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ARROW–DEBREU EQUILIBRIA FOR RANK-DEPENDENT UTILITIES

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We provide conditions on a one-period-two-date pure exchange economy with rank-dependent utility agents under which Arrow–Debreu equilibria exist. When such an equilibrium exists, we show that the state-price density is a weighted marginal rate of intertemporal substitution of a representative agent, where the weight depends on the differential of the probability weighting function. Based on the result, we find that asset prices depend upon agents' subjective beliefs regarding overall consumption growth, and we offer a direction for possible resolution of the equity premium puzzle.

KEY WORDS: rank-dependent utility, probability weighting, Arrow–Debreu equilibrium, state–price density.

1. INTRODUCTION

Rank-dependent utility theory (RDUT) (Quiggin 1982, 1993; Schmeidler 1989; Abdellaoui 2002) is an alternative model of choice under uncertainty to the classical expected utility theory (EUT). Together with Kahneman and Tversky's cumulative prospect theory (CPT),[1] RDUT is among the most highly regarded theories on

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[1]CPT (Tversky and Kahneman 1992) is a modification of the original prospect theory (Kahneman and Tversky 1979). This modification accounts for first-order stochastic dominance, which is violated by the original prospect theory.

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preference and choice that depart from the classical paradigm.[2] RDUT can explain a number of paradoxes that EUT has failed to capture and can provide a better representation of attitudes toward risk. This theory consists of two components: a concave outcome utility function and a probability weighting function. The first component captures the observation that individuals dislike a mean-preserving spread of the distribution of a future payoff (as under EUT). The second component captures a factor that is not accounted for under EUT: the tendency to overweight the tails—both left and right—of the distribution. With a suitable weighting function, the theory also accommodates *simultaneous* risk-averse and risk-seeking behavior, a paradoxical phenomenon often observed in experimental settings.

Previous research on equilibria and asset pricing for non-EUT models has focused primarily on *financial* economies with CPT agents. Barberis and Huang (2008) considered an economy in which agents had identical CPT preferences with risk-free payoff as reference point. Moreover, in addition to multiple risky assets with joint normal distributed return, there is a skewed security that is independent of the other risky assets. Within this setting, Barberis and Huang provided conditions under which an equilibrium existed, and, by assuming a binomial distribution for the skewed security, they computed the equilibrium numerically. In a more general setting, De Giorgi, Hens, and Riegers (2010) showed that, due to discontinuities in CPT agents' demand functions (which were caused by the kink in the CPT value functions), an equilibrium might not exist if there were a finite number of agents. The authors then established the existence of equilibria under the conditions that there was a continuum of agents in the market and that agents' final wealth was constrained to be nonnegative. Azevedo and Gottlieb (2012) obtained a similar negative result concerning the nonexistence of equilibria under CPT. Shefrin (2008, chapter 28) illustrated the qualitative structures of the state-price densities for both the CPT and SP/A economies, under the assumption that equilibrium existed.[3] He and Zhou (2011b) showed that in a one-risky-asset market with general asset return distribution, if there were some CPT agents with the probability weighting function specified in Tversky and Kahneman (1992) and whose reference point coincided with the risk-free return, then there was some expected return for which these agents would be willing to hold any positive amount of the stock. As a result, the market cleared and an equilibrium existed.[4]

In the context of RDUT economies, however, few results have been generated on equilibria despite the fact that the RDUT preference is a special case of the CPT preference and therefore in principle should be more tractable. For concave law-invariant utilities, Carlier and Dana (2008) derived equilibria for a two-agent economy and Dana (2011) proved the existence of equilibria for a multiagent one. Notably, when applied to RDUT preference, the probability weighting functions in these two papers must be assumed to be convex.

The goal of this paper is to establish full equilibria and pricing in the classical Arrow–Debreu sense for a one-period-two-date pure exchange economy with RDUT agents. Our setting is fairly general, especially when applied in a financial market context. We do not assume any particular distribution of the aggregate future endowment, nor do

[2]As Starmer (2000) put it in a survey paper, "the rank-dependent model is likely to become more widely used" because it captures many empirical observations "in a model which is quite amenable to application within the framework of conventional economic analysis."

[3]SP/A stands for the theory of security, potential, and aspiration, which was proposed by Lopes (1987) and developed by Lopes and Oden (1999).

[4]A similar result, lemma 3.1 of De Giorgi et al. (2010), is obtained under the assumption of a complete market. In He and Zhou (2011b), the market is allowed to be incomplete.

we assume the probability weighting function to have a particular shape. Furthermore, we allow agents to have heterogeneous outcome utilities. However, we need to assume homogeneity of both beliefs and probability weighting functions. We provide sufficient conditions on the primitives of the underlying economy under which Arrow–Debreu equilibria exist. Moreover, when such an equilibrium exists, we show that the state-price density is a weighted marginal rate of intertemporal substitution of a representative agent. This weight can be expressed as $w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))$, where w is the probability weighting function and $\tilde{e}_1$ is the aggregate future endowment with its cumulative distribution function (CDF) $F_{\tilde{e}_1}$.

The derived pricing formula is essentially a consumption-based capital asset pricing model (CCAPM). The presence of the weighting function produces markedly different features from the classical, EUT-based CCAPM. For instance, the classical CCAPM stipulates that an asset price depends only on the level of risk aversion and on the "beta" (i.e., the correlation between the asset and the overall economy). The RDUT-based CCAPM, by contrast, displays an additional dependence on agents' subjective beliefs regarding overall consumption growth. As an application of this general finding, we demonstrate, at a qualitative level, how the newly established CCAPM may shed some light on resolving the equity premium puzzle.

The main steps in our approach are as follows. First, we solve the individual consumption problem where the state-price density $\tilde{\rho}$ is exogenously given. Due to the presence of the probability weighting, this is a *nonconcave* maximization problem. Jin and Zhou (2008) and He and Zhou (2011a, 2014) developed a method called "quantile formulation" to overcome this technical obstacle.[5] This formulation leads to a concave maximization problem when the quantile function of the future consumption is chosen as the decision variable. However, the drawback of the method employed by these authors is that one needs to impose a piecewise monotonicity condition on the function $F_{\tilde{\rho}}^{-1}/w'$. This condition is unreasonable for our problem, since we cannot impose any specific assumption on something we are ultimately going to derive. A key step, which is also the main technical contribution of this paper, is to solve the optimal quantile function *explicitly* without any monotonicity condition; this step is accomplished through calculus of variations and introduction of the concave envelope of a certain function.[6] Second, by virtue of the explicit solution to the individual consumption problem, we are able to construct a representative agent for the economy. Third, assuming an equilibrium exists, we derive the state-price density $\tilde{\rho}$ by the market clearing and the anticomonotonicity between $\tilde{\rho}$ and $\tilde{e}_1$. Finally, to establish the existence of equilibria we work under a suitably changed probability measure—termed the "rank-neutral measure"—and make use of the existence of equilibria in the classical EUT economy.

The remainder of this paper is organized as follows. In Section 2, we define the economy and its Arrow–Debreu equilibria. In Section 3, we solve the individual consumption choice problem with an exogenously given state-price density, based on which we construct a representative agent in Section 4. Section 5 contains the main results, namely, the existence and uniqueness of the equilibrium and the pricing formula. In Section 6, we discuss the implications of our results with respect to the equity premium puzzle, and

[5]The idea of quantile formulation was around earlier, e.g., in Schied (2004) and in Carlier and Dana (2006), who studied maximization problems with *concave* criteria.

[6]Carlier and Dana (2006, 2011) applied calculus of variations to characterize the optimal quantile function; however, these authors did not find the explicit solution, which is crucial in deriving the state-price density.

