Rank-Dependent Utility and Risk Taking in Complete Markets*

Xue Dong He[†], Roy Kouwenberg[‡], Xun Yu Zhou[§]

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Abstract

We analyze the portfolio choice problem of investors who maximize rank-dependent utility in a single-period complete market. We propose a new notion of less risk taking: choosing optimal terminal wealth that pays off more in bad states and less in good states of the economy. We prove that investors with a less risk averse preference relation in general choose more risky terminal wealth, receiving a risk premium in return for accepting conditional-zero-mean

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†Corresponding Author. Department of Industrial Engineering and Operations Research, Columbia University, Room 315, Mudd Building, 500 W. 120th Street, New York, NY 10027, US. Email: xh2140@columbia.edu. Telephone: +1-212-854-2936.

‡Mahidol University, College of Management, Bangkok, Thailand; Erasmus University Rotterdam, Erasmus School of Economics, Rotterdam, The Netherlands; Email: roy.kou@mahidol.ac.th.

§Mathematical Institute, The University of Oxford, Woodstock Road, OX2 6GG Oxford, UK, and Oxford–Man Institute of Quantitative Finance, The University of Oxford; Email: zhouxy@maths.ox.ac.uk.

noise (more risk). However, such general comparative static results do not hold for portfolio weights, which we demonstrate with a counter-example in a continuous-time model.

Keywords: rank-dependent utility; portfolio selection; risk aversion; complete markets; less risky terminal wealth; optimal stock holding

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1 Introduction

A fundamental economic question concerns the impact of risk attitude on risk taking in portfolio selection. A well-known result by Arrow (1965) and Pratt (1964) shows that in a simple, single-period market with one risk-free asset and one risky asset (a stock), in which investors maximize expected utility (EU), more risk averse agents will invest less in stocks.[1] In markets with multiple risky assets, or with dynamic trading in continuous-time, however, this basic comparative statics result for the amount of risky assets does not always hold (Dybvig and Wang, 2012, Liu, 2001, 2007). Nonetheless, Dybvig and Wang (2012) show that in complete markets less risk averse EU agents do choose optimal portfolio payoffs that render more noisy payoff distributions (higher risk) in return for higher expected returns (risk premia).

Given the large body of experimental evidence that is at odds with the standard EU framework, it is natural, and indeed important, to try to derive similar results for non-expected utility models, such as rank-dependent utility and prospect theory. In these models investors transform objective probabilities by means of a weighting function, a mechanism that can explain individuals' strong preference for improbable large gains and distaste for improbable large losses. In this paper we focus on rank-dependent utility (RDU), a model that can resolve the Allais paradox frequently observed in laboratory experiments. Further, RDU can explain puzzling investor behavior, such as poor diversification and low stock market participation (Polkovnichenko, 2005).

Chew et al. (1987) show that a preference relation modeled by RDU is more risk averse if and only if the utility function is more concave and the probability weighting function is more convex. Further, these authors show that a more risk averse

[1] Here, the term "more risk averse" is in the classical sense; see the precise definition in Section 2.1.

RDU agent always invests less in the risky asset in a simple, one-period market with one risk-free asset and one risky asset. Similar results are not, however, currently available for a single-period complete market in which the RDU investor can trade a continuum of Arrow-Debreu securities, a gap in the literature that we address in this paper. Further, focusing on the most common implementation of the complete markets concept, we derive for the first time the optimal trading strategy of RDU agents in a continuous-time Black-Scholes economy in which the investor can dynamically trade one riskless asset and one risky stock.

We argue that in complete markets it is not useful nor suitable to define the notion of "less risk taking" as simply investing less in the risky asset at any time and at any wealth level. One reason is that in a general complete market with a continuum of Arrow-Debreu securities, there are infinitely many "risky" contingent claims investors can trade. Secondly, as we will show for the Black-Scholes market, a more risk averse agent does not necessarily invest less in the stock at any time. In this paper, we propose to use instead the notion of less risky *terminal wealth*: namely, terminal wealth X is less risky than terminal wealth Y when X provides more payoff in bad market states and less payoff in good market states, compared to Y.

Our primary result is that a more risk averse RDU agent will *always* choose a less risky terminal wealth as defined above. This result is general, holding for any continuous distribution of the pricing kernel, any increasing and strictly concave utility function, and any strictly increasing probability weighting function.

In literature, a related notion is proposed by Dybvig and Wang (2012): X_1 decreasing-concave stochastically dominates X_2 if X_2 is distributed as $X_1 + Z + \varepsilon$ where ε satisfying $\mathbb{E}[\varepsilon|X + Z] = 0$ represents additional risk and $Z \geq 0$ stands for a risk premium.[2] The authors then show that more risk averse investors with EU preferences choose decreasing-concave stochastic dominating payoffs. The notion of decreasing-concave stochastic dominance and our notion of less risky payoffs are not inclusive to each other: the former is distribution-based with risk premia involved and the latter is scenario-based without concerning risk premia; see the detailed discussions in Section 3.2.3. That said, our second result is that a less risk averse RDU agent indeed chooses a decreasing-concave stochastically dominated payoff. Therefore, we generalize the results for EU provided in Dybvig and Wang (2012) to the

[2]Decreasing-concave stochastic dominance is atually a variant of the so-called monotone-concave stochastic dominance proposed by Dybvig and Wang (2012); see detailed discussions in Section 3.2.2.

case of RDU. One should note that such an extension is nontrivial and of practical relevance, as Dybvig and Wang assume that all agents are risk averse, while the probability weighting functions in our RDU framework allow for (locally) risk seeking attitudes. Indeed, experimental studies have justified inverse-S-shaped probability weighting functions, indicating risk seeking behavior; see for instance Tversky and Kahneman (1992), Camerer and Ho (1994), Abdellaoui (2000), and Bleichrodt and Pinto (2000). Thus, our results show that even when decision makers distort probabilities and thus are locally risk seeking as found in practice, the wealth generated by their investment strategies still displays a sensible trade-off between risk and return.

We then turn our attention to a special case of a single-period complete market that is widely used in the finance literature: a continuous-time Black-Scholes market in which investors can dynamically trade a risk-free asset and one risky stock following a geometric Brownian motion. In this market we can consider the investor's stock holding at any time and any wealth level. For agents with EU preferences, Borell (2007), Xia (2011) and Zariphopoulou and Kallblad (2014) all show that a more risk averse agent *always* holds less stocks than a less risk averse agent at any given time and any wealth level. For RDU agents, because of the inherent *time-inconsistency* due to the presence of the probability weighting, we analyze pre-committed trading strategies and provide closed-form solutions for an RDU agent's optimal terminal wealth and demand for stocks.[3] We then prove that the standard comparative statics result still holds when comparing an EU agent with an RDU agent. However, we illustrate that this result does not hold when both agents are genuine RDU agents. More precisely, we provide a counter-example in which investor A's preference relation (modeled by RDU) is more risk averse than investor B's preference relation, yet investor A holds *more* stocks than B at some point in time.

In related work, Liu (2001, 2007) shows that if the risky asset's dynamics are changed to a stochastic volatility model, a more risk averse EU agent may temporarily hold *more* stocks because of hedging demand. A similar counter-example for EU agents who trade a risky bond and a locally risk-less asset is provided in Dybvig and Wang (2012). We are the first to show that changing the objective function in the Black-Scholes market from EU to RDU will also give rise to cases in which more risk

[3]The optimal strategy derived today at time 0 might not be optimal in the future if the agent reconsiders his planning problem at time t. We assume that the agent *pre-commits* by solving his dynamic portfolio choice problem once, at time 0, and then implements the resulting optimal dynamic trading strategy in the future ($t > 0$).

averse investors hold more stocks.

Hence, we learn that in complete markets with RDU agents general comparative statics results hold for the portfolio payoffs at the end of the planning horizon, rather than for the portfolio weights. The essential reason behind these results is that the stock is merely a device that the investor uses to achieve his desired end wealth profile while satisfying various constraints. Thus, as long as the preference measure is a functional only of the terminal wealth, it is that wealth profile, rather than the stock allocation, that is relevant to the investor's risk appetite. Therefore, our new notion of less risk taking and the existing notion of decreasing-concave monotone dominance, which are based on payoffs, are more appropriate for comparing the portfolio choice of RDU agents with different risk tastes.

The remainder of this paper is organized as follows: In Section 2 we review the definition of more risk averse preferences and the RDU representation of a preference relation. In Section 3 we introduce the portfolio choice problem of an RDU agent in the single-period complete market and propose the notion of less risky terminal wealth. We prove that a more risk averse agent chooses a less risky terminal wealth profile, and we characterize the payoff distribution. In Section 4 we then derive closed-form expressions for the terminal wealth and for the optimal amount of investment in stocks for a pre-committing RDU agent in the Black-Scholes market. Thereafter, we use examples to illustrate that a more risk averse RDU investor can hold *more* stocks at some point in time, even though his wealth profile is less risky. Finally, Section 5 concludes the paper. Proofs are placed in the Appendix.

2 Risk Aversion and Rank-Dependent Utility

2.1 Law-Invariant Preference Relation and Risk Aversion

A preference relation $\succcurlyeq$ is a partial order on a set of random payoffs living on a measurable space. Define $\sim$ the equivalence relation associated with $\succcurlyeq$, i.e., $X \sim Y$ if and only if $X \succcurlyeq Y$ and $Y \succcurlyeq X$. Suppose the measurable space is endowed with a probability measure, then $\succcurlyeq$ is called *law-invariant* if any two random payoffs sharing the same distribution are equivalent. A law-invariant preference relation can be viewed as a relation on a set of distribution functions, and we will take this view in the following analysis.

Recall the following definition from Machina (1982): Given a law-invariant preference relation $\succcurlyeq$, a distribution F differs from another distribution F^* by a *simple compensated spread* with respect to $\succcurlyeq$ if $F \sim F^*$ and there exists $x_0 \in \mathbb{R}$ such that $F(x) \geq F^*(x)$ for all $x < x_0$ and $F(x) \leq F^*(x)$ for all $x \geq x_0$. In particular, when $\succcurlyeq$ is represented by expectation,[4] F is said to differ from F^* by a *simple mean-preserving spread* (Rothschild and Stiglitz, 1970) if F differs from F^* by a simple compensated spread.

A law-invariant preference relation $\succcurlyeq^*$ is *more risk averse* than another one $\succcurlyeq$ if $F^* \succcurlyeq^* F$ for any distributions F and F^* such that F differs from F^* by a simple compensated spread with respect to $\succcurlyeq$. In particular, a law-invariant preference relation $\succcurlyeq^*$ is *risk averse* if it dislikes simple mean-preserving spreads, i.e., $F^* \succcurlyeq^* F$ for any distributions F and F^* such that F differs from F^* by a simple mean-preserving spread; see Rothschild and Stiglitz (1970) and Machina (1982).

2.2 Rank-Dependent Utility

A mapping V from the set of random payoffs to real numbers is said to be a *representation* of $\succcurlyeq$ if $X \succcurlyeq Y$ if and only if $V(X) \geq V(Y)$, in which case $V(X)$ is called the preference value of X. Quiggin (1982) proposed the rank-dependent utility (RDU) representation, which is a generalization of the expected utility (EU) representation (Von Neumann and Morgenstern, 1947). In RDU, the preference value of any random payoff X is defined as follows:

$$V(X) := \int_{\mathbb{R}} U(x) d[1 - w(1 - F_X(x))]. \tag{1}$$

Here, and hereafter, when a probability measure is given and understood, for any random variable X we denote F_X as its cumulative distribution function (CDF). Clearly, V induces a law-invariant preference. The function $U(\cdot)$, which is increasing and continuous in its domain,[5] is called a *utility function*, and the function w, which is an increasing mapping from the unit interval onto itself, is called a *probability weighting function*.

[4]In that case, $F_1(\cdot)$ is preferred to $F_2(\cdot)$ if and only if $\int_{\mathbb{R}} x dF_1(x) \geq \int_{\mathbb{R}} x dF_2(x)$ or, equivalently, any random payoff X is preferred to another random payoff Y if and only if $\mathbb{E}(X) \geq \mathbb{E}(Y)$.

[5]Throughout this paper, by an "increasing" function we mean a "non-decreasing" function, namely f is increasing if $f(x) \geq f(y)$ whenever $x > y$. We say f is "strictly increasing" if $f(x) > f(y)$ whenever $x > y$. Similar conventions are used for "decreasing" and "strictly decreasing" functions.

