ON CHERN-SIMONS GAUGE THEORY

XIAO XIAO

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Abstract

In the chiral anomaly of the Abelian and non-Abelian gauge theory in even dimensional spacetime, it is a curious fact that the anomalous breaking of chiral symmetry introduces a new current, whose charge yields a Chern-Simons term defined on odd dimensional spacetime. This motivates us to probe some interesting properties of Chern-Simons gauge theory and its relations to chiral anomaly.

1 Chiral Anomaly

Remember the spirit of chiral anomaly that the classical chiral symmetry is broken under radiative correction if gauge symmetry is required to be preserved in the quantum level. Correspondingly, the conservation of axial current is violated by a term which is quadratic in field strength:

$$\partial_\mu J^{5\mu} = \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{1}$$

Where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

Even more interesting is the fact that the conservation-violating term can be written as the divergence of a new current:

$$\partial_\mu J^{5\mu} = \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = 4\frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu \partial_\rho A_\sigma = \frac{e^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu(A_\nu\partial_\rho A_\sigma) \tag{3}$$

Thus we can construct a new current which is conserved in the quantum level:

$$\partial_\mu J'^\mu = \partial_\mu(J^{5\mu} - \frac{e^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma) = 0 \tag{4}$$

Where

$$J^\mu_{CS} = -\frac{e^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma \tag{5}$$

If we further integrate one of the component of the current in the other three dimension of spacetime,we will find the emergence of an action defined in 2+1 dimensional spacetime:

$$\int d^3\overrightarrow{x} J^3_{CS} = -\frac{e^2}{4\pi^2}\int d^3\overrightarrow{x}\varepsilon^{\nu\rho\sigma}A_\nu\partial_\rho A_\sigma = S_{CS} \tag{6}$$

Noticing the new antisymmetric tensor is defined in the 2+1 dimensional spacetime, we recognize the action is actually Chern-Simons action which is used to describe the quantum Hall system in 2 spatial dimensions.

This interesting fact is not limited to Ablian anomaly, in the non-Abelian case:

$$\partial_\mu J^{5\mu} = \frac{e^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A^a_\nu\partial_\rho A^a_\sigma + gf^{abc}A^b_\mu A^c_\nu\partial_\rho A^a_\sigma) = \frac{e^2}{4\pi^2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu(A^a_\nu\partial_\rho A^a_\sigma + \frac{1}{3}gf^{abc}A^b_\rho A^c_\nu A^a_\sigma) \tag{7}$$

Such current can also be defined in chiral anomalies in any gauge theory defined in even dimensional spacetime.

Actually, the induced action is proportional to a topological quantity called "Chern-Simons secondary characteristic class" and the term representing anomalous breaking of chiral symmetry is called Pontryagin density.

2 Topologically Massive Theory And Higgs Mechanism

Chern-Simons term, reflecting the topological properties of gauge field, can also contribute a mass to a gauge field which is not coupled to a broken scalar field, thus may be an independent resource of mass of the gauge field.

The Lagrangian of Maxwell-Chern-simons theory is:

$$L_{MCS} = -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{8}$$

Its equation of motion is:

$$\partial_\mu F^{\mu\nu} + \frac{\kappa e^2}{2}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \tag{9}$$

In order to see the mass clearly, we introduce the dual form of the field strength tensor:

$$F^{\mu\nu} = \varepsilon^{\mu\nu\rho}\widetilde{F}_\rho \tag{10}$$

And the equation is:

$$\{\partial_\mu\partial^\mu + (\kappa e^2)^2\}\widetilde{F}^\nu = 0 \tag{11}$$

And Bianchi identity implies:

$$\partial_\mu\widetilde{F}^\mu = 0 \tag{12}$$

Introducing gauge-fixing term

$$L_{gf} = -\frac{1}{2\xi e^2}(\partial_\mu A^\mu)^2 \tag{13}$$

One can deduce the propagator of the gauge field:

$$D_{\mu\nu} = e^2\left(\frac{p^2 g_{\mu\nu} - p_\mu p_\nu - i\kappa e^2\varepsilon_{\mu\nu\rho}p^\rho}{p^2(p^2-\kappa^2 e^4)} + \xi\frac{p_\mu p_\nu}{(p^2)^2}\right) \tag{14}$$

The propagator has a non zero pole indicating a non zero mass.