Section 7 outlines connections between the RDUT and EUT economies. Finally, Section 8 concludes the paper.

2. THE ECONOMY

We consider a one-period-two-date pure exchange economy under uncertainty with a single perishable consumption good. Agents choose their consumption for today, say date $t = 0$, and choose contingent claims on consumption for tomorrow, say date $t = 1$. Without loss of generality, the single consumption good is used as the numeraire throughout the paper. The set of possible states of nature at date 1 is Ω and the set of events at date 1 is a σ-algebra $\mathcal{F}$ of subsets of Ω. There are a finite number of agents indexed by $i = 1, \dots, I$. Each agent i has an endowment $(e_{0i}, \tilde{e}_{1i})$, where e_{0i} is number of units of the good today and $\mathcal{F}$-measurable random variable $\tilde{e}_{1i}$ is the number of units of the good tomorrow. The aggregate endowment is $(e_0, \tilde{e}_1) \triangleq (\sum_{i=1}^{I} e_{0i}, \sum_{i=1}^{I} \tilde{e}_{1i})$. The consumption plan of an agent i is a pair $(c_{0i}, \tilde{c}_{1i})$, where c_{0i} is the number of units of the good consumed today and $\mathcal{F}$-measurable random variable $\tilde{c}_{1i}$ is the number of units of the good consumed tomorrow. The preference $\succsim_i$ of each agent i over consumption plans $(c_{0i}, \tilde{c}_{0i})$ is represented by

$$V_i(c_{0i}, \tilde{c}_{1i}) \triangleq u_{0i}(c_{0i}) + \beta_i \int u_{1i}(\tilde{c}_{1i}) d(w_i \circ \mathbb{P}), \tag{2.1}$$

where

- $\mathbb{P}$ is the belief about the states of the nature;
- u_{0i} is the utility function for consumption today;
- $\int u_{1i}(\tilde{c}_{1i})\, d(w_i \circ \mathbb{P}) \triangleq \int u_{1i}(c)\, d\bar{w}_i(F_{\tilde{c}_{1i}}(c))$ is the rank-dependent utility with outcome utility function u_{1i} for consumption $\tilde{c}_{1i}$ tomorrow and probability weighting function w_i; and
- $\beta_i > 0$ is the time discount factor representing the time impatience for consumption (we can always assume that $\beta_i \le 1$ by appropriately modifying u_{0i} and u_{1i}).

In the above (and hereafter) $F_{\tilde{x}}$ denotes the CDF of a random variable $\tilde{x}$ and $\bar{w}$ denotes the dual of a probability weighting function w, given by

$$\bar{w}(p) \triangleq 1 - w(1 - p) \quad \text{for all } p \in [0, 1].$$

Note that if w_i is continuously differentiable, then

$$\int u_{1i}(\tilde{c}_{1i})\, d(w_i \circ \mathbb{P}) = \int u_{1i}(c) w_i'(1 - F_{\tilde{c}_{1i}}(c))\, dF_{\tilde{c}_{1i}}(c).$$

Hence, we have here an additional term $w'(1 - F_{\tilde{c}_{1i}}(c))$ serving as the weight on every consumption level c when calculating the expected utility. The weight depends on the rank, $1 - F_{\tilde{c}_{1i}}(c)$, of level c over all possible realizations of $\tilde{c}_{1i}$.

We make the following standing assumption on the economy:

ASSUMPTION 2.1.

- *The agents have homogeneous beliefs $\mathbb{P}$ about the states of the nature. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ admits no atom.*

- *For every i, $e_{0i} \geq 0$, $\mathbb{P}(\tilde{e}_{1i} \geq 0) = 1$, and $e_{0i} + \mathbb{P}(\tilde{e}_{1i} > 0) > 0$. The CDF $F_{\tilde{e}_1}$ of $\tilde{e}_1$ is continuous and $\mathbb{P}(\tilde{e}_1 > 0) = 1$. Moreover, $e_0 > 0$.*
- *For every i, the functions $u_{0i}, u_{1i} : [0, \infty) \to \mathbb{R}$ are strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$, and satisfy the Inada condition: $u'_{0i}(0+) = u'_{1i}(0+) = \infty$, $u'_{0i}(\infty) = u'_{1i}(\infty) = 0$. Moreover, without loss of generality, $u_{1i}(\infty) > 0$. The asymptotic elasticity of each u_{1i} is strictly less than one, that is, $\lim_{c\to\infty} \frac{cu'_{1i}(c)}{u_{1i}(c)} < 1$.*
- *The agents have the same probability weighting function w, i.e., $w_1 = w_2 = \ldots = w_I = w$. The probability weighting function $w : [0, 1] \to [0, 1]$ is strictly increasing and continuous on $[0, 1]$ and satisfies $w(0) = 0$, $w(1) = 1$.*

DEFINITION 2.2. For every i, a consumption plan $(c_{0i}, \tilde{c}_{1i})$ is called feasible if $c_{0i} \geq 0$ and $\mathbb{P}(\tilde{c}_{1i} \geq 0) = 1$. The set of all feasible consumption plans is denoted by $\mathcal{C}$.

The above economy is denoted by

$$\mathcal{E} \triangleq \left\{(\Omega, \mathcal{F}, \mathbb{P}),\ (e_{0i}, \tilde{e}_{1i})_{i=1}^{I},\ \mathcal{C},\ \left(V_i(c_{0i}, \tilde{c}_{1i})\right)_{i=1}^{I}\right\}.$$

DEFINITION 2.3. A state-price density[7] is an $\mathcal{F}$-measurable random variable $\tilde{\rho}$ such that $\mathbb{P}(\tilde{\rho} > 0) = 1$, $\mathbb{E}[\tilde{\rho}] < \infty$ and $\mathbb{E}[\tilde{\rho}\tilde{e}_1] < \infty$.

DEFINITION 2.4. An Arrow–Debreu equilibrium of the economy $\mathcal{E}$ is a collection $\left\{\tilde{\rho},\ (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^{I}\right\}$ consisting of a state-price density $\tilde{\rho}$ and a collection $(c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^{I}$ of feasible consumption plans that satisfies the following conditions:

(i) **Individual rationality**: For every i, the feasible consumption plan $(c^*_{0i}, \tilde{c}^*_{1i})$ maximizes the preference $\succsim_i$ of agent i, subject to the budget constraint, that is:

$$V_i(c^*_{0i}, \tilde{c}^*_{1i}) = \max_{(c_{0i}, \tilde{c}_{1i}) \in C} V_i(c_{0i}, \tilde{c}_{1i}) \quad \text{subject to } c_{0i} + \mathbb{E}[\tilde{\rho}\tilde{c}_{1i}] \leq e_{0i} + \mathbb{E}[\tilde{\rho}\tilde{e}_{1i}]. \tag{2.2}$$

(ii) **Market clearing**: $\sum_{i=1}^{I} c^*_{0i} = e_0$ and $\sum_{i=1}^{I} \tilde{c}^*_{1i} = \tilde{e}_1$.

3. INDIVIDUAL CONSUMPTION

In this section, we investigate the individual optimal consumption problem (2.2) when the state-price density $\tilde{\rho}$ is exogenously given. We make the following additional assumption throughout this section *only*:

ASSUMPTION 3.1. *The state-price density $\tilde{\rho}$ has a continuous CDF $F_{\tilde{\rho}}$ and $0 < e_{0i} + \mathbb{E}[\tilde{\rho}\tilde{e}_{1i}] < \infty$.*

Although the outcome utility functions u_{0i} and u_{1i} are concave, the functional $V_i(c_{0i}, \tilde{c}_{1i})$ is not concave in $(c_{0i}, \tilde{c}_{1i})$ unless the probability weighting function w is convex. Thus, (2.2) is a problem of nonconvex programming. The appropriate technique to overcome this difficulty is the so-called "quantile formulation" (Jin and Zhou 2008;

[7] Also sometimes termed "pricing kernel" or "stochastic discount factor (SDF)" in the literature.