The following theorem in Chew et al. (1987) shows that a given RDU preference relation denoted by V_1 is more risk averse than another RDU preference relation V_2 if and only if the utility function of the first agent is more concave (i.e., the utility function $U_1(\cdot)$ is a concave transformation of $U_2(\cdot)$) and the probability weighting function of this same agent is more convex (i.e., the weighting function $w_1(\cdot)$ is a convex transformation of $w_2(\cdot)$):[6]

Theorem 1 *Let $V_i(X)$ be the RDU preference relation with utility function $U_i(\cdot)$ and probability weighting function w_i, $i = 1, 2$, and assume that $U_i(\cdot)$, $i = 1, 2$ are strictly increasing and continuous and that $w_i(\cdot)$, $i = 1, 2$ are strictly increasing, continuous and differentiable on $[0, 1]$. Then, the preference relation represented by $V_1(\cdot)$ is more risk averse than the preference relation represented by $V_2(\cdot)$ if and only if $U_1(\cdot)$ is a concave transformation of $U_2(\cdot)$ and $w_1(\cdot)$ is a convex transformation of $w_2(\cdot)$.*

For a proof of this theorem, see Theorem 1 and Corollary 1 in Chew et al. (1987).[7]

3 Risk Aversion and Risk Taking in a Single-Period Complete Market Portfolio Choice Model

3.1 Portfolio Selection Model

We consider a one-period market with 0 representing today and $T > 0$ representing a future time, which will be referred to as the investor's planning horizon. The set of possible states of nature and the set of events at date T are $(\Omega, \mathcal{F})$, a measurable space. A probability measure $\mathbb{P}$ is given in this measurable space. Any lower-bounded $\mathcal{F}$-measurable random variable X, which is regarded as a contingent claim and will be realized at T, can be traded in the market today. Thus, this market is *complete*. There is a strictly positive random variable ξ serving as the *pricing kernel* that determines

[6]A function $\phi(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a *concave transformation* of another function $\psi(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ if there exists a concave function $h(\cdot) : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ such that $h(-\infty) = h(-\infty)$ and $\phi(\cdot) = h(\psi(\cdot))$. A function $\phi(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a *convex transformation* of another function $\psi(\cdot) : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ if $-\phi(\cdot)$ is more concave than $-\psi(\cdot)$. For an equivalent characterization of concave and convex transformation, see Appendix A in He et al. (2013).

[7]Note that Chew et al. (1987) use the representation $\int_{\mathbb{R}} U(x) d\bar{w}(F_X(x))$, where $\bar{w}(z) := 1 - w(1 - z)$ is the dual of w. Therefore, the statement in Chew et al. (1987, Theorem 1) that $\bar{w}_1(\cdot)$ is a concave transformation of $\bar{w}_2(\cdot)$ translates to the statement that $w_1(\cdot)$ is a convex transformation of $w_2(\cdot)$.

the price of any contingent claim. More precisely, for any contingent claim X, its price today is given as $\mathbb{E}[\xi X]$.

A continuous-time complete market model can be regarded as a special case of the one-period complete market; see Karatzas and Shreve (1998) for details. In particular, the set of terminal payoffs that can be achieved in the Black-Scholes market can be regarded as the tradeable contingent claims in the one-period complete market as defined above. In that special case, the pricing kernel ξ is a lognormally distributed random variable.

We make the following assumption:

Assumption 1 *Both $F_\xi(\cdot)$ and $F_\xi^{-1}(\cdot)$ are continuous and $\mathbb{E}[\xi] < +\infty$.*

Consider an agent with initial capital $x_0 > 0$ at time 0 and a preference relation represented by RDU with utility function $U(\cdot)$ and probability weighting function $w(\cdot)$: i.e., the agent evaluates any random payoff X according to (1). Assuming bankruptcy is not allowed, the agent faces the following portfolio choice problem:

$$\begin{array}{ll} \underset{X}{\text{Max}} & \int_{\mathbb{R}_+} U(x)d[1-w(1-F_X(x))] \\ \text{Subject to} & \mathbb{E}[\xi X] \le x_0, \quad X \ge 0. \end{array} \tag{2}$$

In other words, given the initial capital, the agent chooses the best contingent claim, i.e., the best terminal wealth at T, so as to maximize his RDU.

We make the following assumption regarding $U(\cdot)$ and $w(\cdot)$:

Assumption 2 *$U(\cdot)$ is strictly increasing, strictly concave, continuously differentiable on $(0, +\infty)$ and satisfies the Inada condition: $U'(0+) = +\infty, U'(+\infty) = 0$. $w(\cdot)$ is strictly increasing and continuous in $[0, 1]$ with $w(0) = 0$ and $w(1) = 1$.*

Problem (2) has recently been studied extensively; see Carlier and Dana (2011), He and Zhou (2014), and Xia and Zhou (2013). Here, we cite a general result provided in Xia and Zhou (2013). To this end, we recall the definition of the concave and the convex envelopes of a function:

Definition 1 Let C be a convex set in $\mathbb{R}$ and let $f(\cdot)$ be a real-valued function on C. The *convex envelope* of $f(\cdot)$ is defined as the greatest convex function $g(\cdot)$ on C such that $g(x) \le f(x), x \in C$. Similarly, the *concave envelope* of $f(\cdot)$ is defined as the smallest concave function $g(\cdot)$ on C such that $g(x) \ge f(x), x \in C$.

Theorem 2 *Suppose Assumptions 1 and 2 hold. Then, the optimal solution to (2) is*

$$X^* = (U')^{-1}\left(\lambda^* \hat{N}'(1 - w(F_\xi(\xi)))\right) \tag{3}$$

where $\hat{N}(\cdot)$ *is the concave envelope of*

$$N(z) := -\int_0^{w^{-1}(1-z)} F_\xi^{-1}(t)dt, \quad z \in [0,1] \tag{4}$$

and $\lambda^* > 0$ *is the number such that* $\mathbb{E}[\xi X^*] = x_0$.

Theorem 2 shows that the optimal terminal wealth X^* is a decreasing function of ξ because $\hat{N}(\cdot)$ is concave and $(U')^{-1}(\cdot)$ is decreasing. In other words, the optimal terminal payoff must be anti-comonotonic with respect to the pricing kernel.

3.2 Notions of Less Risk Taking

3.2.1 A new notion of less risk taking

Our goal is to study whether a more risk averse agent takes less risk with his optimal portfolio. To this end, we need to define the meaning of "less risk-taking" in the portfolio selection context. In a single-period market in which investors can only trade one risk-free asset and one risky stock at time 0, one can naturally regard the allocation to the risky stock as the measure of risk taken by the agent. Then, one can study whether a more risk averse agent invests less in the stock in this single-period, single-stock market. For instance, Chew et al. (1987) show that this is the case in the RDU framework; see also Arrow (1965) and Pratt (1964) for this basic comparative statics result for investors who maximize EU.

In the complete market considered here, the notion of "less risk-taking" is not easy to define because agents can trade any contingent claim rather than only one risky stock. We propose the following definition:

Definition 2 Let X and Y be two terminal payoffs, i.e., contingent claims. X is *less risky* than Y if there exists an event $A \in \mathcal{F}$ such that

$$\sup_{\omega\in A} X(\omega) \le \inf_{\omega\in\Omega\setminus A} X(\omega), \quad \sup_{\omega\in A} Y(\omega) \le \inf_{\omega\in\Omega\setminus A} Y(\omega) \tag{5}$$

and almost surely,

$$X(\omega) \geq Y(\omega),\ \omega \in A, \quad X(\omega) \leq Y(\omega),\ \omega \in A^c := \Omega \backslash A. \tag{6}$$

We observe from (5) that A represents the set of bad market scenarios and $\Omega\backslash A$ the set of good scenarios: X and Y have lower outcomes in bad market scenarios A than in good market scenarios A^c. As a result, (6) states that X has higher outcomes than Y in bad market scenarios and lower outcomes than Y in good market scenarios. In real life, people perceive a less risky asset as performing better when the market goes down but performing worse when the market turns up. Thus, Definition 2 agrees with the common perspective regarding a portfolio payoff being less or more risky.

The notion of less risky payoffs defines a relation, which depends on the choice of the event that represents bad market scenarios. In order for this relation to be well defined, we must rule out the possibility that X is less risky than Y with respect to an event A and yet Y is less risky than X with respect to another event B; namely we need to show that the relation is antisymmetric. The following proposition serves this purpose.

Proposition 1 *Let X and Y be two terminal payoffs which do not dominate each other, i.e., neither $X \geq Y$ almost surely and $X > Y$ with positive probability, nor $X \leq Y$ almost surely and $X < Y$ with positive probability. If X is less risky than Y and Y is less risky than X, then $X = Y$ almost surely.*

To exclude arbitrage opportunities, any two terminal payoffs resulting from the same initial wealth cannot dominate each other. Consequently, Proposition 1 shows that the notion of less risky payoffs in Definition 2 is a well-defined relation among all terminal payoffs resulting from the same initial wealth; that is, for any two terminal payoffs, they are less risky to each other if and only if they are the same. Note that this relation is not complete, that is, for some pair of payoffs one cannot tell which one is less risky.[8] However, it is reasonable for this relation to be incomplete because

[8]This is not surprising because the notion of less risky payoffs in Definition 2 is strong in the following sense: market scenarios are divided into only two sets—the set of good scenarios and the set of bad—and payoff X is less risky than payoff Y if the former is larger in the bad scenarios and is smaller in the good scenarios than the latter. One can weaken this notion by allowing intermediate scenarios that are not considered when determining which of the two payoffs is less risky. Such an extension can make the notion of less risky payoffs less incomplete as a relation, at the cost of making it less intuitive.

it is likely in many occasions investors are unable to tell which of two given payoffs is less risky.

Finally, note that the notion of less risky payoffs in Definition 2 is consistent with the risky stock holding in the single-period single-stock market. Indeed, suppose the total return rates of the risk-free asset and the risky stock are R_f and R, respectively, and $X = x_0 R_f + \theta^*(R - R_f)$ and $Y = x_0 R_f + \tilde{\theta}^*(R - R_f)$ are the terminal wealths corresponding to the dollar allocation $\theta^* \geq 0$ and $\tilde{\theta}^* \geq 0$, respectively. Then, it is not difficult to show that X is less risky than Y if and only if $\theta^* \leq \tilde{\theta}^*$, and the set of bad market scenarios is $A = \{\omega \mid R(\omega) - R_f < 0\}$. Therefore, combined with the results in Chew et al. (1987), we conclude that in the single-period single-stock market, a more risk averse agent indeed chooses a less risky terminal payoff per Definition 2.

In general, we have the following result:

Proposition 2 *Let X and Y be two terminal payoffs. Suppose there exist a random variable Z and decreasing functions $f(\cdot)$ and $g(\cdot)$ such that $X = f(Z)$ and $Y = g(Z)$. If there exists z_0 such that $f(z) \leq g(z), z < z_0$ and $f(z) \geq g(z), z \geq z_0$, then X is less risky than Y.*

Proposition 2 shows that when the payoffs X and Y are monotone functions of a random variable (which represents the market condition), determining whether X is less risky than Y boils down to investigating whether the two payoff functions have the *single-crossing property*.

3.2.2 Decreasing-concave stochastic dominance

A related notion is proposed by Dybvig and Wang (2012): For any two random variables Y_1 and Y_2 with finite means, Y_1 *monotone-concave stochastically dominates* Y_2 if $\mathbb{E}[V(Y_1)] \geq \mathbb{E}[V(Y_2)]$ for any concave increasing function $V(\cdot)$. A sufficient and necessary condition for this dominance is that Y_2 is distributed as $Y_1 - Z + \varepsilon$ where $Z \geq 0$ and $\mathbb{E}[\varepsilon | Y_1 - Z] = 0$; see Dybvig and Wang (2012, Theorem 1).

In the same market setting as in Section 3.1, Dybvig and Wang (2012) show that for two investors with EU preferences, the less risk averse investor's optimal payoff X_2^* is distributed as $X_1^* + Z + \varepsilon$, where X_1^* is the optimal payoff of the more risk-averse investor, $Z \geq 0$ and $\mathbb{E}[\varepsilon | X_1^* + Z] = 0$. In other words, $-X_1^*$ monotone-concave stochastically dominates $-X_2^*$. This result suggests us the following notion of less risk taking: X_1 *decreasing-concave stochastically dominates* X_2 if X_2 is distributed as

$X_1 + Z + \varepsilon$ for some $Z \geq 0$ and $\mathbb{E}[\varepsilon | X_1 + Z] = 0$, that is, if $\mathbb{E}[V(X_1)] \geq \mathbb{E}[V(X_2)]$ for any concave decreasing function $V(\cdot)$.[9]

If one regards random noise ε as additional risk and Z as a risk premium, then X_1 decreasing-concave stochastically dominates X_2 if X_2 bears more risk (ε) in return for a risk premium (Z). In this sense, X_1 is less risky.

