Advance Service Reservations with Heterogeneous Customers

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We study a fundamental model of resource allocation in which a finite number of resources must be assigned in an online manner to a heterogeneous stream of customers. The customers arrive randomly over time according to known stochastic processes. Each customer requires a specific amount of capacity and has a specific preference for each of the resources, with some resources being feasible for the customer and some not. The system must find a feasible assignment of each customer to a resource or must reject the customer. The aim is to maximize the total expected capacity utilization of the resources over the horizon. This model has application in services, freight transportation, and online advertising. We present online algorithms with bounded competitive ratios relative to an optimal offline algorithm that knows all stochastic information. Our algorithms perform extremely well compared to common heuristics, as demonstrated on a real data set from a large hospital system in New York City.

Key words: Analysis of algorithms, Approximations/heuristic, Cost analysis

1. Introduction

We study a fundamental model of resource allocation in which a finite number of resources must be assigned in an online manner to a heterogeneous stream of customers. The customers arrive randomly over time according to known stochastic processes. Each customer requires a specific amount of capacity and has a specific preference for each of the resources, with some resources being feasible for the customer and some not. The system must find a feasible assignment of each customer to a resource or must reject the customer. The aim is to maximize the total expected capacity utilization of the resources over the time horizon.

This model has application in multiple areas, including services, online advertising, and freight transportation. We now explain a few of the applications.

Service Reservation. In services such as healthcare, the resources can correspond to service sessions. For example, a resource might be a Monday afternoon session from 1 to 5 PM with Dr. Smith. The customers are patients who arrive to book appointments over time. Based on a patient's urgency, type of visit, arrival time, and preferences, the patient might require a specific length of visit and might be preferably assigned only to a subset of sessions. Upon the arrival of a patient, the system has to reserve a part of a session for the patient. This appointment decision typically takes place immediately. If an appointment cannot be found, the system must reject the patient.

Generalized Adwords. In online advertizing, the resources correspond to advertisers. The capacity of each resource corresponds to the budget of the corresponding advertiser. Ad impressions arrive randomly over time. Each impression, depending on its characteristics, commands a known non-negative bid from each of the advertisers. When an impression occurs, the ad platform must allocate it to an advertiser for use. The ad platform earns the bid, and the budget of the advertiser is depleted by the same amount. The aim of the ad platform is to maximize the expected revenue earned. Our model is more general than adwords models, as we allow bids to have arbitrary sizes, whereas adwords model tend to assume that bid sizes must be very small relative to the budgets, or that each bid must be truncated by the remaining budget (Mehta 2012).

Freight Allocation. Freight carriers such as motor carriers, railroad companies, and shipping companies have fleets of containers that can be deployed to move loads from specific origins to destinations. The assignment of containers to routes are tactical decisions that are performed on a larger time scale. Suppose that we focus on a single route. Each container, with its specific departure and arrival time, corresponds to a resource. Customer demands for the route arrive randomly over time. Each demand unit has a specific size and delivery time line. As each demand unit arrives, the operational decision is how to assign the demand unit to a specific container in the fleet (Spivey and Powell 2004). This assignment generates a quoted time of delivery for the customer, reduces the available capacity in the container, and earns the system an amount that can be roughly proportional to the amount of capacity consumed.

Our model captures most, if not all, of the features of the above applications. Specifically, we consider a continuous-time planning horizon. There are m resources with known capacities. There are n customer types. Each customer type is associated with a known stochastic arrival process. Each customer can be assigned to a known subset of the resources, and consumes a known amount of each resource that it is assigned to. The system aims to assign customers to resources immediately and irrevocably as they arrive in order to maximize the total expected amount of resources used.

A salient feature of our model is that the resources may be perishable. This feature makes the model especially appropriate for service applications. More specifically, each resource may be associated with a known expiry date that falls within or beyond the horizon. The way we capture an expiry date for a specific resource is to make that resource infeasible for all customer types that arrive after its expiry date. That is, to capture the perishability of resources, we equivalently force the composition of customer types that arrive over time to change over time. For this reason, the non-stationarity of arrivals in our model is of especial importance.

Another significant advantage of non-stationary arrivals is the ability to better capture real applications. In real applications, demands can be highly non-stationary, changing with the time of day, time of week, seasons and longer-term trends (Huh et al. 2012). Kim and Whitt (2014) have shown, for example, that call-center and hospital demands are well-modeled by non-homogeneous Poisson processes. For a problem that essentially aims to match demand with supply over time, capturing this non-stationarity in demand arrivals can lead to significant improvements in performance over stationary models.

Our basic model can be adapted as needed to fit various applications. In this paper, we will focus on solution methods for the core model. It is easy to see that the associated dynamic stochastic optimization problem is intractable to solve with dynamic prog ramming. The state of the system grows exponentially with the length of the horizon. Therefore, we aim to develop near-optimal policies that are robust and easy to compute. We will study an online version of the problem. A problem is *online* if at all points in time, the algorithm has to make adaptive decisions based only on past and current information. In contrast, an *offline* algorithm knows all future (stochastic) information up-front. We will use *competitive analysis* to evaluate our algorithms (Borodin and El-Yaniv 2005). We will consider the relative expected performance between an online algorithm and an optimal offline algorithm. We define the minimum ratio between the benefit achieved under the online algorithm and that under the optimal offline algorithm as *competitive ratio* for that online algorithm. An algorithm with a competitive ratio of α is α*-competitive*.

We propose 0.321-competitive online algorithms. Further, we show that an upper bound on the competitive ratio of any algorithm is 1/2. Ours are the first algorithms with performance guarantees for the advance reservation of service with heterogeneous customer needs and preferences. They are also the first algorithms with constant competitive ratios for the adwords problem without any assumption on the bid size and on the stationarity of the arrival process. Despite the conservative performance characterization, we show that our algorithms perform extremely well compared to common heuristics as demonstrated on a real data set from a large hospital system in New York City.

2. Literature Review

Our model is related to many streams of literature, the closest of which are the adwords, the dynamic knapsack, and the appointment-scheduling literature.

2.1. Adwords problems

Our model generalizes adwords problems. Considerable work has been done in this area. If each bid is truncated by the remaining budget, it was shown by Mehta (2012) that a greedy algorithm achieves a worst-case competitive ratio of 1/2 in the adversarial-demand model. For adwords models

Table 1 Results on adwords models.

Reference	Lower bound achieved	Assumption
Mehta (2012)	0.5	adversarial demand, truncated bids
Our work	0.321	stochastic demand
	$1-1/\sqrt{2\pi d}+O(1/d)$	stochastic demand, bid to budget ratio at most $1/d$
Goel and Mehta (2008)	$1-1/e\approx 0.63$	randomly ordered demand, small bids
Mirrokni et al. (2012)	0.76	randomly ordered demand, small bids
Devanur et al. (2012)	$1-1/\sqrt{2\pi d}$	stochastic demands, bid to budget ratio at most $1/d$, $d\geq 2$, truncated bids
Devanur et al. (2011)	$1-1/e\approx 0.63$	i.i.d. demand, truncated bids

in which demands arrive in random orders and bids are small, Goel and Mehta (2008) prove that a greedy algorithm achieves a worst-case ratio of $1-1/e$. Mirrokni et al. (2012) later improve this ratio to 0.76. If demands are i.i.d., but bids are not necessarily small, Devanur et al. (2011) show that a greedy algorithm achieves the worst-case ratio of $1-1/e$. Later, Devanur et al. (2012) show that under stochastic demands, if the bid to budget ratio is at most $1/d$, $d\geq 2$, and if bids can be truncated, then there is an algorithm that achieves a worst-case ratio of $1-1/\sqrt{2\pi d}$. If the bid to budget ratios at most ϵ^2, then the algorithm achieves a worst-case ratio of $1-O(\epsilon)$. Finally, no algorithm can achieve a worst-case ratio that is better than $1-o(1/\sqrt{d})$ when the bid to budget ratios are as large as $1/d$. The main difference between our work and this literature is that we do not make the assumption of truncated bids, small bids, or i.i.d. demand. Furthermore, we study the ratio of expected performance between the online and optimal offline algorithm, rather than the worst-case ratio.

2.2. Dynamic knapsack problems

Our problem is related to multi-constrained dynamic knapsack problems (MKP). In these problems, a set of randomly arriving items must be packed into one or more knapsacks, respecting the capacity constraints of the knapsacks. The goal is to maximize the value of the items packed. Note that our problem is different from these dynamic multi-knapsack problems. In our problem, each customer can be satisfied using one of a subset of resources, rather than any resource, due to preferences, urgency, priorities, etc. These feasibility constraints must be accounted for in the assignment decision. In contrast, a knapsack problems, an object can be placed into any knapsack, as long as the capacity constraints are satisfied.

Dynamic-programming characterizations have been studied in the case of one knapsack (Papastavrou et al. 1996, Kleywegt and Papastavrou 1998, Van Slyke and Young 2000, Lin et al. 2008, Chen and Ross 2014). Some results generalize to the MKP but these results are not sufficient to yield provable approximations (Van Slyke and Young 2000). Many authors have studied online algorithms for the MKP. It is shown in Marchetti-Spaccamela and Vercellis (1995) that no online algorithm for MKP exists with a constant worst-case competitive ratio. Therefore, Marchetti-Spaccamela and Vercellis (1995) and Lueker (1998) study algorithms with bounded additive differences away from the offline optimal. Finally, Chakrabarty et al. (2013) design an algorithm with a bounded worst-case competitive ratio, assuming that the size of each item is very small relative to the capacity, and the value-to-size ratio of each item is upper and lower bounded by two constants. Our model is different from the MKP model because our resources are not interchangeable, as customer preferences for them might be different. Our approach also differs from existing MKP approaches in that we seek to bound the ratio of expected performance between the online and optimal offline algorithm, rather than the worst-case ratio.

2.3. Online resource-allocation problems

Our model falls within the literature on online resource allocation. Adwords and dynamic knapsack problems are subclasses of this literature. Although this is a vast literature, it can be roughly divided into several streams based on the assumptions made about the model. The first stream is focused on designing algorithms for problems in which demands arrive in adversarial fashion (Karp et al. 1990, Kalyanasundaram and Pruhs 1996, Aggarwal et al. 2011, Devanur et al. 2013). The second stream is focused on problems in which demands arrive as a random permutation of a known sequence (Goel and Mehta 2008, Agrawal et al. 2009, Devanur and Hayes 2009, Mahdian and Yan 2011, Karande et al. 2011, Bhalgat et al. 2012). The third stream is focused on problems in which demands are drawn i.i.d. from an unknown distribution (Ghosh et al. 2009, Devanur et al. 2011, Balseiro et al. 2014). The fourth stream is focused on problems in which demands are drawn i.i.d. from an known distribution (Feldman et al. 2009, Agrawal et al. 2009, Feldman et al. 2010, Vee et al. 2010, Jaillet and Lu 2012, Manshadi et al. 2012, Jaillet and Lu 2013). The fifth stream is focused on resource allocation under the small-bid or truncated-bid assumption (Mehta et al. 2007, Buchbinder et al. 2007, Jaillet and Lu 2011, Devanur et al. 2012). Very few papers focus on models with non-stationary stochastic demand as we do, and in these cases, they either assume that bids can be truncated (Ciocan and Farias 2012), or assume that the resource consumption is constant for all demand units, i.e., the problems are matching problems (Alaei et al. 2011, Wang et al. 2016, Gallego et al. 2015).

2.4. Appointment-scheduling problems

Our work is related to the literature on appointment scheduling. Most relevant is the stream of literature focusing on how to assign future appointments to patients. This paradigm is called *advance scheduling*. In the literature of advance scheduling, Truong (2015) studies the analytical properties of a two-class advance-scheduling model and gives efficient methods for computing an optimal scheduling policy. Gocgun and Ghate (2012), Patrick et al. (2008), Feldman et al. (2014), and Gupta and Wang (2008) study structural properties of optimal policies and propose heuristics for several related models, although they do not investigate the theoretical performance of these heuristics. Wang et al. (2016) propose online algorithms with constant competitive ratios for advance scheduling with multiple patient types and with patient preferences. Their model is very close to ours in that the demand processes are allowed to be known non-stationary stochastic processes. They also define the same notion of competitive ratio. However, their model is considerably easier, since they assume that each customer demands a unit amount of capacity. In contrast, we assume that customers have heterogeneous capacity requirements. As we shall show, much of the effort in our algorithms and their analysis is directed towards taking care of these differences in capacity requirement.

The paper is organized as follows. We specify the model and performance metric in Section 3. In Section 4, we prove that 0.5 is an upper bound on the competitive ratio of any online algorithm for this problem. We derive an upper bound on the optimal offline objective in Section 5. In Section 6, we design a basic online algorithm with a competitive ratio of $0.5(1-1/e) \approx 0.316$, which serves to illustrate our key ideas. In Section 7, we refine the algorithm to employ resource sharing in order to obtain an improved competitive ratio of 0.321, as well as an improved empirical performance. In Section 8, we compare the empirical performance of our algorithms against two commonly used heuristics by simulating the algorithms on appointment-scheduling data obtained from a large hospital system in New York City.