He and Zhou 2011a, 2014), namely, to change one of the decision variables of problem (2.2) from the random variable $\tilde{c}_{1i}$ to its quantile function G_i. This formulation recovers the implicit concavity (in terms of quantile functions) of (2.2).

Introducing the quantile formulation requires some preparation. Assume that a random variable $\tilde{x}$ has a CDF $F_{\tilde{x}} : (-\infty, \infty) \to [0, 1]$ which is nondecreasing and right-continuous. The upper quantile $Q^+_{\tilde{x}} : [0, 1) \to [-\infty, \infty]$ and lower quantile $Q^-_{\tilde{x}} : (0, 1] \to [-\infty, \infty]$ of $\tilde{x}$ are defined, respectively, as

$$Q^+_{\tilde{x}}(p) \triangleq \inf\{x \in \mathbb{R} \mid F_{\tilde{x}}(x) > p\}, \quad p \in [0, 1) \quad \text{and}$$
$$Q^-_{\tilde{x}}(p) \triangleq \sup\{x \in \mathbb{R} \mid F_{\tilde{x}}(x) < p\}, \quad p \in (0, 1].$$

It is well known that $Q^+_{\tilde{x}}$ ($Q^-_{\tilde{x}}$) is nondecreasing and right- (left-) continuous. More properties of quantiles can be seen in, e.g., Föllmer and Schied (2011), appendix A.3.

Recall that $\bar{w}$ is the dual of the weighting function w. The inverses of w and $\bar{w}$ are denoted by w^{-1} and $\bar{w}^{-1}$, respectively. Obviously, all of $\bar{w}$, w^{-1}, and $\bar{w}^{-1}$ are strictly increasing and continuous. Moreover, $\bar{w}^{-1}(p) = 1 - w^{-1}(1 - p)$ for all $p \in [0, 1]$.

Notation. *To avoid ambiguity, for $a < b$, we use $\int_a^b$ and $\int_a^{b-}$ to denote, respectively, the integrations over the intervals $[a, b]$ and $[a, b)$; that is, $\int_a^b = \int_{[a,b]}$ and $\int_a^{b-} = \int_{[a,b)}$. Similarly, $\int_{a+}^b = \int_{(a,b]}$ and $\int_{a+}^{b-} = \int_{(a,b)}$.*

We now briefly introduce the general idea of the quantile formulation for (2.2). If G_i is the upper quantile of $\tilde{c}_{1i}$, then

$$\int u_{1i}(\tilde{c}_{1i})\, d(w \circ \mathbb{P}) = \int_0^{1-} u_{1i}(G_i(p))\, d\bar{w}(p).$$

Thus, the objective functional of problem (2.2) can be written as

$$u_{0i}(c_{0i}) + \beta_i \int_0^{1-} u_{1i}(G_i(p))\, d\bar{w}(p).$$

To reformulate the budget constraint of problem (2.2) in terms of G_i, we consider an expenditure minimizing problem as follows:[8] given an (upper) quantile function G_i,

$$\text{(3.1)} \qquad \underset{\tilde{c}_{1i}}{\text{Minimize}}\ \mathbb{E}[\tilde{\rho}\tilde{c}_{1i}] \quad \text{subject to}\ \ \tilde{c}_{1i} \sim G_i.$$

Here, $\tilde{c}_{1i} \sim G_i$ signifies that G_i is the upper quantile of $\tilde{c}_{1i}$. Using the Hardy–Littlewood inequality, it turns out that the solution to problem (3.1) is given by $\tilde{c}_{1i} = G_i(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ and the minimum is $\int_0^{1-} G_i(p) Q^-_{\tilde{\rho}}(1 - p)\, dp$, where $Q^-_{\tilde{\rho}}$ is the lower quantile function of $\tilde{\rho}$ (see Dybvig 1988; Schied 2004; Carlier and Dana 2006; and in particular Jin and Zhou 2008, theorem B.1). Observing that RDU preserves the first-order stochastic dominance, problem (2.2) can be finally reformulated as follows:

$$\text{(3.2)} \qquad \begin{aligned} &\underset{c_{0i}\geq 0,\ G_i \in \mathbb{Q}}{\text{Maximize}} \quad u_{0i}(c_{0i}) + \beta_i \int_0^{1-} u_{1i}(G_i(p))\, d\bar{w}(p) \\ &\qquad \text{subject to} \quad c_{0i} + \int_0^{1-} G_i(p) Q^-_{\tilde{\rho}}(1 - p)\, dp \leq e_{0i} + \mathbb{E}[\tilde{\rho}\tilde{e}_{1i}], \end{aligned}$$

[8]This problem goes back to Dybvig (1988).

where

$$\mathbb{Q} \triangleq \{G : [0, 1) \to [0, \infty] \text{ nondecreasing and right-continuous}\}$$

is the set of upper quantile functions of nonnegative random variables.

Assume $(c_{0i}^*, G_i^*) \in [0, \infty) \times \mathbb{Q}$ solves problem (3.2). Set $\tilde{c}_{1i}^* = G_i^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$. Then, $(c_{0i}^*, \tilde{c}_{1i}^*)$ solves problem (2.2); see proposition C.1 in Jin and Zhou (2008). Therefore, it suffices to solve problem (3.2). Obviously, problem (3.2) is a convex programming, which can be solved by means of two steps:

Step 1. For a fixed Lagrange multiplier $\lambda_i > 0$, solve the following problem:

$$\begin{aligned}\text{(3.3)} \qquad & \underset{c_{0i} \geq 0,\, G_i \in \mathbb{G}}{\text{Maximize}} \quad u_{0i}(c_{0i}) + \beta_i \int_0^{1-} u_{1i}(G_i(p))\, d\bar{w}(p) \\ & -\lambda_i \left(c_{0i} + \int_0^{1-} G_i(p) Q_{\tilde{\rho}}^-(1 - p)\, dp - e_{0i} - \mathbb{E}[\tilde{\rho}\tilde{e}_{1i}] \right),\end{aligned}$$

where

$$\mathbb{G} \triangleq \left\{ G \in \mathbb{Q} \;\middle|\; \int_0^{1-} G(p) Q_{\tilde{\rho}}^-(1 - p)\, dp < \infty \right\}.$$

The strict concavity of u_{0i} and u_{1i} guarantees the uniqueness of the optimal solution, denoted by (c_{0i}^*, G_i^*), which depends implicitly on λ_i.

Step 2. Determine the Lagrange multiplier λ_i. The strict monotonicity of u_{0i} and u_{1i} implies that the budget constraint must be binding at optimality. Thus, the Lagrange multiplier λ_i can be derived from the following equation:

$$c_{0i}^* + \int_0^{1-} G_i^*(p) Q_{\tilde{\rho}}^-(1 - p)\, dp = e_{0i} + \mathbb{E}[\tilde{\rho}\tilde{e}_{1i}].$$

In Step 1, the optimal c_{0i}^* is clearly given by $c_{0i}^* = (u_{0i}')^{-1}(\lambda_i)$. Problem (3.3) then reduces to the following:

$$\text{(3.4)} \qquad \underset{G_i \in \mathbb{G}}{\text{Maximize}} \; U_i(G_i; \lambda_i) \triangleq \int_0^{1-} u_{1i}(G_i(p))\, d\bar{w}(p) - \frac{\lambda_i}{\beta_i} \int_0^{1-} G_i(p) Q_{\tilde{\rho}}^-(1 - p)\, dp.$$

In the following, we first characterize the solution G_i^* of problem (3.4) via calculus of variations, and then derive G_i^* explicitly. The explicit expression of G_i^*, in turn is crucial in establishing the equilibrium.