3.2.3 Comparison of the two notions

The notion of less risky payoffs in Definition 2 and the notion of decreasing-concave stochastic dominance differ in the following aspects: First, a less risky payoff per Definition 2 does not necessarily decreasing-concave stochastically dominate a more risky payoff. For instance, consider a random variable R representing the excess return of certain risky asset and assume that R can take both positive and negative values to exclude arbitrage opportunities. Consider the following two wealth profiles: $X_1 = W_0$ and $X_2 = \theta R + W_0$, where $\theta > 0$ stands for the dollar amount invested in the risky asset and W_0 stands for initial wealth. Using our notion of less risk taking based on payoffs, X_1 is less risky whether $\mathbb{E}[R]$ is positive or negative. This result is intuitive because a portfolio containing some risky asset is always more risky than a portfolio that is fully invested in the risk-free asset. However, X_1 does not decreasing-concave stochastically dominate X_2 when $\mathbb{E}[R] < 0$ (because $-\mathbb{E}[X_1] < -\mathbb{E}[X_2]$).

Second, the notion of less risky payoffs in Definition 2 is scenario-based while the notion of decreasing-concave stochastic dominance is distribution-based. Indeed, the inequality $\mathbb{E}[V(X_1)] \geq \mathbb{E}[V(X_2)]$ that characterizes the latter notion depends only on the distributions of X_1 and X_2. Thus, even when X_1 decreasing-concave stochastically dominates X_2, it is not necessarily that X_1 is less risky than X_2 in the sense of Definition 2.

In conclusion, the notion of less risky payoffs and the notion of decreasing-concave stochastic dominance do not encompass each other. We will therefore study whether a more risk averse RDU agent "takes less risk" under both of these notions.

[9] Indeed, X_1 decreasing-concave stochastically dominates X_2 if and only if $-X_1$ monotone-concave stochastically dominates $-X_2$, i.e., $\mathbb{E}[\tilde{V}(-X_1)] \geq \mathbb{E}[\tilde{V}(-X_2)]$ for any concave increasing function $\tilde{V}(\cdot)$. Define $V(x) := \tilde{V}(-x)$, we immediately conclude that $\mathbb{E}[V(X_1)] \geq \mathbb{E}[V(X_2)]$ for any concave decreasing function $V(\cdot)$.

3.3 More Risk Aversion Leading to Less Risk Taking

Consider two agents whose preference relations are represented by RDU. The utility and probability weighting functions of agent i are $U_i(\cdot)$ and $w_i(\cdot)$, respectively; $i = 1, 2$. Denote by X_i^* the optimal payoff at time T, i.e., the solution to portfolio choice problem (2) for agent i. Next, we present the main result of the paper. To this end, we introduce a notion of extended differentiability.

Definition 3 A function $f(\cdot)$, which maps an open set $C \subseteq \mathbb{R}$ to $\mathbb{R}$, is *extendedly differentiable* in C if

$$\lim_{\delta \downarrow 0} \frac{f(x+\delta) - f(x)}{\delta} = \lim_{\delta \downarrow 0} \frac{f(x-\delta) - f(x)}{-\delta} \in \mathbb{R} \cup \{+\infty, -\infty\}$$

for any $x \in C$.

Theorem 3 *Suppose Assumption 1 holds and that, for each $i = 1, 2$, $U_i(\cdot)$ and $w_i(\cdot)$ satisfy Assumption 2. In addition, assume that $w_i(\cdot)$, $i = 1, 2$, are extendedly differentiable in $(0, 1)$. Let payoff X_i^* be the optimal solution to portfolio choice problem (2) corresponding to $(U_i(\cdot), w_i(\cdot))$, $i = 1, 2$. If there exists a concave function $H(\cdot)$, which is continuously differentiable in the interior of its domain, such that $U_1(x) = H(U_2(x)), x > 0$ and a convex function $T(\cdot)$, which is continuously differentiable in the interior of its domain, such that $w_1(z) = T(w_2(z)), z \in [0, 1]$, then X_1^* is less risky than X_2^*.*

According to Theorem 1, one agent with an RDU preference relation is more risk averse than another agent with an RDU preference relation if and only if the utility and probability weighting functions of the first agent are more concave and more convex, respectively, than those of the second agent. Thus, Theorem 3 shows that in the RDU framework a more risk averse agent will choose a less risky terminal wealth in the single-period complete market. This result agrees with those obtained by Chew et al. (1987) in the setting of a single-period incomplete market with one stock.

We have shown in Theorem 2 that RDU agents' optimal payoffs must be decreasing functions of the pricing kernel ξ. Further, we showed in the proof of Theorem 3 the following single-crossing property: the payoff function of a more risk averse RDU agent is flatter and only once crosses the payoff function of a less risk averse agent. This property has been established by Dybvig and Wang (2012) when the agents have EU preferences. Our RDU investors are markedly different from the strictly risk averse EU agents in Dybvig and Wang (2012), as RDU agents may be locally risk seeking because of the effect of probability weighting.

3.4 The Payoff Distribution of Less Risk Averse Investors

Dybvig and Wang (2012) show that the optimal payoff of a more risk averse EU agent decreasing-concave stochastically dominates the optimal payoff of a less risk averse EU agent: the latter payoff is distributed as the former payoff plus a zero-conditional-mean noise representing risk and a nonnegative random variable providing a risk premium. We now extend this result to the RDU case, where investors apply probability weights and are not necessarily globally risk averse. Indeed, we have the following theorem:

Theorem 4 *Let the same assumptions as in Theorem 3 hold. Then, X_2^* is distributed as $X_1^* + Z + \varepsilon$, where $Z \geq 0$ and $\mathbb{E}[\varepsilon | X_1^* + Z] = 0$. Furthermore, if $X_1^* \neq X_2^*$, then neither Z or ε can be zero.*

Theorem 4 shows that a less risk averse RDU agent chooses a more risky payoff distribution in the sense that a conditional-mean-zero noise ε is added to the payoff of the less risk averse RDU agent. In return, the less risk averse agent receives the risk premium Z. This result generalizes those obtained by Dybvig and Wang (2012) in the EU setting. However, this generalization is nontrivial.

In the EU setting, it is assumed that the agents have concave utility functions and thus are risk averse. Consequently, it is expected that if a less risk averse agent takes more risk, he will be rewarded with a risk premium. In our RDU setting, although the utility functions are still concave, the probability weighting functions are not necessarily convex. It is known that an RDU agent is risk averse if and only if the utility function is concave and the weighting function is convex (Chew et al., 1987, Corollary 2). Thus, the RDU agents in our model are *not necessarily* risk averse. Theorem 4 shows that even when the agents are not strictly risk averse, a less risk averse agent still demands a risk premium when bearing more risk.

4 Stock Holdings in the Black-Scholes Market

In this section we study the impact of risk aversion on stock holdings in the Black-Scholes market, a dynamically complete market in which investors can trade one risk-free asset and one risky stock continuously between time 0 and T and the price of the stock follows a geometric Brownian motion. Previously, we showed that a more risk averse RDU agent chooses less risky terminal wealth in any single-period complete market. In the special case of a Black-Scholes market, we can consider in addition whether a more risk averse agent also invests less in the risky stock at *any* time and at *any* wealth level. In the EU case, an affirmative answer has been provided by Borell (2007), Xia (2011) and Zariphopoulou and Kallblad (2014). However, counter-examples in which this result does not hold exist for stock market processes with stochastic volatility (Liu, 2001, 2007) and for a complete market in which a risky bond is traded (Dybvig and Wang, 2012). It remains to be seen whether similar comparative statics results hold for RDU agents in the Black-Scholes market.

This section first derives closed-form expressions for the optimal terminal wealth and the corresponding (dynamic) amounts invested in stocks by RDU agents, who pre-commit to implementing their optimal investment strategy derived at time 0. We then show that when comparing an RDU agent and an EU agent, the more risk averse agent always holds less stock at any time and at any wealth level. Finally, we show that such comparative statics result does *not* hold when comparing two genuine RDU agents, as illustrated by an explicit example in which the pre-committing RDU agent is more risk averse and thus has more risky terminal wealth, but nonetheless, at some point in time, holds *more* stocks than a less risk averse RDU agent. In line with similar counter-examples cited earlier, this example illustrates that stock holding does not necessarily fully reflect risk preference. Indeed, our notion of risk taking and the notion of decreasing-concave stochastic dominance, both based on payoffs, are more natural because the agent's utility is derived from his terminal wealth, rather than from his dynamic portfolio.

4.1 Market

Consider the Black-Scholes market in which a risk-free asset and a risky asset are traded. The risk-free asset has the constant interest rate r, and the price of the risky asset follows the geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where $W(\cdot)$ is a standard Brownian motion and we denote by $\{\mathcal{F}_t\}_{t\geq 0}$ the augmented filtration generated by $W(\cdot)$.

We consider a portfolio choice problem faced at *time 0* in this market by an agent who is endowed with initial capital x_0 and trades the risk-free and risky assets continuously with the objective of maximizing the RDU of his wealth at a given terminal time T. Denote by $\pi(t)$ the dollar amount invested in the risky asset at time t. Then, the adapted process $\pi(t), t \in [0,T]$, which we call *portfolio*, represents the agent's dynamic trading strategy, that is, $\pi(t)$ stands for the dollar amount invested in the risky asset at time t. The agent's dynamic portfolio choice problem is then formulated as

$$\begin{array}{ll} \underset{\pi}{\text{Max}} & \int_0^\infty u(x)d\left[-w(1-F_{X(T)}(x))\right] \\ \text{subject to} & dX(t) = rX(t)dt + \pi(t)\left[(\mu - r)dt + \sigma dW(t)\right],\, t \in [0,T], \\ & X(0) = x_0, X(t) \geq 0,\ \forall t \in [0,T], \end{array} \tag{7}$$

where $F_{X(T)}(\cdot)$ is the CDF of $X(T)$ viewed at time 0 and $X(t) \geq 0, t \in [0,T]$ is the bankruptcy constraint.

Because of the nonlinear probability weighting function $w(\cdot)$, if the agent uses the same RDU preference relation at future times to construct his optimal portfolio, the issue of *time inconsistency* may arise: the optimal strategy planned today might not be optimal in the future. Here, we assume that the agent only considers the portfolio choice problem once, at time 0, i.e., problem (7), and the resulting optimal solution is called a *pre-commitment* strategy and will be implemented in the future.[10]

Because the standard Black-Scholes market is dynamically complete, meaning that any contingent claim at time T can be replicated by certain trading strategy, the

[10]In the literature, some researchers resolve the issue of time-inconsistency by considering the so-called *equilibrium*, instead of optimal, strategies; see e.g., Ekeland and Lazrak (2006), Björk et al. (2014) and the references therein. However, pre-committed strategies are still important, since they are frequently applied in practice, sometimes with the help of certain commitment devices. For instance, Barberis (2012) finds that the pre-committed strategy of a casino gambler is a stop-loss one (when the model parameters are within reasonable ranges). Many gamblers indeed follow this strategy by applying some commitment measures, such as leaving ATM cards at home or bringing little money to the casino; see Barberis (2012) for a full discussion.

dynamic portfolio choice problem faced by the agent can be translated into (2) with

$$\xi \equiv \xi(T) = e^{-Z(T)}, \quad Z(t) := (r + \frac{\kappa^2}{2})t + \kappa W(t), \quad t \in [0, T], \tag{8}$$

where $\kappa := (\mu - r)/\sigma$ is the *market price of risk*. Without loss of generality, we assume $\kappa > 0$.

4.2 Optimal Portfolio

To present the optimal portfolio, let us first recall the heat kernel:

$$H(s, x, y) := \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-y)^2}{2s}}, \quad s > 0, x, y \in \mathbb{R}.$$

In addition, define

$$h(x) := \hat{N}'(1 - w(1 - F_{Z(T)}(x))), \quad x \in \mathbb{R}.$$

Because $\hat{N}(\cdot)$ is strictly increasing and concave, $h(\cdot)$ is strictly positive and decreasing. Denote $\underline{h} := \inf_{x \in \mathbb{R}} h(x) = \lim_{x \to +\infty} h(x)$ and $\overline{h} := \sup_{x \in \mathbb{R}} h(x) = \lim_{x \to -\infty} h(x)$. To avoid a trivial case, we assume that $\underline{h} < \overline{h}$.

Theorem 5 *Let Assumption 2 hold. Assume that for each $\lambda > 0$, there exist $\alpha \in [1, 2)$ and $C > 0$ such that $(U')^{-1}(\lambda h(x)) \leq C(1 + e^{|x|^\alpha}), x \in \mathbb{R}$. Then,*

(i) the optimal wealth process for (7) is $X^*(t) = u(t, Z(t), \lambda^*)$, where

$$u(t, z, \lambda) := \int_{\mathbb{R}} e^{-(y-z)}(U')^{-1}(\lambda h(y)) H\left(\kappa^2(T - t), z + (r + \frac{\kappa^2}{2})(T - t), y\right) dy$$

and $\lambda^* > 0$ is such that $u(0, 0, \lambda^*) = x_0$.

