Transitional Dynamics in Two-Sector Models of Endogenous Growth

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TRANSITIONAL DYNAMICS IN TWO-SECTOR MODELS OF ENDOGENOUS GROWTH*

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We analyze the steady state and transitional dynamics of two-sector models of endogenous growth. The necessary conditions for endogenous growth imply that transitions depend only on a measure of the imbalance between the two sectors such as the ratio of the two capital stocks. We use the Time-Elimination method to analyze the transitional dynamics. Three main economic forces drive the transition: a Solow effect, a consumption smoothing effect, and a relative wage effect. For plausible parameterizations the consumption smoothing effect tends to dominate the relative wage effect; transition from relatively low levels of physical capital is accomplished through higher work effort rather than higher savings.

I. INTRODUCTION

The transitional dynamics of two-sector models of endogenous growth are not well understood. Following the work of Lucas [1988], much of the recent endogenous growth literature deals with economies with two capital goods. One of the goods is usually physical capital. The other one varies across models: human capital, embodied and disembodied knowledge, public capital, quality of products, number of varieties of products, and financial capital are some examples of stock variables that are accumulated through some investment process. The analysis in all these papers is generally restricted to the steady state; it is always assumed that all the variables in the economy grow at their long-run growth rate.

If there are initial imbalances among the different sectors, however, there may be a transitional period where the relevant variables do not behave as predicted by the steady-state analysis. For instance, the initial ratio of capital stocks may not be the same as the steady-state one because of some recent unusual event such as a war or a large price shock. If, starting from a steady-state position, a war destroys a large fraction of the physical capital stock leaving human capital relatively unaffected, the economy will

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somehow have to get back to the steady-state proportions by having larger (smaller) than steady-state growth rates for the physical (human) capital stock. It is natural to ask how, if at all, this happens. Due to its analytical difficulty, however, these transitional dynamics are always left unexplained. This paper tries to fill this gap in the literature by studying them in detail. Even though we shall be calling the two capital stocks "physical" and "human," our analysis applies to any of the two-sector models mentioned above.

There are several important and interesting reasons to study transitions. First, there is the question of whether there actually are transitional dynamics in real time and, if there are, what they look like. Second, the empirical implications of the transitional paths may be different from the steady-state implications. The exact predictions are needed if the models are to be tested with actual data. Third, the transitional dynamics allow us to understand the behavior of the model in the short run. If two-sector models have rich short-run dynamics as well as positive and endogenous steady-state growth rates, they can be used as integrated theories of business cycles and growth.

The rest of the paper is organized as follows. In Section II we present a general two-sector growth model with two capital goods. The investment in one of the two capital stocks (physical capital) is a perfect substitute for consumption, while the other (human capital) is not. In Section III we characterize the solution. The next section derives necessary conditions for the model to generate positive steady-state growth rates (i.e., endogenous growth). Section V summarizes the methodology used to analyze the transitional dynamics (Mulligan [1991] discusses the Time-Elimination method in more detail). Sections VI and VII examine the transition of the Uzawa [1965] and Lucas [1988] model as well as general two-sector models, respectively. The final section concludes, and a short Appendix provides first-order conditions.

Throughout the paper we find a number of interesting results. We highlight them as we go along.

II. A Model of Human Capital and Growth

IIa. The Setup

We start by describing a general model of human capital and growth. The model is general in that, for now, we do not restrict the parameters. It is not general, however, in that we make specific assumptions about functional forms.

We assume that agents maximize a utility function of the form,

$$\int_0^\infty e^{-\rho t}\left(\frac{c(t)^{1-\theta}-1}{1-\theta}\right)dt, \tag{1}$$

where $c(t)$ is per capita consumption at time t, ρ is the subjective rate of time preference, which includes population growth,[1] and $1/\theta$ is the coefficient that measures the (constant) intertemporal elasticity of substitution. The only consumption good is measured in units of final output. Final output is produced with two capital goods that we call physical and human. Physical capital is assumed to be forgone consumption. Human capital is produced in an alternative sector (which we call the education or learning sector). Households choose a consumption path, and the amount of human and physical capital they use in each sector so as to maximize (1) subject to some accumulation constraints:

$$\dot{k}(t) = f(k_f(t),h_f(t),\hat{k}(t),\hat{h}(t)) - \delta_k k(t) - c(t) \tag{2}$$

$$\dot{k}(t) = e(k_e(t),h_e(t),\hat{h}(t),\hat{k}(t)) - \delta_h h(t), \tag{3}$$

where $k(0) > 0$ and $h(0) > 0$ are given, $\dot{k}()$ and $\dot{h}()$ are the net accumulations of physical and human capital, respectively, and $f()$ and $e()$ are the (flow) productions of final output (f stands for final) and human capital (e stands for education). These two production functions are assumed to exhibit constant returns to the sector-specific capital stocks: h_f and k_f are the effective amounts of human and physical capital employed in the final output sector, and h_e and k_e are the corresponding variables for the education sector.[2] We also allow for the possibility of externalities from the average[3] stocks of capital ($\hat{k}$ and $\hat{h}$) in both sectors. The reason for these externalities is that we want to allow for the production function to

1. That is, $\rho = \rho^* - n$, where n is the exogenous rate of population growth and ρ^* is the pure rate of time preference.

2. Note that we are not allowing for nonreproducible inputs such as raw labor or land. The reason is that we shall end up constraining our analysis to models that display endogenous growth, which limits the role for such inputs. Hence, in order to simplify notation, we decided to neglect them altogether.

3. One could also assume that the externalities apply to the total (not average) stock of physical or human capital. This alternative specification would generate scale effects, for which there is little empirical evidence. Externalities from the aggregate level of investment in one of the two capital goods could also be introduced. Chamley [1991] shows that, at least in the Lucas [1988] specification, they yield the same results as externalities from the stocks.

exhibit increasing (or decreasing) returns to scale, yet we want to have a competitive solution. The modeling of increasing returns through externalities is, after Romer [1986] and Lucas [1988], common practice among endogenous growth theorists. These externalities could be positive, negative, or zero; we have no presumption over their sign.

Note that the key asymmetry between the two capital goods is that the accumulation of one of them, $k(t)$, is a perfect substitute for consumption (that is, consumption subtracts from $\dot{k}(t)$ and not from $\dot{h}(t)$). Hence, even though we are calling k physical and h human capital, technically speaking, the key distinction between capital goods is whether their accumulation is a perfect substitute for consumption or not. Some of the early two-sector neoclassical models, such as Srinivasan [1964], Ryder [1969], Kurz [1968], or Burmeister [1980, Chapter 6], assume that the production processes for consumption and capital are essentially different. Their models are slightly more complicated because they involve an additional control variable and an additional relative price. Our simpler specification, however, seems a good place for us to start.[4]

The depreciation rates δ_k and δ_h are assumed to be constant over time and may include population growth rates (because the model is expressed in per capita terms).

We assume that the two production functions are Cobb-Douglas with constant returns to the private inputs:

$$f(.) = A\,(h_f(t)^{\alpha_h}k_f(t)^{\alpha_k})\,(\hat{h}(t)^{\alpha_{\hat{h}}}\,\hat{k}(t)^{\alpha_{\hat{k}}}) \tag{4}$$

$$e(.) = \phi\,(h_e(t)^{\beta_h}k_e(t)^{\beta_k})\,(\hat{h}(t)^{\beta_{\hat{h}}}\,\hat{k}(t)^{\beta_{\hat{k}}}) \tag{5}$$

with

$$\alpha_k + \alpha_h = 1, \qquad \beta_k + \beta_h = 1.$$

The parameters A and ϕ in (4) and (5) are the levels of technology in each sector. The terms inside the first parentheses are the private capital stocks in each sector, with α_h and α_k being the private shares of human and physical capital in the output sector, and β_h and β_k being the corresponding shares in the education

4. Another key difference with the early literature is that we shall restrict ourselves to parameterizations that can generate endogenous growth. As we will argue in Section IV, this simplifies our problem even further.

Within the endogenous growth literature, Rebelo [1991] uses a two-sector model similar to the one we propose here, but he confines his analysis to the steady state. Jones and Manuelli [1990] show conditions for endogenous growth in models with N capital goods, where all of them are produced with a single technology, which is also used to produce output. That is, $F(K_1,K_2, \ldots, K_N) = K_1 + K_2 + \cdots + K_N + C$. Again, the transition is left unexplained.

sector. The equalities $\alpha_k + \alpha_h = 1$ and $\beta_k + \beta_h = 1$ ensure that there are constant returns to scale at the private level. At the social level, however, there may be increasing, constant, or decreasing returns depending on the signs of the externality parameters, $\alpha_{\hat{k}}$, $\alpha_{\hat{h}}$, $\beta_{\hat{k}}$, $\beta_{\hat{h}}$.