Next we look at the effect of Chern-Simons term on the traditional Higgs mechanism, the whole Lagrangian is:

$$L = -\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + (D_\mu\phi)^* D^\mu\phi - V(\phi) \tag{15}$$

With a non-zero vacuum scalar field expectation value, then quantized in $R(\xi)$gauge, the propagator is:

$$D_{\mu\nu} = \frac{e^2(p^2-m_H^2)}{(p^2-m_+^2)(p^2-m_-^2)}[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2-\xi m_H^2} - i\kappa e^2\frac{\varepsilon_{\mu\nu\rho}p^\rho}{p^2-m_H^2} + e^2\xi\frac{p_\mu p_\nu(p^2-\kappa^2e^4-m_H^2)}{(p^2-m_+^2)(p^2-m_-^2)(p^2-\xi m_H^2)}] \tag{16}$$

With two poles corresponding the total effects of SSB and topological mass,where

$$m_H^2 = 2e^2v^2 \tag{17}$$

And

$$m_\pm^2 = m_H^2 + \frac{(\kappa e^2)^2}{2} \pm \frac{\kappa e^2}{2}\sqrt{\kappa^2e^4 + 4m_H^2} \tag{18}$$

In the limit of $e \to \infty$ the theory become a pure Chern-Simons gauge field coupled to scalar field, and as we have $m_+ \to \infty$ and $m_- \to \frac{2v^2}{\kappa}$, the propagator tends to:

$$D_{\mu\nu} = \frac{1}{p^2 - (\frac{2v^2}{\kappa})^2}[\frac{2v^2}{\kappa}g_{\mu\nu} - \frac{1}{2v^2}p_\mu p_\nu + \frac{i}{\kappa}\varepsilon_{\mu\nu\rho}p^\rho] \tag{19}$$

3 Non-Abelian Chern-Simons Theory

More interested is non-Abelian Chern-Simons theory, whose Lagrangian is:

$$L_{CS} = \kappa\varepsilon^{\mu\nu\rho}tr(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho) \tag{20}$$

Under a gauge transformation:

$$A_\mu \to g(A_\mu + \partial_\mu)g^{-1} \tag{21}$$

The Lagrangian changes by a surface term and a bulk term

$$L_{CS} \to L_{CS} - \kappa\varepsilon^{\mu\nu\rho}\partial_\mu tr(\partial_\nu g g^{-1}A_\rho) - \frac{\kappa}{3}\varepsilon^{\mu\nu\rho}tr(g\partial_\mu g^{-1}g\partial_\mu g^{-1}g\partial_\mu g^{-1}) \tag{22}$$

The quantity

$$w(g) = \frac{1}{24\pi^2}\varepsilon^{\mu\nu\rho}tr(g\partial_\mu g^{-1}g\partial_\mu g^{-1}g\partial_\mu g^{-1}) \tag{23}$$

is winding number and its integration in the spacetime yields an integer, therefore under gauge transformation the action changes by a quantity which is proportional to an integer.

$$S_{CS} \to S_{CS} - 8\pi^2\kappa N \tag{24}$$

For the invariance of the generating functional which defines the quantum theory, the change of the action must be integer times 2π, therefore it is required by gauge invariance of the theory that:

$$4\pi\kappa = integer \tag{25}$$

In the arguments above we simply ignored the surface term by requiring the integration on the boundary of spacetime vanishes, however, there are actually circumstances in which the requirement is not fulfilled. In the Abelian case, examine a variation of gauge field:

$$\delta(\int d^3x\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho) = 2\int d^3x\varepsilon^{\mu\nu\rho}\delta A_\mu\partial_\nu A_\rho + \int d^3x\varepsilon^{\mu\nu\rho}\partial_\nu(A_\mu\delta A_\rho) \tag{26}$$

When the variation is a infinitesimal gauge transformation

$$\delta A_\mu = \partial_\mu\lambda \tag{27}$$

The variation is purely a surface term

$$\delta(\int d^3x\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho) = \int d^3x\varepsilon^{\mu\nu\rho}\partial_\mu(\lambda\partial_\nu A_\rho) \tag{28}$$

