A Dynamic Network Model of Interbank Lending  
— Systemic Risk and Liquidity Provisioning  

Agostino Capponi¹, Xu Sun², and David D. Yao¹  
¹Department of Industrial Engineering and Operations Research, Columbia University  
²Department of Industrial and Systems engineering, University of Florida  

Abstract  
We develop a dynamic model of interbank borrowing and lending activities in which banks are organized into clusters, and adjust their monetary reserve levels to meet prescribed capital requirements. Each bank has its own initial monetary reserve level and faces idiosyncratic risks characterized by an independent Brownian motion; whereas system wide, the banks form a hierarchical structure of clusters. We model the interbank transactional dynamics through a set of interacting measure-valued processes. Each individual process describes the intra-cluster borrowing/lending activities, and the interactions among the processes capture the inter-cluster financial transactions. We establish the weak limit of the interacting measure-valued processes as the number of banks in the system grows large. We then use the weak limit to develop asymptotic approximations of two proposed macro-measures, the liquidity stress index and the concentration index, both capturing the dynamics of systemic risk. We use numerical examples to illustrate the applications of the asymptotics and conduct related sensitivity analysis with respect to various indicators of financial activity.  

Keywords: dynamic interbanking networks, systemic risk, large networks asymptotics  

1 Introduction  
The interbank market plays a critical role in facilitating the provision of liquidity. Yet, this also subjects banks to risk exposures via a complex network of trading relations involving loans and derivatives transactions. Understanding the associated systemic phenomena and their dependence on the topological structure of the network is of critical importance for the design of policies aiming for financial stability.  

Most studies in the literature on interbank networks have focused on static models, where all banks simultaneously clear their liabilities, which are exogenously specified. For instance, the seminal paper by Eisenberg and Noe (2001) develops what is essentially a fixed-point algorithm to derive the clearing vector and hence characterizes how initial shocks spread through the financial network. Such static models provide a useful framework for quantifying the intensity of shocks and the sensitivity of contagion to structural parameters. They fail, however, to capture the often rapidly changing nature of financial networks in which borrowing and lending patterns adapt to the current economic environment and to the evolving idiosyncratic balance-sheet characteristics of the banks. Indeed, active balance sheet management by banks has been widely documented in empirical studies, see for instance Adrian and Shin (2010). In addition, a study by the European Central
Bank (see Halaj and Kok (2013)), using balance sheet data from the banks involved, along with the geographical breakdown of their activities, indicates a pattern of connections via clusters: Most inter-banking transactions are among banks within the same country, hence forming a cluster; on the other hand, some of the largest domestic banks also actively transact with the largest banks in other countries; thus, there is also substantial inter-cluster connectivity.

Motivated by the above reality, we develop a dynamic network model where the financial system is partitioned into several clusters, each consisting of a group of banks actively managing their balance sheets to conform with prescribed target leverage requirements; and we use a system of stochastic differential equations (SDE) to describe the interlinked dynamics of the monetary reserves of the banks in the network.

1.1 Contributions and Organization

A distinct feature of our model is the hierarchical structure of the network where clusters form the top layer; and a set of interacting measure-valued processes, in which each dimension captures the (empirical) reserve distribution of a specific cluster, models both the intra-cluster and the inter-cluster financial activities. We prove that the sequence of measure-valued processes converges weakly, as the total number of banks in the system grows large, to a set of measure-valued functions that can be explicitly characterized.

Our weak convergence analysis is based on the Stroock-Varadhan theory of martingale problems; see Chapter 4 of Ethier and Kurtz (2009). Despite this theory has been extensively applied to prove limit theorems for the empirical measure of an interacting particle system (see, for instance, Giesecke et al. (2013), Giesecke et al. (2015) and Bo and Capponi (2014) for applications in finance), to the best of our knowledge it has never been customized to study the asymptotics of multi-class interacting-particle systems. This introduces technical challenges that are discussed next. Because the dynamics of the interbank system is described via a set of interacting measure-valued processes, the state descriptor lives on the product of measure spaces $S$ for which an appropriate topology needs to be constructed. As opposed to constructing the product topology in the usual way, e.g., via the canonical projections (see, for instance, Chapter 2 of Dudley (2002)), we construct the topology explicitly by designing a novel metric on $S$, namely a multidimensional extension of the Levy-Prokhorov metric. This explicit characterization of the metric allows us to conveniently derive equivalent definitions of weak convergence in $S$ which are useful in analyzing the sample path properties of processes living in the Skorokhod space $D_S[0, \infty)$. In addition, the application of the Stroock-Varadhan theory requires the identification of a class of test functions that is sufficiently rich to generate the space of bounded measurable functions on $S$ (under the bounded pointwise convergence scheme). We introduce a separating class of functions in §3.2, and show that this function class is dense in the space of bounded continuous functions on $S$. We then obtain an explicit characterization of the weak limit of a sequence of interacting empirical measure-valued processes.

We demonstrate how the limiting results can be used to study both transient and steady-state performance measures of the network. In particular, we propose two macro-measures, the liquidity stress index (LSI) and
the concentration index (CI), to characterize the systemic risk dynamics of the network. The LSI measures the proportion of banks in the system each with a reserve level falling below a threshold (a certain percentage of its target level). The CI measures the manner liquidity is distributed (more evenly or highly concentrated) throughout the system.\footnote{It is generally agreed upon that concentration threatens financial stability, primarily because of the government bailout of large financial institutions, see Acharya et al. (2014). Policy makers have designed policies limiting the market share of banking institutions.}

Using numerical examples, we illustrate how our results lead to clear economic insights on the interplay between systemic risk and the network architecture. For instance, our results indicate that the transient response to a liquid shock may lead to “too-interconnected-to-fail” risk in a core-periphery network topology. In particular, suppose an initial shock occurred at a (small) subset of the banks in the network pushing down the value of their reserves below their target levels. If the size of the shock is moderate, its instantaneous amplification may be contained. Higher connectivity may thus serve to mitigate and eventually absorb the shock, and hence enhances robustness. If, however, the size of the shock is higher, connectivity becomes a mechanism that propagates and enhances the shock, leading to a high amplification and severe system-wide liquidity stress. By contrast, a liquidity stress takes longer time to propagate in a ring network, and thus in presence of regulatory intervention (e.g. cash injections by a lender of last resort) it may be possible to limit the contagion effect.

The rest of the paper proceeds as follows. In what remains of this introductory section, we briefly review related theoretical and empirical studies of financial networks. In Section 2, we introduce our dynamic network model for interbank lending and the stochastic differential equations that govern its dynamics. In Section 3, we present our asymptotic analysis of the network model via a set of interacting measure-valued processes. In Section 4, we propose two systemic-risk indicators and develop useful approximations based on a set of measure-valued functions \((\nu_1, \ldots, \nu_N)\). Numerical examples are presented in Section 5, and concluding remarks summarized in Section 6. Proofs of the technical results are delegated to appendix.

1.2 Literature Review

As mentioned above, most studies in the literature concerning systemic risk in the banking system are motivated by the seminal work of Eisenberg and Noe (2001). Important extensions in this direction include the impact of bankruptcy losses as in Rogers and Veraart (2013); the quantification of contagion effects coming from direct counterparty exposures, and their relation to losses generated by inefficient asset liquidation as in Glasserman and Young (2015); the role played by the network topology in amplifying shocks as in the theoretical studies by Acemoglu et al. (2015) and Capponi et al. (2016). We refer the reader to Capponi (2016) and Glasserman and Young (2016) for excellent surveys on financial networks.

Our paper is related to a stream of literature studying network models with mean-field type interactions, in which banks mean-revert to the average monetary value of the system. Those studies include Fouque and Ichiba (2013), who propose a mean-field model where the monetary reserves of banks are modeled as a system of interacting Feller diffusion processes. They investigate how bank growth rates and lending preferences af-
fect default probabilities and provide an interacting particle system algorithm to compute various performance measures of the network. In contrast to Fouque and Ichiba (2013), Bo and Capponi (2015) model the monetary reserves of banks as a system of interacting jump diffusion processes, where the jumps model inflows and outflows of customer deposits. A shortcoming of these models is that they are based on the assumption that the monetary reserves of banks eventually converge to the average monetary value of the system, regardless of the initial size of the bank. This stands in contrast with empirical evidence, suggesting that (I) larger banks have higher liabilities and hence more reserves (Adrian and Shin (2010)); (II) large banks are more actively engaged in the interbank lending market (Cocco et al. (2009)).

In our study, we take the banks’ target reserve levels as exogenous input parameters. Banks in different clusters are allowed to have different speed of adjustments to their target reserve levels. Our framework can be specialized to mimic real-world scenarios, where large banks target higher reserve levels and are more actively engaged in the interbank lending market, providing higher financial intermediation to the system.

Our assumption that banks revert to their target reserve levels is strongly supported by empirical evidence from the last two decades. A study by Berger et al. (2008) reveals that banks in the U.S. hold far more equity than required by their regulatory authorities. They observe that banks, like non-financial firms, adjust their capital ratios to a predetermined target level, and set their capital targets significantly higher than regulatory minimum. A cross-section analysis done on a set of German banks by Memmel and Raupach (2010) reveals that a large portion of banks in the sample follows a target capital level. For these banks, adjusting the ratios via purchasing/selling of assets is less effective than by managing their liabilities. The empirical findings in Gropp and Heider (2010) mirror the findings by Berger et al. (2008) and Memmel and Raupach (2010). By analyzing a sample of large, publicly traded banks in sixteen countries, they conclude that banks have stable capital structures at levels that are specific to each individual bank. In addition, banks’ target leverage/capital ratio is time invariant and bank specific.

A related line of research encompasses the study of trading relationships in the interbank lending market. The findings in Cocco et al. (2009) provide support for the notion that relationships play an important role in the process of liquidity provision in the interbank lending market. Afonso et al. (2013) find that interbank relationships are highly persistent over time and the majority of lending relationships are asymmetric, i.e. one party is providing liquidity while the other is always demanding it. Their analysis also supports the view that banks borrow funds when they lack liquidity and that when they are lending, they lend to banks that have dissimilar businesses. Earlier models proposed by Bo and Capponi (2015) and Fouque and Ichiba (2013) are unable to capture specific trading relationships or the network structure, because banks are assumed to have the same lending preferences. This excludes network topologies such as the core-periphery structure borne out by bilateral interbank data (see, for instance, Craig and Von Peter (2014) for the case of the German interbank market). In contrast to them, our model captures observed lending patterns and incorporates a wide range of topological structures including the core-periphery topology.

Finally, our study also connects to certain measure-valued queueing models; e.g., Gromoll et al. (2008)
and Kaspi and Ramanan (2011). Applying scaling on certain system parameters similar to the scaling we do here, these studies show that a family of measure-valued processes representing the dynamics of the system converges to a fluid limit characterized as the solution of a functional differential equation. More recently, Jennings and Puha (2013) introduce a multi-dimensional measure-valued process to track the system state of a multi-class FIFO queue with customer abandonments and establish a functional law of large numbers (FLLN) for their state descriptor. Their weak convergence proof follows the standard compactness proof. More precisely, after proving the tightness of the prelimit processes, they establish the uniqueness of the (weak) limit by showing that the limit of each converging subsequence can be characterized by the same fluid model solution. By contrast, the characterization step in our approach is done by (i) first showing that any limit point of the sequence of prelimit processes will be a solution of a martingale problem and (ii) proving uniqueness of solutions for the martingale problem.

1.3 Notation and Conventions

We introduce notation and conventions that will be extensively used throughout the paper. We denote by \( \mathbb{R} \) the set of real numbers. Let \( \mathbb{N} \) and \( \mathbb{Z}_+ \) denote, respectively, the set of natural numbers and the set of positive integers. For a row vector \( x \), we use \( x^\top \) to denote its transpose. We denote by \([a_{i,j}]_{m \times n}\) the \( m \)-by-\( n \) matrix whose \((i,j)\)-th entry is \( a_{i,j} \). Let \( e_j \) denote the unit vector with \( j \)th entry equal to one and all remaining entries equal to zero. We use \( \equiv \) and \( \overset{d}{=} \) to denote, respectively, equal by definition and equal in distribution. In addition, let \( \Rightarrow \) denote convergence in distribution. Let \( D \) be the usual function space of right-continuous real-valued functions on the interval \([0, \infty)\) with left limits, as in Whitt (2002). For a measurable space \((M, \mathcal{B}, \mu)\) with the sigma-algebra \( \mathcal{B} \) and measure \( \mu \), we use \( B(M) \) to denote the set of all bounded measurable functions on \( M \), and we say that \( A \in \mathcal{B} \) is a continuity set if \( \mu(\partial A) = 0 \). We use \( 1_A(\cdot) \) to denote the indicator function of the event (set) \( A \). In addition, for a metric space \((E, d)\) with the distance function \( d \), we use \( C(E) \) and \( C_b(E) \) to denote, respectively, the space of continuous functions and the space of bounded continuous functions on \( E \). Similarly, we use \( C^q_b(E) \) to denote the set of continuous functions on \( E \) that have bounded derivatives up to order \( q \). For a given \( x \in E \), the Dirac measure \( \delta_x(\cdot) \) is a probability measure defined by \( \delta_x(A) = 1 \) if \( x \in A \) and \( 0 \) otherwise, for any Borel set \( A \). For a set of metric spaces \( \{E_i\}_{i=1}^n \), we use \( \prod_{i=1}^n E_i \) to denote the product space equipped with the usual product topology. A filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the filtration \( \mathbb{F} := (\mathcal{F}_t)_{t \geq 0} \) supports all stochastic processes defined below.

2 A Model of Interbank Lending

We consider an interbank network that has \( N \in \mathbb{N} \) clusters of banks interacting through borrowing and lending transactions. Each cluster \( j \) consists of \( K_j \) banks. Let \( \Xi \) be the collection of all pairs \((j,k)\), \( k = 1, \ldots, K_j \), \( j = 1, \ldots, N \).

