( j = 1, \ldots, r \).

**Proof of Proposition A.2** The conclusion follows directly from Lemma A.2 below and that

\[
\lim_{\eta \to \infty} \mathbb{E} \left[ \int_{t}^{u} |\mathcal{E}_{s}^\eta| \, ds \right] = 0 \quad \text{for} \quad 0 \leq t < u < +\infty
\]

where \( \mathcal{E} \) is given in (A.12). □

Now turn to the limiting measure-valued process \( \nu \) given in (3.17). First note that for \( f \in C^2_b(O) \)

\[
\langle \nu_j, f \rangle_t = \int_{\mathcal{O}_p} \mathbb{E} \left[ f(p_j, X_j(p_j; t)) \right] \phi_j(dp_j) \quad \text{for} \quad t \geq 0. \quad (A.8)
\]
In what follows, we simply write \( X_j(t) \equiv X_j(p_j; t) \) and \( f(X_j(t)) \equiv f(p_j, X_j(p_j; t)) \). An application of the Ito’s formula implies
\[
f(X_j(t)) = f(x_j) + \ell_j \int_0^t \frac{\partial f}{\partial x}(X_j(s)) (\theta_j - X_j(s)) \, ds - \sum_{h \leq N} \ell_{h,j} \int_0^t \frac{\partial f}{\partial x}(X_j(s)) \left( V_h - Q_h(s) \right) \, ds
+ \sigma_j \int_0^t \frac{\partial f}{\partial x}(X_j(s)) \sqrt{X_j(s)} \, dW_j(s) + \frac{\sigma_j^2}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(X_j(s)) \, ds. \tag{A.9}
\]
Taking expectation on both sides and then first-order derivative with respect to \( t \) yields
\[
\frac{\partial}{\partial t} \mathbb{E} [f(X_j(t))] = \ell_j \mathbb{E} \left[ T_{2}^{dr} f(X_j(t)) \right] - \ell_j \mathbb{E} \left[ T_{1}^{dr} f(X_j(t)) \right] + \mathbb{E} \left[ T^{v} f(X_j(t)) \right] - \sum_{h \leq N} \ell_{h,j} \mathbb{E} \left[ T_{0}^{dr} f(X_j(t)) \right] \left( V_h - Q_h(t) \right)
\]
where the first equality uses the operators given by (A.6) and the second equality is due to (3.11) and by Lemma 3.4. Taking expectation of both sides with respect to \( p_j \), we obtain
\[
\frac{\partial}{\partial t} \int_{\mathcal{O}_p} \mathbb{E} [f(X_j(t))] \phi_j(dq_p) = \int_{\mathcal{O}_p} \left( \ell_j \mathbb{E} \left[ T_{2}^{dr} f(X_j(t)) \right] - \ell_j \mathbb{E} \left[ T_{1}^{dr} f(X_j(t)) \right] + \mathbb{E} \left[ T^{v} f(X_j(t)) \right] \right) \phi_j(dq_p) - \sum_{h \leq N} \ell_{h,j} \int_{\mathcal{O}_p} \mathbb{E} \left[ T_{0}^{dr} f(X_j(t)) \right] \left( V_h - Q_h(t) \right) \phi_j(dq_p)
\]
By Eq. (A.8) and the above equality, we deduce
\[
\frac{d}{dt} \langle \nu_j, f \rangle_t = \ell_j \langle \nu_j, T_{2}^{dr} f \rangle_t - \ell_j \langle \nu_j, T_{1}^{dr} f \rangle_t + \langle \nu_j, T^{v} f \rangle_t + \sum_{h \leq N} \ell_{h,j} \langle \nu_j, T_{0}^{dr} f \rangle_t \left( \langle \nu_h, \Theta \rangle_t - \langle \nu_h, \psi_1 \rangle_t \right). \tag{A.10}
\]
Recall the function \( \Phi(\nu) \) defined by (3.9) and the operator \( \mathcal{A} \) acting on \( \Phi(\nu) \). Using the chain rule and (A.10), we can easily verify that
\[
\frac{d\Phi(\nu(t))}{dt} = \sum_{j=1}^{m} \sum_{n=1}^{m} \frac{\partial \phi}{\partial x_{j,n}} \left[ \ell_j \langle \nu_j, T_{2}^{dr} f_j^n \rangle_t - \ell_j \langle \nu_j, T_{1}^{dr} f_j^n \rangle_t + \langle \nu_j, T^{v} f_j^n \rangle_t \right]
- \sum_{h \leq N} \ell_{h,j} \langle \nu_j, T_{0}^{dr} f_j^n \rangle_t \left( \langle \nu_h, \Theta \rangle_t - \langle \nu_h, \psi_1 \rangle_t \right)
= \mathcal{A} \Phi(\nu(t)),
\]
which can be rearranged to obtain
\[ \Phi(\nu(t)) = \Phi(\nu(s)) + \int_s^t A\Phi(\nu(u))du \quad \text{for} \quad 0 \leq s < t < +\infty. \]