IIb. Point-in-Time Technologies and Point-in-Time Returns

At every point in time, the economywide stocks of capital, $k(t)$ and $h(t)$, are given. Agents can generate a sector-specific capital, $k_f(t)$, $k_e(t)$, $h_f(t)$, and $h_e(t)$, combining the aggregate stocks and effort[5] with what we call point-in-time technologies. Both humans and machines are endowed with one unit of effort, which can be allocated across the two sectors. Thus, if we define $u(t)$ as the human capital effort in the final output sector, and $v(t)$ as the physical capital effort in the final output sector, then the corresponding efforts in the learning sector are $1 - u(t)$ and $1 - v(t)$, respectively. The point-in-time technologies for the sector-specific capital stocks are

$$
\begin{aligned}
h_f(t) &= u(t)^{\alpha_u/\alpha_h} h(t) \\
k_f(t) &= v(t)^{\alpha_v/\alpha_k} k(t) \\
h_e(t) &= (1 - u(t))^{\beta_u/\beta_h} h(t) \\
k_e(t) &= (1 - v(t))^{\beta_v/\beta_k} k(t).
\end{aligned} \tag{6}
$$

If the exponent on an effort variable is less than one, we say that there are decreasing point-in-time returns in the production of that capital stock. If the exponent in an effort variable is exactly equal to one, we say that there are constant point-in-time returns. In the case of constant point-in-time returns in *all sectors*, we can think of $u(t)$ and $v(t)$ as being the fraction of aggregate human and physical capital used in the final output sector at instant t (and, conversely, $(1 - u(t))$ and $(1 - v(t))$ are the fractions used in the education sector). With the sector-specific capital production functions above, the capital accumulation equations become

$$\dot{k}(t) = A(u(t)^{\alpha_u} h(t)^{\alpha_h})(v(t)^{\alpha_v} k(t)^{\alpha_k}) \hat{h}(t)^{\alpha_{\hat{h}}} \hat{k}(t)^{\alpha_{\hat{k}}} - \delta_k k(t) - c(t) \tag{2'}$$

$$\dot{h}(t) = \phi((1 - u(t))^{\beta_u} h(t)^{\beta_h})((1 - v(t))^{\beta_v} k(t)^{\beta_k}) \times \hat{h}(t)^{\beta_{\hat{h}}} \hat{k}(t)^{\beta_{\hat{k}}} - \delta_h h(t). \tag{3'}$$

It is worth noticing at this point that our model includes, as a particular case, that of Uzawa [1965] and Lucas [1988] where the

5. We use the word "effort" rather than "time" because time could be confused with the variable t in our model.

production function of human capital is assumed to use only human capital. This assumption implies that all the physical capital is employed in the output sector, and therefore, the share v is trivially set to one at all points in time. Furthermore, Uzawa and Lucas assume that the production of human capital exhibits constant returns to human capital. That is, our model becomes the Uzawa-Lucas model when the following restrictions are imposed on the parameters: $\alpha_v = \alpha_{\hat{k}} = 0$, $\beta_{\hat{k}} = \beta_{\hat{h}} = \beta_v = \beta_k = 0$, $\alpha_u = \alpha_h = 1 - \alpha_k$, and $\beta_u = \beta_h = 1$. Note that this specification implies constant point-in-time returns in both sectors:

$$\dot{k} = Ak^{\alpha_k}(uh)^{(1-\alpha_k)}\hat{h}^{\alpha_{\hat{h}}} - \delta_k k - c \tag{2''}$$

$$\dot{h} = \phi(1 - u)h - \delta_h h. \tag{3''}$$

III. FIRST-ORDER CONDITIONS, STATE-LIKE, AND CONTROL-LIKE VARIABLES

Agents choose the paths for $c(t)$, $u(t)$, $v(t)$, $k(t)$, and $h(t)$ so as to maximize utility in (1) subject to (2′) and (3′), taking $k(0)$, $h(0)$, $\hat{k}(t)$, and $\hat{h}(t)$ as given. The first-order conditions are well-known and, therefore, are confined to the Appendix (see Sala-i-Martin [1990] for details). To simplify notation, we define the following:

$$\tilde{\alpha}_k = \alpha_k + \alpha_{\hat{k}}, \qquad \tilde{\alpha}_h = \alpha_h + \alpha_{\hat{h}},$$
$$\tilde{\beta}_k = \beta_k + \beta_{\hat{k}}, \qquad \tilde{\beta}_h = \beta_h + \beta_{\hat{h}}. \tag{7}$$

The variables with tildes are the elasticities for the *social* production function when an aggregate consistency condition is imposed. They are the sum of the private elasticity and corresponding externality parameter.

IIIa. Optimal Relation Between u and v

The first-order condition ((A8) in the Appendix) is a relation between the fraction of physical capital, v, and the fraction of human capital, u, used in the final output:

$$v(u(t)) = \frac{u(t)}{\Delta + u(t)(1 - \Delta)} \qquad \text{for all } t, \tag{8}$$

where $\Delta \equiv \alpha_u\beta_v/(\alpha_v\beta_u)$ is a positive constant. Notice that equation (8) implies that when $u = 1$, then $v = 1$; when $u = 0$, then $v = 0$; and the derivative of v with respect to u is positive for all u between zero and one ($v'(u) = 1/[\Delta\{1 - (1 - (1/\Delta))u\}^2] > 0$). If $\alpha_u/\alpha_v = \beta_u/\beta_v$, Δ is equal to one all the time, and therefore v is always equal to u.

That is, if the technology for producing final output is "similar" to the production for producing human capital, then the fraction of physical and human capital used in the production of final goods will be the same. The relation between the growth rates is $\gamma_v(t) = \gamma_u(t)/[1 - u(t)\,(1 - \Delta^{-1})]$.

The monotonic relation between u and v indicates that economic agents never choose to increase human capital in one sector and reduce physical capital in that same sector. In other words, we can think about them deciding how much of their overall resources to spend in either sector and not worry too much about the exact resource (whether physical or human capital) spent since they will both move together. In practice, this means that we can use either one of them as a control variable, since the other one is immediately and uniquely determined by equation (8).

INTERESTING RESULT 1. The optimality conditions require that the two effort variables be monotonically related. We can therefore eliminate one of the control variables of the problem.

IIIb. Concave Production Possibility Frontiers

Consider the production possibilities facing a consumer at a point in time—when the aggregate capital stocks are fixed. Given $h(t)$ and $k(t)$, agents can choose to produce a lot of education by devoting no effort to the final output sector. If $u = 0$ (condition (8) says that the corresponding v is also zero), then $\phi h^{\beta_h} k^{\beta_k} \hat{h}^{\beta_{\hat{h}}} \hat{k}^{\beta_{\hat{k}}}$ units (a flow) of human capital are produced. Alternatively, agents can choose to devote all their effort to the final output sector. For $u = 1$ ($v = 1$ correspondingly), $A h^{\alpha_h} k^{\alpha_k} \hat{h}^{\alpha_{\hat{h}}} \hat{k}^{\alpha_{\hat{k}}}$ units (a flow) of physical capital are produced. Intermediate values for u (and the corresponding optimal choices of v) generate a production possibility frontier (PPF). The concavity of the PPF is determined by the parameters α_u, α_v, β_u, β_v. The algebraic relation between concavity and these parameters is quite complicated, but we can build intuition by discussing some special examples.

EXAMPLE 1. Constant Points-in-Time Returns and Identical Technologies.

Consider the case when there are no externalities and when $\alpha_u = \alpha_h$, $\alpha_v = \alpha_k$, $\beta_u = \beta_h$ and $\beta_v = \beta_k$ so all technologies exhibit constant point-in-time returns. Imagine also that $\alpha_k = \beta_k$ (recall that we have been assuming that $\alpha_k + \alpha_h = 1$ and $\beta_k + \beta_h = 1$ all along) so that the two sectors use the same production functions. The PPF in this case is linear.[6]

EXAMPLE 2. Constant Point-in-Time Returns and Different Technologies.

Suppose now that $\alpha_u = \alpha_h$, $\alpha_v = \alpha_k$, $\beta_u = \beta_h$, and $\beta_v = \beta_k$ so all technologies exhibit constant point-in-time returns, but $\alpha_k \neq \beta_k$ so that the two sectors use different technologies. In this case the PPF is strictly concave.[7] Note that the Uzawa-Lucas model falls into this category: it is a constant point-in-time-returns model with a strictly concave PPF due to different production functions for the two sectors.

EXAMPLE 3. Decreasing Point-in-Time Returns.

Third, suppose that $\alpha_u \leq 1 - \alpha_v$ and $\beta_u \leq 1 - \beta_v$, with one inequality strict. That is, suppose that there are decreasing point-in-time returns somewhere.[8] The PPF is in this case strictly concave.

INTERESTING RESULT 2. For the PPF to be strictly concave, it is sufficient that the point-in-time technologies exhibit nondecreasing returns when the PPF's in the two sectors are different. If the production functions are the same, then we require decreasing point-in-time returns somewhere. If the production functions are the same in the two sectors and all the point-in-time technologies exhibit constant returns, then the PPF is linear.

IIIc. Making our Model Stationary

After eliminating $v(t)$, the rest of the first-order conditions and accumulation constraints entail four nonlinear differential equa-

6. The linearity of the PPF depends on the exponents on u's and v's being the same in $f()$ and in $e()$ rather than on the fact that the exponents on k and h are the same in both technologies. For instance, we can get a linear PPF with decreasing point-in-time returns in h_f (i.e., $\alpha_u < \alpha_h$), if there are offsetting increasing point-in-time returns in k_f (i.e., $\alpha_v > \alpha_k$) and the same is true for h_e and k_e. Whether there are externalities or not has nothing to with the concavity of the PPF.

7. Again, we should note that the concavity of the PPF depends on the fact that the exponents on the effort variables, not the exponents of the capital stocks, are different in the two sectors.