3.1. Calculus of Variations

In this section, we use the Lebesgue–Stieltjes integration over interval $[0, 1)$ of a Borel function f w.r.t. $G \in \mathbb{G}$. To avoid ambiguity, we set $G(0-) \triangleq 0$ for all $G \in \mathbb{G}$. In such a setting, for any Borel function $f : [0, 1) \to \mathbb{R}$, the integration $\int_0^{1-} f(p)\, dG(p)$ is given by

$$\int_0^{1-} f(p)\, dG(p) \triangleq f(0)G(0) + \int_{(0,1)} f(p)\, dG(p).$$

A solution G_i^* of problem (3.4) can be characterized by the following proposition.

PROPOSITION 3.2. *Under Assumption 3.1, let $G_i^* \in \mathbb{G}$. Then, the following statements are equivalent:*

(i) *G_i^* solves problem* (3.4) *and*

$$\int_0^{1-} u_{1i}'(G_i^*(p))G_i^*(p)\,d\bar{w}(p) < \infty; \tag{3.5}$$

(ii) *For all $G \in \mathbb{G}$,*

$$\begin{aligned}&\int_0^{1-}\left(\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp\right)dG(q)\\ &\quad\le \int_0^{1-}\left(\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp\right)dG_i^*(q) < \infty;\end{aligned}$$

(iii) *G_i^* satisfies*

$$\begin{cases}\displaystyle\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp \le 0 \quad \textit{for all} \quad q\in[0,1),\\ \displaystyle\int_0^{1-}\left(\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp\right)dG_i^*(q) = 0.\end{cases} \tag{3.6}$$

Moreover, any of the above conditions implies that $G_i^*(p) > 0$ for all $p \in (0,1)$.

Proof. Let $G_i^* \in \mathbb{G}$.

(i) $\Rightarrow$(ii) Assume (i) holds true. Let $G \in \mathbb{G}$ be arbitrary and fixed. For any $\varepsilon \in (0,1)$, set $G^\varepsilon = (1-\varepsilon)G_i^* + \varepsilon G$. Then, by optimality of G_i^* and concavity of u_{1i}, we have

$$\begin{aligned}0 &\ge \frac{1}{\varepsilon}\left\{\left[\int_0^{1-} u_{1i}(G^\varepsilon(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-} G^\varepsilon(p)Q_{\tilde{\rho}}^-(1-p)\,dp\right]\right.\\ &\quad\left. - \left[\int_0^{1-} u_{1i}(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-} G_i^*(p)Q_{\tilde{\rho}}^-(1-p)\,dp\right]\right\}\\ &\ge \frac{1}{\varepsilon}\left(\int_0^{1-} u_{1i}'(G^\varepsilon(p))(G^\varepsilon(p)-G_i^*(p))\,d\bar{w}(p)\right.\\ &\quad\left.-\frac{\lambda_i}{\beta_i}\int_0^{1-}(G^\varepsilon(p)-G_i^*(p))Q_{\tilde{\rho}}^-(1-p)\,dp\right)\\ &= \int_0^{1-} u_{1i}'(G^\varepsilon(p))(G(p)-G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-}(G(p)-G_i^*(p))Q_{\tilde{\rho}}^-(1-p)\,dp\\ &\xrightarrow{\varepsilon\downarrow 0} \int_0^{1-} u_{1i}'(G_i^*(p))(G(p)-G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-}(G(p)-G_i^*(p))Q_{\tilde{\rho}}^-(1-p)\,dp\end{aligned}$$

by monotone convergence theorem. This unarguably yields

$$\int_0^{1-} u_{1i}'(G_i^*(p))G_i^*(p)\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-} G_i^*(p)Q_{\tilde{\rho}}^-(1-p)\,dp \tag{3.7}$$
$$\geq \int_0^{1-} u_{1i}'(G_i^*(p))G(p)\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-} G(p)Q_{\tilde{\rho}}^-(1-p)\,dp.$$

The right-hand side of (3.7) can be written as

$$\int_0^{1-} u_{1i}'(G_i^*(p))\left(\int_0^p dG(q)\right)d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-}\left(\int_0^p dG(q)\right)Q_{\tilde{\rho}}^-(1-p)\,dp.$$

By Fubini's theorem, it is equal to

$$\int_0^{1-}\left(\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp\right)dG(q).$$

Similarly, the left-hand side of (3.7)

$$\int_0^{1-} u_{1i}'(G_i^*(p))G_i^*(p)\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-} G_i^*(p)Q_{\tilde{\rho}}^-(1-p)\,dp \tag{3.8}$$
$$= \int_0^{1-}\left(\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp\right)dG_i^*(q).$$

Then, (ii) follows from the above analysis.
(ii)⇒(iii) Consider the following problem:

$$\underset{G\in\mathbb{G}}{\text{Maximize}}\ V(G) \triangleq \int_0^{1-}\left(\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp\right)dG(q). \tag{3.9}$$

Let V^* denote the optimal value of problem (3.9), that is, $V^* = \sup_{G\in\mathbb{G}} V(G)$. Assume that (ii) holds true; then $V^* = V(G_i^*) < \infty$. For any given $G \in \mathbb{G}$, we have $\alpha V(G) = V(\alpha G) \leq V^*$ for all $\alpha > 0$, implying $V(G) \leq 0$. Thus, we have $V^* \leq 0$ and

$$\int_q^{1-} u_{1i}'(G_i^*(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\,dp \leq 0 \quad \text{for all} \quad q\in[0,1).$$

On the other hand, $V^* \geq V(\alpha G) = \alpha V(G)$ for any $\alpha > 0$ and $G \in \mathbb{G}$, implying $V^* \geq 0$. Thus, $V(G_i^*) = V^* = 0$. Then, (iii) follows.
(iii)⇒(i) Suppose that (iii) holds true. For any $G \in \mathbb{G}$, by the concavity of u_{1i}, we have

$$\begin{aligned}
&U_i(G_i^*;\lambda_i) - U_i(G;\lambda_i)\\
&= \int_0^{1-}\left[u_{1i}(G_i^*(p)) - u_{1i}(G(p))\right]d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-} Q_{\tilde{\rho}}^-(1-p)(G_i^*(p)-G(p))\,dp\\
&\geq \int_0^{1-} u_{1i}'(G_i^*(p))(G_i^*(p)-G(p))\,d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_0^{1-} Q_{\tilde{\rho}}^-(1-p)(G_i^*(p)-G(p))\,dp
\end{aligned}$$

$$= \int_0^{1-} u_{1i}'(G_i^*(p)) \left(\int_0^p (dG_i^*(q) - dG(q)) \right) d\bar{w}(p)$$
$$-\frac{\lambda_i}{\beta_i} \int_0^{1-} Q_{\tilde{\rho}}^-(1-p) \left(\int_0^p (dG_i^*(q) - dG(q)) \right) dp.$$

It follows from Fubini's theorem and from (iii) that

$$\begin{aligned} & U_i(G_i^*; \lambda_i) - U_i(G; \lambda_i) \\ & \geq \int_0^{1-} \left(\int_q^{1-} u_{1i}'(G_i^*(p))\, d\bar{w}(p) - \frac{\lambda_i}{\beta_i} \int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\, dp \right) (dG_i^*(q) - dG(q)) \\ & \geq 0. \end{aligned}$$