3. Model and Performance Metric

We use $[n]$ to denote the set $\{1,2,\ldots,n\}$ and consider a continuous horizon $[0,T]$. There are m resources and n customer types. Resource $j \in [m]$ has capacity $c_j \in \mathbb{R}_+$. Customers of type $i \in [n]$ arrive according to a non-homogeneous Poisson process with rate $\lambda_i(t)$, for $t \in [0,T]$. Let $\Lambda_i = \int_0^T \lambda_i(t)dt$ be the expected total number of arrivals of type-i customers. The arrival rates of all the customer types are known. When a customer arrives, one of the m resources needs to be immediately allocated to the customer, or the customer must be rejected. If resource j is allocated to a customer of type i, exactly u_{ij} units of resource j must be provided. We assume that the $u_{ij} \in [0, c_j]$, $\forall i \in [n]$ and $j \in [m]$, are known. The reward earned for the assignment of customer type i to resource j is also u_{ij}. The objective is to maximize the total expected reward over the horizon, which equivalently maximizes total resource utilization.

Let I be a sample path of customer arrivals over the entire horizon. We say that an algorithm is *offline* if it knows I at time 0. An algorithm is *online* if at any time t, it only knows future arrival rates and the realization of all the arrivals prior to t.

Let ALG(I) be the total amount of resources allocated by an online algorithm ALG. Let OPT(I) be the total amount of resources allocated by an optimal offline algorithm OPT. We define the competitive ratio of ALG to be

$$\text{Competitive Ratio of } ALG = \frac{\mathbf{E}[\text{ALG}(I)]}{\mathbf{E}[\text{OPT}(I)]}, \tag{1}$$

where the expectation is taken over all sample paths I and over the random realizations of the online algorithm. Our definition of competitive ratio follows previous works including Feldman et al. (2009), Jaillet and Lu (2013), and Wang et al. (2016). It is less conservative than the worst-case ratio $\min_I \frac{\text{ALG}(I)}{\text{OPT}(I)}$ that has been more commonly used for online algorithms.

4. Upper Bound on the Competitive Ratio

In this section, we prove upper bounds on competitive ratios that can be achieved by any online algorithm. We first prove a uniform upper bound on the competitive ratio.

PROPOSITION 1. *The competitive ratio of any online algorithm is at most* 0.5.

Proof. Consider an input with two customer types and a single resource. Assume that the horizon is $[0,1]$. The capacity of the resource is $c_1 = 1$.

- Type-1 customers have a very large arrival rate in time $[0,0.5]$, but their arrival rate is 0 after time 0.5. In particular, $\Lambda_1 = \int_0^{0.5} \lambda_1(t)dt \gg 1$, so that we can ignore the event that no type-1 customer arrives. Their utilization for the single resource is $u_{11} = \epsilon/\Lambda_1$ for some very small value ϵ.
- Type-2 customers arrive in time $(0.5,1]$. They have a very small arrival rate $\Lambda_2 = \int_{0.5}^{1} \lambda_2(t)dt = \epsilon$. Their utilization for the resource is $u_{21} = c_1 = 1$.

Since customers of type 2 request the entire resource, the offline algorithm will allocate the resource to a type-2 customer if there is one. The probability that at least one type-2 customer arrives is $1 - e^{-\Lambda_2} = \Lambda_2 + o(\Lambda_2^2) = \epsilon + o(\epsilon^2)$. With probability $1 - o(\Lambda_2) = 1 - o(\epsilon)$, no type-2 customer will arrive, in which case the optimal offline algorithm will accept as many type-1 customers as possible. The expected total utilization of all type-1 customers is $u_{11} \cdot \Lambda_1 = \epsilon$. Suppose $\epsilon \ll c_1 = 1$. Then all type-1 customer can be accepted. In sum, the expected amount of resource allocated by an optimal offline algorithm is

$$\begin{aligned} &1 \cdot (\epsilon + o(\epsilon^2)) + \epsilon \cdot (1 - o(\epsilon)) \\ =&2\epsilon + o(\epsilon^2). \end{aligned}$$

The decision of an online algorithm is whether to accept type-1 customers during time $[0,0.5]$. If it does accept type-1 customers, the online algorithm earns $u_{11} \cdot \Lambda_1 = \epsilon$ in expectation. Otherwise, with probability $\Lambda_2 + o(\Lambda_2^2)$ it earns u_{21}, which is $u_{21}(\Lambda_2 + o(\Lambda_2^2)) = \epsilon + o(\epsilon^2)$ in expectation. In sum, an online algorithm cannot allocate more than $\epsilon + o(\epsilon^2)$ in expectation. Thus, an upper bound on the competitive ratio is

$$(\epsilon + o(\epsilon^2))/(2\epsilon + o(\epsilon^2)),$$

which tends to 0.5 in the limit as $\epsilon \to 0$. $\square$

Next, we prove upper bounds on competitive ratios for special cases in which the utilization u_{ij} for each resource is bounded away from c_j. Specifically, suppose there is some integer $d \geq 2$ for which $u_{ij} \leq c_j/d$ for all $i \in [n], j \in [m]$. In Proposition 2, we prove an upper bound that depends on any finite d. In Proposition 3, we prove an even tighter upper bound for the asymptotic regime $d \to \infty$. We introduce the following technical lemma for proving the parameter-dependent bounds on competitive ratios.

LEMMA 1. *If N is a Poisson random variable and*

$$\sum_{i=1}^{d-1} P(N=i)\frac{i}{d} + P(N \geq d) = 1 - \frac{1}{2d}$$

for some integer $d \geq 2$, then

$$P(N \geq d) \geq \frac{1}{2}.$$

Proof. It is easy to deduce that

$$\begin{aligned} &\sum_{i=1}^{d-1} P(N=i)\frac{i}{d} + P(N \geq d) && (2)\\ \leq &\sum_{i=0}^{d-1} P(N=i)\frac{d-1}{d} + P(N \geq d) \\ = &1 - (1 - P(N \geq d))\frac{1}{d}. && (3) \end{aligned}$$

Thus, (3) is an upper bound on (2). In order to satisfy (2)$= 1 - \frac{1}{2d}$, we must have (3)$\geq 1 - \frac{1}{2d}$. That is,

$$1 - (1 - P(N \geq d))\frac{1}{d} \geq 1 - \frac{1}{2d}$$
$$\implies P(N \geq d) \geq \frac{1}{2}.$$

$\square$

PROPOSITION 2. *For any given integer $d \geq 2$, if $u_{ij} \leq c_j/d$ for all $i \in [n], j \in [m]$, then the competitive ratio of any online algorithm is at most $\frac{4d-2}{4d-1} \leq 1 - \frac{1}{4d}$.*

Proof. Consider a single resource with capacity $c_1 = 1$. Consider two demand types that arrive over horizon $[0,1]$:

- Type-1 customers only arrive during the first half of the horizon, i.e., $[0,0.5]$. During the first half of horizon, their arrival rate is huge $\Lambda_1 = \int_0^{0.5} \lambda_1(t)dt \gg 2d$, so that we can ignore the event that fewer than $2d$ type-1 customers arrive. Assume $u_{11} = \frac{1}{2d} + \epsilon$ for some infinitesimally small ϵ.
- Type-2 customers arrive only during the second half of the horizon, i.e., $(0.5,1]$. Let N be the total number of arrivals of type-2 customers. Assume that $u_{21} = 1/d$.

The optimal online algorithm has only two choices: (i) accept at least one type-1 customer, in which case the total reward is at most $1 - \frac{1}{2d} + O(\epsilon)$ (because the ϵ allocation forbids the last $\frac{1}{2d}$ unit of the resource from being taken); (ii) do not accept any type-1 customer, in which case the total reward is only collected from type-2 customers

$$\sum_{i=1}^{d-1} P(N=i)i \cdot u_{21} + P(N \geq d)d \cdot u_{21} = \sum_{i=1}^{d-1} P(N=i)\frac{i}{d} + P(N \geq d).$$

Assume that the distribution of N is such that

$$\sum_{i=1}^{d-1} P(N=i)\frac{i}{d} + P(N \geq d) = 1 - \frac{1}{2d}.$$

Then the expected total reward earned by the optimal online algorithm is at most $1 - \frac{1}{2d} + O(\epsilon)$.

The optimal offline algorithm also chooses from the above two options, but now first observes the value of N. If $N \geq d$, the optimal offline algorithm accepts d customers of type 2. Otherwise, the optimal offline algorithm accepts $2d-1$ type-1 customers to achieve total reward $1 - \frac{1}{2d} + O(\epsilon)$. In sum the expected total reward of the offline algorithm is

$$P(N \geq d) + P(N < d)(1 - \frac{1}{2d} + O(\epsilon)).$$

By Lemma 1, we have $P(N \geq d) \geq \frac{1}{2}$. Then the expected total reward of the offline algorithm can be bounded by

$$\begin{aligned}
&P(N \geq d) + P(N < d)(1 - \frac{1}{2d} + O(\epsilon)) \\
=&P(N \geq d) + P(N < d)(1 - \frac{1}{2d}) + O(\epsilon) \\
=&P(N \geq d) + (1 - P(N \geq d))(1 - \frac{1}{2d}) + O(\epsilon) \\
=&P(N \geq d)\frac{1}{2d} + 1 - \frac{1}{2d} + O(\epsilon) \\
\geq&\frac{1}{2} \cdot \frac{1}{2d} + 1 - \frac{1}{2d} + O(\epsilon) \\
=&1 - \frac{1}{4d} + O(\epsilon).
\end{aligned}$$

In sum, the competitive ratio of the optimal algorithm is at most

$$\frac{1 - \frac{1}{2d} + O(\epsilon)}{1 - \frac{1}{4d} + O(\epsilon)} = \frac{4d - 2 + O(\epsilon)}{4d - 1 + O(\epsilon)}.$$

When ϵ tends to 0, the ratio becomes

$$\frac{4d-2}{4d-1} \leq 1 - \frac{1}{4d}.$$

□

PROPOSITION 3. *For large positive integer d, if $u_{ij} \leq c_j/d$ for all $i \in [n], j \in [m]$, then the competitive ratio of any online algorithm is at most $1 - \frac{1}{2\sqrt{\pi d}} + o(1/\sqrt{d})$.*

Proof. We construct a special case of our model as follows. Let $c_j = 1$ for all $j \in [m]$. There are $n = m+1$ customer types.

- For each customer type $i = 1, 2, ..., m$, $u_{ii} = 1/d$ and $u_{ij} = 0$ for all $j \neq i$.
- For customer type $m+1$, $u_{m+1,j} = \epsilon$ for all $j \in [m]$. We will let ϵ tend to 0.
- For each customer type $i = 1, 2, ..., m$,

$$\lambda_i(t) = \begin{cases} 0, & \forall t \in [0, T/2) \\ d/T, & \forall t \in [T/2, T]. \end{cases}$$

As a result, $\Lambda_i = \int_0^T \lambda_i(t)dt = d/2$ for all $i = 1, 2, ..., m$.

- For customer type $m+1$,

$$\lambda_{m+1}(t) = \begin{cases} \frac{m}{\epsilon T}, & \forall t \in [0, T/2] \\ 0, & \forall t \in (T/2, T]. \end{cases}$$

As a result, $\Lambda_{m+1} = \int_0^T \lambda_{m+1}(t)dt = \frac{m}{2\epsilon}$.

We first analyze the optimal online algorithm. For each resource $i \in [m]$, in the second half of the horizon, customers of type i will request $\Lambda_i u_{ii} = 0.5$ amount of resource i, i.e., half of the resource, in expectation. On the other hand, in the first half of the horizon, customers of type $m+1$ can totally take $\Lambda_{m+1} u_{m+1,j} = 0.5m$ units of any resource in expectation. We let ϵ tend to 0, so customers of type $m+1$ will request almost exactly $0.5m$ units of any resource.

It is easy to see that the optimal online strategy is to give type-$(m+1)$ customers 0.5^- unit of each of the m resources, where 0.5^- means infinitesimally approaching 0.5 from below as ϵ tends to 0. This is because of symmetry, or more rigorously, because the marginal reward of adding $1/d$ unit of capacity to each resource is decreasing in the amount of the resource that is remaining at time $T/2$.

Starting from time $T/2$, the optimal online algorithm assigns the remaining 0.5^+ unit of each resource i to customers of type i. Let N_i be the number of arrivals of type-i customers, for $i = 1, 2, ..., m$. We must have $\mathbf{E}[N_i] = \Lambda_i = d/2$. The total expected amount of resource i assigned to customers by the end is

$$0.5^- + \mathbf{E}[\min(0.5, N_i u_{ii})],$$

where 0.5^- is the amount assigned to customers of type $m+1$, and $\mathbf{E}[\min(0.5, N_i u_{ii})]$ is the expected amount assigned to customers of type i.