(ii) For each fixed λ, $u(\cdot, \cdot, \lambda) \in C^\infty([0, T) \times \mathbb{R})$; for each fixed $(t, z) \in [0, T) \times \mathbb{R}$, $u(t, z, \cdot)$ is strictly decreasing; and for each fixed $t \in [0, T), \lambda > 0$, $u(t, \cdot, \lambda)$ is strictly increasing and

$$\lim_{z \to -\infty} u(t, z, \lambda) = e^{-r(T-t)}(U')^{-1}(\lambda \overline{h}), \quad \lim_{z \to +\infty} u(t, z, \lambda) = e^{-r(T-t)}(U')^{-1}(\lambda \underline{h}).$$

(iii) The optimal portfolio $\pi^*(t) = \pi(t, X^*(t)), 0 \le t < T$*, where*

$$\begin{aligned}\pi(t,x) &:= (\sigma^{-1}\kappa)u_z(t, u^{-1}(t,x,\lambda^*), \lambda^*),\\ &\quad x \in (e^{-r(T-t)}(U')^{-1}(\lambda^*\overline{h}), e^{-r(T-t)}(U')^{-1}(\lambda^*\underline{h})), t \in [0,T).\end{aligned} \tag{9}$$

Here, $u^{-1}(t,x,\lambda^*)$ *denotes the spatial inverse of* $u(t,z,\lambda^*)$*, i.e., the inverse function of* $u(t,z,\lambda^*)$ *with respect to* z*, and* $u_z(t,z,\lambda^*)$ *is the first-order derivative of* $u(t,z,\lambda^*)$ *with respect to* z*.*

4.3 Comparing EU and RDU Stock Holdings in the Black-Scholes Market

We show in the following theorem that, keeping the utility function fixed, a pre-committing RDU investor with a concave probability weighting function always holds more stocks than an EU investor. Similarly, a pre-committing RDU investor with a convex probability weighting always holds less stocks than in the EU case.

Theorem 6 *Suppose* $U(\cdot)$ *and* $w(\cdot)$ *satisfy Assumption 2 and there exists* $a > 0$ *such that* $-U'(x)/U''(x) \le a(1+x), x > 0$*. Assume* $\inf_{x\in\mathbb{R}} h(x) = 0$ *and* $\sup_{x\in\mathbb{R}} h(x) = +\infty$ *where* $h(x) = \hat{N}'(1 - w(1 - F_{Z(T)}(x))), x \in \mathbb{R}$*. Further, assume that* $h(\cdot)$ *is continuously differentiable and strictly decreasing, and* $h'(x)/h(x)$ *is bounded. Denote by* $\pi(t,x), (t,x) \in [0,T) \times (0,\infty)$ *the optimal portfolio of the agent with utility function* $U(\cdot)$ *and probability weighting function* $w(\cdot)$*. Denote by* $\bar{\pi}(t,x), (t,x) \in [0,T) \times (0,+\infty)$ *the optimal portfolio of the agent with the same utility* $U(\cdot)$ *and the identity probability weighting function* $\bar{w}(z) = z, z \in [0,1]$*. Then,* $\pi(t,x) \le \bar{\pi}(t,x), (t,x) \in [0,T) \times (0,+\infty)$ *when* $w(\cdot)$ *is convex, and* $\pi(t,x) \ge \bar{\pi}(t,x), (t,x) \in [0,T) \times (0,+\infty)$ *when* $w(\cdot)$ *is concave.*

Let us comment on the technical conditions in Theorem 6. Note that $h(\cdot)$ is always decreasing. The assumptions that $\inf_{x\in\mathbb{R}} h(x) = 0$, $\sup_{x\in\mathbb{R}} h(x) = +\infty$, $h(\cdot)$ is strictly decreasing and that $h'(\cdot)$ is continuous are not essential. The assumption that $-U'(x)/U''(x)$ satisfies the linear growth condition, i.e., is bounded by $a(1+x)$, is technically essential however, because it is needed to apply the comparison principle of parabolic PDEs, as in Xia (2011). Note that the power utility function satisfies this assumption. For the same reason, we need to assume that $h'(x)/h(x)$ is bounded. This assumption is satisfied by the following probability weighting function used by

Jin and Zhou (2008) and by He and Zhou (2014):

$$w(z)=\begin{cases} ke^{(a+b)\Phi^{-1}(\bar{z})+\frac{a^2}{2}}\Phi\left(\Phi^{-1}(z)+a\right), & z\le \bar{z},\\ A+ke^{\frac{b^2}{2}}\Phi\left(\Phi^{-1}(z)-b\right), & z\ge \bar{z},\end{cases} \tag{10}$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable and k and A are given as

$$k=\frac{1}{e^{\frac{b^2}{2}}\Phi\left(-\Phi^{-1}(\bar{z})+b\right)+e^{(a+b)\Phi^{-1}(\bar{z})+\frac{a^2}{2}}\Phi\left(\Phi^{-1}(\bar{z})+a\right)}>0 \text{ and } A=1-ke^{\frac{b^2}{2}},$$

respectively. Because

$$\frac{w''(z)}{w'(z)}=\begin{cases} -\frac{a}{\Phi'(\Phi^{-1}(z))}, & 0<z<\bar{z},\\ \frac{b}{\Phi'(\Phi^{-1}(z))}, & \bar{z}<z<1,\end{cases} \tag{11}$$

this weighting function is concave if $a\ge 0, b\le 0$, is convex if $a\le 0, b\ge 0$ and is inverse-S shaped if $a\ge 0, b\ge 0$. A lengthy calculation shows that when $b<\kappa\sqrt{T}$ and $a>-\kappa\sqrt{T}$,

$$h(x)=\begin{cases} Ce^{-\left(1+\frac{a}{\kappa\sqrt{T}}\right)x}, & x\le x_0,\\ Ce^{\left(-\frac{a+b}{\kappa\sqrt{T}}\right)x_0}e^{-\left(1-\frac{b}{\kappa\sqrt{T}}\right)x}, & x\ge x_0,\end{cases}$$

for some $x_0>0$ and $C>0$. Then, it follows that $h'(x)/h(x)$ is bounded.

A consequence of Theorem 6 is that when comparing an RDU agent and an EU agent, the more risk averse one *always* hold less stocks.

Corollary 1 *Consider two agents with utility functions $U_i(\cdot)$ and probability weighting functions $w_i(\cdot)$, $i=1,2$ and assume one of the agents has EU preferences, i.e., one of $w_i(\cdot)$'s is the identity function. For $i=1,2$, suppose $U_i(\cdot)$ and $w_i(\cdot)$ satisfy Assumption 2 and there exist $a_i>0$ such that $-U_i'(x)/U_i''(x)\le a_i(1+x), x>0$. Assume $\inf_{x\in\mathbb{R}}h_i(x)=0$ and $\sup_{x\in\mathbb{R}}h_i(x)=+\infty$ where $h_i(x)=\hat{N}_i'(1-w_i(1-F_{Z(T)}(x))), x\in\mathbb{R}$. Further, assume that $h_i(\cdot)$ is continuously differentiable and strictly decreasing, and $h_i'(x)/h_i(x)$ is bounded. Denote by $\pi_i(t,x), (t,x)\in[0,T)\times(0,\infty)$ the optimal portfolio of agent i, $i=1,2$. If agent 1 is more risk averse than agent 2, then*

$\pi_1(t,x) \le \pi_2(t,x), (t,x) \in [0,T) \times (0,\infty)$.

4.4 Counter-Example of a More Risk Averse Investor Holding More Stocks in the Black-Scholes Market

We have shown that when comparing an RDU agent with an EU one, the more risk averse one always holds less stocks. Intriguingly, as we will demonstrate numerically now, this comparative statics result does not extend to the case of two *genuine* RDU agents.

Using the same market data as in Mehra and Prescott (1985) and He and Zhou (2014), we set the mean return of the stock equal to $\mu = 7\%$ per year, with volatility $\sigma = 15.34\%$ and risk-free rate $r = 1\%$. Hence, the market price of risk is $\kappa = 0.391$. We assume the evaluation period T of the investor (i.e., the planning horizon) to be exactly one year. The log pricing kernel, $\ln \xi$, then, follows a normal distribution, with mean -8.65% and standard deviation 39.11%.

We consider the power utility function $U(x) = (x^{1-\alpha} - 1)/(1-\alpha), x > 0$, with $\alpha > 0$ as the relative risk aversion parameter. In the special case $\alpha = 1$, the utility function is defined as $U(x) = \ln(x), x > 0$. We choose the probability weighting function to be a power function as well: $w(z) = z^{\gamma}, z \in [0,1]$ for $\gamma > 0$. The special case in which $\gamma = 1$ corresponds to the case of no probability weighting, i.e., to one in which RDU degenerates into the standard expected utility model. For $0 < \gamma < 1$, the probability weighting function is concave, while it is convex for $\gamma > 1$. It is easy to see that $w(\cdot)$ becomes more convex when the parameter γ becomes larger. A more convex probability weighting function implies stronger underweighting of extremely good outcomes and more overweighting of the worst outcomes, which amplifies risk aversion.

In our numerical experiments we focus on the optimal portfolios at *terminal* time T. If the optimal terminal wealth, as a function of $Z(T)$, is strictly increasing and differentiable, then, in a neighborhood of that wealth level, the optimal portfolio is continuous in time t when t approaches T. As a result, the optimal portfolios near the terminal time with different parameters will maintain the same order of stock allocations at the terminal time. Further, if we find that one RDU agent has strictly higher stock holdings at time T than another agent at some particular level of wealth, then we can conclude that the first agent has higher stock holding than the second in a neighborhood of that wealth level and time. On the other hand, under some technical conditions, if one RDU agent, at *any* wealth level, has lower stock holding than another at time T, then, using the comparison theorem from the theory of parabolic partial differential equations, one can conclude that the first agent has lower stock holding at any time; see for instance the example in Theorem 6. Thus, in order to compare the stock holdings of two agents at any time, we only need to compare the holdings at the terminal time T.

We fix $\alpha = 2$ and choose γ to be 0.25, 0.5, 1, 1.5, or 2. Figure 1 shows that optimal terminal wealth is a *strictly* decreasing function of the pricing kernel $\xi(T)$ when the probability weighting function is concave ($\gamma = 0.25, 0.5$) and in the expected utility case ($\gamma = 1$). However, when the probability weighting function is convex ($\gamma =$ 1.5, 2), the optimal wealth of the investor has a ceiling due to the underweighting of very good states, which occur with low probability. Figure 2 shows the stock holding at terminal time T. Note in passing that in the expected utility case ($\gamma = 1$) the stock holding is a linear function of wealth going through the origin, a classical result demonstrating that the fraction of wealth allocated to stocks is constant.

Comparing the case in which $\gamma = 1.5$ and the case $\gamma = 2$ in Figure 2, we find that the stock holding is strictly lower in the first case than in the second case at a range of wealth levels. Recall that the higher γ is, the more risk averse the agent is. This example therefore demonstrates that, in the RDU framework, a more risk averse and pre-committed agent does not necessarily hold less stocks at any time and at any wealth level, in contrast to the result obtained in the standard EU framework. Moreover, because we have shown in Theorem 3 that a more risk averse agent chooses less risky terminal wealth, this example also illustrates that the portfolio replicating a less risky terminal wealth profile does not always necessarily consist of less stocks.

The observations we made in the previous numerical example are not specific to the choice of power probability weighting functions. Indeed, we also considered the probability weighting function (10). From (11), we can see that this weighting function becomes more convex when a becomes smaller or when b becomes larger. Using this probability weighting function, we also find that a more risk averse pre-committed agent does not necessarily hold less stocks at any time or at any wealth level.

A possible economic reason for the above seemingly counter-intuitive examples is the time-inconsistency of the *dynamic* RDU portfolio choice problem, which arises due

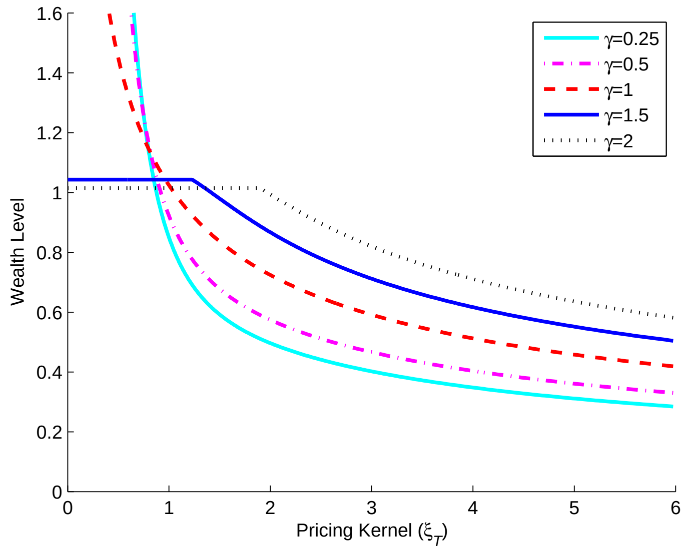

Figure 1: Optimal terminal wealth as a function of the pricing kernel ξ at time T in the Black-Scholes market. The relative risk aversion index is fixed at $\alpha = 2$. The power index of the probability weighting function γ varies: 0.25, 0.5, 1, 1.5, or 2.