Moreover, by (3.6) and (3.8), we have

$$\int_0^{1-} u_{1i}'(G_i^*(p)) G_i^*(p)\, d\bar{w}(p) = \int_0^{1-} \left(\int_q^{1-} u_{1i}'(G_i^*(p))\, d\bar{w}(p) \right) dG_i^*(q)$$
$$= \frac{\lambda_i}{\beta_i} \int_0^{1-} \left(\int_q^{1-} Q_{\tilde{\rho}}^-(1-p)\, dp \right) dG_i^*(q) = \frac{\lambda_i}{\beta_i} \int_0^{1-} G_i^*(p) Q_{\tilde{\rho}}^-(1-p)\, dp < \infty.$$

Thus, (i) follows.
Finally, by (3.6), we have

$$\int_0^{1-} u_{1i}'(G_i^*(p))\, d\bar{w}(p) \leq \frac{\lambda_i}{\beta_i} \int_0^{1-} Q_{\tilde{\rho}}^-(1-p)\, dp = \frac{\lambda_i}{\beta_i} \mathbb{E}[\tilde{\rho}] < \infty,$$

implying $u_{1i}'(G_i^*(p)) < \infty$ for all $p \in (0,1)$, since $\bar{w}$ is strictly increasing and $u_{1i}'(G_i^*(\cdot))$ is nonincreasing. Thus, $G_i^*(p) > 0$ for all $p \in (0,1)$, in view of the Inada condition on u_{1i}. □

3.2. Optimal Solution

Recall that the concave envelope, denoted by $\hat{f}$, of an arbitrarily given function f defined on a nonempty convex subset of an Euclidean space is the smallest concave function that dominates f.

THEOREM 3.3. *Under Assumption 3.1, let N be given by*

$$N(q) = -\int_{\bar{w}^{-1}(q)}^{1-} Q_{\tilde{\rho}}^-(1-p)\, dp, \tag{3.10}$$

where $\hat{N}$ the concave envelope of N, and $\hat{N}'$ the right derivative of $\hat{N}$. If

$$\int_0^{1-} (u_{1i}')^{-1}(\mu \hat{N}'(\bar{w}(p))) Q_{\tilde{\rho}}^-(1-p)\, dp < \infty \tag{3.11}$$

for all $\mu > 0$, then the optimal consumption plan of each agent i is given by

$$\begin{cases} \tilde{c}_{0i}^* = (u_{0i}')^{-1}(\lambda_i^*) \\ \tilde{c}_{1i}^* = (u_{1i}')^{-1}\left(\frac{\lambda_i^*}{\beta_i} \hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big) \right), \end{cases}$$

where the Lagrange multiplier λ_i^* is determined by

$$(u'_{0i})^{-1}(\lambda_i^*) + \mathbb{E}\left[\tilde{\rho}(u'_{1i})^{-1}\left(\frac{\lambda_i^*}{\beta_i}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)\right] = e_{0i} + \mathbb{E}[\tilde{\rho}\tilde{e}_{1i}]. \tag{3.12}$$

Proof. Set

$$\mathbb{Q}_0 \triangleq \{G \in \mathbb{Q} \,|\, G(p) > 0 \text{ for all } \quad p \in (0, 1)\}.$$

By Proposition 3.2, there exists at most one $G_i^* \in \mathbb{G} \cap \mathbb{Q}_0$ satisfying (3.6). We will first derive $G_i^* \in \mathbb{Q}_0$ that satisfies (3.6) and then verify that $G_i^* \in \mathbb{G}$.

A substitution of variables yields

$$\begin{aligned}
&\int_q^{1-} u'_{1i}(G_i^*(p))\, d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q^-_{\tilde{\rho}}(1-p)\, dp \\
&\quad = \int_{\bar{w}(q)}^{1-} u'_{1i}(G_i^*(\bar{w}^{-1}(p)))\, dp - \frac{\lambda_i}{\beta_i}\int_{\bar{w}(q)}^{1-} Q^-_{\tilde{\rho}}(1-\bar{w}^{-1}(p))\, d\bar{w}^{-1}(p)
\end{aligned}$$

for all $q \in [0, 1)$. Moreover,

$$\begin{aligned}
&\int_0^{1-}\left(\int_q^{1-} u'_{1i}(G_i^*(p))\, d\bar{w}(p) - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q^-_{\tilde{\rho}}(1-p)\, dp\right) dG_i^*(q) \\
&= \int_0^{1-}\left(\int_{\bar{w}(q)}^{1-} u'_{1i}(G_i^*(\bar{w}^{-1}(p)))\, dp - \frac{\lambda_i}{\beta_i}\int_{\bar{w}(q)}^{1-} Q^-_{\tilde{\rho}}(1-\bar{w}^{-1}(p))\, d\bar{w}^{-1}(p)\right) dG_i^*(q) \\
&= \int_0^{1-}\left(\int_q^{1-} u'_{1i}(G_i^*(\bar{w}^{-1}(p)))\, dp - \frac{\lambda_i}{\beta_i}\int_q^{1-} Q^-_{\tilde{\rho}}(1-\bar{w}^{-1}(p))\, d\bar{w}^{-1}(p)\right) dG_i^*(\bar{w}^{-1}(q)).
\end{aligned}$$

Set

$$\begin{cases}
H(q) = -u'_{1i}(G_i^*(\bar{w}^{-1}(q))) \\
K(q) = -\displaystyle\int_q^{1-} u'_{1i}(G_i^*(\bar{w}^{-1}(p)))\, dp \\
N(q) = -\displaystyle\int_q^{1-} Q^-_{\tilde{\rho}}(1-\bar{w}^{-1}(p))\, d\bar{w}^{-1}(p)
\end{cases} \tag{3.13}$$

for all $q \in [0, 1)$. Then, (3.6) is equivalent to

$$\begin{cases}
K(q) \geq \frac{\lambda_i}{\beta_i} N(q) \quad \text{for all } \ q \in [0, 1), \\
\int_0^{1-}\left[K(q) - \frac{\lambda_i}{\beta_i} N(q)\right] dG_i^*(\bar{w}^{-1}(q)) = 0;
\end{cases}$$

that is,

$$\begin{cases}
K(q) \geq \frac{\lambda_i}{\beta_i} N(q) \quad \text{for all } \quad q \in [0, 1), \\
K(q) = \frac{\lambda_i}{\beta_i} N(q) \quad \text{for all } \quad q \in \operatorname{supp}(dG_i^*(\bar{w}^{-1}(\cdot))).
\end{cases} \tag{3.14}$$

Here, $\operatorname{supp}(dL(\cdot))$ denotes the support of the Lebesgue–Stieltjes measure $dL(\cdot)$ on $[0, 1)$ generated by a nondecreasing and right-continuous function $L : [0, 1) \to \mathbb{R}$. Notably, the measure of the singleton $\{0\}$ is given by $L(0)$. Moreover, the strict monotonicity of u'_{1i}

implies that[9]

$$\operatorname{supp}(dH(\cdot)) = \operatorname{supp}(dG_i^*(\bar{w}^{-1}(\cdot))$$

and therefore (3.14) is equivalent to

$$\begin{cases} K(q) \geq \frac{\lambda_i}{\beta_i} N(q) & \text{for all} \quad q \in [0, 1), \\ K(q) = \frac{\lambda_i}{\beta_i} N(q) & \text{for all} \quad q \in \operatorname{supp}(dH(\cdot)). \end{cases}$$

Obviously, $G_i^* \in \mathbb{Q}_0$ if and only if $H : [0, 1) \to [-\infty, 0)$ defined by (3.13) is a nondecreasing and right-continuous function such that $H(q) > -\infty$ for all $q \in (0, 1)$. Therefore, we can see that finding a $G_i^* \in \mathbb{Q}_0$ satisfying (3.6) is equivalent to finding a nondecreasing, continuous, and concave function K satisfying

$$\begin{cases} K(q) \geq \frac{\lambda_i}{\beta_i} N(q) & \text{for all} \quad q \in (0, 1), \\ K(q) = \frac{\lambda_i}{\beta_i} N(q) & \text{for all} \quad q \in \operatorname{supp}(dK'(\cdot)), \\ K(1-) = 0. \end{cases} \tag{3.15}$$

Here, K' denotes the right derivative of K.