When d is large, we apply the central limit theorem so that $N_i u_{ii}$ is approximated by a normal distribution with mean 0.5 and variance $\frac{1}{2d}$. Then

$$\mathbf{E}[\min(0.5, N_i u_{ii})] = 0.5 - \frac{1}{\sqrt{2d}} \int_0^\infty \phi(x) x dx + o(1/\sqrt{d}) = 0.5 - \frac{1}{2\sqrt{\pi d}} + o(1/\sqrt{d}),$$

where $\phi(\cdot)$ is the standard normal pdf.

In sum, when d is large so that the distribution of N_i can be approximated by the central limit theorem, the optimal online algorithm earns expected reward

$$0.5^- + 0.5 - \frac{1}{2\sqrt{\pi d}} + o(1/\sqrt{d}) = 1 - \frac{1}{2\sqrt{\pi d}} + o(1/\sqrt{d})$$

from each resource i, and earns total expected reward

$$m(1 - \frac{1}{2\sqrt{\pi d}}) + o(m/\sqrt{d})$$

from all the m resources.

Now we analyze the optimal offline algorithm. The optimal offline algorithm will first fill each resource i with customers of type i, for $i = 1, 2, ..., m$. After that, the total remaining capacity of all the m resources will be

$$\sum_{i=1}^m \max(0, 1 - N_i u_{ii}).$$

We again apply the central limit theorem so that each $N_i u_{ii}$ is approximated by a normal distribution with mean 0.5 and variance $\frac{1}{2d}$. Then the total remaining capacity can be written as

$$m - \sum_{i=1}^m N_i u_{ii} - o(m/\sqrt{d}),$$

where m is the total initial capacity of all the m resources; $N_i u_{ii}$ is the amount of resource requested by customers of type i; $-o(m/\sqrt{d})$ represents the loss from the approximation by the central limit theorem, plus the loss from the (ignorable) events that $N_i u_{ii} > 1$. Now $\sum_{i=1}^m N_i u_{ii}$ is approximated by a normal random variable with mean $m/2$ and variance $\frac{m}{2d}$.

Next, the optimal offline algorithm fills the remaining capacity with customers of type $m+1$, who will totally take $(m/2)^-$ units of any resource. After that, the expected remaining capacity is

$$\begin{aligned}
&\mathbf{E}[\max(0, m - \sum_{i=1}^m N_i u_{ii} - o(m/\sqrt{d}) - (m/2)^-)] \\
=&\mathbf{E}[\max(0, m/2 - \sum_{i=1}^m N_i u_{ii} - o(m/\sqrt{d}))] \\
=&\sqrt{\frac{m}{2d}} \int_0^\infty \phi(x) x dx + o(m/\sqrt{d}) \\
=&\frac{\sqrt{m}}{2\sqrt{\pi d}} + o(m/\sqrt{d}).
\end{aligned}$$

The total expected reward earned by the optimal offline algorithm is

$$m - \frac{\sqrt{m}}{2\sqrt{\pi d}} + o(m/\sqrt{d}),$$

which is the total capacity m less the expected remaining capacity.

Finally, the competitive ratio of the optimal online algorithm is

$$\frac{m(1-\frac{1}{2\sqrt{\pi d}}) + o(m/\sqrt{d})}{m - \frac{\sqrt{m}}{2\sqrt{\pi d}} + o(m/\sqrt{d})} = \frac{1-\frac{1}{2\sqrt{\pi d}} + o(1/\sqrt{d})}{1-\frac{1}{2\sqrt{m\pi d}} + o(1/\sqrt{d})},$$

which tends to $1-\frac{1}{2\sqrt{\pi d}} + o(1/\sqrt{d})$ when $m = d$. □

5. Upper Bound on the Optimal Offline Objective

We derive an upper bound on the optimal offline objective, namely $\mathbf{E}[\mathrm{OPT}(I)]$. Since $\mathbf{E}[\mathrm{OPT}(I)]$ is very hard to analyze due to its complex offline properties, we are interested in developing an upper bound on $\mathbf{E}[\mathrm{OPT}(I)]$, which is more tractable. We will later compare the performance of our online algorithms against this upper bound, rather than directly with $\mathbf{E}[\mathrm{OPT}(I)]$.

Our upper bound can be formulated as a static LP, which allocates the expected demands Λ_i, $i \in [n]$, to the capacities c_j, $j \in [m]$. The decision variable x_{ij} of the LP stands for the average number of customers of type i to be allocated to resource j. The LP produces a fractional assignment.

$$\begin{aligned} V^{LP} = \max_{x_{ij}} & \sum_{i\in[n]} \sum_{j\in[m]} x_{ij} u_{ij} \\ \text{s.t.} & \sum_{i\in[n]} x_{ij} u_{ij} \le c_j, \quad \forall j \in [m] \\ & \sum_{j\in[m]} x_{ij} \le \Lambda_i, \quad \forall i \in [n] \\ & x_{ij} \ge 0, \quad \forall i \in [n], j \in [m]. \end{aligned} \tag{4}$$

By the linearity of assignment problems, it can be shown easily that

PROPOSITION 4. *V^{LP} is an upper bound on $\mathbf{E}[OPT(I)]$.*

Proof of Proposition 4.

Proof. Let $a_i(I)$ be the actual number of arrivals of type-i customers in sample path I. Let $\tilde{x}(I)$ be a corresponding optimal offline (fractional) assignment. Then $\tilde{x}(I)$ must satisfy

$$\sum_{i\in[n]} \tilde{x}_{ij}(I) u_{ij} \le c_j, \quad \forall j \in [m],$$

$$\sum_{j\in[m]} \tilde{x}_{ij}(I) \le a_i(I), \quad \forall i \in [n].$$

Taking expectation on both sides, we obtain

$$\sum_{i\in[n]} \mathbf{E}[\tilde{x}_{ij}(I)]u_{ij} \le c_j, \quad \forall j \in [m],$$

$$\sum_{j\in[m]} \mathbf{E}[\tilde{x}_{ij}(I)] \le \mathbf{E}[a_i(I)] = \Lambda_i, \quad \forall i \in [n].$$

These inequalities imply that $\mathbf{E}[\tilde{x}(I)]$ is a feasible solution to (4). Thus V^{LP} must be an upper bound on $\sum_{i\in[n],j\in[m]} \mathbf{E}[\tilde{x}_{ij}(I)]u_{ij}$, which proves the proposition. □

6. Basic Online Algorithm

As a warm up, we design an online algorithm which we prove to have a competitive ratio of at least $0.5(1-1/e) \approx 0.316$. This algorithm serves to illustrate the following two main ideas, which we will later refine to obtain an improved bound.

- *LP-based random routing.* We make use of an optimal solution x^* to the static LP (4) to route customers to resources. Note that this solution assigns demand to supply at an aggregate level, in the expected sense. Given a solution x^*, for each arriving customer of type $i \in [n]$, we randomly route the customer to each candidate resource $j \in [m]$ independently with probability x^*_{ij}/Λ_i. We say a customer is *routed* to resource j if resource j is chosen as a candidate resource for the customer. By random routing, we can conclude that the arrival process of type-i customers who are routed to resource j is a non-homogeneous Poisson process with rate $\lambda_i(t)\frac{x^*_{ij}}{\Lambda_i}$, for $t \in [0,T]$.
- *Reservation by customer type.* After the random routing stage, we make binary admission decisions about whether to commit each resource j to each customer i who is routed to j. If the decision is 'no', we reject the customer. We make this admission decision as follows. For each resource j, we divide the candidate customer types who will potentially be routed to j into two sets based on utilization u_{ij}. Set $L_j \subseteq [n]$ consists of customer types of which the utilizations u_{ij} are larger than $c_j/2$. Mathematically,

$$L_j = \{i \in [n] : u_{ij} > c_j/2\}.$$

The other set $S_j = [n] - L_j$ consists of customer types with utilization u_{ij} that are at most $c_j/2$.

For each resource j, our algorithm chooses one set, either S_j or L_j, whichever has the higher expected total utilization for resource j. The algorithm exclusively reserves resource j for customers whose types are in the chosen set. The algorithm rejects all customer types in the complementary set. This step is meant to resolve conflict in resource usage among different customer types by restricting use of the resource to the most promising subset of customer types.

Large-or-Small (LS) Algorithm:

1. (Pre-processing step) Solve the LP (4). Let x^* be an optimal solution. For each resource j, define

$$U_j^L \equiv \sum_{i \in L_j} x_{ij}^* u_{ij}$$

as the amount of resource j allocated to customer types in L_j by the static LP. Similarly, define

$$U_j^S \equiv \sum_{i \in S_j} x_{ij}^* u_{ij}$$

as the amount of resource j allocated to customer types in S_j by the LP.

2. (Reservation step) Reserve the resource j for customer types in the set L_j if $U_j^L \geq U_j^S$. Otherwise, reserve resource j for customer types in the set S_j.

3. (Random routing step) Upon an arrival of a type-i customer, randomly pick a resource j with probability x_{ij}^*/Λ_i.

4. (Admission step) If the remaining capacity of resource j is at least u_{ij} and i belongs to the set that is reserved for j then accept the customer. Otherwise, reject the customer.

As a consequence of the random routing process, we can separate the analysis for every resource $j \in [m]$. Define

$$U_j \equiv U_j^L + U_j^S$$

as the total amount of resource j allocated by the LP. We will show that in expectation, at least

$$\frac{1}{2}\left(1 - \frac{1}{e}\right) U_j$$

units of resource j will be occupied in LS.

We will use the following technical lemma, which bounds the tail expectation of demands following a compound Poisson distribution.

LEMMA 2. *Let $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. random variables that take values from $[0, \beta]$, for some given $\beta \in [0, \frac{1}{l}]$ with $l \geq 2$ being an integer. Let N be a Poisson random variable. For any given $\alpha \in [0, 1]$, if*

$$\mathbf{E}\left[\sum_{k=1}^{N} X_k\right] = \alpha,$$

we must have

$$\mathbf{E}\left[\min\left(\sum_{k=1}^{N} X_k, 1 - \beta\right)\right] \geq \frac{1-\beta}{l-1} \mathbf{E}[\min(N', l-1)],$$

where N' is a Poisson random variable with mean $\alpha(l-1)/(1-\beta)$. In particular, when $l = 2$,

$$\mathbf{E}\left[\min\left(\sum_{k=1}^{N} X_k, 1 - \beta\right)\right] \geq (1-\beta)\left(1 - e^{-\alpha/(1-\beta)}\right).$$

Proof of Lemma 2.

Proof. Let $Z_1, Z_2, Z_3, ...,$ be a sequence of i.i.d. random variables each following a uniform distribution over $[0,(1-\beta)/(l-1)]$. For every $k = 1, 2, ...,$ define a function

$$\tilde{X}_k(x) \equiv \frac{1-\beta}{l-1}\mathbf{1}(Z_k < x),$$

where $\mathbf{1}(\cdot)$ denotes an indicator function.

Since $\beta \in [0, 1/l]$, we must have $\beta \le (1-\beta)/(l-1)$. It is then easy to check that for any $x \in [0, \beta]$, we have

$$\mathbf{E}[\tilde{X}_k(x)] = \frac{1-\beta}{l-1} \cdot \frac{x}{(1-\beta)/(l-1)} = x.$$

Thus, we have for every $k = 1, 2, ...,$

$$\mathbf{E}[\tilde{X}_k(X_k)|X_k] = X_k.$$

According to Jensen's inequality, we must have

$$\min\left(\sum_{k=1}^{N} X_k, 1-\beta\right) \ge \mathbf{E}\left[\min\left(\sum_{k=1}^{N} \tilde{X}_k(X_k), 1-\beta\right)|X_1, X_2, \dots\right]$$
$$\Longrightarrow \mathbf{E}\left[\min\left(\sum_{k=1}^{N} X_k, 1-\beta\right)\right] \ge \mathbf{E}\left[\min\left(\sum_{k=1}^{N} \tilde{X}_k(X_k), 1-\beta\right)\right].$$

Since $\tilde{X}_k(X_k)$ is either 0 or $(1-\beta)/(l-1)$, the term $\sum_{k=1}^{N} \tilde{X}_k(X_k)$ has the same distribution as $N'(1-\beta)/(l-1)$ where N' is a Poisson random variable with mean

$$\mathbf{E}[N'] = \frac{l-1}{1-\beta}\mathbf{E}\left[\sum_{k=1}^{N} \tilde{X}_k(X_k)\right] = \frac{l-1}{1-\beta}\mathbf{E}\left[\sum_{k=1}^{N} X_k\right] = \frac{l-1}{1-\beta}\alpha.$$

Therefore,

$$\begin{aligned}\mathbf{E}[\min(\sum_{k=1}^{N} X_k, 1-\beta)] \ge& \mathbf{E}\left[\min\left(\sum_{k=1}^{N} \tilde{X}_k(X_k), 1-\beta\right)\right] \\ =& \mathbf{E}[\min(N'(1-\beta)/(l-1), 1-\beta)] \\ =& \frac{1-\beta}{l-1}\mathbf{E}[\min(N', l-1)].\end{aligned}$$

When $l = 2$, this equals $(1-\beta)\left(1-e^{-\alpha/(1-\beta)}\right)$.