The state variable \( \xi_{j,k}(t) \), representing the monetary reserve of bank \( k \) in cluster \( j \) at time \( t \), satisfies the
following stochastic differential equation (SDE):

\[
\xi_{j,k}(t) = \xi_{j,k}(0) + \int_0^t \ell_j [\theta_{j,k} - \xi_{j,k}(s)] \, ds - \int_0^t \left( \ell_j \pi_{j,k} / K_j \right) \sum_{k'=1}^{K_j} [\theta_{j,k'} - \xi_{j,k'}(s)] \, ds \\
- \int_0^t \sum_{h \neq j} (\ell_h \pi_{h,j} / K_j) \sum_{k'=1}^{K_h} [\theta_{h,k'} - \xi_{h,k'}(s)] \, ds + \sigma_{j,k} \int_0^t \sqrt{\xi_{j,k}(s)} \, dW_{j,k}(s).
\]

(2.1)

In the above equation, the initial state \( \xi_{j,k}(0) \) and the parameters \( \theta_{j,k}, \sigma_{j,k} \), idiosyncratic to each bank \( k \) within the cluster \( j \), are assumed to be non-negative random variables. The remaining parameters \( \ell_j \) and \( \pi_{j,h} \) are assumed to be nonnegative constants. \( W \equiv (W_{j,k}(t); \ t \geq 0), \ k = 1, \ldots, K_j, j = 1, \ldots, N, \) is a set of independent standard Brownian motions. On the right hand side of (2.1), after the initial state \( \xi_{j,k}(0) \), the first integral accounts for the bank’s own (cumulative) input/output up to \( t \), where \( \theta_{j,k} \) is the bank’s required reserve level, and \( \ell_j > 0 \) is an intensity (or “pressure”) factor. Thus, if \( \theta_{j,k} > \xi_{j,k}(t) \) (resp. \( \theta_{j,k} < \xi_{j,k}(t) \)), the bank is more inclined—modulated by the rate \( \ell_j \)—to borrow (resp. to lend) and thereby increasing (resp. decreasing) its reserve. The second and the third integrals represent the bank’s transactions (borrowing and lending) with other banks \( (k') \) within the same cluster \( (j) \) and those from other clusters \( (h) \), respectively. Because each cash outflow for a bank is associated with a commensurate cash inflow for its trading counter-parties, the positive sign of the first integral becomes negative for the second and third integrals. The last integral on the right hand side of (2.1) captures the idiosyncrasy in the banks’ reserves (due, for instance, to the daily deposits and withdraws from retail customers) modeled by Brownian motions \( W_{j,k} \)’s which are independent among the banks. We impose the following assumptions on the system dynamics:

(i) Any bank in cluster \( j \) has the same probability to transact with (a bank in) another cluster \( h \); hence, this probability is denoted by \( \pi_{j,h} \);

(ii) when an inter-cluster \( (j, h) \) transaction occurs, the originating bank in cluster \( j \) chooses one of the \( K_h \) banks in cluster \( h \) with equal probability \( (1/K_h) \).

Hence these probability values form a “routing” matrix \( \Pi \equiv [\pi_{j,h}]_{N \times N} \) of the transactions, also referred to as the transaction probability matrix. We allow banks in the network to transact with other banks outside the system. This implies that our network model is an open system, meaning that \( \Pi \) is a sub-stochastic matrix, i.e., \( \pi_{j,h} \geq 0 \) for any \( j, h \) and each row of the matrix adds up to at most 1, i.e., \( \sum_{h=1}^N \pi_{j,h} \leq 1 \) for each \( j \).

Let \( \tilde{\theta}_j \equiv (1/K_j) \sum_{k=1}^{K_j} \theta_{j,k}, \tilde{\xi}_j \equiv (1/K_j) \sum_{k=1}^{K_j} \xi_{j,k} \) for each \( j \leq N \), and \( \ell_{j,h} \equiv (K_j/K_h) \ell_j \pi_{j,h} \). We can rewrite the above equation (by combining the second and the third integrals) as follows:

\[
\xi_{j,k}(t) = \xi_{j,k}(0) + \int_0^t \ell_j [\theta_{j,k} - \xi_{j,k}(s)] \, ds - \int_0^t \sum_{h=1}^N \ell_{h,j} [\theta_{h} - \xi_{h}(s)] \, ds + \sigma_{j,k} \int_0^t \sqrt{\xi_{j,k}(s)} \, dW_{j,k}(s).
\]

(2.2)

The above formulation is designed to capture certain essential features of interbank lending activities as motivated in the Introduction (such as maintaining a target reserve level and operating in a clustered hierarchy), while modulated with simplifying assumptions (such as the linear “pressure” for borrowing/lending) to maintain tractability.
Throughout the paper, we impose the following multidimensional extension of the Feller condition derived from Condition A in Duffie and Kan (1996).

**Assumption 1** For each \((j, k) \in \Xi\),

\[
\ell_j (1 - \pi_{j,j} / K_j) \theta_{j,k} - (\ell_j \pi_{j,j} / K_j) \sum_{k' \neq k} \theta_{j,k'} - \sum_{h \neq j} (\ell_h \pi_{h,j} / K_j) \sum_{k' = 1}^{K_h} \theta_{h,k'} > \frac{\sigma_{j,k}^2}{2}
\]

(2.3)

for \(\{x_{h,k'}\}_{(h,k') \neq (j,k)} \subset \mathbb{R}_+\).

Applying the main theorem of Duffie and Kan (1996) (see Section 4 therein) we conclude that, under Assumption (1), there exists a unique strong positive solution to the \(K_1 + \cdots + K_N\)-dimensional stochastic differential equation (2.2). This positivity result ensures that the process never hits the zero boundary.

A direct verification of the inequality (2.3) is not straightforward. Because the variables \(x_{h,k'}\) are nonnegative, however, the left-hand-side always admits the lower bound

\[
\ell_j (1 - \pi_{j,j} / K_j) \theta_{j,k} - (\ell_j \pi_{j,j} / K_j) \sum_{k' \neq k} \theta_{j,k'} - \sum_{h \neq j} (\ell_h \pi_{h,j} / K_j) \sum_{k' = 1}^{K_h} \theta_{h,k'} > \frac{\sigma_{j,k}^2}{2},
\]

for all \((j, k) \in \Xi\), then Assumption 1 necessarily holds.

An example (satisfying Assumption 1) can be easily constructed. Assume \(\ell_j = \ell, K_j = K\) for all \(j\), and \(\theta_{j,k} = \theta_j, \sigma_{j,k} = \sigma\) for all \((j, k) \in \Xi\). Then, (2.4) reduces to

\[
\theta_j - \sum_h \pi_{h,j} \theta_h > \frac{\sigma^2}{2}, \quad \text{for} \quad j = 1, \ldots, N.
\]

(2.5)

Let \(\theta \equiv (\theta_1, \ldots, \theta_N)^T\) and \(1\) be the \(N\) dimensional column vector consisting of all entries equal to one. We can then rewrite equation (2.5) in matrix form, yielding the required condition \((I - \Pi^T)\theta > (\sigma^2/2)1\), where \(I\) denotes the \(N \times N\) dimensional identity matrix and > holds component-wise.

### 3 Large Network Asymptotic Analysis

We introduce a set of interacting measure-valued processes that can be viewed as the fluid-scaled state descriptor for the stochastic system in §3.1. In §3.2 we present a topological framework for studying the asymptotic behavior of the set of interacting measure-valued processes. In §3.3 we introduce the associated measure-valued functions which can be viewed as a formal FLLN limit of the system. We then show that under mild assumptions the fluid-scaled state descriptor of the stochastic system converges weakly to the set of measure-valued functions.
3.1 The Interacting Measure-Valued Processes

We introduce the set of measure-valued processes which will be used in the large-network asymptotic analysis. Each process keeps track of the empirical distribution of the type (volatility, target and actual bank’s reserve level) of banks in a cluster, capturing the typical interbank activities. The interaction between these processes captures the macroscopic behavior of the system’s activities. We make the following assumption.

**Assumption 2** The number of banks in cluster \( j \), for every \( j \), is equal to \( K_j^\eta = [\eta \kappa_j] \) where \( \kappa_j > 0 \) is a fixed parameter and the superscript \( \eta \) is used to highlight the dependence of the relevant model quantities on a scaling parameter \( \eta \). We assume that \( \sum_{j=1}^{N} \kappa_j = 1 \), implying that \( \eta \) denotes the total number of banks in the network (up to a roundoff error).

**Remark 3.1** Recall that \( \ell_{j,h} \equiv (K_j^\eta / K_h^\eta) \ell_j \pi_{j,h} \). Assumption 2 then implies that \( \ell_{j,h} = (\kappa_j / \kappa_h) \ell_j \pi_{j,h} \) is independent of the scaling parameter \( \eta \).

Let \( p_{j,k} \equiv (\theta_{j,k}, \sigma_{j,k}) \) be a random vector taking values from \( \mathbb{R}_+^2 \). Define a vector of interacting measure-valued processes

\[
\nu^\eta(\cdot) \equiv (\nu_j^\eta(\cdot), \ldots, \nu_N^\eta(\cdot))
\]

where, for each cluster \( j \) and time \( t \), the empirical measure \( \nu_j^\eta(t) \) is given by

\[
\nu_j^\eta(t) = \frac{1}{K_j^\eta} \sum_{k=1}^{K_j^\eta} \delta_{(p_{j,k}, \xi_{j,k}(t))} \quad \text{for} \quad t \geq 0.
\]

Let \( O \equiv \mathbb{R}_+^3 \) and \( S = \prod_j P_j(O) \), where \( P_j(O) \) represents the space of probability measures on the metric space \( O \). Then \( \nu^\eta(\cdot) \) can be viewed as a \( S \)-valued stochastic process. Note that each component \( \nu_j^\eta(\cdot) \) of the process \( \nu^\eta(\cdot) \) is a standard measure-valued process. For notational brevity, set \( \langle \mu, f \rangle \equiv \int_O f \ d\mu \) for any \( \mu \in P(O) \) and measurable function \( f \). Hence, we obtain

\[
\langle \nu_j^\eta, f \rangle_t = \langle \nu_j^\eta(t), f \rangle = \frac{1}{K_j^\eta} \sum_{k=1}^{K_j^\eta} f(p_{j,k}, \xi_{j,k}(t)).
\]

**Remark 3.2** For \( (p, x) \in O \), where \( p \equiv (\theta, \sigma) \in \mathbb{R}_+^2 \) and \( x \in \mathbb{R}_+ \), define the functions

\[
\psi_1(p, x) = x \quad \text{and} \quad \Theta(p, x) = \theta.
\]

Within each cluster, we can express the average bank’s monetary reserve and required reserve level using the representation (3.3):

\[
\langle \nu_j^\eta, \psi_1 \rangle_t = \bar{\xi}_j(t) \equiv \frac{1}{K_j^\eta} \sum_{k=1}^{K_j^\eta} \xi_{j,k}(t) \quad \text{and} \quad \langle \nu_j^\eta, \Theta \rangle_t = \bar{\theta}_j \equiv \frac{1}{K_j^\eta} \sum_{k=1}^{K_j^\eta} \theta_{j,k} \quad \text{for} \quad t \geq 0.
\]

We will make extensive use of the quantities in (3.5) throughout the paper.
Denote by \( \bar{\phi}_j \) the empirical measure at time zero, i.e., \( \bar{\phi}_j \equiv (1/K_j^n) \sum_{k \leq K_j^n} \delta(p_{j,k}, \xi_{j,k}(0)) \). Because both \( p_{j,k} \) and \( \xi_{j,k}(0) \) are random quantities, \( \bar{\phi}_j \) is a random measure for any \( j = 1, \ldots, N \). In order to obtain a limit theorem, we need to impose convergence on the behavior of the system at the initial time.

**Assumption 3** There exists a probability measure \( \phi_j \in \mathcal{P}(O) \) such that \( \bar{\phi}_j \Rightarrow \phi_j \) for any \( j = 1, \ldots, N \), as the number of banks \( \eta \to \infty \); i.e., \( \mathbb{E}[\Phi(\bar{\phi}_j)] \to \mathbb{E}[\Phi(\phi_j)] \) as \( \eta \to \infty \) for all bounded continuous functions \( \Phi \) with domain \( \mathcal{P}(O) \). In addition, each component of the random vector \( p_{j,k} \) is bounded by a constant \( C_p \) which is independent of \( (j,k) \). Moreover, \( \ell_j \leq C_p \) for \( j = 1, \ldots, N \).

### 3.2 The Topological Metric Space

We aim to show that the sequence of \( S \)-valued processes \( \{\nu^\eta(\cdot) \equiv (\nu^\eta_1(\cdot), \ldots, \nu^\eta_N(\cdot))\} \) indexed by \( \eta \) converges weakly, as \( \eta \to \infty \), to a limit \( \nu(\cdot) \equiv (\nu_1(\cdot), \ldots, \nu_N(\cdot)) \), with respect to an appropriate topology which will be specified in this section.

**One-Dimensional Case**

We measure the distance between two distributions \( \mu, \mu' \in \mathcal{P}(O) \) using the Prokhorov metric, i.e.,

\[
\rho(\mu, \mu') \equiv \inf\{\epsilon > 0 : \mu(A) \leq \mu'(A^\epsilon) + \epsilon \quad \text{for all } Borel \text{ set } A\},
\]

where \( A^\epsilon \equiv \{y \in O : d(x, y) < \epsilon \text{ for some } x \in A\} \) with \( d \) being the Euclidian distance. It is well known that the Prokhorov metric \( \rho \) is topologically equivalent to

\[
\beta(\mu, \mu') \equiv \sup \left\{ \left| \int f \, d(\mu - \mu') \right| : \|f\|_{BL} \leq 1 \right\},
\]

(3.6)

where \( f \) is a bounded Lipschitz function and \( \|f\|_{BL} \equiv \|f\|_L + \|f\|_\infty \) with \( \|f\|_L \equiv \sup_{x \neq y} |f(x) - f(y)|/d(x, y) \) and \( \|f\|_\infty \equiv \sup_x |f(x)| \), e.g. see Chapter 11 of Dudley (2002).