A substitution of variables yields (3.10). Consequently, N is continuous and strictly increasing on $[0, 1)$ with $N(1-) = 0$. The continuity of K and N implies that $\{q \in (0, 1) : K(q) > \frac{\lambda_i}{\beta_i} N(q)\}$ is an open set and can be written as a countable union of open intervals. Thus, condition (3.15) is equivalent to[10]

$$\begin{cases} K(q) \geq \frac{\lambda_i}{\beta_i} N(q) \quad \text{for all} \quad q \in (0, 1), \\ K \text{ is affine on } \left\{q \in (0, 1) : K(q) > \frac{\lambda_i}{\beta_i} N(q)\right\}, \\ (K(0) - \frac{\lambda_i}{\beta_i} N(0))G_i^*(\bar{w}^{-1}(0)) = 0, \\ K(1-) = 0. \end{cases} \tag{3.16}$$

Now, we show that (3.16) implies $K(0) = \frac{\lambda_i}{\beta_i} N(0)$. Otherwise, suppose $K(0) > \frac{\lambda_i}{\beta_i} N(0)$. Then, the continuity of K and N implies that, for some $\varepsilon \in (0, 1)$, $K(q) > \frac{\lambda_i}{\beta_i} N(q)$ on $(0, \varepsilon)$. By (3.16), K is affine on $(0, \varepsilon)$. Then, we know that $K'(0) < \infty$, that is, $u'_{1i}(G_i^*(\bar{w}^{-1}(0))) < \infty$. Therefore, $G_i^*(\bar{w}^{-1}(0)) > 0$. By (3.16), we arrive at $K(0) = \frac{\lambda_i}{\beta_i} N(0)$, a contradiction. Thus, $K(0) > \frac{\lambda_i}{\beta_i} N(0)$ is impossible. Consequently, we can conclude that (3.16) is equivalent to

$$\begin{cases} K(q) \geq \frac{\lambda_i}{\beta_i} N(q) \quad \text{for all} \quad q \in (0, 1), \\ K \text{ is affine on } \left\{q \in (0, 1) : K(q) > \frac{\lambda_i}{\beta_i} N(q)\right\}, \\ K(0) = \frac{\lambda_i}{\beta_i} N(0), \ K(1-) = \frac{\lambda_i}{\beta_i} N(1-). \end{cases} \tag{3.17}$$

[9]Function $H : [0, 1) \to [-\infty, 0)$ defined by (3.13) satisfies the conditions that: (i) $H(q) > -\infty$ for all $q \in (0, 1)$; (ii) H is nondecreasing and right-continuous (assuming $H(0) = -\infty$, we say H is right-continuous at $q = 0$ if $H(q) \downarrow -\infty$ as $q \downarrow 0$); (iii) $H(0-) = -\infty$. Moreover, $0 \in \operatorname{supp}(dH(\cdot))$ if and only if $H(0) > -\infty$.

[10]The point $q = 0$ should be discussed separately. Footnote 9 stipulates that $0 \in \operatorname{supp}(dK'(\cdot))$ if and only if $H(0) > -\infty$. However, $H(0) = -u'_{1i}(G_i^*(\bar{w}^{-1}(0))$; hence, it follows from the Inada condition of u_{1i} that $0 \in \operatorname{supp}(dK'(\cdot))$ if and only if $G_i^*(\bar{w}^{-1}(0)) > 0$. Thus, at point $q = 0$, the second condition in (3.15) is equivalent to $(K(0) - \frac{\lambda_i}{\beta_i} N(0))G_i^*(\bar{w}^{-1}(0)) = 0$.

The stipulation (3.17) is equivalent to the fact that K is the concave envelope of $\frac{\lambda_i}{\beta_i}N$; namely, $K = \frac{\lambda_i}{\beta_i}\hat{N}$. Then,

$$u'_{1i}(G_i^*(1 - w^{-1}(1 - q))) = K'(q) = \frac{\lambda_i}{\beta_i}\hat{N}'(q),$$

or, equivalently,

$$G_i^*(p) = (u'_{1i})^{-1}\left(\frac{\lambda_i}{\beta_i}\hat{N}'(\bar{w}(p))\right)$$

for all $p \in [0, 1)$.

Obviously, (3.11) implies that $G_i^* \in \mathbb{G}$. The desired result then follows from Proposition 3.2. □

REMARK 3.4. If w is continuously differentiable, then, by (3.13), we have

$$N(q) = -\int_q^{1-} \frac{Q_{\tilde{\rho}}^-(1 - \bar{w}^{-1}(p))}{\bar{w}'(\bar{w}^{-1}(p))}\, dp = -\int_q^{1-} \frac{Q_{\tilde{\rho}}^-(w^{-1}(1 - p))}{w'(w^{-1}(1 - p))}\, dp.$$

If N itself is concave, that is, $\frac{Q_{\tilde{\rho}}^-(p)}{w'(p)}$ is increasing, then $\hat{N} = N$ and $\hat{N}'(q) = N'(q) = \frac{Q_{\tilde{\rho}}^-(w^{-1}(1-q))}{w'(w^{-1}(1-q))}$. In this case, we have

$$\tilde{c}_{1i}^* = (u'_{1i})^{-1}\left(\frac{\lambda_i^*}{\beta_i}\frac{\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right),$$

which recovers the solution of the "positive part problem" in Jin and Zhou (2008), where the authors investigated a portfolio choice problem under CPT. The monotonicity of $\frac{Q_{\tilde{\rho}}^-(p)}{w'(p)}$, or, equivalently, the concavity of N imposed by Jin and Zhou (2008), is, however, not reasonable for the purposes of the present paper since we need ultimately to derive the state-price density without any prior assumption imposed on it. Interestingly, however, we will show that this monotonicity is indeed fulfilled automatically by an endogenous state-price density in an equilibrium under mild conditions on the primitives of the economy (see Remark 5.3).

REMARK 3.5. An RDUT agent displays significantly different consumption behavior compared to an EUT agent. For example, suppose $w(p)$ is "sufficiently convex," namely that $\frac{w''(p)}{w'(p)}$ is sufficiently large, when p is close to 1. In this case, it is easy to show that $\frac{Q_{\tilde{\rho}}^-(p)}{w'(p)}$ is decreasing when p is in the neighborhood of 1. Thus, in the same neighborhood, N is convex and hence $\hat{N}$ is affine, implying that $\tilde{c}_{1i}^*$ is a positive constant in the states of nature in which $\tilde{\rho}$ is sufficiently large. Economically, this suggests that if the agent puts too much weight on the left tail, then he would set an endogenous "consumption insurance" that would ensure his "minimal positive consumption" in the bad states of nature. This kind of insurance is not seen in the EUT setting, no matter how "concave" the agent's outcome utility function might be.