□

We are now ready to prove the competitive ratio of LS. The idea is to compare the utilization of each resource j under LS with the utilization of resource j under OFF. The latter is given by $U_j^S + U_j^L$. The former depends on the choice of the set reserved for j, either L_j or S_j. With either choice, we can gauge the total expected utilization, in some cases using Lemma 2, to obtain a lower bound. We then repeat this comparison for all resources j to arrive at a global bound.

THEOREM 1. *LS is at least* $(1-1/e)/2$*-competitive.*

Proof of Theorem 1.

Proof. For each resource $j \in [m]$ there are two cases.

- Case 1: $U_j^L \geq U_j^S$. Let Y_j^L be the total number of customers who are routed to resource j and whose types are in L_j. Y_j^L is a Poisson random variable with mean

$$\mu_j^L \equiv \mathbf{E}[Y_j^L] = \sum_{i \in L_j} x_{ij}^*.$$

Conditional on the value of Y_j^L, the amount of resource j requested by each of the Y_j^L customers is i.i.d. and has mean

$$\bar{u}_j^L \equiv \frac{\sum_{i \in L_j} x_{ij}^* u_{ij}}{\sum_{i \in L_j} x_{ij}^*} = \frac{U_j^L}{\mu_j^L}.$$

If $Y_j^L = 1$, the expected amount of resource j taken by that only customer is just $\bar{u}_j^L$. Thus, we get an expected reward $P(Y_j^L = 1)\bar{u}_j^L$ from the event $Y_j^L = 1$.

If $Y_j^L > 1$, only the first customer can take resource j, and all the other $Y_j^L - 1$ customers will be rejected due to lack of remaining capacity. The expected amount of resource taken by the first customer may not be $\bar{u}_j^L$ since arrivals are non-homogeneous, but must be still greater than $c_j/2$. Thus, we get an expected reward $P(Y_j^L > 1)c_j/2$ from the event $Y_j^L > 1$.

In sum, the expected amount of resource taken by these Y_j^L customers is at least

$$\begin{aligned}
&P(Y_j^L = 1)\bar{u}_j^L + P(Y_j^L > 1)c_j/2 \\
=&\mu_j^L e^{-\mu_j^L} \bar{u}_j^L + \left(1 - e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L}\right) c_j/2 \\
=&U_j^L e^{-\mu_j^L} + \left(1 - e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L}\right) c_j/2. \qquad (5)
\end{aligned}$$

We obtain a lower bound on (5) by minimizing its value with respect to μ_j^L. We can deduce that

$$\begin{aligned}
&\frac{d}{d\mu_j^L}\left[U_j^L e^{-\mu_j^L} + \left(1 - e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L}\right) c_j/2\right] = 0 \\
\Longrightarrow &-U_j^L e^{-\mu_j^L} + \left(1 + e^{-\mu_j^L} - e^{-\mu_j^L} + \mu_j^L e^{-\mu_j^L}\right) c_j/2 = 0 \\
&\Longrightarrow \mu_j^L = 2U_j^L/c_j. \qquad (6)
\end{aligned}$$

It is easy to check that (5) is minimized at solution (6), and the corresponding minimum value of (5) is

$$\begin{aligned}
&U_j^L e^{-2U_j^L/c_j} + \left(1 - e^{-2U_j^L/c_j} - 2U_j^L/c_j e^{-2U_j^L/c_j}\right) c_j/2 \\
=&\left(1 - e^{-2U_j^L/c_j}\right) c_j/2 \\
\geq&\left(1 - e^{-U_j/c_j}\right) c_j/2 \\
\geq&\left(1 - e^{-1}\right) U_j/2.
\end{aligned}$$

The last step follows since $U_j/c_j \leq 1$.

• Case 2: $U_j^L < U_j^S$. Let Y_j^S be the total number of customers who are routed to resource j and whose types are in S_j. Y_j^S is a Poisson random variable with mean

$$\mathbf{E}[Y_j^S] = \sum_{i \in S_j} x_{ij}^*.$$

Let $W_1^S, W_2^S, W_3^S, ...,$ be a sequence of i.i.d. random variables each having distribution

$$P(W_k^S \le x) = \sum_{i \in S_j} \mathbf{1}(u_{ij} \le x) \frac{x_{ij}^*}{\sum_{l \in S_j} x_{lj}^*}, \ \forall k = 1, 2, ...$$

Here each W_k^S can be seen as the random amount of resource j requested by one of the Y_j^S customers conditional on the value of Y_j^S. Then $\sum_{k=1}^{Y_j^S} W_k^S$ represents the total random amount of resource j requested by all the Y_j^S customers. It is easy to check that

$$\mathbf{E}\left[\sum_{k=1}^{Y_j^S} W_k^S\right] = U_j^S \ge \frac{U_j}{2}$$

$$\Longrightarrow \mathbf{E}\left[\sum_{k=1}^{Y_j^S} \frac{W_k^S}{c_j}\right] = \frac{U_j^S}{c_j} \ge \frac{U_j}{2c_j}.$$

If $\sum_{k=1}^{Y_j^S} W_k^S \le c_j$, all the Y_j^S customers will be accepted, and we will get total reward $\sum_{k=1}^{Y_j^S} W_k^S$ from resource j.

If $\sum_{k=1}^{Y_j^S} W_k^S > c_j$, some of the Y_j^S customers must be rejected due to lack of capacity. But whenever a customer is rejected, the remaining available capacity of resource j must be strictly less than $0.5c_j$, since $W_k^S/c_j \in [0, 0.5]$ w.p.1 for every k.

In sum, the total reward we get from resource j is at least

$$\begin{aligned}&\sum_{k=1}^{Y_j^S} W_k^S \cdot \mathbf{1}(\sum_{k=1}^{Y_j^S} W_k^S \le c_j) + 0.5c_j \cdot \mathbf{1}(\sum_{k=1}^{Y_j^S} W_k^S > c_j)\\ \ge &\min(\sum_{k=1}^{Y_j^S} W_k^S, 0.5c_j).\end{aligned}$$

Its expected value can be written as

$$\mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j^S} W_k^S, \frac{c_j}{2}\right)\right] = c_j \mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j^S} \frac{W_k^S}{c_j}, \frac{1}{2}\right)\right].$$

We then apply Lemma 2 to obtain

$$\mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j^S} \frac{W_k^S}{c_j}, \frac{1}{2}\right)\right] \ge \frac{1}{2}\left(1 - e^{-2U_j^S/c_j}\right) \ge \frac{1}{2}\left(1 - e^{-\frac{U_j}{c_j}}\right)$$

$$\Longrightarrow \mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j^S} \frac{W_k^S}{c_j}, \frac{1}{2}\right)\right] \geq \frac{U_j}{2c_j}\left(1-\frac{1}{e}\right)$$

$$\Longrightarrow \mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j^S} W_k^S, \frac{c_j}{2}\right)\right] \geq \frac{U_j}{2}\left(1-\frac{1}{e}\right).$$

In sum, in both cases the expected amount of resource j allocated to customers is at least $U_j(1-1/e)/2$. Summing over every resource $j \in [m]$, we can obtain the performance guarantee of our algorithm

$$\sum_{j\in[m]} U_j \cdot \frac{1}{2}\left(1-\frac{1}{e}\right) = V^{LP} \cdot \frac{1}{2}\left(1-\frac{1}{e}\right) \geq \mathbf{E}\left[\mathrm{OPT}(I)\right] \cdot \frac{1}{2}\left(1-\frac{1}{e}\right).$$

□

Below, we prove that the competitive ratio $(1-1/e)/2$ is tight for LS in the sense that there is at least one problem instance and a corresponding LP solution where LS has relative performance that is bounded above by $(1-1/e)/2$.

PROPOSITION 5. *The competitive ratio $(1-1/e)/2$ is tight for LS.*

Proof. We prove the theorem by constructing a special case where the total expected reward of LS is $\mathbf{E}[\mathrm{OPT}(I)] \cdot (1-1/e)/2$.

Let $n = 2$. Let $\epsilon > 0$ be a small number. Let $u_{11} = 0.1$ and $u_{1j} = 0.1(1-\epsilon)$ for every $j \neq 1$. Let $u_{21} = 0.5+\epsilon$ and $u_{2j} = (0.5+\epsilon)(1-\epsilon)$ for every $j \neq 1$. Let $c_j = 1$ for every $j \in [m]$. The expected number of arrivals is $\Lambda_1 = 5$ and $\Lambda_2 = 0.5/(0.5+\epsilon)$.

We set m to be very large such that (with probability very close to 1) the optimal offline algorithm can assign each demand unit to a distinct resource. Thus we have

$$\mathbf{E}[\mathrm{OPT}(I)] \geq \Lambda_1 u_{12} + \Lambda_2 u_{22} = 5 \cdot 0.1(1-\epsilon) + 0.5/(0.5+\epsilon) \cdot (0.5+\epsilon)(1-\epsilon) = 1-\epsilon.$$

On the other hand, it is easy to verify that the LP has a unique optimal solution, namely $x_{11}^* = \Lambda_1$, $x_{21}^* = \Lambda_2$ and $x_{ij}^* = 0$ for all other i, j. Then LS reserves resource 1 only for type-2 customers. In this way, the probability that LS accepts one customer of type 2 is $1-e^{-\Lambda_2}$. Thus, the expected total reward of LS is

$$u_{21} \cdot (1-e^{-\Lambda_2}) = (0.5+\epsilon) \cdot (1-e^{-0.5/(0.5+\epsilon)}).$$

In sum, the performance ratio of LS can be upper-bounded by

$$\frac{(0.5+\epsilon)\cdot(1-e^{-0.5/(0.5+\epsilon)})}{\mathbf{E}[\mathrm{OPT}(I)]} \leq \frac{(0.5+\epsilon)\cdot(1-e^{-0.5/(0.5+\epsilon)})}{1-\epsilon},$$

which approaches $(1-1/e)/2$ when ϵ tends to 0. Thus, the competitive ratio of LS is at most $(1-1/e)/2$ when ϵ tends to 0.

□

6.1. Parameter-dependent bounds for smaller utilization values

In this section, we modify the *LS* algorithm for cases in which all utilization values u_{ij} are bounded away from c_j. We characterize the performance of the modified algorithm using competitive ratios that depend on model parameters.

Assume that there is some integer $d \geq 2$ for which $u_{ij} \leq c_j/d$ for all $i \in [n], j \in [m]$. The modified *LS* algorithm considers the value of u_{ij} as "large" if $u_{ij} \in (\frac{c_j}{d+1}, \frac{c_j}{d}]$, and "small" if $u_{ij} \in [0, \frac{c_j}{d+1}]$. More precisely, we re-define

$$L_j = \{i \in [n] : u_{ij} > c_j/(d+1)\}$$

and

$$S_j = \{i \in [n] : u_{ij} \leq c_j/(d+1)\}.$$

Also re-define U_j^L and U_j^S based on the modified sets L_j and S_j, respectively, in the same way as before.

In addition to the two options of reserving a resource j for L_j or S_j, the modified *LS* algorithm considers a third option of simply pooling all customer types together. In other words, the third option allows for any customer who is routed to a resource j to take the resource in a first-come first-served fashion.

Intuitively, if we only considered the same (first two) options as in the original *LS* algorithm, the competitive ratio would never exceed 0.5, because in the worst case, the algorithm would always reject half of total demand by the reservation rule. By adding the third option, which potentially utilizes all the demand, we can raise the competitive ratio of the modified *LS* algorithm to 1 as $d \to \infty$. Note that, however, this third option is not helpful in the general case $d = 1$, when it can cause arbitrarily poor performance to pack all customer types together.

Modified Large-or-Small (MLS) Algorithm:

1. (Pre-processing step) Same as for *LS*.
2. (Reservation step) Calculate the following three ratios

$$\text{ratio}_j^L \equiv \frac{1}{d+1}\left[\sum_{k=1}^{d} e^{-\frac{(d+1)U_j^L}{c_j}} \frac{\left(\frac{(d+1)U_j^L}{c_j}\right)^k}{(k-1)!} + \sum_{k=d+1}^{\infty} d \cdot e^{-\frac{(d+1)U_j^L}{c_j}} \frac{\left(\frac{(d+1)U_j^L}{c_j}\right)^k}{k!}\right],$$

$$\text{ratio}_j^S \equiv \frac{1}{d+1}\left[\sum_{k=1}^{d} e^{-\frac{(d+1)U_j^S}{c_j}} \frac{\left(\frac{(d+1)U_j^S}{c_j}\right)^k}{(k-1)!} + \sum_{k=d+1}^{\infty} d \cdot e^{-\frac{(d+1)U_j^S}{c_j}} \frac{\left(\frac{(d+1)U_j^S}{c_j}\right)^k}{k!}\right],$$

$$\text{ratio}_j^{\text{all}} \equiv \frac{U_j}{c_j} \cdot \left(1 - e^{-d} \sum_{i=d}^{\infty} (i-d+1) \frac{d^{i-1}}{i!}\right).$$

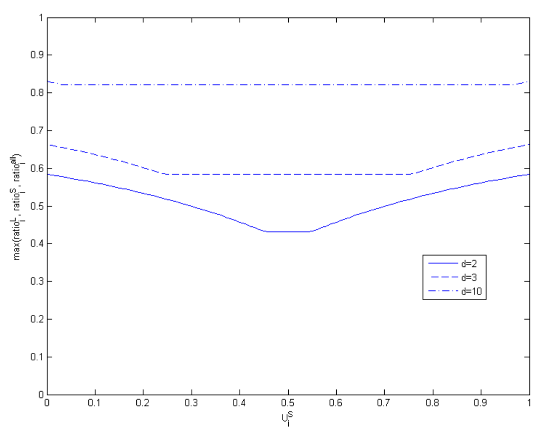
Reserve resource j for customer types in L_j if ratio_j^L is the largest among the three ratios; reserve resource j for customer types in S_j if ratio_j^S is the largest among the three; otherwise, open the resource to all customer types.