In a spacetime $\mathbf{D}\times\mathbf{R}$ in which the space D is a disk and its boundary is a circle, then the integral of the total divergence yields:

$$\int d^3x\varepsilon^{\mu\nu\rho}\partial_\mu(\lambda\partial_\nu A_\rho) = \int_{S^1\times R}\lambda(\partial_0 A_\theta - \partial_\theta A_0) = \int_{S^1\times R}\lambda E_\theta \tag{29}$$

The current is conserved inside the disk, but not on the boundary.

$$J^\mu = \kappa\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho \tag{30}$$

we can see a radial current causes the creation and elimination of particles on the boundary.

$$J_r = \kappa E_\theta \tag{31}$$

This could be seen as anomaly of a fermion theory defined on 1+1 dimensional spacetime with the boundary circle as the spatial dimension.

$$\frac{dQ}{dt} = \frac{n}{2\pi}E \tag{32}$$

4 Induced Chern-Simons Terms

Another interesting point about 2+1 dimensional gauge theory is that, Chern-Simons term can be induced by radiative correction and corrected by radiative process a discrete value, which is quite startling.

First we compute the effective action in a traditional QED model in 2+1 dimensional spacetime

$$S_{eff}[A,m] = N_f \log\det(i\partial\!\!\!/ + A\!\!\!/ + m) \tag{33}$$

The action can be computed perturbatively

$$S_{eff}[A,m] = N_f tr \log(i\partial\!\!\!/ + m) + N_f tr(\frac{1}{i\partial\!\!\!/ + m}A) + \frac{N_f}{2} tr(\frac{1}{i\partial\!\!\!/ + m}A\!\!\!/\frac{1}{i\partial\!\!\!/ + m}A\!\!\!/) + \cdots \tag{34}$$

The term we are interested is the quadratic one, it is possible for Chern-Simons term to appear only in this term

$$S^{[2]}_{eff}[A,m] = \frac{N_f}{2}\int \frac{d^3p}{(2\pi)^3}[A_\mu(-p)\Gamma^{\mu\nu}A_\nu(p)] \tag{35}$$

Where

$$\Gamma^{\mu\nu} = \int \frac{d^3k}{(2\pi)^3} tr[\gamma^\mu \frac{p\!\!\!/ + k\!\!\!/ - m}{(p+k)^2+m^2}\gamma^\nu \frac{k\!\!\!/ - m}{k^2+m^2}] \tag{36}$$

In 3-dim spacetime

$$tr(\gamma^\mu\gamma^\nu\gamma^\rho) = -2\varepsilon^{\mu\nu\rho} \tag{37}$$

Therefore we can extract the term proportional to Levi-Civita tensor

$$\Gamma^{\mu\nu}_{odd}(p,m) = \varepsilon^{\mu\nu\rho}p_\rho \Pi_{odd}(p^2,m) \tag{38}$$

$$\Pi_{odd}(p^2,m) = 2m\int \frac{d^3k}{(2\pi)^3}\frac{1}{[(p+k)^2+m^2][k^2+m^2]} = \frac{1}{2\pi}\frac{m}{|p|}\arcsin(\frac{|p|}{\sqrt{p^2+4m^2}}) \tag{39}$$

In the $p \to 0$ and $m \to \infty$ limit:

$$\Gamma^{\mu\nu}_{odd}(p,m) \sim \frac{1}{4\pi}\frac{m}{|m|}\varepsilon^{\mu\nu\rho}p_\rho + O(\frac{p^2}{m^2}) \tag{40}$$

Inserting this into the effective action we obtain:

$$S^{CS}_{eff}[A,m] = -i\frac{N_f}{2}\frac{1}{4\pi}\frac{m}{|m|}\int d^3x \varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{41}$$

The Chern-Simons term emerges in the loop computation.

The term arises from the large mass limit of the massive fermion theory,such limit appears just in the process of regularization when there are no bare mass term in the Lagrangian, in Pauli-Villars regularization:

$$S_{eff}^{reg}[A, m = 0] = S_{eff}[A, m] - \lim_{M\to\infty} S_{eff}[A, M] \tag{42}$$

Such a regularization scheme preserves gauge invariance, but parity conservation is violated by the mass term introduced to regularize the integrals, such violation of parity is manifest in the emergence of Chern-Simons term.