This shows that \( \nu \) satisfies the martingale problem for \((A, \phi)\) given in (3.9).

By the standard analysis of weak convergence as in Chapter 3 of Ethier and Kurtz (2009), existence of a weak limit for the sequence of measure-valued processes \( \nu^n(\cdot) \) is guaranteed by Lemma A.1 and Proposition A.1; convergence of finite-dimensional distributions follows from Proposition A.2; uniqueness of the martingale problem is ensured by Lemma A.4. This concludes the proof of Theorem 3.1. \( \square \)

**Proof of Proposition 5.1** Our objective is to find functions the \( \alpha(\cdot, u) \) and \( \beta(\cdot, u) \) such that
\[ M(s) \equiv g(s, X(s)) \equiv \exp [\alpha(t - s, u) + \beta(t - s, u)X(s)] \]
is martingale. If this is the case, then
\[ E\left[ e^{uX(t)} \right] = E[M(t)] = M(0) = \exp [\alpha(t, u) + \beta(t, u)X(0)]. \]

Application of the Ito’s formula to \( g(s, X(s)) \) yields
\[ \frac{dg(s,X(s))}{g(s,X(s))} = -\left[ \alpha'(t-s,u) + X(s)\beta'(t-s,u) \right] ds + \beta(t-s,u)\ell(\theta - X(s))ds \]
\[ + q(s)\beta(t-s,u)ds + \beta(t-s,u)X(s)dW(s) + \frac{1}{2}\sigma^2X(s)\beta(t-s,u)^2ds. \]

\( g(s,X(s)) \) is a martingale if the drift term vanishes, i.e. if
\[ \alpha'(t-s,u) + X(s)\beta'(t-s,u) = \beta(t-s,u)\ell(\theta - X(s)) + q(s)\beta(t-s,u) + \frac{1}{2}\sigma^2X(s)\beta(t-s,u)^2 \]
for all possible states of \( X(s) \). Collecting terms, we obtain the following system of non-linear differential equations
\[ \alpha'(s,u) = \ell\theta \cdot \beta(s,u) + q(t-s) \cdot \beta(s,u) \]
\[ \beta'(s,u) = -\ell\theta \cdot \beta(s,u) + \frac{\sigma^2}{2} \beta(s,u)^2 \]
with initial conditions \( \alpha(0,u) = 0 \) and \( \beta(0,u) = u \). The above differential equations admit the solution given by (5.2). \( \square \)

**Proof of Lemma 3.1** Suppose that \( \mu \neq \mu' \). Then we must have \( \beta(\mu, \mu') = \epsilon \) for some \( \epsilon > 0 \). Take a compact set \( K \subset \mathcal{O} \) with \( \mu(K) > 1 - \epsilon/8 \) and \( \mu'(K) > 1 - \epsilon/8 \). The set of functions
\(B \equiv \{f : \|f\|_{BL} \leq 1\}\) restricted to \(K\) forms a compact set of functions for the norm \(\| \cdot \|_\infty\) by the Arzela-Ascoli theorem. Thus there exist \(f^1, f^2, \ldots, f^m \in B\) for some for \(m \in \mathbb{R}\) such that for any \(f \in B\), there is \(f_i\) satisfying \(\sup_{x \in K} |f(x) - f_i(x)| < \epsilon/8\). By the triangular inequality,

\[
|\langle \mu, f \rangle - \langle \mu', f \rangle| \leq \int_O (f - f^i)d\mu + \int_O (f - f^i)d\mu' + \int_O f^i d\mu - \int_O f^i d\mu'.
\]