8. With our Cobb-Douglas production functions, decreasing point-in-time returns impose another condition on the PPF. If the decreasing point-in-time returns are in the physical capital sector, then the marginal rate of transformation must be infinite at $u = v = 0$. If they are in the human capital sector, then (the inverse of) the marginal rate of transformation must be zero at $u = v = 1$. This extra restriction is peculiar to Cobb-Douglas.

tions $(\dot{u},\dot{c},\dot{k},\dot{h})$ in four variables: two controls (c and u) and two states (k and h). Our goal is to find the policy functions that relate the two controls to the two states. Because we want to allow for the possibility of positive steady-state growth rates, it will be convenient to rewrite the first-order conditions in terms of "unscaled" variables (in other words, in terms of variables that remain stationary when the levels of output and the capital stocks grow at a positive rate forever). We do so by defining what Mulligan [1991] calls state-like and control-like variables. State-like variables will be transformations of state variables only, with the property that, unlike the state variables h and k, they remain constant in the steady state. In the present model we use the following two state-like variables:

$$z_1(t) \equiv k(t)h(t)^{\hat{\alpha}_h/(\hat{\alpha}_k-1)} \tag{9}$$

$$z_2(t) \equiv k(t)h(t)^{(\hat{\beta}_h-1)/\hat{\beta}_k}. \tag{10}$$

Both z_1 and z_2 are increasing in k and (if $\hat{\alpha}_k < 1$ and $\hat{\beta}_h < 1$) decreasing in h. One way to think about them is the following: $z_1^{(\hat{\alpha}_k-1)}$ is the output to capital ratio $\bar{f}(.)/k$, when all the capital (both physical and human) is employed in the final output sector, that is when $u = v = 1$.[9] So in some sense, $z_1^{(\hat{\alpha}_k-1)}$ is the ratio of potential output to capital. Similarly, $z_1^{\beta_k}$ is the education output to human capital ratio $\bar{g}(.)/h$, when all the resources are employed in that sector. Hence, $z_1^{\beta_k}$ is the average potential output in the education sector. It is interesting to note that, in the absence of externalities (so $\hat{\alpha}_k = \alpha_k$, $\hat{\alpha}_h = \alpha_h$, $\hat{\beta}_k = \beta_k$, and $\hat{\beta}_h = \beta_h$), both z_1 and z_2 are equal to the ratio of physical to human capital, k/h.[10]

It will also be convenient to define control-like variables as transformations of control variables that, unlike $c(t)$, do not grow in the steady state. For most models, the ratio of consumption to physical capital will work. Thus, we just need to define a as

$$a(t) \equiv c(t)/k(t). \tag{11}$$

The steady-state growth rate of the other control variable u (and v) is zero so we can use it as the second control-like variable.

Our next step is to rewrite the dynamic system describing the solution to our growth model using state-like and control-like

9. An overbar on f and g indicates potential output for that sector (defined as the instantaneous flow of output that the economy could get if all the resources were employed in that sector).

10. Recall that we assumed constant returns to scale to the private capital stocks, $\alpha_k + \alpha_h = 1$ and $\beta_k + \beta_h = 1$.

variables only. To get the growth rates of the newly defined state-like variables, take logarithms and time derivatives of (9) and (10) and get

$$\gamma_{z_1}(t) = \gamma_k(t) - (\tilde{\alpha}_h/(1-\tilde{\alpha}_k))\gamma_h(t) \tag{12}$$

$$\gamma_{z_2}(t) = \gamma_k(t) - ((1-\tilde{\beta}_h)/\tilde{\beta}_k)\gamma_h(t). \tag{14}$$

Rewrite the growth rate of work effort ((A9) in the Appendix) in terms of control-like and state-like variables only:

$$\gamma_u = \frac{e/h(1-u)\,[(1-\alpha_k)u\beta_u + \alpha_u(1-\beta_k)(1-u)]/\alpha_u}{\text{denominator}} \times \frac{-\theta\gamma_c + (\tilde{\alpha}_k - \tilde{\beta}_k)\gamma_k + (\tilde{\alpha}_h - \tilde{\beta}_h)\gamma_h - (\rho + \delta_h)}{\text{denominator}}, \tag{15}$$

where

denominator

$$= (1-u)^{-1}\left((1-\alpha_u) + u(\alpha_u - \beta_u) - \frac{(1-u)\alpha_v + u\beta_v/\Delta}{1-(1-\Delta^{-1})u}\right)$$

and e is defined in (3). Finally, the growth rate of the new control-like variable a is given by

$$\gamma_a(t) = \gamma_c(t) - \gamma_k(t), \tag{16}$$

where

(17)

$$\gamma_c(t) = \frac{Az_1(t)^{\tilde{\alpha}_k-1}u(t)^{\alpha_u-1}v(u(t))^{\alpha_v}(\alpha_k\beta_u u(t) + \alpha_u\beta_k[1-u(t)])/\beta_u - \delta_k - \rho}{\theta},$$

and where

$$\gamma_k(t) = Au(t)^{\alpha_u}v(u(t))^{\alpha_v}z_1(t)^{\tilde{\alpha}_k-1} - a(t) - \delta_k \tag{18}$$

$$\gamma_h(t) = \phi(1-u(t))^{\beta_u}(1-v(u(t)))^{\beta_v}z_2(t)^{\tilde{\beta}_k} - \delta_h \tag{19}$$

(equations (17) and (18) are found by dividing the constraints (4) and (5) by k and h, respectively).

INTERESTING RESULT 3. The dynamic solution to our model can be transformed into a system of four ordinary differential equations ($\dot{z}_1(t)$, $\dot{z}_2(t)$, $\dot{u}(t)$, $\dot{a}(t)$) with two control-like variables ($a(t)$ and $u(t)$) and two state-like variables ($z_1(t)$ and $z_2(t)$) with the property that, in the steady state $\dot{z}_1(t) = \dot{z}_2(t) = \dot{u}(t) = \dot{a}(t) = 0$.

Equations (12), (14), (15), and (16), plus the transversality conditions (A6), and the initial conditions $K(0)$ and $H(0)$ describe the transitional dynamics and the steady state of the model. The next section characterizes the steady state. Section V studies the transition.

IV. Steady-State Analysis

IVa. Necessary Conditions for Endogenous Growth

Define steady state (or constant growth path) as the state where all the variables grow at a constant (possibly zero) rate. Thus, we rule out paths with ever increasing growth rates, but we allow for the possibility of zero steady-state growth rates. We also allow for the possibility of different variables to grow at different rates.

Define endogenous growth models as those sets of parameters for which there exists, for some initial conditions, a constant growth path solution.

Equation (17) says that, in the steady state, z_1 is equal to a bunch of constants. Hence, z_1^* is also constant, and $\gamma_{z_1}^*$ (the steady-state growth rate of z_1) is zero. Equation (18) then says that a^* is equal to constants so γ_a^* is equal to zero. This of course implies that the steady-state growth rate of consumption is equal to that of physical capital $\gamma_c^* = \gamma_k^*$. Equation (19), on the other hand, implies that z_2^* is also equal to a lot of constants so its value is constant and therefore, $\gamma_{z_2}^*$ is equal to zero. That is, we see that, in fact, the growth rate of the control-like and state-like variables is zero:

$$\gamma_{z_1}^* = \gamma_{z_2}^* = \gamma_a^* = \gamma_u^* = 0. \tag{20}$$

The steady-state condition $\gamma_{z_1}^* = \gamma_{z_2}^* = 0$ means that (12) and (14) form a homogeneous system of linear equations in γ_k^* and γ_h^*:

$$\begin{aligned} (1 - \bar{\alpha}_k)\gamma_k^* - \bar{\alpha}_h\gamma_h^* &= 0 \\ \bar{\beta}_k\gamma_k^* - (1 - \bar{\beta}_h)\gamma_h^* &= 0. \end{aligned} \tag{21}$$

A necessary condition for it to have positive solutions for γ_k^* and γ_h^* is that the determinant of the system be zero. In other words, *a necessary condition for the model to display endogenous* *growth* is[11]

$$(1 - \tilde{\alpha}_k)(1 - \tilde{\beta}_h) = \tilde{\alpha}_h \tilde{\beta}_k. \quad (22)$$

Note that this condition involves the elasticities of both capital goods in both *social* production functions (each social elasticity involves both the private elasticity and the externality parameter). In particular, it is independent of the level of technologies (A and ϕ), the taste parameters (ρ and θ), and the point-in-time technologies.

INTERESTING RESULT 4. If we want the two-sector models to display positive steady-state growth rates (endogenous growth), the social capital shares of the two production functions must be related according to condition (22).

IVb. Models That Satisfy Condition 22

Endogenous growth models must satisfy condition (22). We now discuss some special cases of (22) in order to gain some intuition regarding its economic significance.

a. If there are social constant returns to physical capital in the final output sector ($\tilde{\alpha}_k = 1$), then we must have either the production of education independent of physical capital ($\tilde{\beta}_k = 0$) or the final output sector independent of human capital ($\tilde{\alpha}_h = 0$). This latter case corresponds to the linear Ak technology used by Rebelo [1991] or Romer [1986], where output is linear in k and independent of human capital.[12]

b. If there are constant returns to human capital in the education sector ($\tilde{\beta}_h = 1$), then we must have either the final output sector depending on physical capital only ($\tilde{\alpha}_h = 0$) or the education sector depending on human capital only ($\tilde{\beta}_k = 0$). Notice that this latter case corresponds to the Uzawa-Lucas production function in equations (2″) and (3″). We should also realize that in this case, there may be increasing, constant, or decreasing returns to scale in the production of output since the conditions ($\tilde{\beta}_h = 1$) and ($\tilde{\beta}_k = 0$) impose no restrictions on $\tilde{\alpha}_k$ or $\tilde{\alpha}_h$.

11. The sufficient conditions and the bounded utility conditions will entail further restrictions on the size of the parameters. They will require the economy to be sufficiently productive so as to generate permanent growth but not so productive that there is no scarcity.

12. However, note that when $\tilde{\beta}_k = 0$ and $\tilde{\beta}_h$ is different from one, then z_2 is not well defined.

c. If both capital stocks are used in both sectors ($0 < \bar{\alpha}_k, \bar{\alpha}_h, \bar{\beta}_k, \bar{\beta}_h < 1$) and there are social constant returns in one sector ($\bar{\alpha}_h = 1 - \bar{\alpha}_k$), then there must be constant returns in the other sector ($\bar{\beta}_k = 1 - \bar{\beta}_h$).

d. If both capital stocks are used in both sectors ($0 < \bar{\alpha}_k, \bar{\alpha}_h, \bar{\beta}_k, \bar{\beta}_h < 1$) and there are diminishing returns in one sector ($\bar{\alpha}_h < 1 - \bar{\alpha}_k$), then there must be exactly offsetting increasing returns in the other one ($\bar{\beta}_k > 1 - \bar{\beta}_h$).