In non-Abelian theory the Chern-Simons effective action

$$S_{eff}^{CS}[A, m] = -i\frac{N_f}{2}\frac{1}{4\pi}\frac{m}{|m|}\int d^3x\varepsilon^{\mu\nu\rho}tr(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho) \tag{43}$$

can also be generated.

*Coleman and Hill proved that in an Abelian gauge theory, the Chern-Simons term is only corrected in one-loop order, any higher order correction vanishes.

In a pure gauge theory with bare Chern-Simons term, radiative correction can give the coefficient a discrete correction.

$$4\pi\kappa_{ren} = 4\pi\kappa_{bare} + N \tag{44}$$

In Eucilidean space

$$L_{CSYM} = -\frac{1}{2}tr(F_{\mu\nu}F^{\mu\nu}) - im\varepsilon^{\mu\nu\rho}tr(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho) \tag{45}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu] \tag{46}$$

The parameter is required to be

$$4\pi\frac{m}{e^2} = integer \tag{47}$$

The bare propagator is

$$\Delta_{\mu\nu}^{bare}(p) = \frac{1}{p^2 + m^2}(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - m\varepsilon_{\mu\nu\rho}\frac{p^\rho}{p^2}) + \xi\frac{p_\mu p_\nu}{(p^2)^2} \tag{48}$$

$$\Delta_{\mu\nu}^{-1} = (\Delta_{\mu\nu}^{bare})^{-1} + \Pi_{\mu\nu} \tag{49}$$

The contribution of self-energy can be decomposed as

$$\Pi_{\mu\nu}(p) = (\delta_{\mu\nu}p^2 - p_\mu p_\nu)\Pi_{even}(p^2) + m\varepsilon_{\mu\nu\rho}p^\rho\Pi_{odd}(p^2) \tag{50}$$

The renormalized propagator is

$$\Delta_{\mu\nu}(p) = \frac{1}{Z(p^2)[p^2 + m_{ren}^2(p^2)]}(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} - m_{ren}(p^2)\varepsilon_{\mu\nu\rho}\frac{p^\rho}{p^2}) + \xi\frac{p_\mu p_\nu}{(p^2)^2} \tag{51}$$

Where the renormalized mass is

$$m_{ren}(p^2) = \frac{Z_m(p^2)}{Z(p^2)}m \tag{52}$$

And

$$Z(p^2) = 1 + \Pi_{even}(p^2) \tag{53}$$

$$Z_m(p^2) = 1 + \Pi_{odd}(p^2) \tag{54}$$

$$m_{ren} = m_{ren}(0) = \frac{Z_m(0)}{Z(0)} m \tag{55}$$

The charge is also renormalized

$$e_{ren}^2 = \frac{e^2}{Z(0)(\widetilde{Z(0)})^2} \tag{56}$$

After computation we can obtain that

$$Z_m(0) = 1 + \frac{7}{12\pi} N \frac{e^2}{m} \tag{57}$$

$$\widetilde{Z(0)} = 1 - \frac{1}{6\pi} N \frac{e^2}{m} \tag{58}$$

Where N is the dimension of the gauge group

$$(\frac{m}{e^2})_{ren} = (\frac{m}{e^2}) Z_m(0) \widetilde{Z(0)}^2 = (\frac{m}{e^2})\{1 + (\frac{7}{12\pi} - \frac{1}{3\pi}) N \frac{e^2}{m}\} = \frac{m}{e^2} + \frac{N}{4\pi} \tag{59}$$

Finally we get the renormalized coupling constant

$$4\pi\kappa_{ren} = 4\pi\kappa_{bare} + N \tag{60}$$

In a non-Abelian gauge theory whose gauge group is completely broken by vacuum, the integer renormalization is broken to a function that is not necessarily an integer.

$$4\pi\kappa_{ren} = 4\pi\kappa_{bare} + f(\frac{m_{higgs}}{m_{CS}}) \tag{61}$$

However, if the non-Abelian gauge group is broken to a non-Abelian subgroup, say SU(3) to SU(2),the integer renormalization is still robust.

5 Gravity With Topological Mass

The dynamics of gravity can be altered dramatically when topological mass term is included.