The first term on the right hand side

\[
\left| \int_O (f - f^i)d\mu \right| \leq \int_{K^c} (f - f^i)d\mu + \int_K (f - f^i)d\mu < \epsilon/4 + \epsilon/8 = 3\epsilon/8.
\]

Similarly we have \(\int_O (f - f^i)d\mu' < 3\epsilon/8\). The last term is, by definition, \(|\langle \mu, f^i \rangle - \langle \mu', f^i \rangle|\). Thus \(|\langle \mu, f \rangle - \langle \mu', f \rangle| < \epsilon/4 + \max_{i \leq m} |\langle \mu, f^i \rangle - \langle \mu', f^i \rangle|\). Because \(f\) is arbitrarily chosen from \(B\) and \(\beta(\mu, \mu') = \epsilon\), there must exist \(i \in \{1, \ldots, m\}\) such that \(|\langle \mu, f^i \rangle - \langle \mu', f^i \rangle| \geq \epsilon/4\). On the other hand, any globally Lipschitz continuous function can be approximated by a sequence of differentiable functions with bounded derivatives up to order two. We can therefore find \(\tilde{f} \in C_b^2\) satisfying \(|\langle \mu, \tilde{f} \rangle - \langle \mu', \tilde{f} \rangle| \geq \epsilon/8\). The first part of the conclusion follows immediately from the the fact that \(C_b^2(\mathbb{R}^m)\) separates points in \(\mathbb{R}^m\). Since \(D\) forms a subalgebra, an application of Stone-Weierstrass Theorem gives the second part of the conclusion. \(\square\)

**Proof of Lemma 3.2** Non-negativity and symmetry are immediate from the definition. Subadditivity follows from the triangular inequality

\[
\sum_{j=1}^{N} \left| \int f_j \ d(\mu_j - \mu_j') \right| \leq \sum_{j=1}^{N} \left| \int f_j \ d(\mu_j - \mu_j') \right| + \sum_{j=1}^{N} \left| \int f_j \ d(\mu_j - \mu_j') \right|.
\]

To show indiscernibility, suppose that \(\beta(\mu, \mu') = 0\). From the definition (3.8), it follows that for each \(j\)

\[
\sup \left\{ \left| \int f_j \ d(\mu_j - \mu_j') \right| : \|f_j\|_{BL} \leq 1 \right\} = 0.
\]

The above expression simply means that \(\beta(\mu_j, \mu_j') = 0\), which in turn implies \(\rho(\mu_j, \mu_j') = 0\), where we recall that \(\rho\) denotes the Prokhorov metric. Thus \(\mu_j = \mu_j'\) in a sense that \(\mu_j(A) = \mu_j'(A)\) for all Borel set in \(O\). Note that the equality holds for all \(j \leq N\). We therefore conclude that \(\mu = \mu'\). \(\square\)

**Lemma A.2** Let \(0 \leq t < u < +\infty\). It holds that,

\[
\Phi(\nu^0(u)) = \Phi(\nu^0(t)) + \int_t^u D^0(s)ds + \int_t^u E^0(s)ds + \tilde{M}(u) - \tilde{M}(t), \quad \text{(A.11)}
\]

32
where \((\tilde{M}_t; t \geq 0)\) is a \((\mathbb{P}, \mathbb{F})\)-(local) martingale, and
\[
\mathcal{D}^n(t) \equiv \frac{1}{K^n_j} \sum_{k \leq K^n_j} f(\xi_{j,k}(t)) = \frac{1}{K^n_j} \sum_{k \leq K^n_j} f(\xi_{j,k}(0)) + \ell_j \int_0^t \frac{1}{K^n_j} \sum_{k \leq K^n_j} \frac{\partial f(\xi_{j,k}(s))}{\partial x} (\theta_{j,k} - \xi_{j,k}(s)) \, ds + \mathcal{M}_{j,k}(t)
\]
\[
- \sum_{h \leq N} \ell_{h,j} \int_0^t \frac{1}{K^n_j} \sum_{k \leq K^n_j} \frac{\partial f(\xi_{j,k}(s))}{\partial x} (\bar{\theta}_h - \bar{\xi}_h(s)) \, ds + \frac{1}{K^n_j} \sum_{k \leq K^n_j} \mathcal{M}_{j,k}(t)
\]
\[
+ \int_0^t \frac{1}{2K^n_j} \sum_{k \leq K^n_j} \sigma_{j,k}^2 \xi_{j,k}(s) \frac{\partial^2 f(\xi_{j,k}(s))}{\partial x^2} \, ds.
\]