**Multi-Dimensional Extension**

To statistically correlate random measures, we propose a multivariate extension of the metric in (3.6). With a slight abuse of notation, we define

\[
\beta(\mu, \mu') \equiv \sup \left\{ \sum_{j=1}^N \left| \int f \, d(\mu_j - \mu'_j) \right| : \|f\|_{BL} \leq 1 \right\},
\]

(3.7)

where both \( \mu \equiv (\mu_1, \ldots, \mu_N) \) and \( \mu' \equiv (\mu'_1, \ldots, \mu'_N) \) are elements of the space \( S \). We show that the function \( \beta \) is non-negative, indiscernible, symmetric, sub-additive and hence a metric on the product space \( S \equiv \prod_j \mathcal{P}(O) \).

**Proposition 3.1** The function \( \beta : S \times S \to [0, \infty] \) given above is a metric.
Proposition 3.1 leads to the conclusion that $S$ is a Polish space. A topology is generated in the usual way for the Skorokhod space $D_S[0, \infty)$ of $S$-valued càdlàg processes. Convergence in $S$ can be characterized through the following lemma, whose proof follows by a straightforward extension of that in the one-dimensional case ($N = 1$), e.g., see Chapter 11 of Dudley (2002).

**Proposition 3.2** Let $O$ be a separable metric space. For any $\mu^\alpha \equiv (\mu_1^\alpha, \ldots, \mu_N^\alpha)$ and $\mu$ in $S$, the following statements are equivalent:

1. $\beta(\mu^\alpha, \mu) \to 0$;
2. $(\langle \mu_1^\alpha, f \rangle, \ldots, \langle \mu_N^\alpha, f \rangle) \to (\langle \mu_1, f \rangle, \ldots, \langle \mu_N, f \rangle)$ for all $f \in BL(O)$, where $BL$ denotes the collection of bounded Lipschitz functions;
3. $(\langle \mu_1^\alpha, f \rangle, \ldots, \langle \mu_N^\alpha, f \rangle) \to (\langle \mu_1, f \rangle, \ldots, \langle \mu_N, f \rangle)$ for all $f \in C_q^b(O)$, $q \in \mathbb{Z}_+$.

**Remark 3.3** Proposition 3.2 is especially useful in analyzing the sample path properties of processes living in the Skorokhod space $D_S[0, \infty)$. In particular, we will use the equivalent characterization (c) to verify the modulus of continuity condition for the set of interacting processes describing the interbanking activities.

The probability law of an $N$-dimensional diffusion process can be generally characterized as the unique solution of a martingale problem associated with a second-order elliptic differential operator. By analogy, the probability law of an $N$-dimensional measure-valued process can be obtained as the solution to the martingale problem associated with a differential operator $A$ acting on a function $\Phi(\cdot) \in \mathbb{D}$. The set $\mathbb{D}$ is a set of functions on $S$ sufficiently rich to generate the space of bounded measurable functions under the bounded pointwise convergence; see Dawson and Kurtz (1982). This suggests the following choice of the family $\mathbb{D}$:

$$\Phi(\mu) = \phi \left( \langle \mu, f_1 \rangle, \ldots, \langle \mu, f_m \rangle \right),$$

where $\langle \mu, f_n \rangle \equiv (\langle \mu_1, f_1, n \rangle, \ldots, \langle \mu_N, f_N, n \rangle)^\top$, $n = 1, \ldots, m$, for some $m \in \mathbb{Z}_+$; each $f_j, n \in C^\infty(O)$ and $\phi \in C^\infty(\mathbb{R}^{N \times m})$.

**Proposition 3.3** The function class $\mathbb{D}$ separates points in $S$ and is dense in the space of continuous functions defined on any compact subset of $S$.

### 3.3 Weak Convergence in $S$

We will show that the sequence of stochastic processes $\{\nu^\eta(\cdot)\}$ indexed by the scaling factor $\eta$ converges weakly to a limit $\nu(\cdot)$ in the Skorokhod space $D_S[0, \infty)$. The main result is formally stated in Theorem 3.1, and is essentially an FLLN for the sequence $\{\nu^\eta(\cdot)\}$.

For each $j \leq N$, let $z_j \equiv (\theta_j, \sigma_j, x_j)$ be a sample from the limiting distribution $\phi_j$ specified in Assumption 3. For such a $z_j \in O$, define a mean-reverting square-root stochastic integral equation $X_j(z_j; \cdot)$ with
time-varying coefficients:

\[ X_j(z_j; t) = x_j + \int_0^t \ell_j [\theta_j - X_j(z_j; s)] \, ds - \int_0^t \sum_{h=1}^N \ell_{h,j} [V_h - Q_h(s)] \, ds + \int_0^t \sigma_j \sqrt{X_j(z_j; s)} \, dW_j(s), \quad (3.9) \]

where \( W_j \equiv \{W_j(t); t \geq 0\}, \ j = 1, \ldots, N, \) are \( N \) independent standard Brownian motions. With the notation used in the equation above, we are stressing the dependence of the underlying state process \( X_j(z_j; \cdot) \) on the realized parameter set and the initial value \( x_j \). For each \( j \in \{1, \ldots, N\}, V_j \) is a constant that satisfies

\[ V_j = \langle \nu_j, \Theta \rangle_0 \equiv \int_\Theta(p, x) \phi_j(dz_j) = \int_\Theta \theta_j \phi_j(dz_j), \tag{3.10} \]

where we recall that \( \Theta(p, x) = \theta \) with \( p \equiv (\theta, \sigma) \in \mathbb{R}^2_+ \). In addition, let \( \bar{x}_j \equiv \int_\Theta x_j \phi_j(dz_j) \). The time-varying vector-valued function \( Q_j(\cdot) \equiv (Q_1(\cdot), \ldots, Q_N(\cdot))^\top \) satisfies a set of integral equations:

\[
Q_j(t) = \int_\Theta e^{-\ell_j t} \left[ x_j + \int_0^t \left( \ell_j \theta_j - \sum_{h=1}^N \ell_{h,j} (V_h - Q_h(s)) \right) e^{\ell_j s} \, ds \right] \phi_j(dz_j)
\]

\[
= e^{-\ell_j t} \bar{x}_j + (1 - e^{-\ell_j t}) \left( V_j - \sum_{h=1}^N \ell_{h,j} V_h / \ell_j \right) + \sum_{h \leq N} \ell_{h,j} \int_0^t e^{-\ell_j(t-s)} Q_h(s) \, ds,
\tag{3.11}
\]

for \( j = 1, \ldots, N \). Applying Gronwall’s inequality to (3.11), we conclude that each function \( Q_j \) is bounded over any compact interval. Using this result, we deduce that \( Q_j \) has a bounded derivative function and thus it is Lipschitz continuous over any compact interval. It now follows from the existence and uniqueness theorem of solutions to SDEs (see e.g. Theorem 7 in Protter (2005), §5.3, p. 259) that there exists a unique strong solution to the system (3.9) over any compact interval.

Using the state process \( X_j(z_j; t) \) defined by (3.9)-(3.11), we characterize the weak limit of the sequence \( \{\nu_j(\cdot)\} \). For each \( j \in \{1, \ldots, N\}, \) define a measure-valued function \( \nu_j(\cdot) \) via

\[
\langle \nu_j, 1_{A \times B} \rangle_t = \langle \nu_j(t), 1_{A \times B} \rangle \equiv \int_B 1_A(p_j)^P (X_j(z_j; t) \in B) \phi_j(dz_j) \quad \text{for} \quad t \geq 0, \tag{3.12}
\]

where \( A \in B(\mathbb{R}^2_+) \) and \( B \in B(\mathbb{R}) \). Let \( \nu(\cdot) \equiv (\nu_1(\cdot), \ldots, \nu_N(\cdot)) \). The following lemma plays an important role in the proof of the main theorem and in the development of approximations for the systemic-risk indicators studied in §5.

**Lemma 3.1** The time-varying vector-valued function \( Q(t) \equiv (Q_1(t), \ldots, Q_N(t))^\top \) given by the set of integral equations (3.11) equals the collection of measures \( \nu(t) \) acting on the identity function \( \psi_1 \) (recall \( \nu \) is set of measure-valued functions), i.e.,

\[
Q(t) = (\langle \nu_1, \psi_1 \rangle_t, \ldots, (\nu_N, \psi_1) \rangle_t)^\top \quad \text{for} \quad t \geq 0, \tag{3.13}
\]

where \( \nu \) is specified by (3.12) and \( \psi_1 \) given by (3.4).

**Corollary 3.1** If, for each \( j \leq N \), the limiting measure \( \phi_j \) in Assumption 3 is a Dirac measure, i.e., \( \phi_j \equiv \delta_{z_j^*} \) for \( z_j^* = (x_j^*, \theta_{j}^*, \sigma_{j}^*) \), then the vector-valued function \( Q(t) \) is the solution of the following linear system:

\[
\frac{dQ(t)}{dt} = (I - \Lambda^{-1} \Pi^\top \Lambda) L(\theta^* - Q(t)) \, dt \quad \text{and} \quad Q(0) = x^* \equiv (x_1^*, \ldots, x_N^*), \tag{3.14}
\]
where $I$ is the identity matrix, $\Pi$ the transaction probability matrix, $L$ a diagonal matrix whose entries are equal to $\ell_j$, $j = 1, \ldots, N$, $\Lambda$ a diagonal matrix whose entries are $\kappa_j$, $j = 1, \ldots, N$, and $\theta^* \equiv (\theta_1^*, \ldots, \theta_N^*)^\top$. Let $R \equiv (I - \Lambda^{-1}\Pi^\top \Lambda)L$. Then Eq. (3.14) admits a closed-form solution:

$$Q(t) = e^{-Rt}x^* + (I - e^{-Rt})\theta^*, \quad (3.15)$$

where $e^M$ denotes the exponential of the matrix $M$.

**Remark 3.4** The structure of Eq. (3.15) highlights the idiosyncratic effect and, more importantly, the systemic impact of a shock to the initial monetary reserves of a cluster. Recall that component $j$ of $Q(t)$ represents the large-network approximation for the average reserve level of cluster $j$. Suppose that an initial shock occurring to the $j$th cluster of the network pushes the average reserve of cluster $j$ below the average target by $\Delta x$. Using (3.15), we obtain that the total impact of such a shock on the system at time $t$ is negative and given by

$$e^{-Rt}(\Delta xe_j) \approx \Delta x(I - Lt)e_j + \Delta x(\Lambda^{-1}\Pi^\top \Lambda Lt)e_j,$$

where we have used the Taylor approximation to highlight the short-term systemic effects of the exogenous shock. The first term on the right hand side is the idiosyncratic component of the shock. It indicates that if banks in cluster $j$ have a high propensity to transact and adjust to the target level, then they will recover quickly from the shock. The second term on the right hand side captures the network effects through the dependence on the transaction probability matrix $\Pi$. If cluster $j$ has a high propensity to transact (large $\ell_j$) and distributes its transactions uniformly over the network ($\pi_{j,k} \approx \frac{1}{N-1}$), then the short-term impact will be high on all clusters and may result in a systemic breakdown when $\Delta x$ is sufficiently large. On the other hand, if cluster $j$ concentrates its transactions among a few clusters ($\pi_{j,k} \gg \frac{1}{N-1}$ for some values of $k$, and $\pi_{j,k} = 0$ for other values of $k$), then the shock will take a longer time to propagate through those components of the network that have weak connections to cluster $j$.

We are now in a position to state the main result which is formalized through the following theorem.

**Theorem 3.1** Under Assumptions 1 - 3, the sequence of interacting measure-valued processes $\{\nu^\eta(\cdot)\}$ indexed by $\eta$ converges weakly to the limit $\nu(\cdot)$, i.e.,

$$\nu^\eta(\cdot) \Rightarrow \nu(\cdot) \equiv (\nu_1(\cdot), \ldots, \nu_N(\cdot)) \quad \text{in} \quad D_S[0, \infty), \quad \text{as} \quad \eta \to \infty, \quad (3.16)$$

where each coordinate of $\nu(\cdot)$ is a measure-valued function as specified by (3.12).

Theorem 3.1 characterizes the weak limit as a vector of deterministic measure-valued functions, where each dimension describes the asymptotic and transient behavior of the empirical distribution of bank reserves within a cluster. Notice that each component of the vector of measure-valued functions is defined through a diffusion process, namely (3.9); the $N$ diffusion processes, one for each class, are statistically independent; i.e, the dynamics of the $j$-th component of $\nu(\cdot)$ is fully characterized by (3.9) which does not depend on the dynamics of the remaining $N - 1$ diffusion processes.
The proof of the theorem follows the martingale-problem approach as described, for instance, in Stroock and Varadhan (1972). It consists of three major steps. First, we establish the existence of limit points (with respect to the topology of weak convergence of probability measures) by proving the tightness of the sequence of interacting measures-valued processes \( \{ \nu^n(\cdot) \} \). Second, we identify a candidate generator of the limiting measure-valued process, and use it to show that each limit point solves the martingale problem for that generator. Third, we show uniqueness of solutions for the martingale problem. This completes the proof of the main theorem.\(^2\)

4 Systemic Risk Indicators

The objective of this section is to compute asymptotic approximation formulas for systemic performance measures. We focus on two types of systemic risk indicators, namely, the \textit{liquidity stress index} and the \textit{concentration index}. These measures provide not only an overall risk outlook of the network, but also capture excess correlation and volatility in the network. We use the set of measure-valued functions \((\nu_1, \ldots, \nu_N)\) to construct FLLN approximations for these systemic indicators.\(^3\)

Liquidity Stress Index

A bank is said to be experiencing liquidity stress if its reserve level falls short of a certain percent of the target, i.e., \( \xi < \alpha \theta \) for \( \alpha < 1 \). The following quantity, which we call the “liquidity stress index”, is the fraction of banks experiencing liquidity stress at time \( t \):

\[
L^n_j(t) \equiv \frac{1}{K^n_j} \sum_{k=1}^{K^n_j} 1\{\xi_{j,k}(t) < \alpha \theta \},
\]

where we recall that \( \eta \) is the scaling parameter denoting the total number of banks in the network. A larger value of \( L^n_j(t) \) corresponds to a situation when normal banking intermediation at time \( t \) is severely disrupted and the credit supply is reduced with potentially adverse consequences on the real economy. Let \( A \equiv \{ x | x < \alpha \theta \} \). We can then write

\[
L^n(t) \equiv (L^n_1(t), \ldots, L^n_N(t)) = (\langle \nu^n_1, 1_A \rangle_t, \ldots, \langle \nu^n_N, 1_A \rangle_t).
\]

For each \( j \) and a fixed \( t \geq 0 \), \( A \) is a continuity set for \( \nu_j(t) \) (see (3.9) and (3.12)). Thus

\[
\langle \nu^n_j, 1_A \rangle_t \Rightarrow \langle \nu_j, 1_A \rangle_t \quad \text{in} \quad \mathbb{R} \quad \text{as} \quad \eta \to \infty.
\]

We can use the converging-together lemma (see, e.g., Theorem 11.4.3 in Whitt (2002), p. 378) to establish the joint convergence

\[
\left( \langle \nu^n_1, 1_A \rangle_t, \ldots, \langle \nu^n_N, 1_A \rangle_t \right) \Rightarrow \left( \langle \nu_1, 1_A \rangle_t, \ldots, \langle \nu_N, 1_A \rangle_t \right) \quad \text{in} \quad \mathbb{R}^N \quad \text{as} \quad \eta \to \infty,
\]

\(^2\)Uniqueness of solutions for a martingale problem means that any two solutions have the same finite-dimensional distributions.