4. REPRESENTATIVE AGENTS

Assume (3.11) holds for all i and for all $\mu > 0$. Given a state-price density $\tilde{\rho}$ having a continuous CDF $F_{\tilde{\rho}}$, according to Theorem 3.3, the aggregate consumptions are thus

$$\begin{cases} c_0^* = \sum_{i=1}^{I} (u'_{0i})^{-1}(\lambda_i^*) \\ \tilde{c}_1^* = \sum_{i=1}^{I} (u'_{1i})^{-1} \left(\frac{\lambda_i^*}{\beta_i} \hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big) \right). \end{cases}$$

For all $\lambda_1 > 0$, ..., $\lambda_I > 0$, set $\lambda = (\lambda_1, \ldots, \lambda_I)$ and

$$\begin{cases} h_{0\lambda}(y) \triangleq \sum_{i=1}^{I} (u'_{0i})^{-1} (\lambda_i y) \\ h_{1\lambda}(y) \triangleq \sum_{i=1}^{I} (u'_{1i})^{-1} \left(\frac{\lambda_i y}{\beta_i} \right) \end{cases}$$

for all $y \in (0, \infty)$. It is easy to see, for each $t \in \{0, 1\}$, that $h_{t\lambda}$ is strictly decreasing and continuous on $(0, \infty)$ with $h_{t\lambda}(0+) = \infty$ and $h_{t\lambda}(\infty) = 0$, implying that $h_{t\lambda}^{-1}$ is strictly decreasing and continuous on $(0, \infty)$ with $h_{t\lambda}^{-1}(0+) = \infty$ and $h_{t\lambda}^{-1}(\infty) = 0$. We can verify for all $x > 0$ that

$$\begin{aligned} \int_0^x h_{0\lambda}^{-1}(z) dz &= \int_\infty^{h_{0\lambda}^{-1}(x)} y \, dh_{0\lambda}(y) \\ &= \sum_{i=1}^{I} \int_\infty^{h_{0\lambda}^{-1}(x)} y \, d\left((u'_{0i})^{-1}(\lambda_i y)\right) \\ &= \sum_{i=1}^{I} \int_0^{(u'_{0i})^{-1}\left(\lambda_i h_{0\lambda}^{-1}(x)\right)} \frac{u'_{0i}(z)}{\lambda_i} dz \\ &= \sum_{i=1}^{I} \frac{1}{\lambda_i} u_{0i}\Big((u'_{0i})^{-1} \left(\lambda_i h_{0\lambda}^{-1}(x)\right)\Big). \end{aligned}$$

Similarly, for all $x > 0$,

$$\int_0^x h_{1\lambda}^{-1}(z) \, dz = \sum_{i=1}^{I} \frac{\beta_i}{\lambda_i} u_{1i} \left((u'_{1i})^{-1} \left(\frac{\lambda_i}{\beta_i} h_{1\lambda}^{-1}(x) \right) \right).$$

Put

$$u_{t\lambda}(x) = \int_0^x h_{t\lambda}^{-1}(z) \, dz$$

for all $x > 0$, $t = 0, 1$. Then $u'_{t\lambda} = h_{t\lambda}^{-1}$. Obviously, $u_{t\lambda} : [0, \infty) \to \mathbb{R}$ is strictly increasing and strictly concave, is continuously differentiable on $(0, \infty)$, and satisfies the Inada conditions: $u'_{t\lambda}(0+) = \infty$ and $u'_{t\lambda}(\infty) = 0$.

Set $\lambda^* \triangleq (\lambda_1^*, \ldots, \lambda_I^*)$. Then, we have

$$\begin{cases} c_0^* = h_{0\lambda^*}(1) \\ \tilde{c}_1^* = h_{1\lambda^*}\left(\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right). \end{cases}$$

Now, we consider an agent, indexed by λ^*, whose preference over consumption plans $(c_0, \tilde{c}_1)$ is represented by

$$V_{\lambda^*}(c_0, \tilde{c}_1) \triangleq u_{0\lambda^*}(c_0) + \int u_{1\lambda^*}(\tilde{c}_1)\, d(w \circ \mathbb{P}) \tag{4.1}$$

and whose endowment is the aggregate endowment $(e_0, \tilde{e}_1)$. It is easy to verify that (3.11) implies

$$\int_0^{1-} (u'_{1\lambda^*})^{-1}(\mu \hat{N}'(\bar{w}(p)))\, Q^-_{\tilde{\rho}}(1-p)\, dp < \infty$$

for all $\mu > 0$. By the construction of $u_{t\lambda^*}$, we have

$$\begin{cases} c_0^* = (u'_{0\lambda^*})^{-1}(1) \\ \tilde{c}_1^* = (u'_{1\lambda^*})^{-1}\left(\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right). \end{cases} \tag{4.2}$$

Moreover, we have

$$(u'_{0\lambda^*})^{-1}(1) + \mathbb{E}\left[\tilde{\rho}(u'_{1\lambda^*})^{-1}\left(\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)\right] = e_0 + \mathbb{E}[\tilde{\rho}\tilde{e}_1].$$

It follows from Theorem 3.3 that the aggregate consumption plan $(c_0^*, \tilde{c}_1^*)$ is the optimal consumption plan for the agent λ^* and that the corresponding Lagrange multiplier is 1. Thus, we have constructed a representative agent λ^* such that:

- She is still an RDUT agent, with her preference represented by (4.1);
- Her endowment is the aggregate endowment; and
- Her optimal consumption plan is the aggregate consumption plan with Lagrange multiplier being 1.

REMARK 4.1. In a classical EUT economy with concave outcome utility functions, the representative agent can be constructed via solving a linearly weighted expected utility maximization problem,[11]

$$\begin{aligned} &\underset{(c_{0i}, \tilde{c}_{1i}) \in C}{\text{Maximize}} \sum_{i=1}^{I} \lambda_i \left(u_{0i}(c_{0i}) + \beta_i \mathbb{E}[u_{1i}(\tilde{c}_{1i})]\right) \\ &\text{subject to} \quad \sum_{i=1}^{I} (c_{0i}, \tilde{c}_{1i}) = (e_0, \tilde{e}_1), \end{aligned} \tag{4.3}$$

for some $(\lambda_1, \ldots, \lambda_I) \in \mathbb{R}_+^I$. This is the case because an equilibrium allocation must be Pareto optimal and a Pareto optimal allocation can be constructed by solving problem (4.3) owing to the fact that the expected utilities are concave functionals of consumption plans. In an RDUT economy, however, the rank-dependent utilities are generally no

[11] The indirect utility over $(e_0, \tilde{e}_1)$, which is the maximal objective value of (4.3), represents the preference of the representative agent.

longer concave functionals of consumption plans and therefore the agents' preferences are generally not convex, except in special cases in which the probability weighting functions are convex.[12] As a result, a Pareto optimal allocation may generally not solve problem (4.3) for any $(\lambda_1, \ldots, \lambda_I) \in \mathbb{R}_+^I$. The representative agent in an RDUT economy cannot be constructed in the above standard way. However, it can be constructed by summing up the individual optimal consumptions directly, as shown in the preceding analysis. It should be also noted that the time discount factors β_i are now hidden behind the derived utility functions $u_{1\lambda}$ when the individual consumptions are aggregated (see also Example 4.2).