Using the notation introduced earlier, we may rewrite (A.15) as
\[
\langle \nu^n_j, f \rangle_t = \langle \nu^n_j, f \rangle_0 + \ell_j \int_0^t \langle \nu^n_j, T^n_{2,dr} f \rangle_s ds - \ell_j \int_0^t \langle \nu^n_j, T^n_{1,dr} f \rangle_s ds + \frac{1}{K^n_j} \sum_{k \leq K^n_j} \mathcal{M}_{j,k}(t)
\]
\[
- \sum_{h \leq N} \ell_{h,j} \int_0^t \langle \nu^n_j, T^n_{0,dr} f \rangle_s [\langle \nu^n_h, \Theta \rangle_s - \langle \nu^n_h, \psi_1 \rangle_s] ds + \int_0^t \langle \nu^n_j, T^n v f \rangle_s ds,
\]
where the operators \(T^n_{i,dr}, i = 0, 1, 2\) and \(T^n v\) are given by (A.6). Replacing \(\mu\) with \(\nu^n(t)\) in (3.9) and applying Itô’s formula yields (A.11). \(\square\)
Proof of Lemma 3.4  From the definition (3.13), it follows immediately that

\[ \langle \nu_j, \psi_1 \rangle_t \equiv \int_{\mathcal{C}_p} \mathbb{E}[X_j(p_j; t)] \phi_j(d p_j) \quad \text{for} \quad t \geq 0, \quad (A.17) \]

where we recall that the underlying state process \( X_j(p_j) \) follows the SDE as shown in (3.10). Now it follows from the same equation (3.10) that the expectation \( \mathbb{E}[X_j(p_j)] \) satisfies the integral equation below:

\[ \mathbb{E}[X_j(p_j; t)] = x_j + \int_0^t \ell_j(\theta_j - \mathbb{E}[X_j(p_j; s)]) ds - \int_0^t \sum_{h \leq N} \ell_{h,j}(V_h - Q_h(s)) ds \quad \text{for} \quad t \geq 0. \quad (A.18) \]

The integral equation (A.18) admits an explicit solution:

\[ \mathbb{E}[X_j(p_j; t)] = e^{-\ell_j t} \left[ x_j + \int_0^t \left( \ell_j \theta_j - \sum_{h \leq N} \ell_{h,j}(V_h - Q_h(s)) \right) e^{\ell_j s} ds \right]. \quad (A.19) \]

Combining (A.17) and (A.19) completes the proof of the lemma. \( \square \)

Proof of Lemma A.1  The argument below is adapted from the proof of Lemma 6.1. of Giesecke et al. (2013) but takes into account \( \nu^\eta \) being multidimensional. For each \( M > 0 \), define \( K_M \equiv [0, C_p]^3 \times [0, M] \) where \( C_p \) is bound given in Assumption A2. First we note that

\[ \mathbb{E} \left[ \langle \nu^\eta_j, 1_{K^c_{(M+k)^2}} \rangle_t \right] = \nu^\eta_j(t)(K^c_M) = \frac{1}{K^\eta_j} \sum_{k=1}^{K^\eta_j} p(\xi_{j,k} \geq M) \leq \frac{\hat{C}(1,T,C_p)}{M} \]

where the constant \( \hat{C}(1,T,C_p) \equiv C(n,T,C_p)e^{C(n,T,C_p)T} \) is provided in the proof of Lemma A.3. Next define

\[ K^*_M \equiv \left\{ \mu \equiv (\mu_1, \ldots, \mu_N) \in S : \langle \mu_j, 1_{K^c_{(M+k)^2}} \rangle < \frac{1}{\sqrt{M+k}} \text{ for each } j \leq N \text{ and all } k \in \mathbb{N} \right\} \]