\(^3\)The FLLN approximation via the limiting measure-valued process \( \nu \) only represents the first-order approximation of the banks’ monetary reserve dynamics. A more precise approximation would take into account the second-order term given by the fluctuation of the empirical measure-valued process \( \nu^n \) around its law-of-large number limit. This central limit theorem type result is beyond the scope of this paper. We refer reader to the work by Spiliopoulos et al. (2014) for a related analysis. Therein the authors develop a second-order Gaussian approximation, the so-called fluctuation limit, to the distribution of loss from defaults in large portfolios.
where
\[ \langle \nu_j, 1_A \rangle_t = \int \mathbb{P} \left( X_j(z_j; t) < \alpha \theta_j \right) \phi_j(dz_j) \quad (4.2) \]
and \( X_j(z_j; t) \) follows the dynamics given by (3.9).

**Concentration Index**

The “concentration-fragility” view holds that concentrated systems lead to excessive risk-taking, because of moral hazard stemming from the implicit government bail out of too-big-to-fail institutions (O’Hara and Shaw (1990), Acharya et al. (2014)), or the complex and opaque structures that are often associated with large institutions (Cetorelli et al. (2014)). Our analysis of the concentration level of the financial network serves to highlight how the interplay of shocks, volatilities, and inter-dependencies of financial activities can lead to a rise in the concentration of banks’ monetary reserves.

We measure concentration using the Herfindahl index. For a vector of non-negative real numbers \( a \equiv (a_1, \ldots, a_n) \), the Herfindahl index of \( a \) is defined to be \( \mathcal{H}(a) \equiv \frac{\sum_{k=1}^{n} a_i^2}{\left( \sum_{k=1}^{n} a_i \right)^2} \), i.e., the sum of the squares normalized by the square of the sum. It is easy to verify that \( \mathcal{H} \) attains its maximum when all entries of the vector \( a \) are equal; and \( \mathcal{H} \) attains its minimum when all but one entry of the vector \( a \) are zero. This notion can be easily generalized to vector-valued functions.

**Definition 4.1** The concentration index of the interbank network is the sum of the squares of banks’ monetary reserves normalized by the squared aggregate amount of monetary reserves, i.e.,
\[ \mathcal{H}^\eta(t) = \frac{\sum_{(j,k) \in \Xi} \left( \xi_{j,k}(t) \right)^2}{\left( \sum_{(j,k) \in \Xi} \xi_{j,k}(t) \right)^2}. \quad (4.3) \]

The time series \( \{ \mathcal{H}^\eta(t); \ t \geq 0 \} \) defined by (4.3) is a stochastic process adapted to the natural filtration. We scale the process \( \mathcal{H}^\eta \) in a way that the sequence of scaled processes converges weakly to a proper limit. Let
\[ \overline{\mathcal{H}}^\eta(t) = \eta \mathcal{H}^\eta(t), \quad \text{for} \quad t \geq 0. \quad (4.4) \]
Using the definition of \( \mathcal{H}^\eta \), we can write
\[ \overline{\mathcal{H}}^\eta(t) = \frac{\sum_{j=1}^{N} \left( K_j^\eta / \eta \right) \cdot \langle \nu_j^\eta, \psi_1 \rangle_t}{\left( \sum_{j=1}^{N} \left( K_j^\eta / \eta \right) \cdot \langle \nu_j^\eta, \psi_1 \rangle_t \right)^2}, \quad (4.5) \]
where we have defined \( \psi_2(p, x) \equiv x^2 \), and we recall that \( \psi_1(p, x) = x \) is the identity function defined in Eq. (3.4). Using Theorem 3.1 and the moment conditions given in Lemma B.1, we deduce that for each \( j \)
\[ \langle \nu_j^\eta, \psi_1 \rangle_t \Rightarrow \langle \nu_j, \psi_1 \rangle_t \quad \text{and} \quad \langle \nu_j^\eta, \psi_2 \rangle_t \Rightarrow \langle \nu_j, \psi_2 \rangle_t \quad \text{as} \quad \eta \to \infty. \quad (4.6) \]
Because all limits are deterministic, the above convergence can be strengthened to joint convergence by the converging-together lemma (see, e.g., Theorem 11.4.3 of Whitt (2002), p. 378), i.e.,
\[ (\langle \nu_1^\eta, \psi_1 \rangle, \ldots, \langle \nu_N^\eta, \psi_1 \rangle) \Rightarrow (\langle \nu_1, \psi_1 \rangle, \ldots, \langle \nu_N, \psi_1 \rangle) \quad (4.7) \]
and
\[
\left( \langle \nu_1^N, \psi_2 \rangle, \ldots, \langle \nu_N^N, \psi_2 \rangle \right) \Rightarrow \left( \langle \nu_1, \psi_2 \rangle, \ldots, \langle \nu_N, \psi_2 \rangle \right)
\]  \hspace{1cm} (4.8)

as \( \eta \to \infty \). We can then use the continuous mapping theorem (CMT) with continuity of additive functions and the converging-together lemma to obtain
\[
\left( \sum_{j=1}^{N} \left( K_j^N / \eta \right) \langle \nu_j^N, \psi_2 \rangle, \sum_{j=1}^{N} \left( K_j^N / \eta \right) \langle \nu_j^N, \psi_1 \rangle \right) \Rightarrow \left( \sum_{j=1}^{N} \kappa_j \langle \nu_j, \psi_2 \rangle, \sum_{j=1}^{N} \kappa_j \langle \nu_j, \psi_1 \rangle \right)
\]
as \( \eta \to \infty \). Using the CMT with the division operator yields the result below.

Proposition 4.1 In addition to Assumptions 1 - 3, if we have \( \sum_{j=1}^{N} \kappa_j \langle \nu_j, \psi_1 \rangle_t > 0 \) for all \( t \geq 0 \), then
\[
\mathcal{H}_t \Rightarrow \frac{\sum_{j=1}^{N} \kappa_j \langle \nu_j, \psi_2 \rangle_t}{\left( \sum_{j=1}^{N} \kappa_j \langle \nu_j, \psi_1 \rangle_t \right)^2}, \quad \text{as} \quad \eta \to \infty,
\]  \hspace{1cm} (4.9)

where \( \mathcal{H}_t \) is given by (4.4) and set of measure-valued functions \( \nu \equiv (\nu_1, \ldots, \nu_N) \) is given by (3.16).

5 Network Topology and Systemic Risk Dynamics

We now use the asymptotic formulas for the network performance measures derived in the previous section to analyze the interplay between network topology and systemic risk. Furthermore, our dynamic interbanking model allows investigating how the network topology affects both the transient and the steady-state behavior of the liquidity stress and Herfindahl indices.

We work under the assumption that, for each \( j \leq N \), the limiting measure \( \phi_j \) in Assumption 3 is a Dirac measure, i.e., \( \phi_j \equiv \delta_{z_j^*} \) and \( z_j^* \equiv (x_j^*, \theta_j^*, \sigma_j^*) \) as \( \eta \to \infty \).

Using Corollary 3.1 and Lemma 3.1, the mean-reserve processes \( \langle \nu_1^N, \psi_1 \rangle, \ldots, \langle \nu_N^N, \psi_1 \rangle \) are approximated by the solution of the linear system (3.14), i.e.,
\[
Q(t) = e^{-Rt} x^* + (I - e^{-Rt}) \theta^*,
\]
where we recall that \( R \equiv (I - \Lambda^{-1} \Pi^T \Lambda) L \). Using this explicit representation, we can compute the denominator in (4.9). To compute the numerator, however, we need to develop a computational scheme for the time-varying functions \( \langle \nu_1, \psi_2 \rangle, \ldots, \langle \nu_N, \psi_2 \rangle \). Using (A.10) with \( f \) replaced by \( \psi_2 \), we get
\[
\langle \nu_j, \psi_2 \rangle_t = \int_{\Omega} \mathbb{E} \left[ X_j(z_j; \tau) \right] \phi_j(\text{d}z_j) = \mathbb{E} \left[ X_j(z_j^*; \tau) \right] \equiv \epsilon_j(t).
\]

Application of Itô’s formula, along with (3.9), gives immediately the following dynamics
\[
d\epsilon_j(t) = -2 \ell_j \epsilon_j(t) dt + \left( 2 \ell_j \theta_j^* + (\sigma_j^*)^2 - 2 \sum_{h=1}^{N} \ell_{h,j} [\theta_h^* - Q_h(t)] \right) Q_j(t) dt \equiv -2 \ell_j \epsilon_j(t) dt + a_j(t) dt,
\]
where the function \( a_j(\cdot) \) is given by
\[
a_j(\cdot) \equiv \left( 2 \ell_j \theta_j^* + (\sigma_j^*)^2 - 2 \sum_{h=1}^{N} \ell_{h,j} [\theta_h^* - Q_h(\cdot)] \right) Q_j(\cdot).
\]
The above is a first-order, linear nonhomogeneous differential equation, whose solution can be obtained explicitly. In particular,

$$e_j(t) = e^{-2\ell_j t} (x_j^*)^2 + \int_0^t e^{-2\ell_j (t-s)} a_j(s) \, ds.$$  

Unlike in the case of Herfindahl index, the computation of the liquidity stress index requires knowledge of the entire distribution of $X_j(z_j^*; t)$ at any time $t$. Using (4.2) and choosing $\phi_j \equiv \delta_{z_j^*}$, the approximation formula reduces to

$$\int \mathbb{P} \left( X_j(z_j; t) < \alpha \theta_j \right) \phi_j(dz_j) = \mathbb{P} \left( X_j(z_j^*; t) < \alpha \theta_j^* \right).$$

The calculation of these probabilities can be achieved by inverting the moment generating function which admits a closed-form expression. Recall that the dynamics of the underlying state process $X_j(z_j^*; \cdot)$ is given by (3.9), and admits the general form

$$dX(t) = \ell(\theta - X(t)) \, dt + q(t) \, dt + \sigma X(t)^{1/2} dW(t),$$

where $q(\cdot)$ is a deterministic time-varying function. Let

$$\psi(t, u) \equiv \mathbb{E} \left[ e^{uX(t)} \right]$$  

(5.1)

be the moment generating function.

**Proposition 5.1** The moment generating function $\psi(t, u)$ given in (5.1) admits the explicit expression

$$\psi(t, u) = \exp \left[ \alpha(t, u) + \beta(t, u) X(0) \right],$$

where

$$\beta(t, u) = \frac{ue^{-\ell t}}{1 - \frac{\sigma^2}{2\ell} u(1 - e^{-\ell t})} \quad \text{and}$$

$$\alpha(t, u) = -\frac{2\ell \theta}{\sigma^2} \log \left( 1 - \frac{\sigma^2}{2\ell} u(1 - e^{-\ell t}) \right) + \int_0^t \frac{ue^{-\ell s} q(t-s)}{1 - \frac{\sigma^2}{2\ell} u(1 - e^{-\ell s})} \, ds.$$  

(5.2)

Notice that if $q(t) \equiv 0$, then $\psi(t, u)$ is simply the moment generating function of a non-central $\chi^2$-distribution.

### 5.1 Interplay of Network Topology and Systemic Risk

We consider a network consisting of four clusters. Each cluster consists of a dozen banks with identical lending/borrowing preferences. Banks within the same cluster have the same target reserve level and are initially endowed with the same amount of monetary reserves. We considered two network configurations, the core-periphery and the ring structure, and visualize them in Figure 1.

The core-periphery network has been identified as the most accurate description of interbanking activities. Craig and Von Peter (2014) performed an empirical analysis using bilateral interbank data from German banks from 1999 to 2007 and found that the matrix of interbank liabilities follows a core-periphery structure. These findings are in line with the analysis by Fricke and Lux (2015), who employed a dataset of overnight interbank
transactions in the Italian market from 1999 to 2010, and found that a core-periphery structure provides the best fit for these interbank data, with high degree of persistence over time. In the core-periphery model, each core bank transacts with any other core bank in the network, but peripheral banks do not directly interact with each other. In our numerical examples, the model parameters are chosen to match empirical evidence suggesting that core banks are significantly larger and more active than peripheral banks (Craig and Von Peter (2014), Fricke and Lux (2015)). We choose the ring network in representation of sparsely connected networks, to contrast their capacity of absorbing and propagating shocks with the more densely connected core-periphery topologies. Such a choice is quite standard in the literature; see, for instance Acemoglu et al. (2015).