In terms of the Inada conditions, we know that a necessary and sufficient condition for the one-capital-good model to display endogenous growth is that the marginal product of capital be sufficiently bounded away from zero [Jones and Manuelli, 1990]. Our analysis of the two-capital-goods models suggests that the marginal product of either capital good on either sector can approach zero as capital grows without bounds and still get endogenous growth. Condition (22) indicates, however, that the marginal product of at least one of the sectors must be bounded away from zero as physical capital tends to infinity and human capital grows at the corresponding optimal rate. That is, the marginal product of a "broad measure of capital" is bounded above zero.

IVc. One State-Like Variable

Conveniently, the necessary condition for endogenous growth (22) imposes some restrictions on the relation between z_1 and z_2. Namely,

$$z_2 = z_1 \equiv z. \tag{23}$$

This condition means that any two-sector model that is to display positive steady-state growth rates can be expressed in terms of only one state-like predetermined variable, which we call z.

INTERESTING RESULT 5. The dynamic solution of any two-sector model of endogenous growth of the class considered in this paper can be written in terms of one state-like variable and two control-like variables.

The imposition of condition (22) is probably the main difference between the early multisector neoclassical models (such as Kurz [1968], Ryder [1967], Srinivasan [1964], or Burmeister [1980, Chapter 6]) and ours: it allows us to reduce our analysis by one dimension so our models are simpler.

IVd. Steady-State Comparative Statics

The steady-state behavior of our two-sector growth models can be analyzed numerically. Some particular parameterizations (such as that of Uzawa and Lucas) allow for closed-form solutions for the steady-state values of growth rates, effort variables, and so on. For more general parameterizations, closed-form solutions are not available.

In order to save space, in Tables I and II we report the signs of the derivatives of the steady-state values of the effort variable, u^*, the growth rate of output, γ^*, the consumption to capital ratio, a^*, and the state-like variable, z^*, with respect to all of the parameters of the model. We do that for two sets of parameters. Table I does it for the Uzawa-Lucas case where the two capital stocks are used in both sectors and the technologies are the same. The baseline parameters used are reported in the notes to the table. A negative sign in the first column, second row, suggests that an increase in the level of technology in the human capital sector—ϕ—leads to a decrease in the steady-state level of effort. Table II reports the signs of the same derivatives for the general model where both capital goods are used in both sectors (see the notes to the table for the exact parameters used). The key difference in the steady-state behavior of the two models is that the level of technology in the final output sector affects the steady-state growth rate of the economy in the general model but not in the Uzawa-Lucas model. Hence, the conclusion that only the technology in the human

TABLE I
EFFECT OF PARAMETRIC CHANGES ON STEADY-STATE VALUES: UZAWA-LUCAS MODEL

	u^*	γ^*	a^*	z^*
Baseline	0.4583	0.0150	0.1750	7.957
ΔA	0	0	0	+
$\Delta\phi$	−	+	+	−
$\Delta\theta$	+	−	+	+
$\Delta\rho$	+	−	+	+
$\Delta\alpha_k$	0	0	−	+
$\Delta\alpha_{\dot{h}}$	+	+	+	−
$\Delta\alpha_{\dot{k}}$	+	+	+	−

Notes. The table shows the relation between changes in the parameters arrayed horizontally and the steady-state values arrayed vertically. The baseline parameters are $\alpha_k = \alpha_h = \alpha_u = 0.5$, $\alpha_v = 0$, $\beta_k = \beta_v = 0$, $\beta_h = \beta_u = 1$, $\rho = 0.04$, $\theta = 2$, $A = 1$, $\phi = 0.12$, $\delta_k = \delta_h = 0.05$, $\alpha_{\dot{k}} = \alpha_{\dot{h}} = \beta_{\dot{k}} = \beta_{\dot{h}} = 0$.

TABLE II
General Two-Sector Model

	u^*	γ^*	a^*	z^*
Baseline	0.5658	0.0207	0.1504	10.121
ΔA	−	+	+	+
$\Delta \phi$	−	+	+	−
$\Delta \theta$	+	−	+	+
$\Delta \rho$	+	−	+	+
$\Delta \alpha_k$	−	+	−	+
$\Delta \beta_k$	+	+	+	−
$\Delta \alpha_u$	+	−	−	+
$\Delta \alpha_h$	+	−	−	+
$\Delta \beta_u$	−	−	−	+
$\Delta \beta_h$	+	−	−	+
$\Delta \alpha_h'$	+	−	−	+
$\Delta \alpha_k'$	+	+	+	−
$\Delta \beta_h'$	+	−	−	+

Notes. The baseline parameters are $\alpha_k = \alpha_h = 0.5$, $\alpha_u = \alpha_v = 0.48$, $\beta_k = 0.2$, $\beta_h = 0.8$, $\beta_u = 0.18$, $\beta_v = 0.78$, $\rho = 0.04$, $\theta = 2$, $A = 1$, $\phi = 0.12$, $\delta_k = \delta_h = 0.05$, $\alpha_k' = \alpha_h' = \beta_k' = \beta_h' = 0$. Also see the note to Table I.

capital sector matters for growth is specific to the Uzawa-Lucas model where $\dot{h}$ is independent of k.

V. A Methodology to Study Transitions: The Time-Elimination Method

Here we look at the two-sector growth models outside of the steady state. The basis for our analysis is the Time-Elimination method. It provides us with a practical and efficient algorithm for solving these models numerically. Time-Elimination is discussed in detail in Mulligan [1991]. Judd [1990] considers numerical techniques more generally. The idea is to exploit the recursiveness of our problem, even though our solution procedure until now has been the Maximum Principle of optimal control that applies to nonrecursive as well as recursive problems.

Remember that our model is described by an optimal control problem. We derived the Euler equations (A7) and (A9), which, together with budget constraints (2′) and (3′), describe the solution for the optimal control problem (1), (2′), and (3′). In the language of dynamical systems, optimal $c(t)$, $u(t)$, $v(t)$, $k(t)$, and $h(t)$ are the solutions to a boundary value type system of ordinary differential equations in time. The system is (2′), (3′), (A7), and (A9), and the boundary conditions are the Transversality Conditions described in the Appendix as well as the initial capital stocks $k(0)$ and $h(0)$.

We shall distinguish boundary value systems of differential equations from initial value systems. Initial value systems have very special boundary conditions: namely, they take the form of a set of values for all of the dependent variables at a single point in time. For example, our problem would be an initial value problem if we replaced the transversality conditions with values for $c(0)$ and $u(0)$.

We have a boundary value problem: two boundary conditions apply at $t = 0$ and two others apply at $t = \infty$. A standard numerical method for studying boundary value problems is called shooting. For example, this is the methodology employed by King and Rebelo [1990] and Jorgenson and Jun [1990] to examine one-sector growth models. As any practitioner of the shooting algorithm will attest, boundary value problems are much more difficult—both conceptually and computationally—to solve than are initial value problems.[13] In fact, the shooting method becomes unwieldy for systems of more than two or three dimensions. One key advantage of the Time-Elimination method is that it transforms the boundary value type problem described by (2′), (3′), (A7), (A9), and the TVC's into an initial value problem.

The Time-Elimination method is a four-step algorithm. First, we transform our system (2′), (3′), (A7), (A9) for which there exists a constant growth path to one for which there exists a stationary point or steady state. This is exactly what we did in Sections III and IV: we defined state-like variables, $z_1(t)$ and $z_2(t)$, and control-like variables, $a(t)$, $u(t)$, and $v(t)$. We found that the original Euler equations can be expressed in terms of $z_1(t)$, $z_2(t)$, $a(t)$, $u(t)$, and $v(t)$ only. An additional simplification was obtained in Section IV where we imposed a constant growth condition. That condition required z_2 to be equal to $z_1 \equiv z$. All together, we succeeded in transforming a system of differential equations for c, u, k, and h—a system for which there exists a constant growth path—into a system of differential equations for a, u, and z for which there

13. On a more pragmatic level, computer math packages are much more likely to include routines that solve initial value problems than to include routines that solve boundary value problems. We use MATLAB's ODE23 routine to solve initial value problems. We can therefore worry about economics rather than numerical mathematics (we believe that we have a comparative advantage in the former). See Press et al. [1990] for a comparison of initial value and boundary value problems.

exists a stationary point. We now denote the resulting differential equations (12), (15), and (16) as system (24):

$$\begin{aligned} \dot{a}(t) &= \kappa_1(a(t),u(t),z(t)) \\ \dot{u}(t) &= \kappa_2(a(t),u(t),z(t)) \\ \dot{z}(t) &= \kappa_3(a(t),u(t),z(t)), \end{aligned} \tag{24}$$

where κ_i are complicated nonlinear functions.

Our second step is to argue that the stationary point (a^*,u^*,z^*) of system (24) represents an optimal solution for our optimal control problem for *some* feasible initial conditions. This was shown in the last section: the economy will exhibit constant growth for some initial conditions $k(0)$ and $h(0)$.[14] Since some economies will be characterized by the stationary point of (24), we focus on the stable manifold of that stationary point. By definition, the stable manifold is the locus of points in the $[a,u,z]$ space which, when allowed to evolve according to (24), asymptotically approach the stationary point.[15] Notice that since the stationary point satisfies the Transversality Conditions, so do all economies that lie in the stable manifold. Therefore, the stable manifold describes optimal solutions to our optimal control problem.

Third, we appeal to the recursiveness of our problem to derive an alternative representation of the stable manifold of (24). Namely, we intend to represent the stable manifold—a locus of points in the $[a,u,z]$ space—by a pair of functions $a(z)$ and $u(z)$. To do so, we begin by noticing that, in the absence of externalities, solutions to our model are solutions to a social planning problem and that the social planning problem can be represented by a dynamic program. Solutions to that dynamic program can be represented by policy functions $a(z)$ and $u(z)$. Since the stable manifold describes solutions and the policy functions describe solutions, it must be that projections of the stable manifold into the $[a,z]$ and $[u,z]$ planes are graphs of the policy functions.