Here we use \( A^c \) to denote the complement of a set \( A \) and \( \langle \mu_j, 1_A \rangle \) equals the probability \( \mu_j(A) \). Clearly the set \( K^*_M \) is a compact subset of \( S \). In addition, we have

\[
\mathbb{P} \left( \nu^\eta(t) \notin K^*_M \right) \leq \sum_{j=1}^{N} \sum_{k=1}^{\infty} \mathbb{P} \left( \langle \nu^\eta_j, 1_{K^c_{(M+k)^2}} \rangle_t > \frac{1}{\sqrt{M+k}} \right) \\
\leq \sum_{j=1}^{N} \sum_{k=1}^{\infty} \mathbb{E} \left[ \langle \nu^\eta_j, 1_{K^c_{(M+k)^2}} \rangle_t \right] 1/\sqrt{M+k} \\
\leq \sum_{j=1}^{N} \sum_{k=1}^{\infty} \frac{\hat{C}(1,T,C_p)}{(M+k)^2/\sqrt{M+k}} \to 0 \quad \text{as} \quad M \to \infty.
\]

34
The convergence to zero is independent of the index \( \eta \). Hence for any \( \epsilon > 0 \), by choosing \( M \) large enough, one gets
\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^\eta(t) \in \mathcal{K}_M) > 1 - \epsilon
\]
as desired. \( \square \)

**Lemma A.3** If Assumption (A2) is satisfied, then for any finite \( T > 0 \) and \( n \in \mathbb{N} \), we have
\[
\sup_{\eta} \sup_{0 \leq t \leq T} \sum_{j=1}^{N} \frac{K_j^n}{j} \sum_{k=1}^{K^n_j} \mathbb{E} ||\xi_{j,k}(t)||^n < +\infty. \tag{A.20}
\]

**Proof of Lemma A.3** Recall that
\[
\xi_{j,k}(t) = \xi_{j,k}(0) + \int_0^t \ell_j [\theta_{j,k} - \xi_{j,k}(s)] \, ds - \int_0^t \sum_{h \leq N} \ell_{h,j} [\bar{\theta}_h - \bar{\xi}_h(s)] \, ds + \sigma_{j,k} \int_0^t \sqrt{\xi_{j,k}(s)} \, dW_{j,k}(s).
\]
An application of the Ito’s formula gives
\[
(\xi_{j,k}(t))^n = (\xi_{j,k}(0))^n + n \int_0^t \ell_j (\xi_{j,k}(s))^{n-1} [\theta_{j,k} - \xi_{j,k}(s)] \, ds + \frac{n(n-1)}{2} \int_0^t \sigma_{j,k}^2 (\xi_{j,k}(t))^{n-1} \, ds
\]
\[- n \int_0^t \sum_{h \leq N} \ell_{h,j} (\xi_{j,k}(t))^{n-1} [\bar{\theta}_h - \bar{\xi}_h(s)] \, ds + n \int_0^t \sigma_{j,k} (\xi_{j,k}(t))^{n-1/2} \, dW_{j,k}(s). \tag{A.21}
\]
An application of Young’s inequality yields
\[
(\xi_{j,k}(s))^{n-1} \leq (n - 1) \frac{||\xi_{j,k}(s)||^n}{n} + 1/n
\]
and
\[
(\xi_{j,k}(s))^{n-1} \xi_{h,k'}(s) \leq (n - 1) \frac{||\xi_{j,k}(s)||^n}{n} + \frac{1}{n} \sum_{k' \leq K^n_h} ||\xi_{h,k'}(s)||^n / n.
\]
It then follows that
\[
(\xi_{j,k}(s))^{n-1} \bar{\xi}_h(s) \leq (n - 1) \frac{||\xi_{j,k}(s)||^n}{n} + (1/K^n_h) \sum_{k' \leq K^n_h} ||\xi_{h,k'}(s)||^n / n.
\]
Combining the preceding inequalities with (A.21), we have
\[
\mathbb{E} ||\xi_{j,k}(t)||^n \leq \mathbb{E} ||\xi_{j,k}(0)||^n + C_n \int_0^t \mathbb{E} ||\xi_{j,k}(s)||^n \, ds + C_n \int_0^t \sum_{j \leq N} \frac{1}{K^n_j} \sum_{k \leq K^n_j} \mathbb{E} ||\xi_{j,k}(s)||^n \, ds + C_n t, \tag{A.22}
\]
where \( C_n \) is a constant that only depends on \( C_p \) and \( n \). We can therefore conclude that
\[
\sum_{j \leq N} \frac{1}{K^n_j} \sum_{k \leq K^n_j} \mathbb{E} ||\xi_{j,k}(t)||^n \leq \sum_{j \leq N} \frac{1}{K^n_j} \sum_{k \leq K^n_j} \mathbb{E} ||\xi_{j,k}(0)||^n + C_n \int_0^t \sum_{j \leq N} \frac{1}{K^n_j} \sum_{k \leq K^n_j} \mathbb{E} ||\xi_{j,k}(s)||^n \, ds
\]
\[+ NC_n \int_0^t \sum_{j \leq N} \frac{1}{K^n_j} \sum_{k \leq K^n_j} \mathbb{E} ||\xi_{j,k}(s)||^n \, ds + NC_n t. \]