(a)The Einstein theory acquires a spin-2, propagating massive degree of freedom when the topological mass term is present

(b)The topological term has third time derivative, but the propagation is also causal.

(c)The topological mass contribution is the three dimensional analog of Weyl tensor.

The Einstein theory is a symmetric tensor gauge theory, the excitations are decribed by the transverse traceless part of the spatial components, which has $\frac{1}{2}d(d-3)$ degrees of freedom. In 3-dim theory, the dynamic is trivial.

$$R^{\alpha}{}_{\beta\gamma\sigma} = \partial_\gamma \Gamma^{\alpha}_{\beta\sigma} - \partial_\sigma \Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\gamma\mu}\Gamma^{\mu}_{\beta\sigma} - \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\mu}_{\beta\gamma} \tag{62}$$

In 3 dimensional spacetime, $C^{\alpha}{}_{\beta\gamma\sigma}$ vanishes. And the Ricci tensor has all the components of Riemann tensor.

$$R_{\alpha\beta\gamma\sigma} = g_{\alpha\gamma}\widetilde{R}_{\beta\sigma} + g_{\beta\sigma}\widetilde{R}_{\alpha\gamma} - g_{\alpha\sigma}\widetilde{R}_{\beta\gamma} - g_{\beta\gamma}\widetilde{R}_{\alpha\sigma} \tag{63}$$

Where

$$\widetilde{R}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}R \tag{64}$$

Another way to see this is to establish the equivalence between Riemann tensor and Einstein tensor.

$$R_{\gamma\sigma}{}^{\alpha\beta} = -\varepsilon^{\alpha\beta\mu}\varepsilon_{\gamma\sigma\nu}G^{\nu}{}_{\mu} \tag{65}$$

Where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} \tag{66}$$

The equivalence of the Riemann tensor and Einstein tensor states that in the empty space where there are no matter fields, the spacetime is flat, no gravitational waves and gravitons.

In 3-dim Einstein theory, a non vanishing Weyl tensor can be constructed.

$$C^{\mu\nu} = \frac{1}{\sqrt{g}}\varepsilon^{\mu\alpha\beta}\nabla_\alpha \widetilde{R}^{\nu}_{\beta} \tag{67}$$

$C^{\mu\nu}$ satisfies a Bianchi identity, i.e. it is covariantly conserved, therefore it is a functional derivative of a geometrical invariant, the invariant is Chern-Simons characteristic, obtained from Hirzebruch-Pontryagin density:

$$^{*}RR = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R^{\rho\sigma}_{\alpha\beta} = \partial_\mu X^\mu \tag{68}$$

Integrate X^3 we can obtain the characteristic class:

$$I_{CS} = -\frac{1}{4}\int X^3 = -\frac{1}{4}\int dx\varepsilon^{\mu\nu\alpha}[R_{\mu\nu ab}\omega^{ab}_{\alpha} + \frac{2}{3}\omega^{c}_{\mu b}\omega^{a}_{\nu c}\omega^{b}_{\alpha a}] \tag{69}$$

In order to express the Pontryagin density of gravity in a total divergence, we apply vielbein description of gravity. Where

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} + \omega^{c}_{\mu a}\omega_{\nu cb} - (\mu \longleftrightarrow \nu) \tag{70}$$

The total Lagrangian is

$$I = \frac{1}{\kappa^2} \int dx \sqrt{g} R + \frac{1}{\kappa^2 \mu} I_{CS} \tag{71}$$

The equation of motion is

$$G^{\mu\nu} + \frac{1}{\mu} C^{\mu\nu} = 0 \tag{72}$$

Written as a second order equation, we have

$$(\nabla^\alpha \nabla_\alpha + \mu^2) R_{\mu\nu} = -g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} + 3 R^\alpha_\mu R_{\alpha\nu} \tag{73}$$

To see the degree of freedom of graviton, in the linearized gravity, we have

$$I^L_E = -\frac{1}{2} \int dx h_{\mu\nu} G^{\mu\nu}_L \tag{74}$$

Where

$$G^{\mu\nu}_L = R^{\mu\nu}_L - \frac{1}{2} \eta^{\mu\nu} R_L \tag{75}$$

and

$$R^{\mu\nu}_L = \frac{1}{2}(-\Box h^{\mu\nu} + \partial^\mu \partial_\alpha h^{\alpha\nu} + \partial^\nu \partial_\alpha h^{\alpha\mu} - \partial^\mu \partial^\nu h) \tag{76}$$