3. (Random routing step) Same as for LS.
4. (Admission step) Same as for LS.

Theorem 2 states that the performance of the MLS algorithm is determined by $\max\{\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}}\}$, which is essentially a function of d, U_j^L and U_j^S. Each of the three ratios represents the competitive ratio when the algorithm uses the corresponding reservation strategy. Figure 1 illustrates when each of the three ratios is the largest. For example, when $d = 2$, $U_j = c_j$ and $U_j^S = 0.2U_j$, we have $\text{ratio}_j^S \approx 0.18$, $\text{ratio}_j^L \approx 0.54$ and $\text{ratio}_j^{\text{all}} \approx 0.43$, while the upper bound given by Proposition 2 is about 0.86.

We have ratio_j^L increases in U_j^L, and ratio_j^S increases in U_j^S, because intuitively, the reservation strategies perform better when there are more customers reserved. When U_j^L and U_j^S are balanced, $\text{ratio}_j^{\text{all}}$ is the largest (for $d \geq 2$), because this option pools all customers together and does not discard half of total demand.

Figure 1 **Competitive ratio $\max(\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}})$ of MLS as a function of $U_j^S \in [0, U_j]$ when $U_j = c_j = 1$. In particular, $\text{ratio}_j^{\text{all}}$ is independent of U_j^S as we fix $U_j = 1$. ratio_j^L depends on U_j^S as $U_j^L = U_j - U_j^S$. The flat part in the middle of each curve corresponds to $\text{ratio}_j^{\text{all}}$ being larger than ratio_j^L and ratio_j^S. The decreasing part of each curve corresponds to ratio_j^L being the largest (when U_j^S is small and U_j^L is large). The increasing part corresponds to ratio_j^S being the largest.**



THEOREM 2. *Suppose there is some integer $d \geq 2$ for which $u_{ij} \leq \frac{c_j}{d}$ for all i and j. From each resource j, the modified LS algorithm earns expected reward*

$$c_j \max(ratio_j^L, ratio_j^S, ratio_j^{all}).$$

Proof. Based on the modified sets S_j and L_j, re-define variables such as Y_j^S, Y_j^L, μ_j^L, $\bar{u}_j^L$ and $\{W_k^S\}_{k\geq 1}$ in the same way as in the proof of Theorem 1.

• Case 1: Suppose the modified algorithm reserves resource j for customer types in L_j, i.e., $\text{ratio}_j^L = \max(\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}})$.

The algorithm will allocate resource j to exactly $\min(d, Y_j^L)$ customers whose types are in L_j. Conditioned on $Y_j^L \leq d$, each of these $\min(d, Y_j^L) = Y_j^L$ customers will take $\bar{u}_j^L = U_j^L/\mu_j^L$ unit of resource j in expectation. Conditioned on $Y_j^L > d$, each of these $\min(d, Y_j^L) = d$ customers must take at least $\frac{c_j}{d+1}$ unit of resource j. Thus, We expect to earn at least

$$\begin{aligned}
&\sum_{k=1}^{d} k\bar{u}_j^L \cdot P(Y_j^L = k) + \frac{dc_j}{d+1} \cdot P(Y_j^L > d) \qquad (7)\\
=&\sum_{k=1}^{d} k\frac{U_j^L}{\mu_j^L} \cdot e^{-\mu_j^L} \frac{(\mu_j^L)^k}{k!} + \sum_{k=d+1}^{\infty} \frac{dc_j}{d+1} \cdot e^{-\mu_j^L} \frac{(\mu_j^L)^k}{k!}\\
=&\sum_{k=1}^{d} U_j^L \cdot e^{-\mu_j^L} \frac{(\mu_j^L)^{k-1}}{(k-1)!} + \sum_{k=d+1}^{\infty} \frac{dc_j}{d+1} \cdot e^{-\mu_j^L} \frac{(\mu_j^L)^k}{k!}\\
=&U_j^L P(Y_j^L < d) + \frac{dc_j}{d+1} P(Y_j^L > d).
\end{aligned}$$

We want to find the $\mu_j^L = \mathbf{E}[Y_j^L]$ that minimizes this expected reward. We can deduce that

$$\begin{aligned}
&\frac{\partial}{\partial \mu_j^L}[U_j^L P(Y_j^L < d) + \frac{dc_j}{d+1} P(Y_j^L > d)]\\
=&U_j^L P(Y_j^L < d-1) - U_j^L P(Y_j^L < d) + \frac{dc_j}{d+1} P(Y_j^L > d-1) - \frac{dc_j}{d+1} P(Y_j^L > d)\\
=&\frac{dc_j}{d+1} P(Y_j^L = d) - U_j^L P(Y_j^L = d-1)\\
=&\frac{dc_j}{d+1} e^{-\mu_j^L} \frac{(\mu_j^L)^d}{d!} - U_j^L e^{-\mu_j^L} \frac{(\mu_j^L)^{d-1}}{(d-1)!}\\
=&e^{-\mu_j^L} \frac{(\mu_j^L)^{d-1}}{(d-1)!} \left(\frac{dc_j}{d+1} \cdot \frac{\mu_j^L}{d} - U_j^L \right).
\end{aligned}$$

Setting the derivative to zero, we obtain $\mu_j^L = (d+1)\frac{U_j^L}{c_j}$. It is easy to check that (7) is minimized at $\mu_j^L = (d+1)\frac{U_j^L}{c_j}$, when it is equal to

$$\begin{aligned}
&\sum_{k=1}^{d} k\frac{U_j^L}{\mu_j^L} \cdot e^{-\mu_j^L} \frac{(\mu_j^L)^k}{k!} + \sum_{k=d+1}^{\infty} \frac{dc_j}{d+1} \cdot e^{-\mu_j^L} \frac{(\mu_j^L)^k}{k!}\\
=&\sum_{k=1}^{d} \frac{kc_j}{d+1} \cdot e^{-\frac{(d+1)U_j^L}{c_j}} \frac{\left(\frac{(d+1)U_j^L}{c_j}\right)^k}{k!} + \sum_{k=d+1}^{\infty} \frac{dc_j}{d+1} \cdot e^{-\frac{(d+1)U_j^L}{c_j}} \frac{\left(\frac{(d+1)U_j^L}{c_j}\right)^k}{k!}\\
=&\frac{c_j}{d+1} \left[\sum_{k=1}^{d} e^{-\frac{(d+1)U_j^L}{c_j}} \frac{\left(\frac{(d+1)U_j^L}{c_j}\right)^k}{(k-1)!} + \sum_{k=d+1}^{\infty} d \cdot e^{-\frac{(d+1)U_j^L}{c_j}} \frac{\left(\frac{(d+1)U_j^L}{c_j}\right)^k}{k!} \right]
\end{aligned}$$

$$=c_j \text{ratio}_j^L.$$

Therefore, when the algorithm reserves resource j for L_j, $c_j \text{ratio}_j^L = c_j \max(\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}})$ lower bounds the expected reward from resource j.

- Case 2: Next, suppose the algorithm reserves resource j for customer types in S_j, i.e., $\text{ratio}_j^S = \max(\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}})$.

Recall that $\{W_k^S\}_{k\geq 1}$ is a sequence of i.i.d. random variables, each representing the random amount of resource j requested by one of the Y_j^S customers, conditional on the value of Y_j^S. Since $u_{ij} \leq c_j/(d+1)$ for $i \in S_j$, we must have

$$W_k^S/c_j \in [0, \frac{1}{d+1}].$$

Conditioned on Y_j^S, if $\sum_{k=1}^{Y_j^S} W_k^S > c_j$, the remaining capacity of resource j must be strictly less than $\frac{c_j}{d+1}$ by the end. Thus the total reward we expect to earn from resource j is

$$\begin{aligned}
&\mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j^S} W_k^S, c_j - \frac{c_j}{d+1}\right)\right] \\
=&c_j \cdot \mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j^S} \frac{W_k^S}{c_j}, 1 - \frac{1}{d+1}\right)\right] \\
\geq &c_j \cdot \frac{1}{d+1}\left[\sum_{k=1}^{d} e^{-\frac{(d+1)U_j^S}{c_j}} \frac{\left(\frac{(d+1)U_j^S}{c_j}\right)^k}{(k-1)!} + \sum_{k=d+1}^{\infty} d \cdot e^{-\frac{(d+1)U_j^S}{c_j}} \frac{\left(\frac{(d+1)U_j^S}{c_j}\right)^k}{k!}\right] \\
=&c_j \text{ratio}_j^S.
\end{aligned}$$

Here the inequality is by Lemma 2, when $\beta = 1/(d+1)$, $l = d+1$ and $\alpha = U_j^S/c_j$. Therefore, when the algorithm reserves resource j for S_j, $c_j \text{ratio}_j^S = c_j \max(\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}})$ lower bounds the expected reward from resource j.

- Case 3: Finally, suppose the algorithm opens resource j to all customer types. Let Y_j be the total number of customers who are routed to resource j. Then Y_j is a Poisson random variable with mean

$$\mathbf{E}[Y_j] = \sum_{i\in[n]} x_{ij}^*.$$

Let $W_1, W_2, W_3, ...,$ be a sequence of i.i.d. random variables each having distribution

$$P(W_k \leq x) = \sum_{i\in[n]} \mathbf{1}(u_{ij} \leq x) \frac{x_{ij}^*}{\sum_{l\in[n]} x_{lj}^*}, \;\; \forall k = 1, 2, ...$$

Here each W_k can be seen as the random amount of resource j requested by one of the Y_j customers conditional on the value of Y_j. Then $\sum_{k=1}^{Y_j} W_k$ represents the total random amount of resource j requested by all the Y_j customers, conditioned on Y_j. It is easy to check that $\mathbf{E}\left[\sum_{k=1}^{Y_j} \frac{W_k}{c_j}\right] = \frac{U_j}{c_j}$.

Since $W_k/c_j \in [0, 1/d]$ for every k, the expected amount of resource j taken by these Y_j customers is at least

$$\mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j} W_k, c_j - c_j/d\right)\right] = c_j \mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j} \frac{W_k}{c_j}, 1 - 1/d\right)\right].$$