35
An application of the Gronwall’s inequality then gives
\[ \sum_{j \leq N} \frac{1}{K_j^n} \sum_{k \leq K_j^n} \mathbb{E}[|\xi_{j,k}(t)|^n] \leq C(n, T, C_p) e^{C(n, T, C_p) T}. \]
The result immediately follows due to fact that the bound on the right hand side is independent of \( t \) and \( \eta \).

\[ \square \]

Lemma A.4 The uniqueness of the martingale problem of the generator \( A \) given by (A.7) holds.

**Proof.** Our proof follows a duality argument, see e.g., Chapter 4.4 of Ethier and Kurtz (2009). Similar techniques have been adopted by Giesecke et al. (2013) and recently expanded by Bo and Capponi (2015). Particularly, uniqueness of the original problem can be implied by existence of the dual problem.

We extend the duality argument to a multidimensional setting. Here a more carefully crafted treatment is needed due to the interactions between dimensions. We start off by looking at the function \( \Phi \) given by (3.9), i.e.,
\[ \Phi(\mu) = \phi(\langle \mu, f^1 \rangle, \ldots, \langle \mu, f^m \rangle) \]
Using an argument similar to Giesecke et al. (2013) allows us to approximate the function \( \Phi(\mu) \) by linear combinations of functions of the form
\[ \Gamma_G(\mu) = \int_{O^{N \times m}} G(z) \prod_{j \leq N} \prod_{n \leq m} \mu_j(dz^n_j), \]
where \( G \in C^* = \bigcup_{m \in \mathbb{N}} C^2_b(O^{N \times m}) \) and \( z^n_j = (p^n_j, x^n_j) \in O \).

Consider a \( S \)-valued process \( \nu \) solving the martingale problem \( (A, \phi) \). Then for \( G \in C^2_b(O^{N \times m}) \), we have that
\[ \Gamma_G(\nu(t)) = \int_0^t A \Gamma_G(\nu(s))ds + \mathcal{N}_1(t), \quad (A.23) \]
where \( \mathcal{N}_1 \) is a martingale and the function
\[ A \Gamma_G(\mu) = \sum_{j=1}^N \sum_{n=1}^m \int_{O^{N \times m}} \left( \ell_j T^{dr}_{2,(j,n)} - \ell_j T^{dr}_{1,(j,n)} + T^{v}_{(j,n)} \right) G(z) \prod_{j \leq N} \prod_{n \leq m} \mu_j(dz^n_j) \]
\[ - \sum_{j=1}^N \sum_{n=1}^m \sum_{h \leq N} \sum_{l \leq N} \ell_{h,j} \left[ \langle \mu_h, \Theta \rangle - \langle \mu_h, \psi_1 \rangle \right] T^{dr}_{0,(j,n)} G(z) \prod_{j \leq N} \prod_{n \leq m} \mu_j(dz^n_j) \quad (A.24) \]
where, for an operator \( T \), \( T_{(j,n)} \) means that \( T \) operates on the \((j, n)\)th coordinate of the function \( G \). Now let \( J \) be a linear mapping from \( C(O) \) to \( C(O^{N+1}) \) and \( J^n_j G \) denote the action of \( J \) acting
on \( G \) as a function of the \((j, n)\)th variable:

\[
\mathcal{J}_j^n G(z) \equiv m \left( \sum_{h \leq N} (x_h^{m+1} - \theta_h^{m+1}) \right) \frac{\partial G(z)}{\partial z_{j,n}}.
\] (A.25)