We first test the asymptotic accuracy of the proposed systemic indicators, by comparing the values obtained from the large network approximation with the corresponding Monte-Carlo estimates. Intuitively, we expect that as the size of the network increases, the asymptotic approximation gets closer to the Monte-Carlo estimate. This statement is visually confirmed from Figure 2 and Figure 3, which reports the 95% error band of the LSI for $\eta = 220$, together with the large-network approximation given by (4.2).

### 5.2 Transient and Steady-State Network Performance

We analyze the transient and steady-state performance of the network, measured in terms of liquidity stress and Herfindahl indices. At time zero, we apply an exogenous shock to all banks in cluster 1, which leads to a downward deviation from the target reserve level for each bank in the cluster. We consider two shock regimes to highlight the qualitatively different behavior of the core-periphery and ring networks in amplifying
Figure 2: LSI under the core-periphery network topology. We compare the asymptotic approximation formula with the Monte-Carlo estimates obtained for a finite number of banks. We choose $\alpha = 0.95$. We choose $(K_1, K_2, K_3, K_4) = \eta \times (2/11, 3/11, 3/11, 3/11)$, the speed of adjustment $\ell_1 = 2\ell_2 = 2\ell_3 = 2\ell_4 = 2$, and the loading factor $\sigma_{j,k} = 0.25$ identically equal for all banks. All banks in cluster $j$ have the same initial reserve $x^*_j$, and $(x^*_1, x^*_2, x^*_3, x^*_4) = (100, 24, 30, 27)$; all banks in cluster $j$ have the same target reserve level $\theta^*_j$, where $(\theta^*_1, \theta^*_2, \theta^*_3, \theta^*_4) = (120, 24, 30, 27)$. The transaction probability matrix $\Pi^1$ is specified in Figure 1.

In Figure 4, the shock yields a downward deviation from the target level by 10 units for the banks in the first cluster. While the liquidity stress index ramps up instantaneously in the core-periphery network, it propagates at a slower speed in the ring network. Noticeably, the core-periphery network recovers more rapidly from the shock relative to the more sparsely connected ring network. This suggests that connectivity improves the ability for a banking network to absorb shocks over a long term, in line with the existing literature of one-period models of network contagion (e.g. Acemoglu et al. (2015)). However, our analysis highlights an important effect which is absent in static models: the instantaneous response to a shock is higher in a more densely connected network. While a shock of moderate size may not lead to a systemic distress of the network (e.g. the LSI in clusters 3 and 4 does not peak to very high values), a shock of larger size may have more serious consequences. For instance, Figure 5 considers a similar setup, but applies a larger shock to the monetary reserves of banks in the first cluster, leading to a downward deviation of 20 units.
Figure 3: LSI under the ring network topology. We compare the asymptotic approximation formula with the Monte-Carlo estimates obtained for a finite number of banks. We choose $\alpha = 0.95$. We choose $(K_1, K_2, K_3, K_4) = \eta \times (2/11, 3/11, 3/11, 3/11)$, the speed of adjustment $\ell_1 = 2\ell_2 = 2\ell_3 = 2\ell_4 = 2$, and the loading factor $\sigma_{j,k} = 0.25$ identically equal for all banks. All banks in cluster $j$ have the same initial reserve $x_j^*$, and $(x_1^*, x_2^*, x_3^*, x_4^*) = (100, 24, 30, 27)$; all banks in cluster $j$ have the same target reserve level $\theta_j^*$, where $(\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*) = (120, 24, 30, 27)$. The transaction probability matrix $\Pi^2$ is specified in Figure 1.

The systemic consequences are stronger: the shock wipes out 16.67\% of the total reserves in cluster 1. The core-periphery network experiences a system-wide liquidity stress almost immediately after the shock, which leaves over 90\% of the banks in the network under liquidity stress. In contrast, the shock propagates at a much lower speed in the ring network. For instance, in the ring network, the LSI of cluster 2 reaches its maximum at $t_2 = 0.1$, while the LSIs of clusters 3 and 4 reach their peak at a later time, respectively $t_3 = 2.5$ and $t_4 = 4.2$. Even though the core-periphery network always recovers better in the long run, the transient behavior of the network in response to a large shock raises serious concerns for financial stability. It is unlikely that any form of government intervention would be able to mitigate the severe shortage of liquidity arising in a densely connected core-periphery network. In contrast, the effects of liquidity stress take more time to propagate in the ring network and thus allow for the possibility of restoring financial stability through say, liquidity injections.

Figure 6 suggests that higher idiosyncratic risk (higher $\sigma$) leads to a greater concentration in monetary
reserves. This is intuitively expected because higher volatility increases the variability of the sample paths of the interbanking network, and thus raises the probability of observing higher heterogeneity in the distribution of monetary reserves in the network. In line with intuition, the Herfindahl index is generally higher in the ring network, because a more sparsely connected network reduces the amount of risk sharing in the network. A larger shock to the initial monetary reserves of a cluster leads to a higher concentration index both for the ring and core-periphery network.

6 Concluding Remarks

In this paper, we have developed a dynamic network model driven by empirically observed banking behavior. Banks manage their trading activities to maintain a desired target level of reserve capital, and the network structure is organized according to a hierarchical structure. We have modeled the intra-cluster and inter-cluster transactional activities using a vector of interacting measure-valued processes, each component of which captures the trading characteristics of the banks in a specific cluster. We have established the weak
Figure 5: Asymptotic approximations for the liquidity-stress-index at level $\alpha = 0.95$. We choose $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (2/11, 3/11, 3/11, 3/11), \quad (x_1^*, x_2^*, x_3^*, x_4^*) = (100, 24, 30, 27), \quad (\theta_1^*, \theta_2^*, \theta_3^*, \theta_4^*) = (120, 24, 30, 27), \quad \ell_1 = 2\ell_2 = 2\ell_3 = 2\ell_4 = 2$, and $\sigma_{j,k} = 0.25$ for all banks. The transaction probability matrices $\Pi^1$ and $\Pi^2$ are specified in Figure 1.

We have provided an explicit characterization of the limit vector process in which each component tracks the typical behavior of a bank in the cluster. The explicit analytical form of the limit allows us to obtain tractable representations for statistical measures of systemic performance. We have analyzed in detail two important indicators of systemic distress—the liquidity stress index and the concentration index. Through illustrative numerical examples, we have analyzed the sensitivity of these systemic risk indicators with respect to network parameters, including banks’ volatilities and target leverages.

We expect our approach to be applicable to a broader class of stochastic systems than interbank networks. Those include, for example, queueing networks with heterogeneous pools of non-exponential servers. We expect this system to be fully characterized by a vector of interacting measure-valued processes, in which each component tracks the evolution of the empirical distribution of the elapsed/residual service times within a pool. Such an analysis would extend existing studies such as Kaspi and Ramanan (2011) (see also Kaspi and Ramanan (2013) for a second-order refinement via martingale measures), from a one-dimensional to a high-dimensional setting, with multiple heterogeneous server pools and appropriate routing rules.
Figure 6: Concentration indices computed using the analytic approximation. We choose $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (2/11, 3/11, 3/11, 3/11)$, $\ell_1 = 2\ell_2 = 2\ell_3 = 2\ell_4 = 2$, $(\theta^*_1, \theta^*_2, \theta^*_3, \theta^*_4) = (120, 26, 28, 30)$. The loading factors are identical for all banks. The transaction matrices $\Pi^1$ and $\Pi^2$ are specified in Figure 1.

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Appendices

We provide the proof of the main theorem in Appendix A. All other proofs are relegated to Appendix B.

A Proof of Theorem 3.1

The proof consists of three technical steps. The first step is to prove tightness via Lemma A.1 and Proposition A.1; the second step is to identify the weak limit through Propositions A.2 and A.3; the third step is to prove the uniqueness of the weak limit through Lemma A.2. Taken together, these steps complete the proof of the main theorem.

Tightness of the Sequence of Measure-Valued Processes. The proof of tightness for the sequence of measure-valued processes $\nu^\omega(\cdot)$ is implied by (i) the compact containment condition (CCC) and (ii) the modulus of continuity condition (MCC). The CCC holds if and only if for each $\epsilon > 0$ there exists a compact subset
\( \mathcal{K} \) of \( S \) such that, for an arbitrarily fixed \( T > 0 \),
\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^n(t) \in \mathcal{K} \text{ for all } t \in [0, T]) > 1 - \epsilon.
\] (A.1)

The CCC is often difficult to verify. However, a weaker condition which we will refer to as pointwise containment condition (PCC) can often be used in conjunction with the MCC to establish the CCC; see, e.g., Ledger (2016). The PCC holds if for all \( \epsilon > 0 \) and \( t \in [0, T] \), there exists a compact set \( \mathcal{K}(\epsilon, t) \) that depends on both \( \epsilon \) and \( t \) such that
\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^n(t) \in \mathcal{K}(\epsilon, t)) > 1 - \epsilon.
\] (A.2)

By Proposition 3.1, \((S, \beta)\) is a complete metric space. Then by Theorem 17 of Ledger (2016), if the family of \( S\)-valued processes \( \nu^n \) satisfy both the MCC and the PCC, then the CCC holds. The forthcoming lemma verifies the PCC.

**Lemma A.1** For each \( \epsilon \) and \( t \geq 0 \), there exists a compact subset \( \mathcal{K}^* \) of \( S \) such that
\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^n(t) \in \mathcal{K}^*(\epsilon, t)) > 1 - \epsilon.
\]

**Proof of Lemma A.1** The argument below is adapted from the proof of Lemma 6.1 of Giesecke et al. (2013), but takes into account that \( \nu^n \) is multidimensional. For each \( M > 0 \), define \( \mathcal{K}_M \equiv [0, C_p]^2 \times [0, M] \) where \( C_p \) is the bound given in Assumption 3. Further, denote by \( A^c \) the complement of a set \( A \). We then have, for each \( j = 1, \ldots, N \),
\[
\mathbb{E}\left[ \langle \nu^n_j, 1_{\mathcal{K}^c_M}\rangle_t \right] = \nu^n_j(t) (\mathcal{K}^c_M) = \frac{1}{K^n_j} \sum_{k=1}^{K^n_j} \mathbb{P}(\xi_{j,k}(t) \geq M) \leq \frac{\hat{C}(1, T, C_p)}{M},
\]
where the constant \( \hat{C}(1, T, C_p) \equiv C(1, T, C_p)e^{C(1, T/C_p)T} \) is provided in the proof of Lemma B.1. Next we define
\[
\mathcal{K}_M^* \equiv \left\{ \mu \equiv (\mu_1, \ldots, \mu_N) \in S : \langle \mu_j, 1_{\mathcal{K}^c_{(M+k)^2}} \rangle < \frac{1}{\sqrt{M+k}} \text{ for all } j = 1, \ldots, N \right\}.
\]
Note that, in the above expression, \( \langle \mu_j, 1_A \rangle \) equals the probability \( \mu_j(A) \). For each \( j \), the collection of probability measures
\[
\mathcal{K}_{M,j}^* \equiv \left\{ \mu_j : \langle \mu_j, 1_{\mathcal{K}^c_{(M+k)^2}} \rangle < \frac{1}{\sqrt{M+k}} \right\}
\]
is tight (by the definition of tightness). From Prokhorov’s theorem (see, e.g., Theorem 11.5.4 in Dudley (2002)), it follows that \( \mathcal{K}_{M,j}^* \) is a compact subset of \( \mathcal{P}(\mathcal{O}) \). Applying Tychonoff’s Theorem (see, e.g., Theorem 2.2.8 in Dudley (2002)), we conclude that the set \( \mathcal{K}_M^* \) is a compact subset of \( S \). In addition, we have
\[
\mathbb{P}(\nu^n(t) \notin \mathcal{K}_M^*) \leq \sum_{j=1}^N \sum_{k=1}^\infty \mathbb{P}(\langle \nu^n_j, 1_{\mathcal{K}^c_{(M+k)^2}} \rangle_t > \frac{1}{\sqrt{M+k}}) \leq \sum_{j=1}^N \sum_{k=1}^\infty \mathbb{E}\left[ \langle \nu^n_j, 1_{\mathcal{K}^c_{(M+k)^2}} \rangle_t \right] \leq \sum_{k=1}^\infty \frac{N\hat{C}(1, T, C_p)}{(M+k)^2/\sqrt{M+k}} \to 0 \text{ as } M \to \infty.
\]
The convergence to zero is independent of the index $\eta$. Hence for any $\epsilon > 0$, by choosing $M$ large enough, one gets
\[
\inf_{\eta \in \mathbb{N}} \mathbb{P}(\nu^n(t) \in K^*_M) > 1 - \epsilon,
\]
as desired. $\square$

The PCC will be strengthened to CCC if MCC holds. The following proposition uses Proposition 3.2 to verify the MCC.

**Proposition A.1** Let $g(x, y) = \|x - y\|_2 \wedge 1$ for any $x, y \in \mathbb{R}^N$ and define $\mathbb{E}_t[\cdot] \equiv \mathbb{E}[\cdot | \mathcal{F}_t]$ for $t \geq 0$. Then for each $\gamma \geq 0$, there exists a positive random variable $a^n(\gamma)$ that depends on $\gamma$ with $\lim_{\gamma \to 0} \sup_{\eta} \mathbb{E}[a^n(\gamma)] = 0$ such that for all $0 \leq t \leq T, 0 \leq u \leq \gamma \text{ and } 0 \leq v \leq \gamma \wedge t$,
\[
\mathbb{E}_t \left[ g^2 (\langle \nu^n, f \rangle_{t+u}, \langle \nu^n, f \rangle_t) g^2 (\langle \nu^n, f \rangle_t, \langle \nu^n, f \rangle_{t-v}) \right] \leq \mathbb{E}_t [a^n(\gamma)],
\]
where $f \in C^2_0(\Omega)$ and $\langle \nu^n, f \rangle_t \equiv (\langle \nu^n_1, f \rangle_t, \ldots, \langle \nu^n_N, f \rangle_t)^\top$.