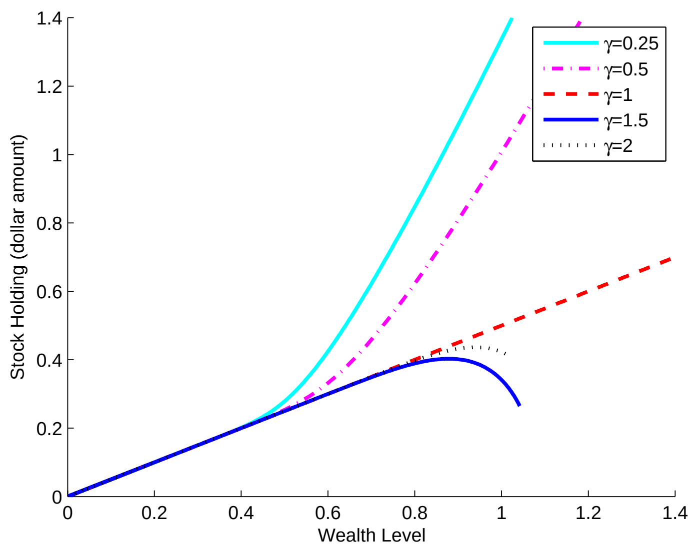

Figure 2: Optimal stock holding as a function of wealth level at time T in the Black-Scholes market. The relative risk aversion index is fixed at $\alpha = 2$. The power index of the probability weighting function γ varies: 0.25, 0.5, 1, 1.5, or 2.

to the presence of probability weighting. The replicating strategy of the "optimal" terminal wealth, which is derived from a single-period portfolio choice problem, is optimal only for $t = 0$, and *not* optimal (in the traditional sense of "optimality") for any $t > 0$. Indeed, since the underlying problem is time inconsistent, there is simply no trading strategy that is optimal for all $t > 0$. One either has to resort to some different notion of "optimality" (such as "equilibrium strategies") or simply to pre-committed strategies (which are widely adopted in practice). If we take the latter, then the stock allocations after $t = 0$ are obtained from pre-commitment rather than from optimality, and a more risk averse agent may *need* to take riskier positions in order to pre-commit.

Similar counter-examples, however, are also available for portfolio choice problems in which time-inconsistency plays no role at all, see Liu (2001, 2007) and Dybvig and Wang (2012). The standard comparative statics results, therefore, do not hold for risky asset holdings, regardless of whether the problem concerned is time-consistent or inconsistent. The key message is that, in complete markets, risky assets are simply a means that investors use to achieve desired payoffs. Thus, in our view, the new notion of less risk taking and the notion of decreasing-concave monotone dominance, which are based on payoffs, are more appropriate for comparing the portfolio choice of agents with different risk tastes.

5 Conclusion

In this paper we studied the optimal portfolio choice problem of an investor who maximizes RDU in a single-period complete market. In particular, we studied whether an investor with a more risk averse RDU preference ordering will always choose a less risky investment strategy. For this purpose, we first introduced a new notion of less risk taking for portfolio choice in single-period complete markets based on the investor's terminal wealth at planning horizon T. A less risky terminal wealth profile returns more in bad market states and less in good states. We showed that our notion of less risk taking is consistent with less stock holdings in a simple discrete-time one-period market with one stock and one risk-free asset as in Chew et al. (1987).

Then, we proved that in a general single-period complete market with a continuum of Arrow-Debreu securities, a more risk averse RDU agent will always choose a less risky terminal wealth. Further, the payoff distribution of a less risk averse RDU agent differs from the distribution of a more risk averse agent by a risk premium and some conditional-mean-zero noise. Hence, RDU agents trade off expected return and risk, defined in terms of their terminal payoff distribution. Our result is general, holding for any strictly concave utility function and any strictly increasing probability weighting function, under mild technical conditions. We extended earlier results for globally risk averse EU agents by Dybvig and Wang (2012) to a non-expected utility framework that accommodates probability weighting and locally risk seeking behavior.

Our general result for optimal payoffs, however, does not necessarily imply that a more risk averse RDU investor will hold less risky assets at all times and all wealth levels. We investigated this issue in the continuous-time Black-Scholes market in which the investor can dynamically trade one stock and a riskless asset. In this setting we first derived a closed-form expression for the time 0 optimal investment strategy of an RDU investor who pre-commits to implementing this strategy after time 0. Then, we showed that between an EU investor and RDU investor, the more risk averse one always holds less risky assets at any time and any wealth level. For two genuine RDU agents, however, we illustrated by a numerical counter-example that a more risk averse RDU investor may need to hold *more* stocks for a period of time to replicate his less risky terminal wealth, compared to an investor who is less risk averse. We argued that this phenomenon may arise from the inherent time-inconsistency of the RDU portfolio selection problem in a dynamic setting. Other counter-examples are available for standard portfolio choice problems with EU agents (Dybvig and Wang, 2012, Liu, 2001, 2007). The underlying issue is that, in complete markets, risky asset holdings are merely a device to achieve desired payoffs, and general comparative statics results hold only for these terminal payoffs.

Appendix A Some Results on Convex Envelopes

Recall the notion of extended differentiability defined in Section 3.3. For any function $f(\cdot)$ on a convex set C, recall the definition of its convex envelope $\hat{f}(\cdot)$ in Section 3.1. It is well known that the convex envelope $\hat{f}(\cdot)$ satisfies

$$\hat{f}(x) = \inf\Big\{\sum_{i=1}^{m}\alpha_i f(x_i) \mid x_i \in C, \alpha_i \geq 0, i = 1, \ldots, m, \sum_{i=1}^{m}\alpha_i = 1, \sum_{i=1}^{m}\alpha_i x_i = x\Big\}. \tag{A.1}$$

In the following, we need to use Lemma B.6 in Xia and Zhou (2013). For readers' convenience, we reproduce this lemma here (with change of notations):

Lemma 1 *Assume function $f : [0,1] \to \mathbb{R}$ is continuous. Then, its convex envelop $\hat{f}$ satisfies the following:*

(i) $\hat{f}$ is continuous on $[0,1]$;

(ii) $\hat{f}$ is affine on $\{x \in [0,1] : \hat{f}(x) < f(x)\}$; and

(iii) $\hat{f}(0) = f(0)$ and $\hat{f}(1) = f(1)$.

The following two lemmas introduce further properties of convex envelopes.

Lemma 2 *Let $f(\cdot)$ be a continuous function on $[0,1]$. Denote $\hat{f}(\cdot)$ as its convex envelop. Then, the following are true:*

(i) If $f(\cdot)$ is strictly increasing, so is $\hat{f}(\cdot)$.

(ii) If $f(\cdot)$ is extendedly differentiable on $(0,1)$, then $\hat{f}(\cdot)$ is continuously differentiable in $(0,1)$. In addition, for any $z \in (0,1)$ such that $\hat{f}(z) = f(z)$, $f(\cdot)$ is differentiable at z and $\hat{f}'(z) = f'(z)$.

Proof We first prove statement (i). Recall that $\hat{f}$ is the convex envelope of f, so $\hat{f}(z) \le f(z), z \in [0,1]$. Fix any $0 \le z_1 < z_2 \le 1$. Define

$$x_i := \sup\{z \le z_i | \hat{f}(z) = f(z)\}, \quad y_i := \inf\{z \ge z_i | \hat{f}(z) = f(z)\}, \quad i = 1, 2.$$

From Lemma 1-(iii), we have $f(0) = \hat{f}(0)$ and $f(1) = \hat{f}(1)$; so $x_i \in [0, z_i]$ and $y_i \in [z_i, 1]$ are well defined. Because both $\hat{f}$ and f are continuous due to Lemma 1-(i), we must have $\hat{f}(x_i) = f(x_i)$ and $\hat{f}(y_i) = f(y_i)$. Finally, by the definition of x_i and y_i, we conclude that $\hat{f}(z) < f(z)$ for any $z \in (x_i, y_i)$. Furthermore, $x_i < y_i$ if and only if $\hat{f}(z_i) < f(z_i)$, in which case $x_i < z_i < y_i$.

If $x_i = y_i$, $i = 1, 2$, we have $\hat{f}(z_1) = f(z_1) < f(z_2) = \hat{f}(z_2)$, where the inequality is the case because f is strictly increasing. If $x_1 = y_1$ and $x_2 < y_2$, we have $x_1 = y_1 = z_1$ and $\hat{f}(z_1) = f(z_1)$. By the definition of x_2, we must have $z_1 \le x_2$. Because $\hat{f}(z) < f(z), z \in (x_2, y_2)$, Lemma 1-(ii) shows that $\hat{f}$ is affine on $[x_2, y_2]$. Because f is strictly increasing, we have $\hat{f}(x_2) = f(x_2) < f(y_2) = \hat{f}(y_2)$, showing that $\hat{f}$

is strictly increasing in $[x_2, y_2]$. Because $z_1 \le x_2 < z_2 < y_2$, we conclude that $\hat{f}(z_1) = f(z_1) \le f(x_2) = \hat{f}(x_2) < \hat{f}(z_2)$. Using a similar argument, we can show that $\hat{f}(z_1) < \hat{f}(z_2)$ when $x_1 < y_1$ and $x_2 = y_2$. Finally, we consider the case in which $x_i < y_i, i = 1, 2$. Again, because f is strictly increasing and $\hat{f}$ is affine on $[x_i, y_i]$, we conclude that $\hat{f}(z_1) < \hat{f}(y_1) = f(y_1)$ and $f(x_2) = \hat{f}(x_2) < \hat{f}(z_2)$. If $y_1 \le x_2$, we immediately conclude that $\hat{f}(z_1) < \hat{f}(z_2)$. Otherwise, by the definition of $x_i, y_i, i = 1, 2$, we must have $y_1 = y_2$, in which case $x_1 < z_1 < z_2 < y_2$ and $\hat{f}$ is affine on $[x_1, y_2]$. Consequently, $\hat{f}$ is strictly increasing on $[x_1, y_2]$ and $\hat{f}(z_1) < \hat{f}(z_2)$.

Next, we prove statement (ii). First, thanks to Lemma 1, the set $\{z \in [0,1] | \hat{f}(z) < f(z)\}$ does not contain $\{0, 1\}$ and is open. In addition, $\hat{f}(\cdot)$ is affine on this set. Consequently, $\hat{f}(\cdot)$ is differentiable on this set. For any $z \in (0, 1)$ with $\hat{f}(z) = f(z)$, we have

$$\hat{f}'(z-) = \lim_{\delta \downarrow 0} \frac{\hat{f}(z-\delta) - \hat{f}(z)}{-\delta} \ge \lim_{\delta \downarrow 0} \frac{f(z-\delta) - f(z)}{-\delta} = f'(z).$$

Similarly, $\hat{f}'(z+) \le f'(z)$. Because $+\infty > \hat{f}'(z+) \ge \hat{f}'(z-) > -\infty$ from the convexity of $\hat{f}(\cdot)$, we conclude that $\hat{f}'(z+) = \hat{f}'(z-) = f'(z) \in \mathbb{R}$. Thus, both $\hat{f}(\cdot)$ and $f(\cdot)$ are differentiable at z. Consequently, $\hat{f}(\cdot)$ is differentiable on $(0, 1)$. Because $\hat{f}(\cdot)$ is convex, it must be continuously differentiable on $(0, 1)$. $\square$

Lemma 3 *Let $f(\cdot)$ and $g(\cdot)$ be two strictly increasing and continuous functions on $[0, 1]$ with extended differentiability on $(0, 1)$. Denote $\hat{f}(\cdot)$ and $\hat{g}(\cdot)$ as the convex envelopes of $f(\cdot)$ and $g(\cdot)$, respectively. Suppose that there exist a strictly increasing, continuous and convex function $H(\cdot)$ on $[0, 1]$ with differentiability on $(0, 1)$ and $H(0) = 0$ and $H(1) = 1$ such that $f(z) = g(H(z)), z \in [0, 1]$. Then, $\frac{\hat{f}'(z)}{\hat{g}'(H(z))}$ is increasing on $(0, 1)$.*

Proof From Lemmas 1 and 2, $\hat{f}(\cdot)$ and $\hat{g}(\cdot)$ are strictly increasing and continuous on $[0, 1]$ and continuously differentiable on $(0, 1)$. Because of convexity, we must have $\hat{f}'(z) > 0, \hat{g}'(z) > 0, z \in (0, 1)$. Thus, the function $\frac{\hat{f}'(z)}{\hat{g}'(H(z))}$ is well-defined on $(0, 1)$.