We then apply Lemma 2 to obtain

$$c_j \mathbf{E}\left[\min\left(\sum_{k=1}^{Y_j} \frac{W_k}{c_j}, 1 - 1/d\right)\right] \geq c_j \frac{1 - 1/d}{d - 1} \mathbf{E}[\min(N', d-1)] = \frac{c_j}{d} \mathbf{E}[\min(N', d-1)],$$

where N' is a Poisson random variable with mean $\frac{U_j}{c_j} \cdot \frac{d-1}{1-1/d} = \frac{U_j}{c_j} \cdot d$. Let N be a Poisson random variable with mean $\mathbf{E}[N] = d$ that is independent of N'. We have $\mathbf{E}[N] \geq \mathbf{E}[N']$ since $U_j/c_j \leq 1$. We can further deduce that

$$\begin{aligned}
&\frac{c_j}{d}\mathbf{E}[\min(N', d-1)] \\
=&\frac{c_j}{d}\left[\sum_{i=1}^{d-1} i \cdot e^{-\mathbf{E}[N']} \frac{(\mathbf{E}[N'])^i}{i!} + \sum_{i=d}^{\infty}(d-1)e^{-\mathbf{E}[N']}\frac{(\mathbf{E}[N'])^i}{i!}\right] \\
=&\frac{c_j}{d}\mathbf{E}[N']\left[\sum_{i=0}^{d-2} e^{-\mathbf{E}[N']}\frac{(\mathbf{E}[N'])^i}{i!} + \sum_{i=d-1}^{\infty}\frac{d-1}{i+1}\cdot e^{-\mathbf{E}[N']}\frac{(\mathbf{E}[N'])^i}{i!}\right] \\
=&\frac{c_j}{d}\mathbf{E}[N']\mathbf{E}[\min(1, \frac{d-1}{N'+1})] \\
\geq&\frac{c_j}{d}\mathbf{E}[N']\mathbf{E}[\min(1, \frac{d-1}{N+1})] \\
=&\frac{c_j}{d}\frac{\mathbf{E}[N']}{\mathbf{E}[N]}\cdot\mathbf{E}[N]\mathbf{E}[\min(1, \frac{d-1}{N+1})] \\
=&\frac{c_j}{d}\frac{\mathbf{E}[N']}{\mathbf{E}[N]}\cdot\mathbf{E}[N]\left[\sum_{i=0}^{d-2} e^{-\mathbf{E}[N]}\frac{(\mathbf{E}[N])^i}{i!} + \sum_{i=d-1}^{\infty}\frac{d-1}{i+1}\cdot e^{-\mathbf{E}[N]}\frac{(\mathbf{E}[N])^i}{i!}\right] \\
=&\frac{c_j}{d}\frac{\mathbf{E}[N']}{\mathbf{E}[N]}\cdot\left[\sum_{i=1}^{d-1} i \cdot e^{-\mathbf{E}[N]}\frac{(\mathbf{E}[N])^i}{i!} + \sum_{i=d}^{\infty}(d-1)e^{-\mathbf{E}[N]}\frac{(\mathbf{E}[N])^i}{i!}\right] \\
=&\frac{c_j}{d}\frac{\mathbf{E}[N']}{\mathbf{E}[N]}\mathbf{E}[\min(N, d-1)] \\
=&U_j \cdot \frac{1}{d}\mathbf{E}[\min(N, d-1)] \\
=&U_j \cdot \frac{1}{d}(\mathbf{E}[N] - \mathbf{E}[\max(N-d+1, 0)]) \\
=&U_j \cdot \left(1 - e^{-d}\sum_{i=d}^{\infty}(i-d+1)\frac{d^{i-1}}{i!}\right)
\end{aligned}$$

$$=c_j \text{ratio}_j^{\text{all}}.$$

Therefore, when the algorithm reserves resource j for all customer types, $c_j \text{ratio}_j^{\text{all}} = c_j \max(\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}})$ lower bounds the expected reward from resource j.

□

As illustrated in Figure 1, $\text{ratio}_j^{\text{all}}$ increases in d. In the next corollary, we prove that $\text{ratio}_j^{\text{all}} = 1 - O(1/\sqrt{d})$. This matches the best possible dependence on d according to Proposition 3.

COROLLARY 1. *If there is some integer $d \geq 2$ for which $u_{ij} \leq \frac{c_j}{d}$ for all i and j, then the competitive ratio of the modified LS is at least $1 - \frac{1}{\sqrt{2\pi d}} + O(1/d)$.*

Proof. The expected amount of resource j allocated to customers is at least

$$\begin{aligned}
&c_j \max(\text{ratio}_j^L, \text{ratio}_j^S, \text{ratio}_j^{\text{all}}) \\
\geq &c_j \text{ratio}_j^{\text{all}} \\
= &U_j \cdot \left(1 - e^{-d} \sum_{i=d}^{\infty} (i - d + 1) \frac{d^{i-1}}{i!}\right) \\
= &U_j \cdot \left(1 - e^{-d} \frac{d^d}{d!} - \frac{1}{d} e^{-d} \sum_{i=d}^{\infty} \frac{d^i}{i!}\right) \\
\geq &U_j \cdot \left(1 - e^{-d} \frac{d^d}{d!} - \frac{1}{d} e^{-d} \sum_{i=0}^{\infty} \frac{d^i}{i!}\right) \\
= &U_j \cdot \left(1 - e^{-d} \frac{d^d}{d!} - \frac{1}{d}\right) \\
= &U_j \cdot \left(1 - \frac{1}{\sqrt{2\pi d}} + O(1/d)\right).
\end{aligned}$$

The last step above follows by Stirling's formula.

Summing over every resource $j \in [m]$, we can obtain the performance guarantee of the modified LS algorithm

$$\begin{aligned}
&\sum_{j \in [m]} U_j \cdot \left(1 - \frac{1}{\sqrt{2\pi d}} + O(1/d)\right) \\
= &V^{LP} \cdot \left(1 - \frac{1}{\sqrt{2\pi d}} + O(1/d)\right) \\
\geq &\mathbf{E}\left[\text{OPT}(I)\right] \cdot \left(1 - \frac{1}{\sqrt{2\pi d}} + O(1/d)\right).
\end{aligned}$$

□

7. Improving the Bound

In this section, we derive an algorithm with an improved competitive ratio compared to LS. This algorithm also groups customer types based on the utilization u_{ij}, but in a more sophisticated way than LS. Moreover, this algorithm also relaxes the random routing step in order to allow customers more opportunities to be assigned to resources. This strategy of allowing greater resource sharing among customer types greatly improves the empirical performance of the algorithm.

We will prove that the competitive ratio of the new algorithm is

$$r^* = \max\left\{r \in (0, 0.5) : r \leq \max_{z\in(0,0.5)} h(z,r)\right\}, \tag{8}$$

where

$$h(z,r) \equiv z - \left[z - \frac{1}{2}\left(1 - \frac{1}{1-2r}\cdot\frac{1}{e^2}\right)\right](1-2r)\left(\frac{1-z}{1-z-r}\right)^{2(1-z)}. \tag{9}$$

We can numerically solve (8) to find that $r^* \approx 0.321$.

For every resource j, we first divide all customer types into two sets S_j and L_j in the same way that LS does. Then, we further partition the customers in S_j into two sets, "medium small" and "tiny", depending on their utilization of resource j. Let

$$M_j = \{i \in S_j : u_{ij} \geq z^* \cdot c_j\}, \tag{10}$$

$$T_j = \{i \in S_j : u_{ij} < z^* \cdot c_j\}, \tag{11}$$

where

$$z^* \equiv \arg\max_{z\in(0,0.5)} h(z, r^*) \approx 0.42. \tag{12}$$

It is easy to check that there is only one maximizer.

Recall that $U_j = \sum_{i\in[n]} x^*_{ij}u_{ij}$, $U^L_j = \sum_{i\in L_j} x^*_{ij}u_{ij}$ and $U^S_j = \sum_{i\in S_j} x^*_{ij}u_{ij}$, where x^* is an optimal solution to the LP (4). We further define $U^M_j \equiv \sum_{i\in M_j} x^*_{ij}u_{ij}$ and $U^T_j \equiv \sum_{i\in T_j} x^*_{ij}u_{ij}$ analogously.

Intuitively, the load values $U^T_j, U^M_j, U^S_j, U^L_j$ serve as estimates for how much capacity of resource j is expected to be utilized by customers of types in the sets T_j, M_j, S_j and L_j, respectively. For a given resource j, if any of the load values dominates the others, it might be a good strategy to reserve resource j exclusively for customers in the corresponding set.

We next categorize every resource j into one of two types based on the load values.

Definition 1. Resource j is a type-A resource if

$$U^S_j \geq -0.5c_j \log(1 - 2r^*U_j/c_j)$$

$$\text{or} \quad U^T_j \geq -(1-z^*)c_j \log\left(1 - \frac{r^*U_j}{c_j(1-z^*)}\right).$$

Otherwise, resource j is a type-B resource.

The motivation for the above definition is as follows. If resource j is of type A, then U_j^S or U_j^T are relatively large compared to other load values, which implies that customers that are routed to resource j by the LP (4) tend to have relatively small utilization u_{ij}. On the other hand, if resource j is of type B, then U_j^L is relatively larger, which implies that customers that are routed to resource j by the LP tend to have relatively large utilization u_{ij}.

Depending on the type of resource, we will reserve each resource wholly for a certain set of customer types. We say that a customer of type i is *admissible* to resource j if this customer can be assigned to resource j by our algorithm. The following definition defines the reservation criteria of the algorithm.

DEFINITION 2. A customer of type i, $i \in [n]$, is admissible to resource j, $j \in [m]$, if and only if at least one of the following criteria holds:

- Resource j is of type A,
- or $i \in M_j \cup L_j$.

We are now ready to specify the improved algorithm.

Refined Large-or-Small Algorithm (RLS):

1. (Pre-processing step) Same as for LS.
2. (Random routing step) Same as for LS.
3. (Admission step) If a customer is admissible to resource j and there is enough remaining capacity in j, then assign the customer to resource j.
4. (Resource-sharing step) If a customer is rejected in the Admission Step, but there is another resource with enough remaining capacity and to which the customer is admissible, then assign the customer to any such resource. Otherwise, reject the customer.

The idea of the algorithm is as follows. If a resource j is of type A, then we can bound from above the left-over capacity of the resource, because a type-A resource tends to be used by a sufficiently high number of customers in sets S_j and T_j. Whenever one such customer is rejected, we know that the remaining capacity is small. Furthermore, since it is not disadvantageous to turn away customers of other types, their utilization being higher than those in S_j and T_j, we will admit customers of all types to a type-A resource. On the other hand, if resource j is of type B, then customers who are admissible to the resource have large utilization values. We can allocate a large enough amount of the resource as soon as one such customer arrives. We do not admit customers with small utilization values to type-B resources in order to leave enough space for customer types in M_j and L_j.

For each resource j, let N_j denote the total number of customers who are routed to resource j in Step (2) of RLS. Note that N_j does not include customers who are assigned resource j in Step

(4) of *RLS*. Let $W_{j1}, W_{j2}, ...,$ be a sequence of i.i.d. random variables each having a distribution that is given by

$$P(W_{j1} \le x) = \sum_{i \in [n]} \mathbf{1}(u_{ij} \le x) \frac{x^*_{ij}}{\sum_{k \in [n]} x^*_{kj}}.$$

That is, each variable W_{j1} can be seen as the utilization u_{ij} of a single random customer who is routed to resource j during the horizon. Since the probability that such a customer has type i is $\frac{x^*_{ij}}{\sum_{k \in [n]} x^*_{kj}}$, W_{j1} takes value u_{ij} with this probability. The following lemma gives a lower bound on the expected amount of capacity of a type-A resource that will be allocated by *RLS*.

LEMMA 3. *If resource j is of type-A, then the expected amount of resource j allocated by RLS is at least*

$$\max \left\{ \mathbf{E} \left[\min(\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \le 0.5c_j), 0.5c_j) \right], \mathbf{E} \left[\min(\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \le z^* c_j), (1 - z^*)c_j) \right] \right\}.$$

Proof. $\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \le 0.5c_j)$ has the same distribution as the total (random) amount of resource j requested by customers who are routed to resource j and whose types are in S_j. If the actual amount of resource j allocated to customers is less than $\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \le 0.5c_j)$, it must be that at least one customer with type in S_j is rejected due to lack of remaining capacity of resource j. In such a case, the actual amount of resource j allocated to customers must be at least $0.5c_j$, because otherwise the customer with type in S_j would not have been rejected. Thus, the total amount of resource j allocated by our algorithm is at least $\min(\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \le 0.5c_j), 0.5c_j)$.

A similar argument applies to customers with types in T_j. $\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \le z^* c_j)$ has the same distribution as the total amount of resource j requested by customers who are routed to resource j and whose types are in T_j. If at least one of these requests is rejected, the remaining capacity of resource j must be at most $z^* c_j$. Thus, the total amount of resource j allocated to customers is at least $\min(\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \le z^* c_j), (1 - z^*)c_j)$.

The proof follows when we take expectation of the lower bounds. □

Let $\mu_j^M = \sum_{i \in M_j} x^*_{ij}$ and $\mu_j^L = \sum_{i \in L_j} x^*_{ij}$ be the expected number of customers who are routed to resource j and whose types are in M_j and L_j, respectively. The following lemma gives a lower bound on the expected amount of capacity of a type-B resource that will be allocated by *RLS*.

LEMMA 4. *If resource j is of type B, then the expected amount of resource j allocated to customers in RLS is at least*

$$\min\{z^* c_j, e^{-\mu_j^M} [U_j^L e^{-\mu_j^L} + 0.5c_j(1 - e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L})] + (1 - e^{-\mu_j^M}) z^* c_j\}.$$

Proof. If any customer is assigned to resource j in Step (4) of the algorithm, then at least z^*c_j of resource j is allocated, since every customer type i that is admissible to resource j satisfies $u_{ij} \geq z^*c_j$.

If no customer is assigned to resource j by Step (4), then resource j can only be allocated to customers who are directly routed to this resource, i.e. in Step (3) of RLS. We consider three cases:

- At least one customer with type in M_j is routed to resource j. This event occurs with probability $1 - e^{-\mu_j^M}$. In such a case, we use z^*c_j as the lower bound on the amount of resource j taken by customers.

- No customer with type in M_j is routed to resource j, and exactly one customer with type in L_j is routed to resource j. This event occurs with probability $e^{-\mu_j^M} \cdot \mu_j^L e^{-\mu_j^L}$. Conditional on this event, the expected amount of resource j taken by the only customer with type in L_j is

$$\frac{\sum_{i \in L_j} x_{ij}^* u_{ij}}{\sum_{i \in L_j} x_{ij}^*} = \frac{U_j^L}{\mu_j^L}.$$

- No customer with type in M_j is routed to resource j, and more than one customer with type in L_j are routed to resource j. This event occurs with probability $e^{-\mu_j^M}(1 - e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L})$. In this event, we use $0.5c_j$ as the lower bound on the amount of resource j taken by the customer in L_j, by definition of L_j.