Once we allow for externalities, our problem cannot necessarily be described by a social planner's dynamic program. Nevertheless, we shall slightly abuse terminology and call projections of the

14. Of course, there will not be constant growth for all initial conditions $k(0)$ and $h(0)$. That is why we are interested in transitional dynamics!

15. In the case where the stable manifold is one dimensional—as it will be in our problems—it is informally referred to as the "stable arm." We find it helpful to think of our stable manifold as a human arm suspended in the $[a,u,z]$ space with the elbow at the point (a^*,u^*,z^*).

stable manifold into the $[a,z]$ and $[u,z]$ planes policy functions for the control-like variables:

$$\begin{aligned} a(t) &= a(z(t)) \\ u(t) &= u(z(t)). \end{aligned} \tag{25}$$

Note that the policy functions map values for the state-like variable z into values for the control-like variables a and u.

Using (25) and the chain rule of calculus, the transformed equations of motion for z, a, and u (system (24)) can be manipulated to compute slopes of the policy functions:

$$\begin{aligned} a'(z) &= \frac{\dot{a}(t)}{\dot{z}(t)} = \frac{\kappa_1(a,u,z)}{\kappa_3(a,u,z)} \equiv \xi_1(a,u,z) \\ u'(z) &= \frac{\dot{u}(t)}{\dot{z}(t)} = \frac{\kappa_2(a,u,z)}{\kappa_3(a,u,z)} \equiv \xi_2(a,u,z). \end{aligned} \tag{26}$$

The reader will notice at this point that the chain rule of calculus allows us to "eliminate time": the system of differential equations in time (24) is used to derive a system of differential equations in z (26). The equations (26) yield the slope of the policy functions for all values of (a,u,z) (except a^*,u^*,z^*). From step two we have some boundary conditions for system (26): the stationary point "satisfies" the policy functions $a(z)$ and $u(z)$:

$$\begin{aligned} a^* &= a(z^*) \\ u^* &= u(z^*). \end{aligned} \tag{27}$$

With one modification, (26)–(27) is an initial value type system of ordinary differential equations. The required modification is to specify the slopes of the policy functions at the steady state (note that at the steady state $\dot{a} = \dot{u} = \dot{z} = 0$ so equations (26) cannot be applied directly). There are two ways to find these slopes. The first is to apply L'Hôpital's rule to (26).[16] Alternatively, one can

16. L'Hôpital's rule will yield three slopes: one corresponding to the stable manifold and two corresponding to the unstable manifold. This is because both the stable and unstable manifolds satisfy (26) and (27). One must therefore have some intuition about the shape of these manifolds in order to choose the correct one of the three slopes yielded by L'Hôpital's rule. Once the correct steady-state slope is specified, (26) and (27) uniquely determine the stable manifold of the stationary point of (24). Our "eigenvector" method for finding the steady-state slopes avoids this confusion.

linearize (24) around the steady state and study the eigenvectors of the matrix describing the linearized version of the system.[17] The eigenvectors are tangent to the stable and unstable manifolds of the nonlinear system at the steady state, and the eigenvalues can be used to distinguish stable from unstable manifolds.[18] We used this "eigenvector" procedure to determine the slopes of our policy functions at the steady state.

The fourth step of the Time-Elimination algorithm consists of using MATLAB's subroutine ODE23 to solve the system of two ordinary differential equations (26)—augmented with the steady-state slopes computed above—subject to the "initial" values (27).

Additionally, we may be interested in finding the time path for z. This can be done by substituting the numerically computed policy functions into (14) and (numerically) integrating with respect to t.[19] Most interesting economic questions, however, can be answered from knowledge of the policy functions alone.

Two of the best features of the method used in this paper are the speed at which the computer vomits the answers and the simplicity of the programs needed: on an IBM 16 MHz 386SX, we usually can find policy functions for the problem (1) to (3) in less than 30 seconds! We do not have the patience to try shooting, but guess that shooting would take somewhere on the order of one hour to solve the same models.

In summary, the Time-Elimination constitutes a very simple numerical method for studying dynamic models. First, transform the system of differential equations into one that has a stationary point. Second, argue that the stationary point satisfies the transversality conditions. Third is the Time-Elimination step: apply the chain rule of calculus to construct a system of differential equations for policy functions rather than time paths. The stationary point is the appropriate boundary condition. Finally, ask MATLAB to solve this initial value type system of ordinary differential equations for the policy functions. To our surprise, the most

17. The linearized (around a steady state) version of a nonlinear system is fully described by a matrix that is the Jacobian of the nonlinear system evaluated at the steady state.

18. The stable manifold theorem from the theory of dynamical systems guarantees that the slope of the stable eigenvector is exactly equal to the slope of the stable manifold of the nonlinear system (24) at the steady state.

For example, suppose that for a particular parameterization the stable eigenvector of the Jacobian matrix of the right-hand side of (24) (evaluated at the steady state) is (3,2,1). Then $a'(z)$ and $u'(z)$ evaluated at the steady state are 3 and 2, respectively.

19. The integration of (10) subject to $z(0)$ is another initial value problem.

difficult part of the algorithm is computing the steady state (a^*,u^*,z^*)! Once the steady state is known, computation of transitional dynamics is handled quite easily by MATLAB.

The dynamics of our system are fully determined by one state-like variable, z, and two control-like variables, u and a. The evolution of the economy can be described by a phase diagram in the $[a,u,z]$ space. The stable arm (the models do turn out to be saddle-path stable) will be a one-dimensional curve in a three-dimensional space, a curve that goes through the steady state. The stable arm can be represented by two "policy functions" $a(z)$ and $u(z)$: projections of the stable arm into the $[u,z]$ two-dimensional space and the $[a,z]$ two-dimensional space.

VI. Transitional Dynamics in the Uzawa-Lucas Model

We start by applying the methodology just described to study the transition of the Uzawa-Lucas model. Recall that one of the special features of this model is that the education sector uses human capital as the only input of production. The two accumulation constraints in this special case are (2″) and (3″):[20]

$$\dot{k} = Ak^{\alpha_k}(uh)^{(1-\alpha_k)} - \delta_k k - c \tag{2″}$$

$$\dot{h} = \phi(1 - u)h - \delta_h h. \tag{3″}$$

VIa. Stability

From (9) and (10) the only state-like variable z for the Uzawa-Lucas model without externalities is equal to the ratio of the two capital stocks, k/h.[21] Hence, in what follows, we use z and k/h without distinction. As part of the Time-Elimination method just described, we need to compute the eigenvalues of the linearized system around the steady state. We always find that there are one negative and two positive eigenvalues so that the model is locally saddle-path stable. This is true even when we include small positive externalities in the final output sector so that it exhibits social increasing returns.

We then apply the time-elimination method, and we are able to calculate the policy functions for any value of z between zero and any (arbitrarily large) positive number. This means that the model

20. Although our numerical method can be used to analyze models with externalities, in this paper we concentrate our simulations on models with no externalities. That is, from now on we set $\alpha_h = 0$.

21. This is not the case when there are externalities (see equation (9)).

is *globally saddle-path stable*.[22] Through independent research and drastically different methodologies, Faig [1991] and Caballé and Santos [1991] have arrived at the same conclusion.[23]

INTERESTING RESULT 6. The Uzawa [1965]-Lucas [1988] model of endogenous growth is globally saddle-path stable.[24]

VIb. Policy Functions

But interesting as they are, the stability properties of the model are not our ultimate goal. We want to study the *economic forces* that lead the economy from any arbitrary initial ratio z_0 to the steady-state ratio z^*. We are also interested in how some economically interesting and observable variables (such as the saving rate, the growth rates of output, consumption, and physical and human capital, or the interest rate) behave along such transition, so we can compare the model with actual data.

The slopes of the two policy functions $u(z)$ and $a(z)$ depend on the relation between θ and α_k. For $\theta > \alpha_k$, the two functions are downward sloping; and for $\theta < \alpha_k$, the two are upward sloping. For the knife-edge case of $\theta = \alpha_k$, the two policy functions are horizontal.

We can give economic interpretations to these optimal choices of u and a. Consider the situation where physical capital is relatively scarce (so $z = k/h$ is low). Because the system is stable, $z < z^*$ should be associated with a positive growth rate of z, which is given by the difference between the growth rates of k and h:

$$\gamma_z = Az^{(\alpha_k-1)}u^{(1-\alpha_k)} - c/k - \delta_k - \phi(1-u) - \delta_h. \tag{28}$$

Imagine that c/k and u are constant (as is the case when $\theta = \alpha_k$). Low values of z are associated with high growth rates of z, simply because the average product of physical capital is high (as is the case in the one-sector Solow model with constant saving rates, the growth rates of k are high when k is low, simply because the

22. We never find a vertical asymptote in the policy functions either. Note that, since the policy functions are unique-valued, this implies that the steady state is unique. The policy functions are unique-valued because the Hamiltonian is concave in the choice variables.

23. Our methodology allows us to calculate the speed at which the economy converges to the steady state. In Mulligan and Sala-i-Martin [1992a, 1992b] we show that, for very plausible parameterizations, the Uzawa-Lucas model entails half-lives of 25 or more years. Hence, transitions can be long and important.

24. This result comes from extensive experimentation with all kinds of parameters and is not based on a formal proof. Strictly speaking, Interesting Result 6 should say, "We have not been able to find parameters for which the Uzawa-Lucas model was not globally saddle-path stable.

average product of capital is high). This is what we call the *average product of capital* or *Solow effect.*

A low value of z (that is, a relatively scarce stock of physical capital) can also be raised through an increase in savings (a decrease in a, the consumption to capital ratio) or through an increase in the allocation, u, of the existing human capital to the production of final output. Because agents like to smooth consumption, they do not like the first choice. In fact, they dislike this possibility more, the higher the value of θ. This is a *wealth* or *consumption smoothing* effect. A low k/h ratio, on the other hand, implies a low wage in the final output sector. This motivates agents to go to the education sector.[25] That is, they do not like the possibility of increasing u when z is low because the opportunity cost of schooling is then low. This *substitution* or *relative wage rate* effect is more important the larger the value of α_k (the wage rate is proportional to z^{α_k} for a given value of u).