Define $\mathcal{A}_f := \{z \in [0,1] | \hat{f}(z) < f(z)\}$ and $\mathcal{B}_f := [0,1] \backslash \mathcal{A}_f$. Define $\mathcal{A}_g$ and $\mathcal{B}_g$ similarly. Then, both $\mathcal{B}_f$ and $\mathcal{B}_g$ are closed sets containing $\{0, 1\}$ because of Lemma 1. Next, we show that $\mathcal{A}_f \subseteq H^{-1}\mathcal{A}_g$. For any $z_0 \in \mathcal{A}_f$, define $x_0 := \sup\{z \le z_0 | \hat{f}(z) = f(z)\}$ and $y_0 := \inf\{z \ge z_0 | \hat{f}(z) = f(z)\}$. Again, by Lemma 1, we have

$x_0 < z_0 < y_0$, $\hat{f}(x_0) = f(x_0)$ and $\hat{f}(y_0) = f(y_0)$. Thanks to Lemma 1-(ii), we have $\hat{f}(\cdot)$ is affine on $[x_0, y_0]$; so

$$f(z_0) > \hat{f}(z_0) = \frac{\hat{f}(y_0) - \hat{f}(x_0)}{y_0 - x_0}(z_0 - x_0) + \hat{f}(x_0) = \frac{f(y_0) - f(x_0)}{y_0 - x_0}(z_0 - x_0) + f(x_0).$$

Recalling that $f(z) = g(H(z)), z \in [0, 1]$, we have

$$\begin{aligned} g(H(z_0)) = f(z_0) &> \frac{f(y_0) - f(x_0)}{y_0 - x_0}(z_0 - x_0) + f(x_0) \\ &= \frac{g(H(y_0)) - g(H(x_0))}{y_0 - x_0}(z_0 - x_0) + g(H(x_0)) \\ &= \frac{g(H(y_0)) - g(H(x_0))}{H(y_0) - H(x_0)}(H(z_0) - H(x_0))\frac{\frac{H(y_0)-H(x_0)}{y_0-x_0}}{\frac{H(z_0)-H(x_0)}{z_0-x_0}} + g(H(x_0)) \\ &\geq \frac{g(H(y_0)) - g(H(x_0))}{H(y_0) - H(x_0)}(H(z_0) - H(x_0)) + g(H(x_0)), \end{aligned}$$

where the last inequality is due to the convexity of $H(\cdot)$. By the characterization of convex envelop (A.1), $H(z_0) \in \mathcal{A}_g$.

Finally, we show that for any $0 < z_1 < z_2 < 1$,

$$\frac{\hat{f}'(z_1)}{\hat{g}'(H(z_1))} \leq \frac{\hat{f}'(z_2)}{\hat{g}'(H(z_2))}.$$

Define $y_1 := \inf\{z \geq z_1 \mid \hat{g}(H(z)) = g(H(z))\}$ and $x_2 := \sup\{z \leq z_2 \mid \hat{g}(H(z)) = g(H(z))\}$. By Lemma 1 and the continuity and strict monotonicity of H, we conclude that $y_1 \in [z_1, 1]$ and $x_2 \in [0, z_2]$ are well defined, $\hat{g}(H(z)) < g(H(z))$ for $z \in [z_1, y_1) \cup (x_2, z_2]$, $y_1 > z_1$ if and only if $\hat{g}(H(z_1)) < g(H(z_1))$, $x_2 < z_2$ if and only if $\hat{g}(H(z_2)) < g(H(z_2))$, and $\hat{g}(H(y_1)) = g(H(y_1)), \hat{g}(H(x_2)) = g(H(x_2))$, i.e., $y_1, x_2 \in H^{-1}\mathcal{B}_g$.

We first consider the case in which $y_1 \geq z_2$. In this case, $H(z_1) < H(z_2) \leq H(y_1)$ and $\hat{g}(\tilde{z}) < g(\tilde{z})$ for $\tilde{z} \in [H(z_1), H(y_1))$. By Lemma 1-(ii), $\hat{g}$ is affine on $[H(z_1), H(y_1)]$, yielding $\hat{g}'(H(z_1)) = \hat{g}'(H(z_2))$. Because $\hat{f}(\cdot)$ is convex, $\hat{f}'(\cdot)$ is increasing. As a result,

$$\frac{\hat{f}'(z_1)}{\hat{g}'(H(z_1))} = \frac{\hat{f}'(z_1)}{\hat{g}'(H(z_2))} \leq \frac{\hat{f}'(z_2)}{\hat{g}'(H(z_2))}.$$

The case in which $x_2 \leq z_1$ can be treated similarly.

Finally, we consider the case in which $y_1 < z_2$ and $x_2 > z_1$. Because $\hat{g}(H(z)) <$

$g(H(z))$ for $z \in [z_1, y_1)$ and $\hat{g}(H(x_2)) = g(H(x_2))$, we must have $y_1 \le x_2$, i.e., $0 < z_1 \le y_1 \le x_2 \le z_2 < 1$. On the one hand, $\hat{g}'(H(z_1)) = \hat{g}'(H(y_1))$ because $\hat{g}(\cdot)$ is affine on $[H(z_1), H(y_1)]$. As a result,

$$\frac{\hat{f}'(z_1)}{\hat{g}'(H(z_1))} = \frac{\hat{f}'(z_1)}{\hat{g}'(H(y_1))} \le \frac{\hat{f}'(y_1)}{\hat{g}'(H(y_1))}$$

where the inequality is due to the convexity of $\hat{f}(\cdot)$. For a similar reason, we have

$$\frac{\hat{f}'(z_2)}{\hat{g}'(H(z_2))} \ge \frac{\hat{f}'(x_2)}{\hat{g}'(H(x_2))}.$$

On the other hand, because $\mathcal{A}_f \subseteq H^{-1}\mathcal{A}_g$ and $y_1 \in H^{-1}\mathcal{B}_g$, we conclude that $y_1 \in \mathcal{B}_f$. As a result, by Lemma 2-(ii), $g(\cdot)$ is differentiable at $H(y_1)$ with $\hat{g}'(H(y_1)) = g'(H(y_1))$ and $f(\cdot)$ is differentiable at y_1 with $\hat{f}'(y_1) = f'(y_1)$. Because $f(y_1) = g(H(y_1))$, $H(\cdot)$ is differentiable on $(0,1)$ and $g(\cdot)$ is differentiable at $H(y_1)$, we have $\hat{f}'(y_1) = f'(y_1) = g'(H(y_1))H'(y_1)$. For a similar reason, $g(\cdot)$ is differentiable at $H(x_2)$, $\hat{g}'(H(x_2)) = g'(H(x_2))$ and $\hat{f}'(x_2) = f'(x_2) = g'(H(x_2))H'(x_2)$. Then, we have

$$\frac{\hat{f}'(x_2)}{\hat{g}'(H(x_2))} = \frac{g'(H(x_2))H'(x_2)}{g'(H(x_2))} = H'(x_2) \ge H'(y_1) = \frac{g'(H(y_1))H'(y_1)}{g'(H(y_1))} = \frac{\hat{f}'(y_1)}{\hat{g}'(H(y_1))},$$

and this completes the proof. $\square$

Appendix B Proofs

Proof of Theorem 2 This is a direct consequence of Theorem 3.3 in Xia and Zhou (2013). Note that we do not have consumption at time 0. In addition, the function $N(\cdot)$, defined as in Eq. (3.10) in Xia and Zhou (2013), is the same as in (4). $\square$

Proof of Proposition 1 Suppose X is less risky than Y when the event A represents the bad market scenarios, but Y is less risky than X when the bad event is B. Because $X \ge Y$ on A almost surely (abbreviated as 'a.s.') and $Y \ge X$ a.s. on B, we conclude that $X = Y$ a.s. on $A \cap B$. Using a similar argument, we can conclude that $X = Y$ a.s. on $A^c \cap B^c$. Next, we first consider the case in which $A \cap B^c$ has zero probability. In this case, because $X \le Y$ a.s. on A^c, we have $X \le Y$ a.s. on $A^c \cap B$. Because

$A \cap B^c = \Omega/((A \cap B) \cup (A^c \cap B^c) \cup (A^c \cap B))$ has zero probability, we conclude that $X \le Y$ a.s. Since X and Y do not dominate each other, it follows that $X = Y$ a.s. Similarly, when $B \cap A^c$ has zero probability, we also have $X = Y$ a.s.

Finally, we consider the case in which both $A \cap B^c$ and $B \cap A^c$ have nonzero probability. Because $\sup_{\omega \in A} X(\omega) \le \inf_{\omega \in A^c} X(\omega)$, we have $\sup_{\omega \in A \cap B^c} X(\omega) \le \inf_{\omega \in A^c \cap B} X(\omega)$. On the other hand, since $\sup_{\omega \in B} X(\omega) \le \inf_{\omega \in B^c} X(\omega)$, we conclude that $\sup_{\omega \in B \cap A^c} X(\omega) \le \inf_{\omega \in B^c \cap A} X(\omega)$. As a result, we have

$$\inf_{\omega \in A \cap B^c} X(\omega) \le \sup_{\omega \in A \cap B^c} X(\omega) \le \inf_{\omega \in A^c \cap B} X(\omega) \le \sup_{\omega \in B \cap A^c} X(\omega) \le \inf_{\omega \in B^c \cap A} X(\omega),$$

from which it follows that X is constant on $(A \cap B^c) \cup (B \cap A^c)$. Similarly, Y is constant on $(A \cap B^c) \cup (B \cap A^c)$. However, $X \ge Y$ on A and $X \le Y$ on A^c a.s., we deduce $X = Y$ on $(A \cap B^c) \cup (B \cap A^c)$. Since we already showed that $X = Y$ a.s. on $(A \cap B) \cup (A^c \cap B^c)$, we finally arrive at $X = Y$ a.s. □

Proof of Proposition 2 Choose $A = \{\omega \mid Z(\omega) < z_0\}$ and the conclusion follows immediately. □

Proof of Theorem 3 By Theorem 2, we have, for each $i = 1, 2$,

$$X_i^* = (U_i')^{-1} \left(\lambda_i^* \hat{N}_i'(1 - w_i(F_\xi(\xi))) \right)$$

where $\hat{N}_i(\cdot)$ is the concave envelope of

$$N_i(z) := -\int_0^{w_i^{-1}(1-z)} F_\xi^{-1}(t) dt, \quad z \in [0, 1]$$

and $\lambda_i^* > 0$ is the number such that $\mathbb{E}[\xi X_i^*] = x_0$. It is clear that X_i^* is a function of ξ and this function is decreasing because of the concavity of $U_i(\cdot)$ and $\hat{N}_i(\cdot)$.

To proceed, we first show that

$$\frac{\hat{N}_1'(1 - w_1(F_\xi(x)))}{\hat{N}_2'(1 - w_2(F_\xi(x)))}$$

is a decreasing function, which is equivalent to showing that

$$\frac{\hat{N}_1'(1-w_1(w_2^{-1}(z)))}{\hat{N}_2'(1-z)} = \frac{\hat{N}_1'(1-T(z))}{\hat{N}_2'(1-z)}, \quad 0<z<1$$

is decreasing. Denote $M_i(z) := -N_i(1-z), z \in [0,1]$ and $\hat{M}_i(\cdot)$ the convex envelop of $M_i(\cdot)$, $i=1,2$. Then we have $\hat{N}_i(z) = -\hat{M}_i(1-z), z \in [0,1]$. As a result, we only need to show that

$$\frac{\hat{M}_1'(T(z))}{\hat{M}_2'(z)}, \quad 0<z<1$$

is decreasing.

Denote $\varphi(z) := \int_0^z F_\xi^{-1}(t)dt, z \in [0,1]$. By Assumption 1, $\varphi(\cdot)$ is strictly increasing, continuous on $[0,1]$ and continuously differentiable in $(0,1)$ with $\varphi'(z) = F_\xi^{-1}(z) > 0, z \in (0,1)$. On the other hand, because we assume that $w_i(\cdot)$ is extendedly differentiable in $(0,1)$, so is $w_i^{-1}(\cdot)$, $i=1,2$. Noting $M_i(z) = -N_i(1-z) = \varphi(w_i^{-1}(z)), z \in [0,1]$, we conclude that $M_i(\cdot)$ is strictly increasing and continuous on $[0,1]$ and is extendedly differentiable in $(0,1)$, $i=1,2$.

On the other hand, since $w_1(z) = T(w_2(z)), z \in [0,1]$, we have $w_1^{-1}(y) = w_2^{-1}(T^{-1}(y))$, $y \in [0,1]$. As a result,

$$M_1(T(z)) = \varphi(w_1^{-1}(T(z))) = \varphi(w_2^{-1}(T^{-1}(T(z)))) = M_2(z), \quad z \in [0,1].$$

Now, applying Lemma 3 in Appendix A, we conclude that $\frac{\hat{M}_1'(T(z))}{\hat{M}_2'(z)}$ is decreasing in $0<z<1$; so $\frac{\hat{N}_1'(1-w_1(F_\xi(x)))}{\hat{N}_2'(1-w_2(F_\xi(x)))}$ is decreasing in $x>0$.