**Proof of Proposition A.1** For notational brevity, in what follows we write $f(x) \equiv f(p, x)$ whenever it is clear from the context. In view of the equation (B.10), we have
\[
\langle \nu^n_j, f \rangle_t = \langle \nu^n_j, f \rangle_0 + A_j(t) + B_j(t) + E_j(t) + F_j(t), \quad (A.3)
\]
where we have defined
\[
A_j(t) \equiv \ell_j \int_0^t \frac{1}{K^\eta_j} \sum_{k=1}^{K^\eta_j} \frac{\partial f(\xi_{j,k}(s))}{\partial x} (\theta_{j,k} - \xi_{j,k}(s)) \, ds,
\]
\[
B_j(t) \equiv - \sum_{h \leq N} \ell_{h,j} \int_0^t \frac{1}{K^\eta_j} \sum_{k=1}^{K^\eta_j} \frac{\partial f(\xi_{j,k}(s))}{\partial x} (\theta_{h} - \xi_{h}(s)) \, ds,
\]
\[
E_j(t) \equiv \int_0^t \frac{1}{2K^\eta_j} \sum_{k=1}^{K^\eta_j} \sigma_{j,k}^2 \xi_{j,k}(s) \frac{\partial^2 f(\xi_{j,k}(s))}{\partial x^2} \, ds,
\]
\[
F_j(t) \equiv \int_0^t \frac{1}{K^\eta_j} \sum_{k=1}^{K^\eta_j} \sigma_{j,k} \frac{\partial f(\xi_{j,k}(s))}{\partial x} (\xi_{j,k}(s))^{1/2} dW_{j,k}(s).
\]

Using the definition of $g$ and (A.4), it follows that
\[
g^2 (\langle \nu^n, f \rangle_{t+u}, \langle \nu^n, f \rangle_t) \leq \sum_{j=1}^N \left| \langle \nu^n_j, f \rangle_{t+u} - \langle \nu^n_j, f \rangle_t \right|^2
\]
\[
\leq 4 \sum_{j=1}^N \left( |A_j(t + u) - A_j(t)|^2 + |B_j(t + u) - B_j(t)|^2 + |E_j(t + u) - E_j(t)|^2 + |F_j(t + u) - F_j(t)|^2 \right).
\]

(A.5)
Below, we analyze in turn the terms $A_j, B_j, E_j$ and $F_j$. First, for $0 \leq u \leq \gamma$, we have

$$|A_j(t + u) - A_j(t)|^2 \leq C_p^2 \left\| \frac{\partial f}{\partial x} \right\|^2 u \int_t^{t+u} \left( \frac{1}{K_j} \sum_{k=1}^{K_j} (\theta_{j,k} - \xi_{j,k}(s)) \right)^2 ds$$

$$\leq C_p^2 \left\| \frac{\partial f}{\partial x} \right\|^2 \gamma \int_0^T \frac{1}{K_j} \sum_{k=1}^{K_j} (\theta_{j,k} - \xi_{j,k}(s))^2 ds \equiv a_{j,1}^\eta(\gamma),$$

where both the first and the second inequality follow by applying the Cauchy-Schwartz inequality. Similarly, we have

$$|B_j(t + u) - B_j(t)|^2 \leq C_p^2 \left\| \frac{\partial f}{\partial x} \right\|^2 \gamma \sum_{h=1}^N \frac{1}{K_h} \int_0^T \sum_{k'=1}^{K_{h,k'}} (\theta_{h,k} - \xi_{h,k}(s))^2 ds \equiv a_{j,2}^\eta(\gamma)$$

$$|E_j(t + u) - E_j(t)|^2 \leq C_p^4 \left\| \frac{\partial^2 f}{\partial x^2} \right\|^2 \gamma \int_0^T \frac{1}{K_j} \sum_{k=1}^{K_j} |\xi_{j,k}(s)|^2 ds \equiv a_{j,3}^\eta(\gamma) \quad \text{for} \quad 0 \leq u \leq \gamma.$$

Finally,

$$\mathbb{E}_t \left[ |F_j(t + u) - F_j(t)|^2 \right] \leq \tilde{C} \mathbb{E}_t \left[ \int_t^{t+u} \left( \frac{1}{K_j} \sum_{k=1}^{K_j} \sigma_{j,k} \frac{\partial f(\xi_{j,k}(s))}{\partial x} (\xi_{j,k}(s))^{1/2} \right)^2 ds \right]$$

$$\leq \tilde{C} \mathbb{E}_t \left[ C_p^2 \left\| \frac{\partial f}{\partial x} \right\|^2 \int_t^{t+u} \frac{1}{K_j} \sum_{k=1}^{K_j} \xi_{j,k}(s) ds \right]$$

$$\leq \mathbb{E}_t \left[ \tilde{C} C_p^2 \left\| \frac{\partial f}{\partial x} \right\|^2 \gamma^{1/4} \left( 1 + \int_0^T \frac{1}{K_j} \sum_{k=1}^{K_j} |\xi_{j,k}(s)|^2 ds \right) \right] \equiv \mathbb{E}_t \left[ a_{j,4}^\eta(\gamma) \right],$$

where the first inequality follows by applying the Burkholder-Davis-Gundy inequality with $\tilde{C}$ being a universal constant and the third inequality uses the technical estimate in (A.6)

$$\int_s^t \xi_{j,k}(u) du \leq \frac{1}{2} (t - s)^{1/4} \left( 1 + \int_0^T |\xi_{j,k}(u)|^2 du \right) \quad \text{for} \quad 0 \leq s \leq t \leq T,$$

which follows directly from inequality (6.1) in Giesecke et al. (2013), which is in turn an application of the Cauchy-Schwartz inequality combined with the geometric/harmonic mean inequality. Let

$$a_0^\eta(\gamma) \equiv \sum_{j=1}^N \left( a_{j,1}^\eta(\gamma) + a_{j,2}^\eta(\gamma) + a_{j,3}^\eta(\gamma) + a_{j,4}^\eta(\gamma) \right).$$

By Lemma B.1 in Appendix B, we conclude $\lim_{\gamma \to 0} \sup_\eta \mathbb{E}[a^\eta(\gamma)] = 0$. The proof is then complete by noting that $g^2(\langle \nu^0, f \rangle, t, \langle \nu^0, f \rangle \wedge t) \leq 1$, for $v \in [0, \gamma \wedge t]$. □

**Identification of the Limit.** We formulate and solve the martingale problem that pins down the limiting measure-valued process. We start by introducing the following operators on the space $C^2(O)$:

$$\mathcal{T}_0 f = \partial f/\partial x, \quad \mathcal{T}_1 f = x(\partial f/\partial x), \quad \mathcal{T}_2 f = \theta(\partial f/\partial x), \quad \mathcal{T}_v f = (\sigma^2/2) (\partial^2 f/\partial x^2).$$
Recall that \( \Phi \) is an element of the function class \( \mathcal{D} \) defined in (3.8). We identify the generator of the limiting process as the operator \( \mathcal{A} \) acting on \( \Phi(\cdot) \) defined by

\[
\mathcal{A}\Phi(\mu) \equiv \sum_{n=1}^{m} \sum_{j=1}^{N} \frac{\partial \Phi}{\partial x_{j,n}} \left[ \ell_{j}(\mu_{j}, T_{2}^{dr} f_{j,n}) - \ell_{j}(\mu_{j}, T_{1}^{dr} f_{j,n}) + \langle \mu_{j}, T^{v} f_{j,n} \rangle \right]
- \sum_{h \leq N} \ell_{h,j}(\mu_{j}, T_{0}^{dr} f_{j,n}) \langle \mu_{h}, \Theta - \psi_{1} \rangle
\quad \text{for} \quad \mu \equiv (\mu_{1}, \ldots, \mu_{N}) \in S.
\]  

(A.8)

**Proposition A.2** The operator \( \mathcal{A} \) is the generator of our limit martingale problem in the sense of

\[
\lim_{\eta \to \infty} \mathbb{E} \left[ \left( \Phi(\nu^{\eta}(t_{r+1})) - \Phi(\nu^{\eta}(t_{r})) - \int_{t_{r}}^{t_{r+1}} \mathcal{A}\Phi(\nu^{\eta}(u))du \right) \prod_{i=1}^{r} \Psi_{i}(\nu^{\eta}(t_{j})) \right] = 0,
\]

where \( 0 \leq t_{1} < \ldots < t_{r+1} < +\infty \) with \( r \in \mathbb{N} \), and \( \Psi_{i} \in \mathcal{B}(S) \) (the set of all bounded measurable functions on \( S \), \( i = 1, \ldots, r \)).

**Proof of Proposition A.2** The conclusion follows directly from Lemma B.2 and the fact that

\[
\lim_{\eta \to \infty} \mathbb{E} \left[ \int_{t}^{u} |\mathcal{E}^{\eta}(s)| ds \right] = 0 \quad \text{for} \quad 0 \leq t < u < +\infty,
\]

where \( \mathcal{E} \) is given in (B.9). \( \square \)

Next, we turn to the measure-valued functions \( \nu \equiv (\nu_{1}, \ldots, \nu_{N}) \) given by (3.16). Our next result shows that \( \nu \) solves the martingale problem for \( \mathcal{A} \).

**Proposition A.3** For the measure-valued functions \( \nu \equiv (\nu_{1}, \ldots, \nu_{N}) \) given in (3.16) and the operator \( \mathcal{A} \) specified by (A.8), it holds that

\[
\Phi(\nu(t)) = \Phi(\nu(s)) + \int_{s}^{t} \mathcal{A}\Phi(\nu(u))du \quad \text{for} \quad 0 \leq s < t < +\infty.
\]  

(A.9)

**Proof of Proposition A.3** First note that for \( f \in C^{2}(\mathcal{O}) \) we have

\[
\langle \nu_{j}, f \rangle_{t} = \int_{\mathcal{O}} \mathbb{E} \left[ f(p_{j}, X_{j}(z_{j}; t)) \right] \phi_{j}(dz_{j}) \quad \text{for} \quad t \geq 0.
\]  

(A.10)

where \( X_{j}(z_{j}; t) \) is defined by (3.9). In what follows, we simply write \( X_{j}(t) \equiv X_{j}(z_{j}; t) \) and \( f(X_{j}(t)) \equiv f(p_{j}, X_{j}(t)) \). An application of the Ito’s formula yields

\[
f(X_{j}(t)) = f(x_{j}) + \ell_{j} \int_{0}^{t} \frac{\partial f}{\partial x}(X_{j}(s))(\theta_{j} - X_{j}(s))ds - \sum_{h=1}^{N} \ell_{h,j} \int_{0}^{t} \frac{\partial f}{\partial x}(X_{j}(s)) \left( V_{h} - Q_{h}(s) \right) ds
+ \sigma_{j} \int_{0}^{t} \frac{\partial f}{\partial x}(X_{j}(s)) \sqrt{X_{j}(s)} dW_{j}(s) + \frac{\sigma_{j}^{2}}{2} \int_{0}^{t} \frac{\partial^{2} f}{\partial x^{2}}(X_{j}(s)) X_{j}(s) ds.
\]  

(A.11)
Taking expectation on both sides and then first-order derivative with respect to $t$ yields

$$
\frac{\partial}{\partial t} \mathbb{E}[f(X_j(t))] = \ell_j \mathbb{E}\left[T_2^{dr} f(X_j(t))\right] - \ell_j \mathbb{E}\left[T_1^{dr} f(X_j(t))\right] + \mathbb{E}[T^v f(X_j(t))]
$$

$$
- \sum_{h=1}^N \ell_{h,j} \mathbb{E}\left[T_0^{dr} f(X_j(t))\right] \left(V_h - Q_h(t)\right)
$$

$$
= \ell_j \mathbb{E}\left[T_2^{dr} f(X_j(t))\right] - \ell_j \mathbb{E}\left[T_1^{dr} f(X_j(t))\right] + \mathbb{E}[T^v f(X_j(t))]
$$

$$
- \sum_{h=1}^N \ell_{h,j} \mathbb{E}\left[T_0^{dr} f(X_j(t))\right] \left(\langle \nu_h, \Theta \rangle_t - \langle \nu_h, \psi_1 \rangle_t\right),
$$

where the first equality uses the operators given by (A.7) and the second equality follows from (3.10) and Lemma 3.1. Taking expectation of both sides with respect to the probability distribution $\phi_j$, we obtain

$$
\frac{\partial}{\partial t} \int_{\mathbb{O}} \mathbb{E}[f(X_j(t))] \phi_j(dz_j) = \int_{\mathbb{O}} \left(\ell_j \mathbb{E}\left[T_2^{dr} f(X_j(t))\right] - \ell_j \mathbb{E}\left[T_1^{dr} f(X_j(t))\right] + \mathbb{E}[T^v f(X_j(t))]\right) \phi_j(dz_j)
$$

$$
- \sum_{h=1}^N \ell_{h,j} \int_{\mathbb{O}} \mathbb{E}\left[T_0^{dr} f(X_j(t))\right] \left(V_h - Q_h(t)\right) \phi_j(dz_j)
$$

$$
= \int_{\mathbb{O}} \left(\ell_j \mathbb{E}\left[T_2^{dr} f(X_j(t))\right] - \ell_j \mathbb{E}\left[T_1^{dr} f(X_j(t))\right] + \mathbb{E}[T^v f(X_j(t))]\right) \phi_j(dz_j)
$$

$$
- \sum_{h=1}^N \ell_{h,j} \int_{\mathbb{O}} \mathbb{E}\left[T_0^{dr} f(X_j(t))\right] \left(\langle \nu_h, \Theta \rangle_t - \langle \nu_h, \psi_1 \rangle_t\right) \phi_j(dz_j).
$$

By Eq. (A.10) and the above equality, we deduce

$$
\frac{d\langle \nu_j, f \rangle_t}{dt} = \ell_j \langle \nu_j, T_2^{dr} f \rangle_t - \ell_j \langle \nu_j, T_1^{dr} f \rangle_t + \langle \nu_j, T^v f \rangle_t + \sum_{h=1}^N \ell_{h,j} \langle \nu_j, T_0^{dr} f \rangle_t \left(\langle \nu_h, \Theta \rangle_t - \langle \nu_h, \psi_1 \rangle_t\right). \tag{A.12}
$$

Recall the function $\Phi(\nu)$ defined by (3.8) and the operator $\mathcal{A}$ acting on $\Phi(\nu)$. Using the chain rule and (A.12), we obtain

$$
\frac{d\Phi(\nu(t))}{dt} = \sum_{n=1}^m \sum_{j=1}^N \frac{\partial \Phi}{\partial x_{j,n}} \left[\ell_j \langle \nu_j, T_2^{dr} f_{j,n} \rangle_t - \ell_j \langle \nu_j, T_1^{dr} f_{j,n} \rangle_t + \langle \nu_j, T^v f_{j,n} \rangle_t\right]
$$

$$
- \sum_{h=1}^N \ell_{h,j} \langle \nu_j, T_0^{dr} f_{j,n} \rangle_t \left(\langle \nu_h, \Theta \rangle_t - \langle \nu_h, \psi_1 \rangle_t\right) \equiv \mathcal{A}\Phi(\nu(t)),
$$

which can be rearranged to obtain (A.9).

\[\square\]

**Lemma A.2** The uniqueness of the martingale problem of the generator $\mathcal{A}$ given by (A.8) holds.

**Proof.** Our proof follows a duality argument; see e.g., §4.4 in Ethier and Kurtz (2009), p. 182-195. In particular, duality means that the existence of a solution to the dual problem ensures uniqueness of a solution to the original problem.