The final outcome depends on the relation between θ and α_k. If $\theta = \alpha_k$, then the consumption smoothing and the relative wage effects cancel out so the steady-state capital ratio z^* is restored through the high average product of physical capital or Solow effect.[26] Other things held constant, higher values of θ increase the agent's willingness to smooth, which tends to raise the ratio of consumption to physical capital, a. In other words, if $\theta > \alpha_k$, then the consumption smoothing effect dominates so a and u are high for low values of z. The policy functions are downward sloping.

A lower value of θ leads people not to worry too much about departures from a smooth path of consumption. They are ready to increase the stock of physical capital through higher savings. This allows them to take advantage of the low wage situation by increasing the allocation of human capital to the education sector with a corresponding reduction in work effort. In other words, if $\theta < \alpha_k$, then the relative wage effect dominates so a and u are low for low values of z.

INTERESTING RESULT 7. In the Uzawa-Lucas model, the transition from low k/h ratios involves high or low work effort and

25. The wage differential cannot be reduced by reallocating physical capital between the sectors because, for the Uzawa-Lucas model, physical capital is not used in the h sector.

26. This implies that $\dot{h}/h$ is constant at all points in time. If we call this constant growth rate x, then we can write the production function of final output as $y = Bk^{\alpha_k}(e^{xt})$, where B is a constant equal to $Au^{\alpha_u}h_0$. Note that this production function resembles that of the neoclassical model with productivity growing at an exogenous constant rate x.

consumption depending on the relative size of θ and α_k. If $\theta > \alpha_k$, the wealth effect dominates, and physical capital is restored through high work effort (and the two policy functions are downward sloping in k/h). If $\theta < \alpha_k$, the substitution effect dominates so physical capital is restored through low consumption (and the two policy functions are upward sloping). If $\theta = \alpha_k$, the two effects offset (and the two policy functions are flat). The symmetric result applies for transitions from high k/h ratios.

VIc. Transitional Behavior of Some Interesting Variables

The Case when $\alpha_k < \theta$. We think that, empirically, this is the most relevant case: the share of physical capital in the final output sector, α_k, is a number between zero and one while the inverse of the intertemporal elasticity of substitution, θ, is often estimated to be larger than one. Hence, we shall concentrate our numerical simulations on the case when $\alpha_k < \theta$. The numerical simulations use the following values for the rest of the parameters: $\theta = 2$, $\rho = 0.04$, $\alpha_k = 0.5$, $\delta_k = \delta_h = 0.05$, and $A = 1$. Finally, we choose the level of technology in the education sector, $\phi = 0.12$, so as to get an annual steady-state growth rate, $\gamma^* = (1/\theta)(\phi - \delta_h - \rho)$ equal to 0.015. Note that these parameters imply a steady-state rate of return of $\phi - \delta_h = 0.07$.

As we just argued, when $\alpha_k < \theta$ the two policy functions $u(z)$ and $a(z)$ are downward sloping. They are depicted in the first two panels of Figure I. The horizontal axes in Figure I measures the distance between k/h and the steady-state value of $(k/h)^*$. The vertical line at zero corresponds to the steady state. If we think of a less developed country that starts with a relatively low stock of human capital ($z > z^*$), then z falls over time, and u and c/k increase over time. In other words, the country devotes a relatively low but rising fraction of resources to consumption (c/k is low), but spends a substantial but falling fraction of time in education ($1 - u$ is high).[27]

Dividing the budget constraint (2′) by k and using the policy functions found above, we find the behavior of the growth rate of k along the transition. This growth rate (reflected in panel (iii) of Figure I) is downward sloping. Along a transition where $z = k/h$ is rising (that is, when $z > z^*$), the growth rate of physical capital will

27. Note that for low values of z, $u > 1$. At this point, investment in human capital is negative.

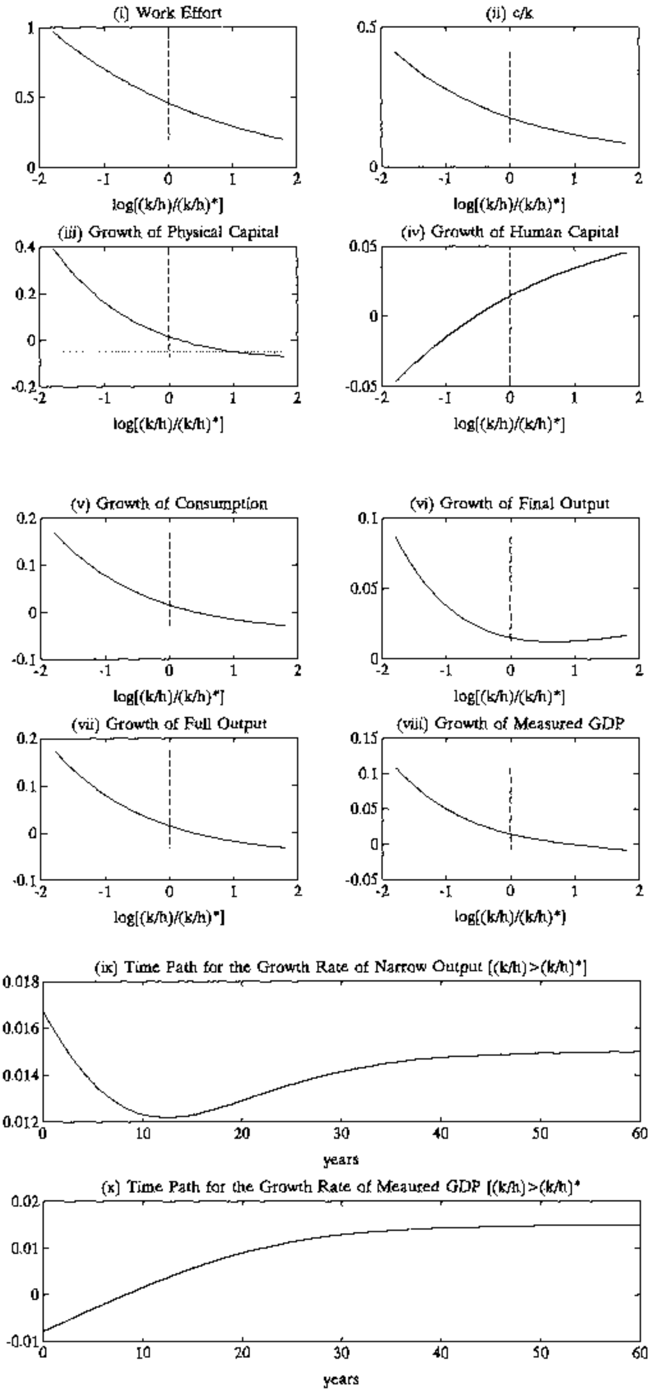

FIGURE I
Transitional Dynamics in the Uzawa-Lucas Model

be falling.[28] Panels (iv) and (v) show that along the same transition the growth rates of consumption and human capital are falling and rising, respectively.

The relation between the growth rate of the final output and z can be U-shaped, and the minimum growth rate can occur either to the left or to the right of the steady state. Holding constant the rest of our benchmark parameters, the minimum occurs at the steady state when $\theta = 3$, to the right of the steady state when $\theta < 3$, and to the left of the steady state when $\theta > 3$. That is, the imbalance effect could have a symmetric effect on the growth rate, with higher growth rates of output emerging if either k or h is in relatively short supply, or asymmetric, with growth rising with one type of imbalance and falling with the other type in the neighborhood of the steady state. The growth rate of output for $\theta = 2$ is depicted in panel (vi) of Figure I. It is a downward sloping function when it goes through the steady state, but it becomes an increasing function of z for larger values of z.

The concept of output we just discussed does not include the production of human capital. We can construct a broad measure of output by adding the production of human capital, multiplied by the shadow price of human capital in units of goods ($\nu/\lambda = (1 - \alpha)(A/\phi)z^{\alpha}u^{-\alpha}$). Full output is then given by

$$Y^{\text{full}} = y + (\nu/\lambda)\phi(1 - u)h. \tag{29}$$

Using this definition and the formula for the relative price ν/λ, we can compute the growth rate of this broad measure of output, $\gamma_{Y^{\text{full}}} = \gamma_y - (\dot{u}/u)(1 - \alpha)/(1 - \alpha + \alpha u)$. This growth rate is depicted in panel (vii). Unlike the narrow measure of output reported in panel (vi), this one is unambiguously downward sloping for all relevant values of k/h. Hence, transitions involving falling ratios of k/h entail increasing growth rates of full output.

Conventional measures of national income account for some positive fraction of the activity in the human capital sector (for instance, the wages of professors are counted). Other parts of the activity are excluded from the national accounts (the wages of the students are usually not counted). The growth rate of measured national income behaves somewhere between the growth rates reported in panels (v) and (vi). If we imagine that measured national income includes 25 percent of the education sector, then

28. Panel (iii) also displays a horizontal line at δ_k. The intersection between this line and γ_k shows the point where the inequality constraint $\dot{k} + \delta_k k \geq 0$ would be binding.

the growth rate of national income is a negative function of k/h (panel (viii) displays the transition for this case).[29]

The negative relation between the growth rate of output and z, is an interesting implication of transition of the Uzawa-Lucas model. It implies an asymmetric response to losses of physical and human capital. For example, a country or region that loses a lot of physical infrastructure in a war or natural disaster will recover quickly. A country that loses a large fraction of the human capital, on the other hand, will suffer low growth rates along the transition to the steady state.