Next, from $U_1(x) = H(U_2(x)), x>0$, it follows that $U_1'(x) = H'(U_2(x))U_2'(x), x > 0$. Consequently,

$$(U_1')^{-1}(y) = (U_2')^{-1}\left(\frac{y}{H'(U_2((U_1')^{-1}(y)))}\right), \quad y>0.$$

Denote

$$\psi_1(x) := \frac{\lambda_1^*\hat{N}_1'(1-w_1(F_\xi(x)))}{H'(U_2((U_1')^{-1}(\lambda_1^*\hat{N}_1'(1-w_1(F_\xi(x))))))}, \quad \psi_2(x) := \lambda_2^*\hat{N}_2'(1-w_2(F_\xi(x))).$$

Then, $X_i^* = f_i(\xi)$, where $f_i(x) := (U_2')^{-1}(\psi_i(x))$, $i = 1, 2$. Note that $f_i(\cdot)$ is decreasing and continuous because $\hat{N}_i'(\cdot)$ is continuous as a result of Lemma 2-(ii) in Appendix. Because $(U_2')^{-1}(\cdot)$ is decreasing, to finish the proof, it suffices to show, in view of Proposition 2, that there exists x_0 such that $\psi_1(x) \geq \psi_2(x)\ \forall x < x_0$ and $\psi_1(x) \leq \psi_2(x)\ \forall x \geq x_0$.

Note that $\hat{N}_i'(z) > 0, z \in (0, 1)$ following Lemma 2; so $\psi_i(x) > 0$. We have

$$\frac{\psi_1(x)}{\psi_2(x)} = \frac{\lambda_1^*}{\lambda_2^*} \frac{\hat{N}_1'(1 - w_1(F_\xi(x))}{\hat{N}_2'(1 - w_2(F_\xi(x)))} \frac{1}{H'(U_2((U_1')^{-1}(\lambda_1^* \hat{N}_1'(1 - w_1(F_\xi(x))))))}.$$

We have shown that $\frac{\hat{N}_1'(1-w_1(F_\xi(x))}{\hat{N}_2'(1-w_2(F_\xi(x)))}$ is decreasing in x. On the other hand, because $H(\cdot)$, $U_1(\cdot)$ and $\hat{N}_1(\cdot)$ are concave and $U_2(\cdot)$, $w_1(\cdot)$ and $F_\xi(\cdot)$ are increasing, we have that $H'(U_2((U_1')^{-1}(\lambda_1^* \hat{N}_1'(1 - w_1(F_\xi(x))))))$ is increasing in x. Consequently, $\psi_1(x)/\psi_2(x)$ is decreasing in x. Because of the initial budget constraint, we have $\mathbb{E}[\xi(U_2')^{-1}(\psi_1(\xi))] = \mathbb{E}[\xi(U_2')^{-1}(\psi_2(\xi))]$. Thus, neither $\psi_1(x)/\psi_2(x) > 1\ \forall x \in \mathbb{R}$ nor $\psi_2(x)/\psi_2(x) < 1\ \forall x \in \mathbb{R}$ holds. However, $\psi_1(x)/\psi_2(x)$ is decreasing in x; hence there must exist x_0 such that $\psi_1(x)/\psi_2(x) \geq 1$ for $x < x_0$ and $\psi_1(x)/\psi_2(x) \leq 1$ for $x \geq x_0$. This completes the proof. $\square$

Proof of Theorem 4 We first show that $\mathbb{E}[X_2^*] \geq \mathbb{E}[X_1^*]$ and that the inequality becomes an equality if and only if $X_1^* = X_2^*$. By Theorem 2, $X_i^* = f_i(\xi)$ where $f_i(\cdot)$ is a decreasing function, $i = 1, 2$. From the proof of Theorem 3, there exist $x_0 > 0$ such that $f_2(x) \geq f_1(x)$ when $x < x_0$ and $f_2(x) \leq f_1(x)$ when $x > x_0$. Consequently, $(f_2(x) - f_1(x))(x - x_0) \leq 0$ for any x, so we have

$$\begin{aligned} 0 &\geq \mathbb{E}[(f_2(\xi) - f_1(\xi))(\xi - x_0)] = \mathbb{E}[(X_2^* - X_1^*)(\xi - x_0)] \\ &= \mathbb{E}[\xi X_2^*] - \mathbb{E}[\xi X_1^*] - x_0(\mathbb{E}[X_2^*] - \mathbb{E}[X_1^*]) = -x_0(\mathbb{E}[X_2^*] - \mathbb{E}[X_1^*]) \end{aligned}$$

where the last equality is due to the initial budget constraint. As a result, $\mathbb{E}[X_2^*] \geq \mathbb{E}[X_1^*]$. Noting that the only inequality in the above derivation becomes an equality if and only if $f_2(\xi) - f_1(\xi) = 0$ almost surely, we conclude that $\mathbb{E}[X_2^*] = \mathbb{E}[X_1^*]$ if and only if $X_1^* = X_2^*$.

Next, following steps similar to those presented in Dybvig and Wang (2012), we show that X_2^* is distributed as $X_1^* + Z + \varepsilon$. To this end, we only need to show that X_1^* decreasing-concave stochastically dominates X_2^*, i.e., $\mathbb{E}[V(X_1^*)] \geq \mathbb{E}[V(X_2^*)]$ for any decreasing concave function $V(\cdot)$. Fix any such function $V(\cdot)$. For any x in the domain of $V(\cdot)$, let $V'(x)$ be any selection from the sub-gradient of $V(\cdot)$ at x. Because of concavity, $V'(\cdot)$ is decreasing. Recall that $X_i^* = f_i(\xi)$, $i = 1, 2$. For any $x > x_0$,

$$V(f_2(x)) - V(f_1(x)) \le V'(f_1(x))(f_2(x) - f_1(x)) \le V'(f_1(x_0))(f_2(x) - f_1(x))$$

where the last inequality is valid because $V'(\cdot)$ is decreasing and $f_1(x_0) \ge f_1(x) \ge f_2(x)$ for $x > x_0$. On the other hand, for any $x < x_0$,

$$V(f_2(x)) - V(f_1(x)) \le V'(f_1(x))(f_2(x) - f_1(x)) \le V'(f_1(x_0))(f_2(x) - f_1(x))$$

where the last inequality is due to $V'(\cdot)$ being decreasing and $f_1(x_0) \le f_1(x) \le f_2(x)$ for $x < x_0$. Therefore, we have $V(X_2^*) - V(X_1^*) \le V'(f_1(x_0))(X_2^* - X_1^*)$ almost surely. Consequently,

$$\mathbb{E}[V(X_2^*)] - \mathbb{E}[V(X_1^*)] \le V'(f_1(x_0))\mathbb{E}[X_2^* - X_1^*] \le 0,$$

where the last inequality follows from the fact that $\mathbb{E}[X_1^*] \le \mathbb{E}[X_2^*]$ and $V(\cdot)$ is decreasing.

Finally, if $X_1^* \ne X_2^*$, then $\mathbb{E}[X_1^*] < \mathbb{E}[X_2^*]$. Because $\mathbb{E}[X_2^*] = \mathbb{E}[X_1^*] + \mathbb{E}[Z]$, we conclude that $Z \ne 0$. Next, we show that $\epsilon \ne 0$; otherwise, X_2^* strictly first-order stochastically dominates X_1^* because $Z \ge 0$ and $Z \ne 0$. Because the utility function $U_1(\cdot)$ and probability weighting function $w_1(\cdot)$ (of, say, agent 1) are strictly increasing, X_2^* is strictly preferred to X_1^* by agent 1, which contradicts the optimality of X_1^*. $\square$

Proof of Thoerem 5 We first prove statement (i). According to Theorem 2, the optimal terminal wealth

$$X^*(T) = (U')^{-1}\left(\lambda^* \hat{N}'(1 - w(F_\xi(\xi)))\right) = (U')^{-1}(\lambda^* h(Z(T))).$$

According to standard portfolio selection theory (Karatzas and Shreve, 1998), we have

$$X^*(t) = e^{Z(t)}\mathbb{E}\left[e^{-Z(T)}X^*(T)|\mathcal{F}_t\right], \quad 0 \le t \le T.$$

Straightforward calculation shows that $X^*(t) = u(t, Z(t), \lambda^*)$. In particular, $x_0 =$

$X^*(0) = u(0, Z(0), \lambda^*) = u(0, 0, \lambda^*)$.

Statement (ii) can be checked directly using the dominance convergence theorem and monotone convergence theorem and the assumption that $(U')^{-1}(\lambda h(z))$ are properly bounded.

Finally, we prove statement (iii). Because $X^*(\cdot)$ represents the wealth process associated with the optimal portfolio $\pi^*(\cdot)$, we have the following wealth equation:

$$dX^*(t) = rX^*(t)dt + \pi^*(t)\left[(\mu - r)dt + \sigma dW(t)\right].$$

On the other hand, applying Itô lemma to $X^*(t) = u(t, Z(t), \lambda^*)$ yields

$$dX^*(t) = a(t)dt + u_z(t, Z(t), \lambda^*)\kappa dW(t)$$

for some process $a(\cdot)$. As a result, the optimal portfolio must be

$$\pi^*(t) = (\sigma^{-1}\kappa)u_z(t, Z(t), \lambda^*).$$

Consequently,

$$\pi^*(t) = (\sigma^{-1}\kappa)u_z(t, Z(t), \lambda^*) = (\sigma^{-1}\kappa)u_z(t, u^{-1}(t, X^*(t), \lambda^*), \lambda^*) = \pi(t, X^*(t)), \quad t \in [0, T). \quad \square$$

Proof of Theorem 6 First, we derive some bound of $(U')^{-1}(\lambda h(x))$ in order to apply Theorem 5.

Because h'/h is bounded and $h(\cdot)$ is strictly decreasing, there exist $C > 0$ such that $-C \leq h'(x)/h(x) < 0, x \in \mathbb{R}$. As a result, for any $x < 0$,

$$\ln(h(0)/h(x)) = \int_x^0 h'(u)/h(u)du \geq Cx,$$

which shows that $h(x) \leq h(0)e^{-Cx}$. Similarly, for any $x > 0$, we have $h(x) \geq h(0)e^{-Cx}$.

On the other hand, because $-U'(x)/U''(x) \leq a(1+x)$ and $\lim_{y\to 0}(U')^{-1}(y) = +\infty$, we have

$$-\frac{1}{(U')^{-1}(y)U''((U')^{-1}(y))} \leq a\frac{1+(U')^{-1}(y)}{y(U')^{-1}(y)} \leq \frac{2a}{y}$$

for sufficiently small y. Then, for sufficiently small y and y_0 such that $y < y_0$, we have

$$\begin{aligned}
(U')^{-1}(y) &= (U')^{-1}(y_0)\exp\left[\ln(U')^{-1}(y) - \ln(U')^{-1}(y_0)\right] \\
&= (U')^{-1}(y_0)\exp\left[\int_y^{y_0} -\frac{1}{(U')^{-1}(x)U''((U')^{-1}(x))}dx\right] \\
&\leq (U')^{-1}(y_0)\exp\left[\int_y^{y_0} \frac{2a}{x}dx\right] \\
&= (U')^{-1}(y_0)y_0^{2a}y^{-2a}.
\end{aligned}$$

Therefore, for any $\lambda > 0$ and sufficiently large x,

$$(U')^{-1}(\lambda h(x)) \leq (U')^{-1}(\lambda h(0)e^{-Cx}) \leq (U')^{-1}(y_0)y_0^{2a}(\lambda h(0))^{-2a}e^{-2aCx}.$$

Now, we can apply Theorem 5 and obtain that, in the optimal portfolio,

$$\pi(t,x) = (\sigma^{-1}\kappa)u_z(t, u^{-1}(t,x,\lambda^*), \lambda^*).$$

By the Feyman-Kac representation formula, u satisfies the following linear parabolic partial differential equation (PDE):

$$\begin{cases} u_t + \frac{\kappa^2}{2}u_{zz} + (r + \frac{\kappa^2}{2})u_z - ru = 0, \\ u(T,z,\lambda^*) = (U')^{-1}\left(\lambda^* h(z)\right). \end{cases} \tag{B.1}$$

From $u(t, u^{-1}(t,x,\lambda^*), \lambda^*) = x$, we obtain

$$\frac{\partial}{\partial t}u^{-1}(t,x,\lambda^*) = -\frac{u_t(t,u^{-1}(t,x,\lambda^*),\lambda^*)}{u_z(t,u^{-1}(t,x,\lambda^*),\lambda^*)}, \quad \frac{\partial}{\partial x}u^{-1}(t,x,\lambda^*) = \frac{1}{u_z(t,u^{-1}(t,x,\lambda^*),\lambda^*)}.$$