Let $C^* \equiv \bigcup_{m \in \mathbb{N}} C^\infty(O^{N \times m})$. We start by defining a flow on $C^*$, as in the proof of Lemma 7.1 in Giecke et al. (2013). Suppose $G \in C^*$. Then there must exist some $m \in \mathbb{N}$ such that $G \in C^\infty(O^{N \times m})$.  

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To begin, let \( z_{1:m} \equiv (z_1, \ldots, z_m) \in O^{N \times m} \), where for each \( n \in \{1, \ldots, m\} \), \( z_n \equiv (z_{1,n}, \ldots, z_{N,n})^\top \) with \( z_{j,n} \equiv (\theta_{j,n}, \sigma_{j,n}, x_{j,n}) \in O \), \( j = 1, \ldots, N \). Define the spatial diffusion semigroup:

\[
S_t : G(z_{1:m}) \to (S_t G)(z_{1:m}) \equiv (Nm)^{-1}E[G(Y(t))] \quad \text{for} \quad Y(t) \equiv (Y_1(t), \ldots, Y_m(t)),
\]

where for each \( n \in \{1, \ldots, m\} \), \( Y_n(t) \equiv (Y_{1,n}(t), \ldots, Y_{N,n}(t))^\top \) with \( Y_{j,n}(t) \equiv (\theta_{j,n}, \sigma_{j,n}, X_{j,n}(t)) \), \( j = 1, \ldots, N \); \( X_{j,n} \) is a diffusion process defined by

\[
X_{j,n}(t) = x_{j,n} + \ell_j \int_0^t (\theta_{j,n} - X_{j,n}(s)) \, ds + \sigma_{j,n} \int_0^t X_{j,n}(s) \, d\tilde{W}_{j,n}(s).
\]

\( \tilde{W}_{j,n}, j = 1, \ldots, N, n = 1, \ldots, m \), are Brownian motions independent of each other, and of those in (3.9). Note that each \( Y_{j,n} \) (as well as \( X_{j,n} \)) depends on \( z_{1:m} \equiv (\theta_{j,n}, \sigma_{j,n}, x_{j,n}) \). Hence for \( t \geq 0 \), \( S_t G \) is indeed a function of \( z_{1:m} \). Let us also define the operator \( J_{j,n} \) acting on \( G \in C^* \) as a function of its \((j, n)\)-th argument:

\[
J_{j,n} : G(z_{1:m}) \to (J_{j,n} G)(z_{1:m+1}) \equiv \left( \sum_{h=1}^N \ell_{h,j}(x_{h,m+1} - \theta_{h,m+1}) \right) \frac{\partial G(z_{1:m})}{\partial x_{j,n}}.
\]

(A.13)

Let us introduce a \( C^* \)-valued Markov jump process \( \chi \) defined through the following procedure:

(i) \( \chi \) jumps from \( C^\infty(O^{N \times m}) \) to \( C^\infty(O^{N \times (m+1)}) \) at a rate \( 1/(Nm) \); at the time of the jump let \( \chi(t) \) be \( J_{j,n} \chi(t-), \) where the operator \( J_{j,n} \) is specified by (A.13).

(ii) between jumps, \( \chi(\cdot) \) evolves deterministically on \( C^\infty(O^{N \times m}) \) according to the transformation semigroup \( S \) with infinitesimal generator given by

\[
\mathcal{G} \equiv (Nm)^{-1} \sum_{n=1}^m \sum_{j=1}^N \left( \ell_j \mathcal{T}_{2,j,n}^{dr} - \ell_j \mathcal{T}_{1,j,n}^{dr} + \mathcal{T}_{j,n}^v \right).
\]

(A.14)

In particular, using \( \{\Lambda_{j,n}; j = 1, \ldots, N, n = 1, \ldots, m\} \) to denote \((Nm)\) independent Poisson processes, each with rate \( 1/(Nm) \), \( \chi \) can be expressed as the strong solution to the stochastic differential equation

\[
d\chi(t) = d(\chi_c(t)) + \sum_{n=1}^m \sum_{j=1}^N [J_{j,n} \chi(t-) - \chi(t-)] \, d\Lambda_{j,n}(t) \quad \text{for} \quad t \geq 0,
\]

(A.15)

where we have used the subscript \( c \) to denote the continuous part. To proceed, define

\[
\Gamma(\mu, G) \equiv \int_{O^{N \times m}} G(z_{1:m}) \prod_{n=1}^m \prod_{j=1}^N \mu_j(dz_{j,n})
\]

for \( G \in C^\infty(O^{N \times m}) \) and \( \mu \equiv (\mu_1, \ldots, \mu_N) \in S \). It is easily checked that

\[
\Gamma(\mu, G) = (Nm)^{-1} \sum_{n=1}^m \sum_{j=1}^N \langle \mu_j, G_{j,n}(z_{j,n}) \rangle
\]

(A.16)

where

\[
G_{(j,n)}(z_{j,n}) \equiv \int G(z_{1:m}) \prod_{(i,j) \neq (j,n)} \mu_i(dz_{i,i})
\]

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is obtained by integrating out all except the \((j, n)\)-th coordinate of \(z_{1:m}\). Now consider an \(S\)-valued process \(\nu\) solving the martingale problem for \(\mathcal{A}\). Using (A.16) and (A.8) (with \(\phi\) and \(f_{j,n}\) there being \(\phi([a_{j,n}]_{N \times m}) = (Nm)^{-1} \sum_{n=1}^{m} \sum_{j=1}^{N} a_{j,n}\) and \(G_{(j,n)}\) respectively), we get

\[
\Gamma(\nu(t), G) = \int_0^t \mathcal{A}\Gamma(\nu(s), G) \, ds + \mathcal{N}_1(t),
\]

where \(\mathcal{N}_1\) is a martingale and the operator \(\mathcal{A}\) is specified as follows:

\[
\mathcal{A}\Gamma(\mu, G) \equiv (Nm)^{-1} \sum_{n=1}^{m} \sum_{j=1}^{N} \int_{\mathcal{O}_{N \times m}} (\ell_j \mathcal{T}_{2,(j,n)} - \ell_j \mathcal{T}_{1,(j,n)} + \mathcal{T}_{(j,n)}^{\nu}) G(z_{1:m}) \prod_{n=1}^{m} \prod_{j=1}^{N} \mu_j \, (dz_{j,n})
\]

\[
- (Nm)^{-1} \sum_{n=1}^{m} \sum_{j=1}^{N} \int_{\mathcal{O}_{N \times m}} \ell_{h,j} [\langle \mu_h, \Theta \rangle - \langle \mu_h, \psi_1 \rangle] \mathcal{T}_{0,(j,n)}^{\nu} G(z_{1:m}) \prod_{n=1}^{m} \prod_{j=1}^{N} \mu_j \, (dz_{j,n}),
\]

where for an operator \(\mathcal{T}\), the notation \(\mathcal{T}_{(j,n)}\) means that \(\mathcal{T}\) operates on the \((j, n)\)-th argument of the function \(G\). By (A.13) and (A.14), we can rewrite (A.18) as

\[
\mathcal{A}\Gamma(\mu, G) = \Gamma(\mu, \mathcal{G}G) + \sum_{n=1}^{m} \sum_{j=1}^{N} (Nm)^{-1} [\Gamma(\mu, \mathcal{J}_{j,n}G) - \Gamma(\mu, G)] + \Gamma(\mu, G).
\]

On the other hand, from our construction of the Markov jump process \(\chi\), i.e., (A.15), it follows

\[
\Gamma(\mu, \chi(t)) = \int_0^t \mathcal{A}\#\Gamma(\mu, \chi(s)) \, ds + \mathcal{N}_2(t),
\]

where \(\mathcal{N}_2\) is a martingale and

\[
\mathcal{A}\#\Gamma(\mu, G) \equiv \Gamma(\mu, \mathcal{G}G) + \sum_{n=1}^{m} \sum_{j=1}^{N} (Nm)^{-1} [\Gamma(\mu, \mathcal{J}_{j,n}G) - \Gamma(\mu, G)].
\]

Combining (A.19) and (A.21) yields

\[
\mathcal{A}\Gamma(\mu, G) = \mathcal{A}\#\Gamma(\mu, G) + \Gamma(\mu, G),
\]

which completes the proof.

By the standard analysis of weak convergence (see §3.7, §3.8, §3.9 in Ethier and Kurtz (2009), p. 127-146), existence of a weak limit for the sequence of measure-valued processes \(\nu^N(\cdot)\) is guaranteed by Lemma A.1 and Proposition A.1; convergence to a solution of the martingale problem follows from Propositions A.2 and A.3; uniqueness of the martingale problem is ensured by Lemma A.2. This concludes the proof of Theorem 3.1.

\[\square\]

B Proofs of other technical results

Proof of Proposition 3.1 Non-negativity and symmetry are immediate from the definition. To show subadditivity, pick arbitrarily \(\mu, \mu', \mu''\) from \(S\). By the triangular inequality,

\[
\sum_{j=1}^{N} \left| \int f \, d(\mu_j - \mu_j'') \right| \leq \sum_{j=1}^{N} \left| \int f \, d(\mu_j - \mu_j') \right| + \sum_{j=1}^{N} \left| \int f \, d(\mu_j' - \mu_j'') \right|.
\]
To show indiscernibility, suppose $\beta(\mu, \mu') = 0$. From the definition (3.7), it follows that for each $j$

$$\sup \left\{ \left| \int f \, \mathrm{d}(\mu_j - \mu'_j) \right| : \|f\|_{BL} \leq 1 \right\} = 0.$$ 

The above expression simply means that $\beta(\mu_j, \mu'_j) = 0$, which in turn implies $\rho(\mu_j, \mu'_j) = 0$, where we recall that $\rho$ denotes the Prokhorov metric. Thus $\mu_j = \mu'_j$ in a sense that $\mu_j(A) = \mu'_j(A)$ for all Borel set in $\mathcal{O}$. Note that the equality holds for all $j \leq N$. We therefore conclude that $\mu = \mu'$.

\[ \square\]

\textbf{Proof of Proposition 3.3} Denote by $\mathbb{D}_0$ the class of functions of the form

$$\Phi(\mu) = \phi \left( \langle \mu, f_1 \rangle, \ldots, \langle \mu, f_m \rangle \right),$$

where $\langle \mu, f_n \rangle \equiv (\langle \mu_1, f_n \rangle, \ldots, \langle \mu_N, f_n \rangle)^\top$, $n = 1, \ldots, m$, for some $m \in \mathbb{Z}_+$, $f_n \in C^\infty(\mathcal{O})$ and $\phi \in C^\infty(\mathbb{R}^{N \times m})$. Clearly, $\mathbb{D}_0$ is a subset of the function class $\mathbb{D}$ given by (3.8). Thus, in order to prove the proposition, it suffices to show that the function class $\mathbb{D}_0$ has the desired properties. To this purpose, pick arbitrarily $\mu, \mu'$ from the product space $S$ and suppose $\mu \neq \mu'$. Then we must have $\beta(\mu, \mu') = \epsilon$ for some $\epsilon > 0$. Setting $\bar{\epsilon} \equiv \epsilon/N$, take a compact set $K \subset \mathcal{O}$ such that $\mu_j(K) > 1 - \bar{\epsilon}/8$ and $\mu'_j(K) > 1 - \bar{\epsilon}/8$ for all $j = 1, \ldots, N$. By the Arzela-Ascoli theorem, the set of functions $\mathbb{B} \equiv \{ f : \|f\|_{BL} \leq 1 \}$ restricted to $K$ forms a compact set of functions for the norm $\| \cdot \|_\infty$. Thus there exist $f_1, f_2, \ldots, f_m \in \mathbb{B}$ for some $m \in \mathbb{N}$ such that for any $f \in \mathbb{B}$, there exists $f_i$ satisfying $\sup_{x \in K} |f(x) - f_i(x)| < \bar{\epsilon}/8$. By the triangular inequality,

$$|\langle \mu_j, f \rangle - \langle \mu'_j, f \rangle| \leq |\langle \mu_j, f \rangle - \langle \mu_j, f_i \rangle| + |\langle \mu'_j, f_i \rangle - \langle \mu_j, f_i \rangle| + |\langle \mu_j, f_i \rangle - \langle \mu'_j, f_i \rangle|$$

for $j = 1, \ldots, N$. The first term on the right hand side

$$|\langle \mu_j, f \rangle - \langle \mu_j, f_i \rangle| \leq |\langle \mu_j, (f - f_i)1_{K^c} \rangle| + |\langle \mu_j, (f - f_i)1_K \rangle| < \bar{\epsilon}/4 + \bar{\epsilon}/8 = 3\bar{\epsilon}/8,$$

where the second inequality uses the fact that $\|f - f_i\| \leq 2$ and that $\mu_j(K^c) \leq \bar{\epsilon}/8$. Similarly for the second term we have $|\langle \mu'_j, f \rangle - \langle \mu'_j, f_i \rangle| < 3\bar{\epsilon}/8$. Combining the above yields

$$|\langle \mu_j, f \rangle - \langle \mu'_j, f \rangle| < 3\bar{\epsilon}/4 + \max_{i \leq m} |\langle \mu_j, f_i \rangle - \langle \mu'_j, f_i \rangle|.$$  \hspace{1cm} (B.1)

Adding up (B.1) over $j = 1, \ldots, N$, we get

$$\sum_{j=1}^N |\langle \mu_j, f \rangle - \langle \mu'_j, f \rangle| < 3\epsilon/4 + \sum_{j=1}^N \max_{i \leq m} |\langle \mu_j, f_i \rangle - \langle \mu'_j, f_i \rangle|. $$

Because $f$ is arbitrarily chosen from $\mathbb{B}$ and $\beta(\mu, \mu') = \epsilon$, the right-hand side of (B.1) must be greater or equal to $\epsilon$. Hence, there exist $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, N\}$ such that $|\langle \mu_j, f_i \rangle - \langle \mu'_j, f_i \rangle| \geq \epsilon/(4N) \equiv \bar{\epsilon}/4$.