For our benchmark parameters, the *saving rate* (defined as $s = 1 - c/y$, where y the production of the final output sector) is negatively related to k/h (not shown in the figure). Hence, consumption is higher relative to output when k/h is high. The behavior of the saving rate depends a lot on the exact parameters of the model. Everything else held constant, for sufficiently large values of θ, the saving rate will be upward sloping. Furthermore, there is always a value of θ for which the saving rate along the transition is constant. This value is given by

$$\theta^* = \frac{(\rho + \delta_k)\alpha_k}{\alpha_k\delta_k - (1 - \alpha_k)(\phi + \delta_k - \delta_h)}, \tag{30}$$

and the corresponding saving rate is

$$s^* = 1 - (\theta^* - 1)\alpha_k/\theta^*. \tag{31}$$

Note that, because the parameters ϕ, δ_h, and ρ must satisfy $\phi > (\delta_h + \rho)$, in order for the steady-state growth rate to be positive, equation (30) implies that $\alpha_k > 1/2$ must hold if we want $\theta \geq 0$. Note that a capital share larger than 1/2 is unlikely to be satisfied if we interpret k as a strict measure of physical capital.

Finally, we construct and report the time paths for the growth rates of an economy whose log of the initial k/h ratio is about twice its steady-state value. In panel (ix) we report the time path of the growth rate of our narrow measure of output. The growth rate starts at 1.7 percent, and it falls for about twelve years. It then starts increasing, and it reaches its steady-state value in about 40 years. Panel (x) reports the time path for the growth rate of measured output (which we assume to account for 25 percent of human capital investment only). Note that the growth rate is

29. Elsewhere we have argued that 25 percent is a reasonable number [Mulligan and Sala-i-Martin, 1992b].

actually negative for about ten years. The growth rate recovers very slowly over time. It reaches its steady-state value in about 40 years. As we suggested above, therefore, an economy that loses a substantial fraction of its human capital stock takes a long time to recover.

The Case when $\alpha_k > \theta$. As we argued above, this case entails low willingness to smooth consumption and very low wages for low values of z. The two policy functions are upward sloping. The qualitative behavior of the rest of the variables is similar to the previous case with two exceptions.[30] First, the growth rate of narrow output is no longer a U-shaped function of z, but instead it is an unambiguously decreasing function of z. Second, contrary to the case when $\alpha_k < \theta$, the growth rate of human capital is a decreasing function of z. This means that a transition from high z's will entail a rising growth rate of human capital.

The Case when $\alpha_k = \theta$. This is the case when the consumption smoothing effect and the wage effect exactly cancel so the policy functions $u(z)$ and $a(z)$ are horizontal. As was the case with the $\alpha_k > \theta$, the qualitative behavior of the rest of the interesting variables along the transition is very similar to the case when $\theta < \alpha_k$ with two notable exceptions. The growth rate of narrow output is an unambiguously decreasing function of z, and the growth rate of human capital and the relative price are constant functions of z.

VII. Transitional Dynamics in the General Model

VIIa. Linear PPF's

The Uzawa-Lucas model is a particular case where the education sector does not use physical capital as an input of production. We now want to analyze the transition of the more general model, starting with the case when the production functions in the two sectors are identical and all the point-in-time technologies exhibit constant returns to scale. As we showed in Section III, the PPF associated with such a case is linear. The main finding here is that the policy functions $u(z)$ and $a(z)$ are vertical lines at z^*. Thus, if the initial stock of physical capital is low relative to human capital so that $z < z^*$, then agents choose to invest (disinvest) at an infinite rate in the physical (human) capital sector by setting $v = u = \infty$. This implies a discrete transformation of physical capital into

30. To economize on space, we do not report the figures corresponding to these particular parameterizations.

human capital. The opposite is true if $z < z^*$. In either case, the transition takes no real time.

The result of instantaneous transition (or no transitional dynamics) applies to any model that yields a linear PPF regardless of the values of α_k and β_k. Remember from Section III that parameterizations that satisfy $\alpha_v = \beta_v = 1 - \alpha_u = 1 - \beta_u$ yield linear PPF's.

VIIb. Models with Strictly Concave PPF's

In this section we study the stability properties of the general model.[31] We know from Section III that there are different ways to get strictly concave PPF's. One of them is to have different productions in the two sectors. Another is to postulate diminishing point-in-time returns in some of the technologies. One common thing about them, however, is that the linearization of the dynamic system (24) around the steady state always gives one negative and two positive real eigenvalues. Hence, all the models are locally saddle-path stable. Since we are able to calculate the policy functions $u(z)$ and $a(z)$ for any value of z between zero and any arbitrarily large positive number, the models are globally saddle-path stable.

INTERESTING RESULT 8. If the point-in-time PPF of a two-sector model of endogenous growth is linear, the model entails no transition (i.e., the economy "jumps" to the steady state at time zero). Thus, in order to get transitions in real time, we must have strictly concave PPF's.

We now describe the form of the two policy functions, $u(z)$ and $a(z)$, in the general two-sector model with a concave PPF. As was the case for the Uzawa-Lucas model, the two policy functions are downward sloping when $\theta \geq 1$, which we think is the most plausible case.

An example of such policy functions is pictured in panels (i) and (ii) of Figure II. They correspond to a case when the technologies in the two sectors are different and there are diminishing point-in-time returns in all sectors. The exact parameters used are reported in the notes to Table II. Note that since $\alpha_v < \alpha_k$, $\alpha_u < \alpha_h$, $\beta_v < \beta_k$, and $\beta_u < \beta_h$, all technologies exhibit diminishing point-in-time returns (see equation (6) and the subsequent discussion in Section II).

31. Again, we concentrate our simulations on models with no externalities.

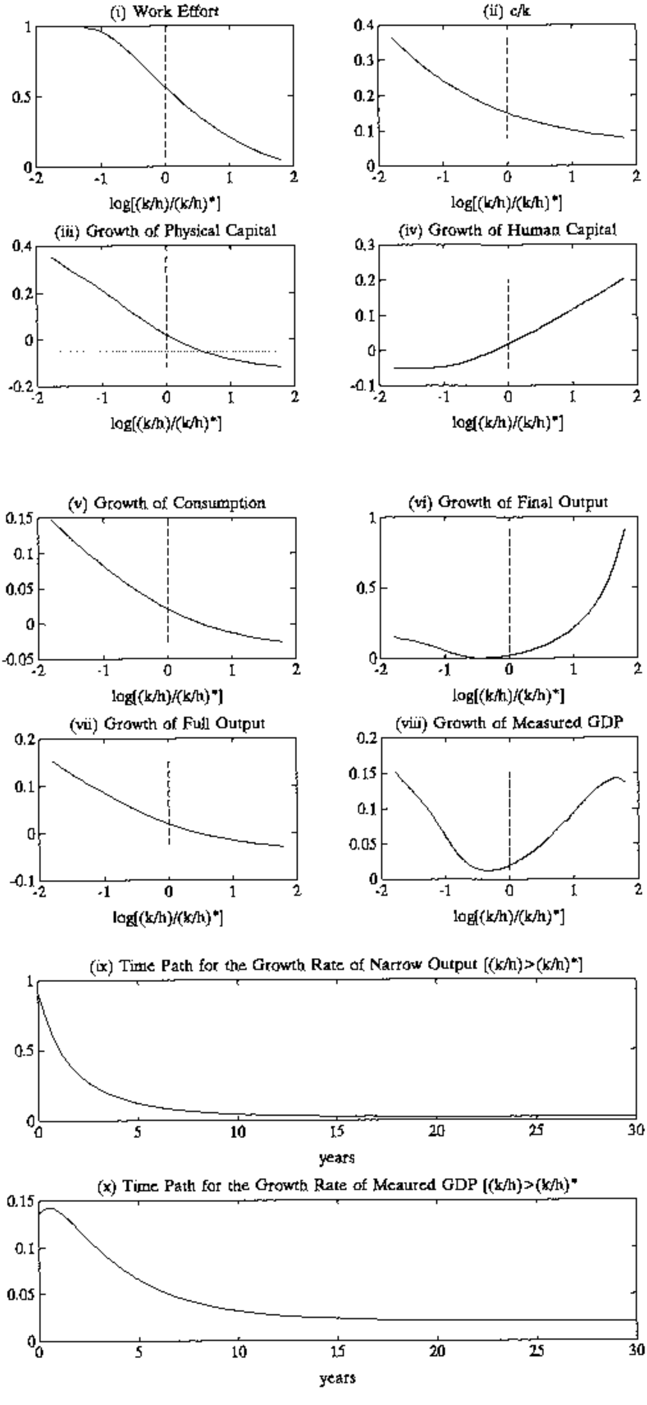

FIGURE II
Transitional Dynamics in the General Two-Sector Model

In the first two panels of Figure II, we note that both functions are downward sloping. The function $u(z)$ displays an inverse S-shape, and it takes values strictly between zero and one. The reason is that we assumed decreasing point-in-time returns. In a Cobb-Douglas framework such as ours, diminishing point-in-time returns also imply infinite marginal products when u and v take values of zero or one. Hence, it is never optimal for our economies to devote none or all of the resources to the final output sector so u always takes values between zero and one.[32] This result, however, does not hold in general. In particular, if we have constant point-in-time returns in all sectors but the two technologies are different (we know from Interesting Result 2 that the PPF in this case is also strictly concave), then the policy function $u(z)$ is not bounded by zero and one.

One of the key differences between this general model and the Uzawa-Lucas model is that here the policy function for work effort, $u(z)$, is always downward sloping, even for low values of θ. In other words, the relative wage effect cannot dominate in this model. When z is low, the marginal product of physical capital in the output sector is large so agents shift some of the physical capital to that sector. This increases the wage rate (the marginal product of human capital in the final output sector) which leads agents to shift human capital to that sector also. In the terminology used in the previous section, the relative wage effect disappears, so low levels of physical capital relative to human capital are associated with high levels of activity in the final output sector.