As a result,

$$\begin{aligned}
\frac{\pi_t(t,x)}{\sigma^{-1}\kappa} &= u_{zt}(t,u^{-1}(t,x,\lambda^*),\lambda^*) - \frac{u_{zz}(t,u^{-1}(t,x,\lambda^*),\lambda^*)u_t(t,u^{-1}(t,x,\lambda^*),\lambda^*)}{u_z(t,u^{-1}(t,x,\lambda^*),\lambda^*)},\\
\frac{\pi_x(t,x)}{\sigma^{-1}\kappa} &= \frac{u_{zz}(t,u^{-1}(t,x,\lambda^*),\lambda^*)}{u_z(t,u^{-1}(t,x,\lambda^*),\lambda^*)},\\
\frac{\pi_{xx}(t,x)}{\sigma^{-1}\kappa} &= \frac{1}{u_z(t,u^{-1}(t,x,\lambda^*),\lambda^*)^2}\left[u_{zzz}(t,u^{-1}(t,x,\lambda^*),\lambda^*) - \frac{u_{zz}(t,u^{-1}(t,x,\lambda^*),\lambda^*)^2}{u_z(t,u^{-1}(t,x,\lambda^*),\lambda^*)}\right].
\end{aligned}$$

From (B.1), we have

$$u_{zt} = ru_z - (r+\frac{\kappa^2}{2})u_{zz} - \frac{\kappa^2}{2}u_{zzz}, \quad u_t = ru - (r+\frac{\kappa^2}{2})u_z - \frac{\kappa^2}{2}u_{zz}.$$

Consequently,

$$\begin{aligned}
\frac{\pi_t}{\sigma^{-1}\kappa} &= ru_z - (r+\frac{\kappa^2}{2})u_{zz} - \frac{\kappa^2}{2}u_{zzz} - \frac{ruu_{zz}}{u_z} + (r+\frac{\kappa^2}{2})u_{zz} + \frac{\kappa^2}{2}\frac{u_{zz}^2}{u_z}\\
&= ru_z - \frac{\kappa^2}{2}(u_{zzz} - \frac{u_{zz}^2}{u_z}) - \frac{ruu_{zz}}{u_z}\\
&= ru_z - \frac{\kappa^2}{2}\frac{\pi_{xx}}{\sigma^{-1}\kappa}u_z^2 - ru\frac{\pi_x}{\sigma^{-1}\kappa},
\end{aligned}$$

where the argument in π_t, π_x, π_{xx} is (t,x) and the argument in $u, u_t, u_{zt}, u_z, u_{zz}, u_{zzz}$ is $(t,u^{-1}(t,x,\lambda^*),\lambda^*)$. Recalling that $\pi = (\sigma^{-1}\kappa)u_z$, we conclude that π satisfies the following nonlinear parabolic PDE:

$$\pi_t + \frac{\sigma^2}{2}\pi^2\pi_{xx} + rx\pi_x - r\pi = 0, \quad x \in (0,+\infty),\, t \in [0,T).$$

Next, we derive the boundary condition for π. On the one hand, one can use the dominance convergence theorem to show that

$$\lim_{z\to-\infty} u_z(t,z,\lambda^*) = 0, \quad \lim_{z\to-\infty} u(t,z,\lambda^*) = 0.$$

As a result, we have

$$\lim_{x\to 0}\pi(t,x) = 0.$$

On the other hand, because $h(\cdot)$ is continuously differentiable, $\pi(t,x)$ is continuous when t goes to T. Because

$$u_z(T,z,\lambda^*) = \frac{\lambda^* h'(z)}{U''((U')^{-1}(\lambda^* h(z)))},$$

we obtain the following boundary condition:

$$\begin{aligned}\pi(T,x) &= (\sigma^{-1}\kappa)u_z(T,u^{-1}(T,x,\lambda^*),\lambda^*) = (\sigma^{-1}\kappa)\frac{\lambda^* h'\left(h^{-1}\left(\frac{U'(x)}{\lambda^*}\right)\right)}{U''(x)} \\ &= (\sigma^{-1}\kappa)\left[-\frac{h'\left(h^{-1}\left(\frac{U'(x)}{\lambda^*}\right)\right)}{h\left(h^{-1}\left(\frac{U'(x)}{\lambda^*}\right)\right)}\right]\left[-\frac{U'(x)}{U''(x)}\right], \quad x \in (0,+\infty).\end{aligned}$$

To summarize, $\pi(t,x),(t,x) \in [0,T]\times(0,+\infty)$ satisfies the following nonlinear parabolic PDE:

$$\begin{cases} \pi_t + \frac{\sigma^2}{2}\pi^2\pi_{xx} + rx\pi_x - r\pi = 0, & (t,x)\in[0,T)\times(0,+\infty), \\ \pi(t,0) = 0, & t\in[0,T), \\ \pi(T,x) = (\sigma^{-1}\kappa)\left[-\frac{h'\left(h^{-1}\left(\frac{U'(x)}{\lambda^*}\right)\right)}{h\left(h^{-1}\left(\frac{U'(x)}{\lambda^*}\right)\right)}\right]\left[-\frac{U'(x)}{U''(x)}\right], & x\in(0,+\infty). \end{cases} \tag{B.2}$$

When the probability weighting function $\bar{w}(\cdot)$ is the identity function, i.e., when $\bar{w}(z) = z, z\in[0,1]$, the corresponding $\bar{h}(\cdot)$ is

$$\bar{h}(x) = \hat{\bar{N}}'(1-\bar{w}(1-F_{Z(T)}(x))) = \hat{\bar{N}}'(F_{Z(T)}(x)),$$

where $\hat{\bar{N}}(\cdot)$ is the concave envelop of

$$\bar{N}(z) = -\int_0^{\bar{w}^{-1}(1-z)} F_\xi^{-1}(t)dt = -\int_0^{1-z} e^{-F^{-1}_{Z(T)}(1-t)}dt = -\int_z^1 e^{-F^{-1}_{Z(T)}(t)}dt.$$

Then, $\bar{N}'(z) = e^{-F^{-1}_{Z(T)}(z)}$, which is a decreasing function. Consequently, $\bar{N}(\cdot)$ is concave and $\hat{\bar{N}}(\cdot) = \bar{N}(\cdot)$. As a result,

$$\bar{h}(x) = \bar{N}'(F_{Z(T)}(x)) = e^{-x}, \quad x\in\mathbb{R}.$$

Because $\bar{h}'(x)/\bar{h}(x) = -1$, we conclude that the portfolio $\bar{\pi}$, which is optimal to an agent with utility function $U(\cdot)$ and identity probability weighting function $\bar{w}(\cdot)$, satisfies

$$\begin{cases} \bar{\pi}_t + \frac{\sigma^2}{2}\bar{\pi}^2\bar{\pi}_{xx} + rx\pi_x - r\bar{\pi} = 0, & (t,x) \in [0,T) \times (0,+\infty), \\ \bar{\pi}(t,0) = 0, & t \in [0,T), \\ \bar{\pi}(T,x) = (\sigma^{-1}\kappa)\left[-\frac{U'(x)}{U''(x)}\right], & x \in (0,+\infty). \end{cases}$$

When $w(\cdot)$ is a concave, $\bar{w}(\cdot)$ is a convex transformation of $w(\cdot)$. From the proof of Theorem 3, we have that $\hat{\bar{N}}'(1-\bar{w}(F_\xi(x)))/\hat{N}'(1-w(F_\xi(x)))$ is decreasing in x. On the other hand, $F_\xi(x) = 1 - F_{Z(T)}(-\ln x)$. Thus, we conclude that $\bar{h}(x)/h(x)$ is an increasing function in x. This implies in turn that $h(x)/\bar{h}(x)$ is decreasing in x, which, since $\bar{h}(x) = e^{-x}$, means that

$$h'(x)e^x + h(x)e^x \leq 0,$$

whence we conclude that $h'(x)/h(x) \leq -1, x \in \mathbb{R}$. Similarly, when $w(\cdot)$ is convex, $h(x)e^x$ is increasing, so we can conclude that $h'(x)/h(x) \geq -1, x \in \mathbb{R}$. As a result, when $w(\cdot)$ is concave, $\pi(T,x) \geq \bar{\pi}(T,x), x \in (0,+\infty)$. When $w(\cdot)$ is convex, $\pi(T,x) \leq \bar{\pi}(T,x), x \in (0,+\infty)$.

Define $V(\cdot)$ through

$$-\frac{V'(x)}{V''(x)} = \frac{\pi(T,x)}{\sigma^{-1}\kappa} = \varphi(x)\left[-\frac{U'(x)}{U''(x)}\right],$$

i.e.,

$$V'(x) = V'(x_0)\exp\left[-\int_{x_0}^{x} \frac{1}{\varphi(s)}\left(-\frac{U''(s)}{U'(s)}\right) ds\right]$$

where

$$\varphi(x) := -\frac{h'\left(h^{-1}\left(\frac{U'(x)}{\lambda^*}\right)\right)}{h\left(h^{-1}\left(\frac{U'(x)}{\lambda^*}\right)\right)}.$$

Because $0 < -h'(y)/h(y) \le C < +\infty$, we have $\frac{1}{\varphi(x)} \ge 1/C$. As a result, for $x > x_0$,

$$\int_{x_0}^{x} \frac{1}{\varphi(s)} \left(-\frac{U''(s)}{U'(s)}\right) ds \ge 1/C \int_{x_0}^{x} \left(-\frac{U''(s)}{U'(s)}\right) ds$$
$$= -1/C[\ln(U'(x)) - \ln(U'(x_0))].$$

Thus, for $x > x_0$, we have

$$V'(x) \le V'(x_0)(U'(x_0))^{-1/C}(U'(x))^{1/C}.$$

For $x < x_0$, we have

$$\begin{aligned}\int_{x_0}^{x} \frac{1}{\varphi(s)} \left(-\frac{U''(s)}{U'(s)}\right) ds &= -\int_{x}^{x_0} \frac{1}{\varphi(s)} \left(-\frac{U''(s)}{U'(s)}\right) ds \\ &\le -1/C \int_{x}^{x_0} \left(-\frac{U''(s)}{U'(s)}\right) ds \\ &= -1/C[\ln(U'(x)) - \ln(U'(x_0))].\end{aligned}$$

As a result,

$$V'(x) \ge V'(x_0)(U'(x_0))^{-1/C}(U'(x))^{1/C}.$$

Because $\lim_{x\to 0} U'(x) = +\infty, \quad \lim_{x\to +\infty} U'(x) = 0$, we conclude that

$$\lim_{x\to 0} V'(x) = +\infty, \quad \lim_{x\to +\infty} V'(x) = 0.$$

On the other hand, it is easy to check that $V'(\cdot)$ is strictly decreasing because $\varphi(x) > 0, x > 0$. Thus, $V(\cdot)$ satisfies Assumption 2. In addition, $-\frac{V'(x)}{V''(x)} \le C(-\frac{U'(x)}{U''(x)}) \le Ca(1+x), x > 0$.

Finally, when $w(\cdot)$ is concave, $-\frac{V'(x)}{V''(x)} \ge -\frac{U'(x)}{U''(x)}, x > 0$. Because both V and U satisfy Assumption 2.2 and the conditions in Definition 2.1 of Xia (2011), we can apply Theorem 4.2 of Xia (2011) to conclude that $\pi(t,x) \ge \bar{\pi}(t,x), (t,x) \in [0,T) \times (0,+\infty)$. Similarly, when $w(\cdot)$ is convex, we conclude that $\pi(t,x) \le \bar{\pi}(t,x), (t,x) \in [0,T) \times (0,+\infty)$. □

Proof of Corollary 1 Without loss of generality, we assume that agent 2 has EU preferences, i.e., $w_2(x) \equiv x$. Consider another EU agent, say agent 3, with utility function $U_1(\cdot)$ and denote his portfolio as $\pi_3(t,x)$. Because agent 1 is more risk averse than agent 2, $U_1(\cdot)$ is a concave transformation of $U_2(\cdot)$ and $w_1(\cdot)$ is a convex transformation of $w_2(\cdot)$. Clearly, $w_1(\cdot)$ is convex since $w_2(x)\equiv x$. Thus, we can apply Theorem 6 to conclude that $\pi_1(t,x)\le\pi_3(t,x),(t,x)\in[0,T)\times(0,\infty)$. On the other hand, Xia (2011, Theorem 4.2) shows that $\pi_3(t,x)\le\pi_2(t,x),(t,x)\in[0,T)\times(0,\infty)$ because both agents 2 and 3 have EU preferences with utility functions $U_2(\cdot)$ and $U_1(\cdot)$, respectively, and $U_1(\cdot)$ is a concave transformation of $U_2(\cdot)$. Therefore, we conclude that $\pi_1(t,x)\le\pi_2(t,x),(t,x)\in[0,T)\times(0,\infty)$. $\square$

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