On the other hand, any globally Lipschitz continuous function can be approximated by a sequence of infinitely differentiable functions. We can therefore find $\tilde{f} \in C^\infty(\mathcal{O})$ satisfying $|\langle \mu_j, \tilde{f} \rangle - \langle \mu'_j, \tilde{f} \rangle| \geq \bar{\epsilon}/8$. The first part of the conclusion follows immediately from the the fact that $C^\infty(\mathbb{R}^{N \times m})$ separates points in $\mathbb{R}^{N \times m}$. Since $\mathbb{D}_0$ forms a subalgebra, an application of Stone-Weierstrass Theorem gives the second part of the conclusion.

\[ \square\]
Proof of Lemma 3.1 From the definition (3.12), it follows immediately that

\[ \langle \nu_j, \psi_1 \rangle_t \equiv \int_0^t \mathbb{E} [X_j(z_j; t)] \phi_j (dz_j) \quad \text{for} \quad t \geq 0, \]  

(B.2)

where we recall that the underlying state process \( X_j(z_j; t) \) follows the SDE given in (3.9). It follows from the same equation (3.9) that the expectation \( \mathbb{E}[X_j(z_j; t)] \) satisfies the integral equation below:

\[ \mathbb{E} [X_j(z_j; t)] = x_j + \int_0^t \ell_j (\theta_j - \mathbb{E} [X_j(z_j; s)]) ds - \int_0^t \sum_{h=1}^N \ell_{h,j} (V_h - Q_h(s)) ds \quad \text{for} \quad t \geq 0. \]  

(B.3)

The integral equation (B.3) admits an explicit solution:

\[ \mathbb{E} [X_j(z_j; t)] = e^{-\ell_j t} \left[ x_j + \int_0^t \left( \ell_j \theta_j - \sum_{h=1}^N \ell_{h,j} (V_h - Q_h(s)) \right) e^{\ell_j s} ds \right]. \]  

(B.4)

Combining (B.2) and (B.4) completes the proof of the lemma. \( \square \)

Proof of Proposition 5.1 Our objective is to find functions \( \alpha(\cdot, u) \) and \( \beta(\cdot, u) \) with \( \alpha(0, u) = 0 \) and \( \beta(0, u) = u \) such that

\[ \mathcal{M}(s) \equiv g(s, X(s)) \equiv \exp [\alpha(t - s, u) + \beta(t - s, u) X(s)], \quad \text{for} \quad 0 \leq s \leq t, \]

is martingale. If this is the case, then

\[ \mathbb{E} [e^{uX(t)}] = \mathbb{E} [\mathcal{M}(t)] = \mathcal{M}(0) = \exp [\alpha(t, u) + \beta(t, u) X(0)]. \]

An application of the Ito’s formula to \( g(s, X(s)) \) yields

\[ \frac{d g(s, X(s))}{g(s, X(s))} = - \left[ \alpha'(t - s, u) + X(s) \beta'(t - s, u) \right] ds + \beta(t - s, u) \ell(\theta - X(s)) ds \]

\[ + q(s) \beta(t - s, u) ds + \beta(t - s, u) X(s) dW(s) + \frac{1}{2} \sigma^2 X(s) \beta(t - s, u)^2 ds, \]

where we have used \( \alpha' \) and \( \beta' \) to denote, respectively, the first-order derivatives of \( \alpha \) and \( \beta \) with respect to the first argument. Then \( g(s, X(s)) \) is a martingale if the drift term vanishes, i.e., if

\[ \alpha'(t - s, u) + X(s) \beta'(t - s, u) = \beta(t - s, u) \ell(\theta - X(s)) + q(s) \beta(t - s, u) + \frac{1}{2} \sigma^2 X(s) \beta(t - s, u)^2 \]

for all possible states of \( X(s) \). By setting the drift term to zero, we obtain immediately the following system of non-linear differential equations

\[ \alpha'(s, u) = \ell \theta \beta(s, u) + q(t - s) \beta(s, u) \quad \text{and} \]

\[ \beta'(s, u) = - \ell \theta \beta(s, u) + \frac{\sigma^2}{2} \beta(s, u)^2 \]

with initial conditions \( \alpha(0, u) = 0 \) and \( \beta(0, u) = u \). The set of differential equations admits the solution specified in (5.2). \( \square \)

Lemma B.1 If Assumption 3 is satisfied, then for any finite \( T > 0 \) and \( n \in \mathbb{N} \), we have

\[ \sup_{\eta} \sup_{0 \leq t \leq T} \sum_{j=1}^N \frac{1}{K^n_{j}} \sum_{k=1}^{K^n_{j}} \mathbb{E} [\xi_{j,k}(t)^n] < +\infty. \]  

(B.5)
Proof of Lemma B.1  Recall from Eq. (2.2) that

\[ \xi_{j,k}(t) = \xi_{j,k}(0) + \int_0^t \ell_j [\theta_{j,k} - \xi_{j,k}(s)] \, ds - \int_0^t \sum_{h=1}^N \ell_{h,j} [\theta_h - \bar{\xi}_h(s)] \, ds + \sigma_{j,k} \int_0^t \sqrt{\xi_{j,k}(s)} \, dW_{j,k}(s). \]

An application of the Ito’s formula to the smooth function \( f(\xi_{j,k}(t)) \), where \( f(x) = x^n \), gives

\[
\begin{align*}
(\xi_{j,k}(t))^n &= (\xi_{j,k}(0))^n + n \int_0^t \ell_j (\xi_{j,k}(s))^{n-1} [\theta_{j,k} - \xi_{j,k}(s)] \, ds + \frac{n(n-1)}{2} \int_0^t \sigma_{j,k}^2 (\xi_{j,k}(t))^{n-1} \, ds \\
&\quad - n \int_0^t \sum_{h=1}^N \ell_{h,j} (\xi_{j,k}(t))^{n-1} [\theta_h - \bar{\xi}_h(s)] \, ds + n \int_0^t \sigma_{j,k} (\xi_{j,k}(t))^{n-1/2} \, dW_{j,k}(s) \\
&\quad \leq (\xi_{j,k}(0))^n + n \int_0^t \ell_j \theta_{j,k} (\xi_{j,k}(s))^{n-1} \, ds + \frac{n(n-1)}{2} \int_0^t \sigma_{j,k}^2 (\xi_{j,k}(t))^{n-1} \, ds \\
&\quad + n \int_0^t \sum_{h=1}^N \ell_{h,j} (\xi_{j,k}(t))^{n-1} \bar{\xi}_h(s) \, ds + n \int_0^t \sigma_{j,k} (\xi_{j,k}(t))^{n-1/2} \, dW_{j,k}(s).
\end{align*}
\]

(B.6)

where the inequality follows from the positivity of those \( \xi_{j,k} \). An application of Young’s inequality yields

\[
(\xi_{j,k}(s))^{n-1} \leq (n-1) (\xi_{j,k}(s))^n / n + 1/n
\]

and

\[
(\xi_{j,k}(s))^{n-1} \xi_{h,k'}(s) \leq (n-1) (\xi_{j,k}(s))^n / n + (\xi_{h,k'}(s))^n / n.
\]

It then follows that

\[
(\xi_{j,k}(s))^{n-1} \bar{\xi}_h(s) \leq (n-1) (\xi_{j,k}(s))^n / n + \frac{1}{K_{j,k}^n} \sum_{k'=1}^K \xi_{h,k'}(s)^n / n.
\]

Combining the preceding inequalities with (B.6), we have

\[
\mathbb{E} [(\xi_{j,k}(t))^n] \leq \mathbb{E} [(\xi_{j,k}(0))^n] + C_n \int_0^t \mathbb{E} [(\xi_{j,k}(s))^n] \, ds + C_n \int_0^t \sum_{j=1}^N \frac{1}{K_{j,k}^n} \sum_{k=1}^K \mathbb{E} [(\xi_{j,k}(s))^n] \, ds + C_n t, \quad (B.7)
\]

where \( C_n \) is a constant that only depends on \( C_p \) (given in Assumption 3) and \( n \). We can therefore conclude that

\[
\sum_{j=1}^N \frac{1}{K_{j,k}^n} \sum_{k=1}^K \mathbb{E} [(\xi_{j,k}(t))^n] \leq \sum_{j=1}^N \frac{1}{K_{j,k}^n} \sum_{k=1}^K \mathbb{E} [(\xi_{j,k}(0))^n] + C_n \int_0^t \sum_{j=1}^N \frac{1}{K_{j,k}^n} \sum_{k=1}^K \mathbb{E} [(\xi_{j,k}(s))^n] \, ds \\
+ NC_n \int_0^t \sum_{j=1}^N \frac{1}{K_{j,k}^n} \sum_{k=1}^K \mathbb{E} [(\xi_{j,k}(s))^n] \, ds + NC_n t.
\]

An application of the Gronwall’s inequality then gives

\[
\sum_{j=1}^N \frac{1}{K_{j,k}^n} \sum_{k=1}^K \mathbb{E} [(\xi_{j,k}(t))^n] \leq C(n,T,C_p) e^{C(n,T,C_p)T} \quad \text{for all} \quad 0 \leq t \leq T.
\]

The result follows from the fact that the bound on the right hand side of the above inequality is independent of \( t \) and \( \eta \). \( \square \)
Lemma B.2  Let $0 \leq t < u < +\infty$. It holds that,
\[ \Phi(\nu^n(u)) = \Phi(\nu^n(t)) + \int_t^u D^n(s)\,ds + \int_t^u E^n(s)\,ds + \widetilde{\mathcal{M}}(u) - \widetilde{\mathcal{M}}(t), \]
where $(\widetilde{\mathcal{M}}_t; \ t \geq 0)$ is a $(\mathbb{P}, \mathbb{F})$-(local) martingale, and
\[
D^n(t) \equiv \sum_{n=1}^m \sum_{j=1}^N \frac{\partial \phi}{\partial x_{j,n}} \left[ \ell_j(\nu^n_j, T^{dr}_2 f_{j,n})_t - \ell_j(\nu^n_j, T^{dr}_1 f_{j,n})_t + (\nu^n_j, T^{v} f_{j,n})_t, \right.
\]
\[- \sum_{h \leq N} \ell_h,j(\nu^n_j, T^{dr}_0 f_{j,n})_t \left( (\nu^n_h, \Theta)_t - (\nu^n_h, \psi_1)_t \right) \right],
\]
\[
E^n(t) \equiv \sum_{l,n=1}^m \sum_{j=1}^N \frac{\partial^2 \phi}{\partial x_{j,l} \partial x_{j,n}} \cdot \frac{1}{2(K^\eta_j)^2} \sum_{k=1}^{K^\eta_j} \left[ \sigma_{j,k}^2 \frac{\partial f_{j,l}(\xi_{j,k}(t))}{\partial x} \frac{\partial f_{j,n}(\xi_{j,k}(t))}{\partial x} \xi_{j,k}(t) \right].
\]

Proof of Lemma B.2  First applying Itô’s formula to $f(\xi_{j,k}(t))$ and then taking the average over $k \in \{1, \ldots, K_j\}$ yields
\[
(\nu^n_j, f)_t = (\nu^n_j, f)_0 + \ell_j \int_0^t (\nu^n_j, T^{dr}_2 f)_s\,ds - \ell_j \int_0^t (\nu^n_j, T^{dr}_1 f)_s\,ds + \frac{1}{K^\eta_j} \sum_{k=1}^{K^\eta_j} \mathcal{M}_{j,k}(t)
\]
\[ \quad - \sum_{h=1}^N \ell_{h,j} \int_0^t (\nu^n_j, T^{dr}_0 f)_s \left[ (\nu^n_h, \Theta)_s - (\nu^n_h, \psi_1)_s \right] \,ds + \int_0^t (\nu^n_j, T^{v} f)_s\,ds,
\]
where the operators $T^{dr}_i, i = 0, 1, 2$ and $T^{v}$ are given by (A.7), and the $(\mathbb{P}, \mathbb{F})$-(local) martingale is given by
\[
\mathcal{M}_{j,k}(t) \equiv \int_0^t \sigma_{j,k} \frac{\partial f(\xi_{j,k}(s))}{\partial x}(\xi_{j,k}(s))^{1/2}\,dW_{j,k}(s).
\]
Replacing $\mu$ with $\nu^n(t)$ in (3.8) and applying Itô’s formula yields (B.8). \hfill \square

References