In summary, if no customer is assigned to resource j in Step (3), the expected amount of resource j taken by customers routed to the resource is at least

$$\begin{aligned}
&(1 - e^{-\mu_j^M}) z^* c_j + e^{-\mu_j^M} \cdot \mu_j^L e^{-\mu_j^L} \cdot \frac{U_j^L}{\mu_j^L} + e^{-\mu_j^M}(1 - e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L}) 0.5 c_j \\
=& e^{-\mu_j^M} [U_j^L e^{-\mu_j^L} + 0.5 c_j (1 - e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L})] + (1 - e^{-\mu_j^M}) z^* c_j.
\end{aligned}$$

We complete the proof by combining this result with the lower bound z^*c_j for the case that a customer is assigned to resource j by Step (4) of the algorithm. □

We combine the previous two lemmas to prove the performance guarantee of the algorithm.

THEOREM 3. *For each resource j, the expected amount of resource j allocated to customers is at least r^*U_j.*

Proof. First consider the case that resource j is of type A. Since $\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \leq 0.5c_j)$ and $\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \leq z^* c_j)$ has compound Poisson distribution with mean

$$\mathbf{E}\left[\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \leq 0.5c_j)\right] = U_j^S, \quad \mathbf{E}\left[\sum_{k=1}^{N_j} W_{jk} \mathbf{1}(W_{jk} \leq z^* c_j)\right] = U_j^T,$$

we can apply Lemma 2 to get

$$\begin{aligned}\mathbf{E}\left[\min(\sum_{k=1}^{N_j} W_{jk}\mathbf{1}(W_{jk}\le 0.5c_j), 0.5c_j)\right] =& c_j\mathbf{E}\left[\min(\sum_{k=1}^{N_j} \frac{W_{jk}}{c_j}\mathbf{1}(W_{jk}\le 0.5c_j), 0.5)\right]\\ \ge& c_j 0.5\left(1-e^{-2U_j^S/c_j}\right),\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}\left[\min(\sum_{k=1}^{N_j} W_{jk}\mathbf{1}(W_{jk}\le z^*c_j), (1-z^*)c_j)\right] =& c_j\mathbf{E}\left[\min(\sum_{k=1}^{N_j} \frac{W_{jk}}{c_j}\mathbf{1}(W_{jk}\le z^*c_j), 1-z^*)\right]\\ \ge& c_j(1-z^*)\left(1-e^{-\frac{U_j^T}{c_j(1-z^*)}}\right).\end{aligned}$$

Then according to Definition 1, we have by the definition of type-A resources

$$U_j^S \ge -0.5c_j\log(1-2r^*U_j/c_j) \quad\Longrightarrow\quad c_j 0.5(1-e^{-2U_j^S/c_j}) \ge r^*U_j,$$

or

$$U_j^T \ge -(1-z^*)c_j\log(1-\frac{r^*U_j}{c_j(1-z^*)}) \quad\Longrightarrow\quad c_j(1-z^*)(1-e^{-\frac{U_j^T}{c_j(1-z^*)}}) \ge r^*U_j.$$

In sum, we have

$$\begin{aligned}&\max\left\{\mathbf{E}\left[\min(\sum_{k=1}^{N_j} W_{jk}\mathbf{1}(W_{jk}\le 0.5c_j), 0.5c_j)\right], \mathbf{E}\left[\min(\sum_{k=1}^{N_j} W_{jk}\mathbf{1}(W_{jk}\le z^*c_j), (1-z^*)c_j)\right]\right\}\\ &\ge \max\left\{c_j 0.5(1-e^{-2U_j^S/c_j}), c_j(1-z^*)(1-e^{-\frac{U_j^T}{c_j(1-z^*)}})\right\}\\ &\ge r^*U_j.\end{aligned}$$

This proves the theorem for type-A resources.

Now we consider the case that resource j is of type B. Starting from this point, we will assume without loss of generality that $c_j = 1$. Based on Lemma 4, we need to show

$$\min\{z^*, e^{-\mu_j^M}[U_j^L e^{-\mu_j^L} + 0.5(1-e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L})] + (1-e^{-\mu_j^M})z^*\} \ge r^*U_j.$$

Since $z^* > r^*$ as we numerically checked, it suffices to show

$$e^{-\mu_j^M}\left[U_j^L e^{-\mu_j^L} + 0.5\left(1-e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L}\right)\right] + \left(1-e^{-\mu_j^M}\right)z^* \ge r^*U_j$$

based on Lemma 4.

By examining the first and second derivatives of $U_j^L e^{-\mu_j^L} + 0.5(1-e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L})$ with respect to μ_j^L, it is easy to check that

$$\begin{aligned}e^{-\mu_j^M}\left[U_j^L e^{-\mu_j^L} + 0.5\left(1-e^{-\mu_j^L} - \mu_j^L e^{-\mu_j^L}\right)\right] + \left(1-e^{-\mu_j^M}\right)z^* \ge&\quad e^{-\mu_j^M}\cdot 0.5\left(1-e^{-2U_j^L}\right) + \left(1-e^{-\mu_j^M}\right)z^*\\ =&\quad z^* - e^{-\mu_j^M}\left[z^* - 0.5\left(1-e^{-2U_j^L}\right)\right].\end{aligned}$$

If $z^* < 0.5(1-e^{-2U_j^L})$, we must have $z^* - e^{-\mu_j^M}[z^* - 0.5(1-e^{-2U_j^L})] > z^* > r^* = r^* c_j \geq r^* U_j$, which proves the theorem for this case.

Now suppose $z^* \geq 0.5(1-e^{-2U_j^L})$. Since $\mu_j^M = \sum_{i\in M_j} x_{ij}^* \geq \sum_{i \in M_j} x_{ij}^* \cdot 2u_{ij} = 2U_j^M$, we have

$$\begin{aligned} z^* - e^{-\mu_j^M}\left[z^* - 0.5\left(1-e^{-2U_j^L}\right)\right] \geq & z^* - e^{-2U_j^M}\left[z^* - 0.5\left(1-e^{-2U_j^L}\right)\right] \\ = & z^* - e^{-2(U_j^S - U_j^T)}\left[z^* - 0.5\left(1-e^{-2(U_j - U_j^S)}\right)\right]. \end{aligned} \tag{13}$$

It is easy to see that (13) is decreasing in U_j^T. We next show that, given U_j and U_j^T, (13) is also decreasing in U_j^S.

$$\begin{aligned} & \frac{\partial}{\partial U_j^S}\left[z^* - e^{-2(U_j^S - U_j^T)}[z^* - 0.5(1-e^{-2(U_j - U_j^S)})]\right] \\ = & 2e^{-2(U_j^S - U_j^T)}[z^* - 0.5(1-e^{-2(U_j-U_j^S)})] - e^{-2(U_j^S - U_j^T)} e^{-2(U_j - U_j^S)} \\ = & e^{-2(U_j^S - U_j^T)}(2z^* - 1) \\ < & 0. \end{aligned}$$

According to Definition 1, we must have

$$U_j^S < -0.5 \log(1 - 2r^* U_j) \tag{14}$$

and

$$U_j^T < -(1-z^*)\log\left(1 - \frac{r^* U_j}{1-z^*}\right). \tag{15}$$

Since (13) is decreasing in U_j^S and U_j^T, we can plug in (14) and (15) and obtain

$$\begin{aligned} & z^* - e^{-2(U_j^S - U_j^T)}[z^* - 0.5(1-e^{-2(U_j - U_j^S)})] \\ \geq & z^* - e^{-2(-0.5\log(1-2r^*U_j) + (1-z^*)\log(1-\frac{r^*U_j}{1-z^*}))}[z^* - 0.5(1-e^{-2(U_j + 0.5\log(1-2r^*U_j))})] \\ = & z^* - \left[z^* - \frac{1}{2}\left(1 - \frac{1}{1-2r^*U_j}\cdot\frac{1}{e^{2U_j}}\right)\right](1-2r^*U_j)\left(\frac{1-z^*}{1-z^*-r^*U_j}\right)^{2(1-z^*)}. \end{aligned} \tag{16}$$

Since z^* and r^* are constants, (16) is a function of a single variable U_j. It is easy to check that this function is increasing and concave in U_j for $U_j \leq c_j = 1$. Moreover, it equals 0 at $U_j = 0$. Therefore,

$$\begin{aligned} & z^* - \left[z^* - \frac{1}{2}\left(1 - \frac{1}{1-2r^*U_j}\cdot\frac{1}{e^{2U_j}}\right)\right](1-2r^*U_j)\left(\frac{1-z^*}{1-z^*-r^*U_j}\right)^{2(1-z^*)} \\ \geq & U_j\left[z^* - \left[z^* - \frac{1}{2}\left(1 - \frac{1}{1-2r^*}\cdot\frac{1}{e^2}\right)\right](1-2r^*)\left(\frac{1-z^*}{1-z^*-r^*}\right)^{2(1-z^*)}\right] \\ = & U_j h(z^*, r^*) \\ = & U_j r^*. \end{aligned}$$

This completes the proof for the theorem. □

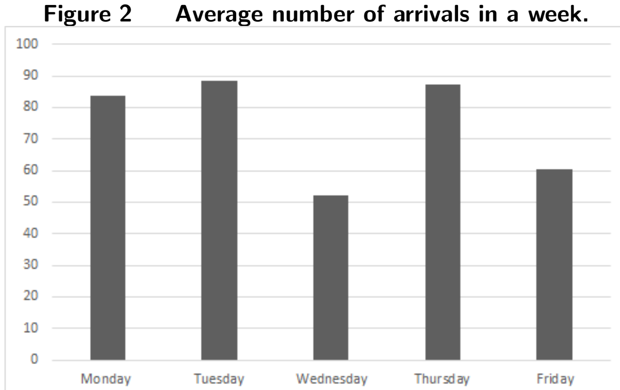
8. Numerical Study

We compare the empirical performance of our algorithms against two commonly used heuristics by simulating the algorithms on appointment-scheduling data obtained from a large hospital system in New York City.

We obtain our data set from an Allergy department in the hospital system. The data set contains more than 20000 appointment entries recorded in the year 2013. Each entry in the data records information about one appointment. The entry includes the date that the patient makes the appointment, the exact time of the appointment, whether the patient eventually showed up to the original appointment, canceled the appointment some time later, or missed the appointment.

The average total number of patients who arrive to make appointments on each day is shown in Figure 2. It can be readily seen that the arrival pattern is highly non-stationary, as the average total number of arrivals on Thursday is 60% more than that on Wednesday.

Figure 2 Average number of arrivals in a week.



We simulate a discrete horizon of 200 days. In each day, a random number of patients arrive to make appointments. Each patient needs to be assigned an appointment of 15 min, 30 min, 45 min, depending on his or her condition. By medical necessity, some patients must be assigned same-day appointments if at all. We call these patients *urgent patients*. Other patients can be assigned to any day in the future. We call these patients *regular patients*. The relative proportions of patients in each priority category are summarized in Table 2. We impose a requirement that regular patients must be assigned an appointment that is no more than 20-days away from the date of his or her first request for an appointment. Although this hard deadline is not strictly enforced in reality, consideration for patient satisfaction often impels the administration to limit as much as possible the number of days that each patient must be made to wait. Our deadline mimics this effect.

We assume a 5-day work week. We estimate the expected number of patients arriving per day of the week as shown in Figure 2. We assume that each patient randomly and independently falls into one of the six categories shown in Table 2. All patients, whether urgent or regular, arrive at the

Table 2 Percentage of patients in different categories.

	15 min	30 min	45 min
urgent	27%	1%	0%
regular	45%	14%	9%

beginning of a day. In our model, the type of a patient is defined by both the time of arrival (one of these 200 days) and one of the six conventional "types" as defined in Table 2. Overall, there are 1,200 patient types in our model. Moreover, each $\lambda_i(t)$ is only non-zero for one day of the horizon.

We assume that there are multiple sessions on each day. Each session corresponds to a resource in our model. We vary the session length among 1, 1.5, 2, 3, or 4 hours. We assume that a patient can be assigned to any appointment within a day, as long as there is enough service time remaining and the day falls within the deadline to serve the patient. We vary the number of sessions that are available per day.

We test the following two algorithms

- Our basic online algorithm (LS).
- Our modified LS algorithm (MLS).
- Our refined algorithm (RLS).
- A greedy heuristic (GRD) that tries to assign every patient to the most recent session that is available and falls within his deadline.
- A heuristic ($RSRV$) that reserves for each category an amount of capacity that is approximately equal to the average utilization of that category. This reservation is nested in the sense that higher-priority patients have access to their reserved capacity, as well as the reserved capacity of all lower-priority categories. The heuristic then assigns patients greedily to the reserved capacity.
- The primal-dual algorithm (PD) given by Buchbinder et al. (2007).