Next using (A.25) to rewrite (A.24) we obtain

\[
A \Gamma G(\mu) = \sum_{j=1}^{N} \sum_{n=1}^{m} \int_{\Omega} \left( \ell_j \mathcal{T}_{2,(j,n)}^{dr} - \ell_j \mathcal{T}_{1,(j,n)}^{dr} + \mathcal{T}_{(j,n)}^{\nu} \right) G(z) \prod_{j \leq N} \prod_{n \leq m} \mu_j (d \mu_j^n) \\
- \sum_{j=1}^{N} \sum_{n=1}^{m} \left( \frac{1}{m} \right) \left[ \Gamma_j^n G(\mu) - \Gamma G(\mu) \right] + N \Gamma G(\mu)
\] (A.26)

Let \( F \) denote the linear space of functions on \( C^* \) which contains functions of the form:

\[
\Gamma(\mu, G) = \Gamma_{\mu}(G) \equiv \Gamma G(\mu).
\]

Then (A.26) becomes

\[
A \Gamma G(\mu) = A^\# \Gamma_{\mu}(G) + N \Gamma_{\mu}(G)
\] (A.27)

where for \( \Gamma_{\mu}(G) \in F \),

\[
A^\# \Gamma_{\mu}(G) = \Gamma_{\mu}(\mathcal{G}G) - \sum_{j=1}^{N} \sum_{n=1}^{m} \left( \frac{1}{m} \right) \left[ \Gamma_{\mu}(\mathcal{J}_j^n G) - \Gamma_{\mu}(G) \right] + N \Gamma_{\mu}(G)
\] (A.28)

and the generator \( \mathcal{G} \) given by

\[
\mathcal{G} \equiv \sum_{j=1}^{N} \sum_{n=1}^{m} \left( \ell_j \mathcal{T}_{2,(j,n)}^{dr} - \ell_j \mathcal{T}_{1,(j,n)}^{dr} + \mathcal{T}_{(j,n)}^{\nu} \right).
\] (A.29)

The application of the duality argument emerges from the observation that \( A^\# \) has the structure of an infinitesimal generator of a \( C^* \)-valued Markov jump process \( \chi(\cdot) \) (if exists) restricted to functions in \( F \). The dynamics of the \( C^* \)-valued Markov process with the infinitesimal generator \( A^\# \) involves two basic mechanisms:

(i) Jump mechanism: \( G \rightarrow J_j^n G \) with jump rate \( 1/m \); 

(ii) Spacial diffusion semigroup: between jumps, \( \chi(\cdot) \) involves in a deterministic and continuous fashion, namely

\[
G \rightarrow S_t G
\]

where \( S \) represents the semigroup of transformation on \( C^* \) with infinitesimal generator \( \mathcal{G} \).
The rest of the proof is dedicated to the construction of such a $C^*$-valued Markov process $\chi(\cdot)$.

First define for each $(j, n)$ a diffusion process

$$X_j^n(t) = x_j^n + \ell_j \int_0^t (\theta^n_j - X_j^n(s)) \, ds + \sigma_j^n \int_0^t X_j^n(s) \, d\tilde{W}_j^n(s)$$

where the $m$ reference Brownian motions $\tilde{W}_j^n$ are independent of each other and of those in the original model. Next consider a set of (deterministic) measure-valued processes $\zeta(\cdot) \equiv (\zeta_1(\cdot), \ldots, \zeta_N(\cdot))$ such that for $t$ that is not a jump time

$$\chi(t) = \Gamma_{\zeta(t)}(G) = \Gamma_G(\zeta(t)) \equiv E \left[ G \left( Y_1^1(t), \ldots, Y_m^m(t) \right) \right]$$

where the superscript $c$ represents the continuous part and $Y^n(t) \equiv [(p^n_1, X^n_1(t)), \ldots, (p^n_N, X^n_N(t))]^\top$; the jump part is described through

$$d\chi(t) = d(\chi(t))^c + \sum_{j \leq N} \sum_{n \leq m} \left[ J^n_j G - \chi(t-) \right] \, d\Lambda^n_j(t), \quad \text{for} \quad t \geq 0,$$

where $\Lambda^n_j$ denotes a Poisson processes with rate $1/m$. An application of Ito’s formula yields

$$\chi(t) = \int_0^t A\# \Gamma_{\zeta(s)}(G) \, ds + \mathcal{N}_2(t), \quad \text{for} \quad t \geq 0,$$

where $\mathcal{N}_2$ is a martingale. It is straightforward to verify that $A\Gamma_{G}(\tilde{\mu}) = A\# \Gamma_{\tilde{\mu}}(G) + N\Gamma_{\tilde{\mu}}(G)$, which completes the proof. \qed