Linearized mass term is

$$\frac{1}{\mu} I^L_{CS} = \frac{1}{2\mu} \int dx \varepsilon_{\mu\alpha\beta} G^{\alpha\nu}_L \partial^\mu h^\beta_\nu \tag{77}$$

The functional derivative with respect to gravitational field is

$$-\frac{1}{\mu} \frac{\delta I^L_{CS}}{\delta h_{\mu\nu}} = \frac{1}{\mu} C^{\mu\nu}_L \tag{78}$$

This is consistent with former arguments.And the Lagrangian contains two parts: linearized Einstein term and linearized Chern-Simons term.

$$I^L = I^L_E + \frac{1}{\mu} I^L_{CS} \tag{79}$$

We decompose the field into components

$$h^{\mu\nu} = (h^{00} \equiv N, h^{0i} = N^i, h^{ij}) \tag{80}$$

It can be decomposed into transver-trace part and other parts

$$h^{ij} = (\delta^{ij} + \partial^i \partial^j)\varphi - \partial^i \partial^j \chi + (\partial^i \xi^j_T + \partial^j \xi^i_T) \tag{81}$$

$$\overrightarrow{N} = \overrightarrow{N}_T + \nabla N_L \tag{82}$$

$$\nabla \cdot \overrightarrow{N}_T = 0 = \nabla \cdot \overrightarrow{\xi}_T \tag{83}$$

Plugging in the decomposition in the action

$$I_L = -\frac{1}{2}\int dx\{\varphi\Box\varphi + \lambda\varphi + \sigma^2 + \frac{1}{\mu}\sigma\lambda\} \tag{84}$$

where

$$\lambda = \nabla^2(N + 2\dot{N}_L) + \ddot{\chi} - \Box\varphi \tag{85}$$

and

$$\sigma = \varepsilon^{ij}\partial_j(N_T^i + \dot{\xi}_T^{\,i}) \tag{86}$$

Eliminating the constriant, we get

$$I = -\frac{1}{2}\int dx\varphi(\Box + \mu^2)\varphi \tag{87}$$

which describes a single propagating degree of freedom

6 APPENDIX

6.1 DERIVATION OF (10)

Plug the definition of dual form into the equation:

$$\varepsilon^{\mu\nu\rho}\partial_\mu\widetilde{F}^\rho + \frac{\kappa e^2}{2}\varepsilon^{\nu\alpha\beta}\varepsilon_{\alpha\beta\sigma}\widetilde{F}^\sigma = \varepsilon^{\mu\nu\rho}\partial_\mu\widetilde{F}^\rho + \frac{\kappa e^2}{2}2\delta^\nu{}_\sigma\widetilde{F}^\sigma = 0 \tag{88}$$

Contract the equation with a antisymmetric tensor and differential operator and use the equation again we obtain

$$-\varepsilon^{\lambda\sigma}{}_\nu\varepsilon^{\mu\rho\nu}\partial^\lambda\partial_\mu\widetilde{F}^\rho - (\kappa e^2)^2\widetilde{F}^\sigma = 0 \tag{89}$$

We know the result of contracting the indexes of the antisymmetric tensors

$$(g^{\lambda\mu}g^{\sigma\rho} - g^{\lambda\rho}g^{\sigma\mu})\partial_\lambda\partial_\mu\widetilde{F}_\rho + (\kappa e^2)^2\widetilde{F}^\sigma = 0 \tag{90}$$

This is indeed

$$\partial^\mu\partial_\mu\widetilde{F}^\sigma - \partial^\sigma(\partial_\mu\widetilde{F}^\mu) + (\kappa e^2)^2\widetilde{F}^\sigma = 0 \tag{91}$$

Because we have Bianchi identity

$$\partial_\mu\widetilde{F}^\mu = 0 \tag{92}$$

Then we get a Klein-Gordon like massive equation

$$[\partial^\mu\partial_\mu + (\kappa e^2)^2]\widetilde{F}^\sigma = 0 \tag{93}$$

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