The wealth or consumption-smoothing effect, on the other hand, still exists. In fact, we find that, for low enough values of θ, the willingness to intertemporally substitute consumption over time is so large that agents are willing to increase the low physical capital stock by giving up consumption relative to capital. Low values of θ are therefore associated with upward sloping $a(z)$ policy functions. Hence, unlike the Uzawa-Lucas model this model can have upward sloping $a(z)$ and, at the same time, downward-sloping $u(z)$ policy functions.

VIIc. Transitional Behavior of Some Interesting Variables

The transitional dynamics of some interesting variables is presented in Figure II. As was the case in the Uzawa-Lucas

32. One implication is that the speed at which the economy converges to the steady state is lower when diminishing point-in-time returns are present.

specification, the growth rates of physical capital, consumption, and full output are downward sloping (panels (iii), (v), and (vii)). The growth rate of human capital is upward sloping, as was the case in the Uzawa-Lucas model (panel (iv)).

The main difference between this model and that of Uzawa and Lucas is that the growth rate of final output (and therefore the measured output where only a fraction of the human capital investment is accounted for) is much more likely to display a U-shape with a minimum *to the left* of the steady state (panels (vi) and (viii)). The rest of the variables (and in particular the saving rate) have the same qualitative features as the ones in Uzawa and Lucas.

The time paths for the growth rate of narrow output and measured output are reported in the last two panels of Figure II, respectively. The paths reported correspond to an economy whose initial log of k/h is about twice as large as its steady-state value. Note that the growth rate of narrow output is enormous for the first ten years. The steady-state growth rate is reached in about 25 years. The growth rate of our intermediate measure of output is 14 percent at year zero. It then increases to about 15 percent in the next couple of years. It then falls smoothly toward its steady-state value of 2 percent, which is reached in about 25 years.

VIII. Conclusions

In this paper we studied the transitional dynamics of two-sector models of endogenous growth. Unlike the neoclassical growth models, transitions in these types of models arise because of imbalances in the stocks of capital in the two sectors, not because the levels of capital are different from the steady state. This means that the transitional dynamics can be analyzed in terms of a variable which reflects imbalances. This is what we called the state-like variable z, which in most parameterizations used in this paper, corresponds to the ratio of the two capital stocks, k/h.

Since an economy's growth can be predicted by its sectorial imbalance, growth regressions should include proxies for the imbalance as an explanatory variable. In another paper [Mulligan and Sala-i-Martin, 1992b] we use data for U. S. states and find that the ratio of output to human capital is important in explaining growth rates.

We found that if the point-in-time PPF is linear, the transition takes no real time. On the other hand, if the point-in-time PPF is strictly concave, there is transition in real time. In this latter case, the models are always globally saddle-path stable.[33]

The transition involves three effects. First, when physical capital is relatively low, the average product of capital is high so the ratio of physical to human capital grows rapidly (this effect is the same as in the Solow model with a constant saving rate). Second, there is a substitution or relative wage effect that leads people to reduce work effort when physical capital is relatively low (low wages). And third, there is a wealth or consumption smoothing effect that leads people to high consumption relative to physical capital, when physical capital is low. We find that for plausible intertemporal elasticities of substitution ($\theta \geq 1$), the wealth effect dominates so the transition entails downward-sloping policy functions $u(z)$ (the fraction of labor used in the final output sector) and $a(z)$ (the ratio of consumption to physical capital).

One important empirical implication of the transitions studied in this paper is that the growth rate of the economy reacts in an asymmetric way to losses of human and physical capital. Imagine that a war or a natural disaster destroys a large fraction of the inputs of the economy. The growth rate predicted by the models depends on how large the loss of human capital is relative to physical capital. The Uzawa-Lucas model says that if the loss of human capital is larger, then the growth rate will be very low for a long period of time. If the loss of physical capital is larger, the growth rate will be large so the economy will recover quickly.

Hirshleifer [1963] provides some empirical support for this prediction. He analyzes several historical episodes where an economy suffered a large disaster of some sort. The cases of Japan and Germany during and after the Second World War, Russia after the Soviet Revolution, the black death in Europe, and the American Civil War lead him to conclude: "The speed and success of recovery in the observed historical instances have been due in large part to the proportionally smaller destruction of population than of material resources" [p. 121].

After centuries of being one of the world's leading economies, China suffered a long period of economic slowdown after the Mongol invasion that destroyed a substantial fraction of the stock of human capital. The asymmetric adjustment to destruction of physical as opposed to human capital is further supported by other

33. The last two statements were found to be true in the absence of externalities.

historical examples. Melos and Babylon, for instance, failed to recover both in terms of size and in terms of economic prosperity after losing a substantial fraction of the population while keeping their physical infrastructure intact. The Roman city of Carthage came back to life only after a century of desolation.

Appendix: First-Order Conditions for the General Model

The first-order conditions to the general program with respect to c, k, h, u, and v are, respectively,

$$e^{-\rho t}c^{-\theta} = \lambda \tag{A1}$$

$$-\dot{\lambda} = \lambda(A\alpha_k k^{\alpha_k - 1 + \alpha_{\hat{k}}} h^{\alpha_h + \alpha_{\hat{h}}} u^{\alpha_u} v^{\alpha_v} - \delta_k) + \nu(\phi\beta_k k^{\beta_k - 1 + \beta_{\hat{k}}} h^{\beta_h + \beta_{\hat{h}}} (1-u)^{\beta_u} (1-v)^{\beta_v}) \tag{A2}$$

$$-\dot{\nu} = \lambda(Ak^{\alpha_k + \alpha_{\hat{k}}} \alpha_h h^{\alpha_h - 1 + \alpha_{\hat{h}}} u^{\alpha_u} v^{\alpha_v}) + \nu(\phi k^{\beta_k + \beta_{\hat{k}}} \beta_h h^{\beta_h - 1 + \beta_{\hat{h}}} (1-u)^{\beta_u} (1-v)^{\beta_v} - \delta_h) \tag{A3}$$

$$\lambda(Ak^{\alpha_k + \alpha_{\hat{k}}} h^{\alpha_h + \alpha_{\hat{h}}} \alpha_u u^{\alpha_u - 1} v^{\alpha_v}) = \nu(\phi k^{\beta_k + \beta_{\hat{k}}} h^{\beta_h + \beta_{\hat{h}}} \beta_u (1-u)^{\beta_u - 1} (1-v)^{\beta_v}) \tag{A4}$$

$$\lambda(Ak^{\alpha_k + \alpha_{\hat{k}}} h^{\alpha_h + \alpha_{\hat{h}}} u^{\alpha_u} \alpha_v v^{\alpha_v - 1}) = \nu(\phi k^{\beta_k + \beta_{\hat{k}}} h^{\beta_h + \beta_{\hat{h}}} (1-u)^{\beta_u} \beta_v (1-v)^{\beta_v - 1}), \tag{A5}$$

where the aggregate consistency conditions $\hat{h} = h$ and $\hat{k} = k$ have been used.[34] The two limiting transversality conditions are

$$\lim_{t\to\infty} \lambda(t)k(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \nu(t)h(t) = 0. \tag{A6}$$

We can substitute the ratio of shadow prices in (A4) into equation (A2) to get an expression for the growth rate of the shadow price of physical capital $\dot{\lambda}/\lambda$. Also, by taking logarithms and derivatives of (A1), we get the growth rate of consumption as a function of $\dot{\lambda}/\lambda$. If we put the two together, we get an expression for consumption growth:

$$\gamma_c(t) = \frac{Ak^{\tilde{\alpha}_k - 1} h^{\tilde{\alpha}_h} u^{\alpha_u - 1} v^{\alpha_v} (\alpha_k \beta_u u(t) + \alpha_u \beta_k [1 - u(t)])/\beta_u - \delta_k - \rho}{\theta}, \tag{A7}$$

where γ_c is defined as the growth rate of consumption, $\gamma_c = \dot{c}/c$.

34. The representative agent has the representative or average amount of both types of capital goods. Hence, it must be the case that $\hat{h} = h$ and $\hat{k} = k$.

To find a relation between the fraction of physical capital used in the final output sector and the fraction of human capital used in the final output sector between the shares of capital employed in each sector (u and v), divide equation (A4) by equation (A5) and get

$$\text{(A8)} \qquad v(u(t)) = \frac{u(t)}{\Delta + u(t)(1 - \Delta)} \qquad \text{for all } t,$$

where $\Delta \equiv \alpha_u \beta_v / (\alpha_v \beta_u)$ is a constant. This is equation (8) in the text. Now take logarithms and time derivatives of both sides of equation (A4) and use (A8) to get a relation between the growth rate of the relative shadow price of the two capital goods ($\gamma_\lambda - \gamma_\nu$), the growth rates of the two capital stocks and work effort (γ_k, γ_h, and γ_u), and the level of work effort, u. We can also plug the relative shadow price from (A5) into (A3) to get a value for the growth rate of the shadow price of human capital, $\dot{\nu}/\nu$. Using these last two equations and (A7), we get

$$\text{(A9)} \quad \gamma_u = \frac{e/h(1 - u)\,[(1 - \alpha_k)u\beta_u + \alpha_u(1 - \beta_k)(1 - u)]/\alpha_u}{\text{denominator}} \times \frac{-\theta\gamma_c + (\tilde{\alpha}_k - \tilde{\beta}_k)\gamma_k + (\tilde{\alpha}_h - \tilde{\beta}_h)\gamma_h - (\rho + \delta_h)}{\text{denominator}},$$

where

denominator

$$= (1 - u)^{-1}\left((1 - \alpha_u) + u(\alpha_u - \beta_u) - \frac{(1 - u)\alpha_v + u\beta_v/\Delta}{1 - (1 - \Delta^{-1})u}\right)$$

and e is defined in (3) in the text. Equation (A9) corresponds to (14) in the text.

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