For each algorithm and each test case, we simulate the total length of appointments made during the entire 200 periods and calculate the average total length over 1000 replicates. We report the ratio of this average number relative to the optimal objective value of the upper bound given in (4). Note that in this numerical setting, the LP (4) can be solved by a simple greedy approach. First, we pack all urgent patient types into the same-day appointment sessions. Then, for period t from 1 to 200, we pack regular patient types which arrive in period t into the earliest available sessions. This yields an optimal solution to LP (4). In more general settings for which simple heuristics do not give optimal LP solutions, one can always apply efficient packing LP solvers (Allen-Zhu and Orecchia 2018).

Tables 3 to 7 summarize the performance of the algorithms. The *scale* is the ratio of total capacity to total demand. In each cell, the first number is the performance of the algorithm relative to the upper bound (4); the second number is the average number of days that admitted regular patients need to wait under the algorithm. We make several observations:

Table 3 **Algorithm performance and average waiting time of regular patients. The length of each session is 1 hour.**

Number of sessions	Scale	LS	RLS	GRD	RSRV	PD
18	70.5%	69.6%, 18.6	94.7%, 16.3	98.4%, 14.4	80.2%, 17.5	97.9%, 16.7
19	74.4%	69.2%, 18.3	94.2%, 15.4	98.4%, 11.0	78.8%, 17.4	97.6%, 16.9
20	78.3%	69.3%, 18.1	94.3%, 13.5	98.1%, 6.4	77.4%, 17.2	97.1%, 16.9
21	82.2%	69.4%, 17.7	94.4%, 11.1	96.2%, 2.4	76.2%, 16.9	96.4%, 16.7
22	86.2%	69.9%, 17.2	94.5%, 10.8	96.0%, 1.3	75.1%, 16.6	95.5%, 16.2
23	90.1%	70.3%, 16.2	94.5%, 11.0	96.0%, 0.9	74.0%, 16.2	94.3%, 15.5
24	94.0%	70.8%, 13.8	94.2%, 10.2	95.5%, 0.7	73.1%, 15.6	92.5%, 15.6
25	97.9%	70.7%, 9.0	93.7%, 6.8	94.5%, 0.5	72.2%, 14.6	90.5%, 15.9
26	101.8%	70.7%, 4.2	93.4%, 3.3	94.7%, 0.4	73.0%, 12.8	90.3%, 16.1
27	105.7%	71.0%, 1.6	95.3%, 1.3	95.7%, 0.3	74.8%, 9.7	91.3%, 16.0
28	109.7%	71.0%, 0.9	96.6%, 0.8	96.6%, 0.2	76.6%, 6.3	92.4%, 15.7
29	113.6%	70.9%, 0.6	97.4%, 0.6	97.2%, 0.2	77.4%, 3.2	93.4%, 15.4
30	117.5%	71.0%, 0.5	97.8%, 0.4	97.7%, 0.2	77.8%, 1.4	94.2%, 15.1
31	121.4%	71.0%, 0.4	98.3%, 0.3	98.2%, 0.1	78.0%, 0.9	95.1%, 14.4
32	125.3%	71.0%, 0.3	98.6%, 0.3	98.4%, 0.1	78.1%, 0.6	96.4%, 12.9
33	129.2%	70.9%, 0.2	98.8%, 0.2	98.7%, 0.1	78.1%, 0.4	97.2%, 12.0

Table 4 **Algorithm performance and average waiting time of regular patients. The length of each session is 1.5 hours.**

Number of sessions	Scale	MLS	RLS	GRD	RSRV	PD
12	70.5%	76.2%, 18.6	98.2%, 15.9	98.4%, 14.4	92.7%, 17.7	97.9%, 17.5
13	76.4%	76.3%, 18.3	97.9%, 12.8	98.4%, 8.8	91.5%, 17.5	97.4%, 17.5
14	82.2%	76.5%, 17.7	97.7%, 10.8	96.3%, 2.4	90.2%, 17.3	96.7%, 17.4
15	88.1%	77.1%, 16.8	97.5%, 11.5	96.1%, 1.1	88.9%, 17.0	95.7%, 17.3
16	94.0%	77.5%, 13.9	96.7%, 10.8	95.6%, 0.7	87.6%, 16.5	93.9%, 17.2
17	99.9%	77.5%, 6.8	95.7%, 5.5	93.7%, 0.5	86.5%, 15.7	91.5%, 17.1
18	105.7%	77.6%, 1.7	97.2%, 1.4	95.7%, 0.3	90.3%, 14.3	93.7%, 16.8
19	111.6%	77.7%, 0.8	98.0%, 0.7	96.8%, 0.2	94.2%, 11.0	95.0%, 16.7
20	117.5%	77.7%, 0.5	98.4%, 0.5	97.6%, 0.2	97.9%, 6.3	95.9%, 16.5
21	123.4%	77.7%, 0.4	98.7%, 0.3	98.1%, 0.1	99.2%, 2.2	96.5%, 16.5
22	129.2%	77.7%, 0.3	99.0%, 0.3	98.5%, 0.1	99.6%, 0.9	97.1%, 16.4

Table 5 **Algorithm performance and average waiting time of regular patients. The length of each session is 2 hours.**

Number of sessions	Scale	MLS	RLS	GRD	RSRV	PD
9	70.5%	78.0%, 18.6	99.1%, 16.0	98.5%, 14.4	91.1%, 17.9	97.9%, 17.7
10	78.3%	77.3%, 18.1	99.0%, 11.2	98.4%, 6.4	89.2%, 17.7	97.3%, 17.6
11	86.2%	77.9%, 17.2	98.6%, 11.6	96.2%, 1.4	87.2%, 17.4	96.3%, 17.5
12	94.0%	78.3%, 13.9	97.3%, 11.0	95.7%, 0.7	85.4%, 17.1	94.5%, 17.4
13	101.8%	78.3%, 4.4	96.4%, 3.7	94.4%, 0.4	85.3%, 16.4	93.0%, 17.3
14	109.7%	78.9%, 1.0	98.0%, 0.9	96.5%, 0.3	90.2%, 15.0	95.1%, 17.2
15	117.5%	79.2%, 0.5	98.6%, 0.5	97.6%, 0.2	95.2%, 10.8	96.3%, 17.1
16	125.3%	79.1%, 0.3	99.0%, 0.3	98.2%, 0.1	99.4%, 4.8	97.0%, 16.9

Table 6 **Algorithm performance and average waiting time of regular patients. The length of each session is 3 hours.**

Number of sessions	Scale	MLS	RLS	GRD	RSRV	PD
6	70.5%	84.3%, 18.6	99.3%, 16.3	98.6%, 14.3	93.9%, 17.7	98.0%, 18.0
7	82.2%	84.5%, 17.8	98.9%, 11.6	96.5%, 2.4	91.8%, 17.2	96.9%, 17.7
8	94.0%	84.9%, 13.9	97.4%, 11.6	95.7%, 0.7	89.5%, 16.3	94.8%, 17.6
9	105.7%	85.3%, 1.7	97.5%, 1.5	95.6%, 0.3	92.4%, 13.3	94.7%, 17.5
10	117.5%	85.4%, 0.5	98.8%, 0.5	97.6%, 0.2	99.6%, 4.3	96.5%, 17.4
11	129.2%	85.4%, 0.3	99.4%, 0.3	98.5%, 0.1	99.9%, 0.7	97.4%, 17.2

Table 7 **Algorithm performance and average waiting time of regular patients. The length of each session is 4 hours.**

Number of sessions	Scale	MLS	RLS	GRD	RSRV	PD
5	78.3%	86.0%, 18.1	99.2%, 11.9	98.6%, 6.2	93.7%, 17.4	97.4%, 17.8
6	94.0%	86.7%, 13.9	97.3%, 11.9	95.8%, 0.7	91.1%, 16.0	94.9%, 17.6
7	109.7%	87.1%, 1.0	98.2%, 0.9	96.5%, 0.3	97.0%, 8.2	95.6%, 17.5
8	125.3%	87.5%, 0.3	99.3%, 0.3	98.3%, 0.1	99.8%, 0.8	97.1%, 17.3

Table 8 **Algorithm performance and average waiting time of regular patients. Regular patients arrive only on Mondays, and same-day patients arrive only on the other weekdays. The length of each session is 1 hour.**

Number of sessions	Scale	LS	RLS	GRD	RSRV	PD
16	74.4%	75.5%, 17.4	94.7%, 6.3	93.9%, 0.6	86.1%, 1.1	86.7%, 7.3
17	79.1%	75.7%, 17.5	94.3%, 8.1	92.2%, 0.6	85.3%, 1.0	85.3%, 3.2
18	83.7%	76.0%, 17.2	94.2%, 9.8	90.6%, 0.5	84.0%, 0.9	84.6%, 2.1
19	88.4%	76.1%, 16.3	94.0%, 11.0	89.0%, 0.5	82.3%, 0.9	83.7%, 1.9
20	93.0%	76.1%, 11.1	93.9%, 8.4	88.0%, 0.5	80.8%, 0.8	83.0%, 1.7
21	97.7%	76.2%, 3.5	94.4%, 2.7	88.6%, 0.4	80.6%, 0.8	83.6%, 1.6
22	102.3%	76.1%, 2.0	94.7%, 1.7	89.5%, 0.4	80.7%, 0.7	84.4%, 1.5
23	107.0%	75.7%, 1.5	94.5%, 1.3	90.1%, 0.4	80.8%, 0.7	85.2%, 1.4
24	111.6%	75.6%, 1.2	95.0%, 1.0	91.9%, 0.3	82.0%, 0.6	87.0%, 1.3
25	116.3%	75.6%, 1.0	95.7%, 0.8	93.8%, 0.3	83.4%, 0.6	88.9%, 1.3
26	120.9%	75.6%, 0.8	96.2%, 0.7	95.4%, 0.3	84.8%, 0.6	90.5%, 1.2
27	125.6%	75.6%, 0.6	96.6%, 0.6	96.7%, 0.2	86.2%, 0.5	91.8%, 1.2

Table 9 **Performance relative to the upper bound given in** (4)**, when parameters are randomly generated.**

Number of sessions	LS	RLS	GRD	RSRV	PD
Worst Setting	44.3%	68.4%	66.3%	43.7%	67.6%
Average Setting	65.2%	96.3%	95.9%	85.2%	95.9%

- The refined algorithm RLS is never more than 7% worse than the upper bound on average in each of the scenarios tested. The reservation heuristic $RSRV$ could be as much as 16% worse than the upper bound on average. The greedy heuristic GRD could be as much as 5.7% worse than the upper bound on average.
- Predictably, the refined algorithm RLS dominates the basic algorithms LS and MLS. This performance gain comes from better resource sharing.

- The greedy heuristic *GRD* also performs consistently better than the static reservation heuristic *RSRV*, except when the scale is high. Most likely, the greedy heuristic allows greater resource sharing among different customer types, which results in better resource utilization. However, when the scale is high, there is an abundance of capacity, so that resource sharing is less important.
- The greedy heuristic *GRD* tends to be good when the scale is either very large or very small. These are situations in which it is easier to do well. When there is little capacity, the utilization can be kept high even with a naive algorithm because there is relative very high demand. When there is an abundance of capacity, the utilization can be close to optimal because a high proportion of demand can be accommodated. Therefore, an algorithm offers the most value relative to a naive heuristic when the scale is moderate.
- Similar to *GRD*, the Primal-Dual algorithm performs well when the scale is either large or small. However, its performance is slightly worse than *GRD* in most cases. This might be because the Primal-Dual algorithm is specially designed to improve the worst-case performance, whereas we report the average-case performance.
- The refined algorithm *RLS* performs significantly better than, or is very close to, the better of the two heuristics. It performs much better than the heuristics when the scale is moderate, which is when an algorithm offers the most value relative to a naive heuristic.
- The average number of days that admitted regular patients need to wait under the greedy heuristic is the smallest among all algorithms. This is because the greedy heuristic allows regular patients to take sessions that could have been reserved for same-day patients. In Table 8, we test scenarios that better illustrate the outcomes of making greedy assignments. We let regular patients arrive only on Mondays, and let same-day patients arrive only on the other four weekdays. We find that the greedy heuristic results in extremely short waiting times compared to other algorithms, but the performance of the greedy heuristic is much worse due to the fact that it does not reserve the right amount of resources for same-day patients.

We also test the algorithms under randomly generated settings. In Table 9, we report the worst performance and the average performance of all the algorithms over 100 random settings. The performance of algorithms in each setting is calculated by simulating 1000 replicates. Each of the 100 random settings is generated by

- uniformly generating the percentages in Table 2;
- uniformly picking a deadline for all regular patients between 5 and 30 days;
- uniformly setting the capacity of all resources to be between 45 and 150 minutes;
- uniformly picking a scale between 70% and 130%.

Again, our RLS algorithm consistently performs well in these test cases.

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