EXAMPLE 4.2. Consider constant relative risk aversion (CRRA) utility functions

$$u_{ti}(c) = \frac{c^{1-\alpha_i}}{1-\alpha_i}$$

where $\alpha_i \in (-\infty, 0) \cup (0, 1)$ is the relative risk aversion coefficient of agent i. By an obvious calculation, we have

$$\begin{cases} h_{0\lambda}(y) = \sum_{i=1}^{I} \left(\frac{1}{\lambda_i y}\right)^{\frac{1}{\alpha_i}} \\ h_{1\lambda}(y) = \sum_{i=1}^{I} \left(\frac{\beta_i}{\lambda_i y}\right)^{\frac{1}{\alpha_i}} \end{cases}.$$

We now consider two special cases:

(a) Agents have homogeneous utilities, i.e., $\alpha_i = \alpha$ for all i. In this case, we have

$$\begin{cases} h_{0\lambda}(y) = \sum_{i=1}^{I} \left(\frac{1}{\lambda_i}\right)^{\frac{1}{\alpha}} y^{-\frac{1}{\alpha}}, \\ h_{1\lambda}(y) = \sum_{i=1}^{I} \left(\frac{\beta_i}{\lambda_i}\right)^{\frac{1}{\alpha}} y^{-\frac{1}{\alpha}}. \end{cases}$$

Therefore,

$$u_{1\lambda}(c) = \int_0^c h_{1\lambda}^{-1}(x)\, dx = \left[\sum_{i=1}^{I} \left(\frac{\beta_i}{\lambda_i}\right)^{\frac{1}{\alpha}}\right]^{\alpha} \frac{c^{1-\alpha}}{1-\alpha}.$$

Similarly,

$$u_{0\lambda}(c) = \left[\sum_{i=1}^{I} \left(\frac{1}{\lambda_i}\right)^{\frac{1}{\alpha}}\right]^{\alpha} \frac{c^{1-\alpha}}{1-\alpha}.$$

So both $u_{0\lambda}$ and $u_{1\lambda}$ are still CRRA. Moreover, since $0 < \beta_i \le 1$, we have

$$\left[\sum_{i=1}^{I} \left(\frac{\beta_i}{\lambda_i}\right)^{\frac{1}{\alpha}}\right]^{\alpha} \le \left[\sum_{i=1}^{I} \left(\frac{1}{\lambda_i}\right)^{\frac{1}{\alpha}}\right]^{\alpha};$$

[12]In these cases, the representative agents can be constructed in the same way as in an EUT economy (see Dana 2011).

that is, the time impatience of the representative agent for consumptions is preserved.

(b) Agents have heterogeneous utilities. For simplicity, assume $I = 2$, $\alpha_1 = \frac{1}{2}$, and $\alpha_2 = \frac{1}{4}$. In this case, we have

$$h_{1\lambda}(y) = \left(\frac{\beta_1}{\lambda_1 y}\right)^2 + \left(\frac{\beta_2}{\lambda_2 y}\right)^4.$$

Therefore,

$$u'_{1\lambda}(c) = h_{1\lambda}^{-1}(c) = \frac{\sqrt{2}\beta_2^2}{\lambda_2^2\sqrt{\sqrt{4\left(\frac{\beta_2}{\lambda_2}\right)^4 c + \left(\frac{\beta_1}{\lambda_1}\right)^4} - \left(\frac{\beta_1}{\lambda_1}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\sqrt{4\left(\frac{\beta_2}{\lambda_2}\right)^4 \frac{1}{c} + \left(\frac{\beta_1}{\lambda_1}\right)^4 \frac{1}{c^2}} + \left(\frac{\beta_1}{\lambda_1}\right)^2 \frac{1}{c}}.$$

Similarly,

$$u'_{0\lambda}(c) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{4\left(\frac{1}{\lambda_2}\right)^4 \frac{1}{c} + \left(\frac{1}{\lambda_1}\right)^4 \frac{1}{c^2}} + \left(\frac{1}{\lambda_1}\right)^2 \frac{1}{c}}.$$

So the utility functions of the representative agent are in general no longer CRRA. However, if $\beta_1 = \beta_2 = \beta \leq 1$, then $u'_{1\lambda}(c) = \beta u'_{0\lambda}(c)$, which still captures the time impatience of the representative agent for consumption.

5. EQUILIBRIA

In this section, we introduce the following assumption:

ASSUMPTION 5.1. *In addition to Assumption 2.1, w is continuously differentiable on* (0, 1).

For any $\lambda = (\lambda_1, \ldots, \lambda_I)$, let function $\Psi_\lambda : [0, 1] \to [0, \infty)$ be given by

$$\Psi_\lambda(p) \triangleq w'(p)\, u'_{1\lambda}\left(Q^+_{\tilde{e}_1}(1-p)\right),$$

where $u_{1\lambda}$ is defined in Section 4.

5.1. Rank-Dependent Asset Pricing

In this subsection, we derive an explicit expression relating the endogenous state-price density $\tilde{\rho}$, the marginal rate of substitution between the initial and the end-of-period consumption of a representative agent, and the probability weighting function, for the economy $\mathcal{E}$, under the assumption that an equilibrium exists where $\tilde{\rho}$ has a continuous CDF. The existence of such an equilibrium will be studied in the next subsection.

THEOREM 5.2. *Under Assumption 5.1, if there exists an equilibrium of economy $\mathcal{E}$ where the state-price density $\tilde{\rho}$ has a continuous CDF and condition* (3.11) *is satisfied for all i and*

$\mu > 0$, *then the function* Ψ_{λ^*} *is strictly increasing on* $[0, 1)$ *(here* λ^* *is given in Section 4). Moreover,*

$$Q^-_{\tilde{\rho}}(p) = \Psi_{\lambda^*}(p) \tag{5.1}$$

and

$$\tilde{\rho} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1)) \frac{u'_{1\lambda^*}(\tilde{e}_1)}{u'_{0\lambda^*}(e_0)} \quad a.s. \tag{5.2}$$

Proof. Recall that $(u'_{1\lambda^*})^{-1}(\cdot)$ is strictly decreasing and continuous on $(0, \infty)$. Moreover, $\hat{N}'(1 - w(\cdot))$ is nondecreasing and left-continuous. Therefore, (4.2) indicates that $\tilde{c}^*_1$ is a nonincreasing and left-continuous function of $F_{\tilde{\rho}}(\tilde{\rho})$. By Föllmer and Schied (2011) appendix A.3, we know that the upper quantile of $\tilde{c}^*_1$ is given by

$$(u'_{1\lambda^*})^{-1}\left(\hat{N}'(1 - w(1 - p))\right).$$

At equilibrium, by the market clearing condition, we have $\tilde{c}^*_1 = \tilde{e}_1$. Thus, the upper quantiles of $\tilde{c}^*$ and $\tilde{e}_1$ coincide; that is,

$$(u'_{1\lambda^*})^{-1}\left(\hat{N}'(1 - w(1 - p))\right) = Q^+_{\tilde{e}_1}(p).$$

Observe that $Q^+_{\tilde{e}_1}(\cdot)$ is strictly increasing due to the continuity of $F_{\tilde{e}_1}$. Then, $\hat{N}'$ must be strictly decreasing, and hence $\hat{N}$ is *strictly* concave. Therefore, $N = \hat{N}$, implying that (see Remark 3.4)

$$\hat{N}'(p) = \frac{Q^-_{\tilde{\rho}}(w^{-1}(1 - p))}{w'(w^{-1}(1 - p))}.$$

An obvious substitution yields that

$$(u'_{1\lambda^*})^{-1}\left(\frac{Q^-_{\tilde{\rho}}(1 - p)}{w'(1 - p)}\right) = Q^+_{\tilde{e}_1}(p),$$

which implies (5.1). Moreover, since $Q^-_{\tilde{\rho}}(\cdot)$ is strictly increasing, function Ψ_{λ^*} must also be strictly increasing.

On the other hand, recalling that $\tilde{e}_1 \equiv \tilde{c}^*_1$ is a nonincreasing and left-continuous function of $\tilde{\rho}$, we have

$$F_{\tilde{e}_1}(\tilde{e}_1) = g(\tilde{\rho})$$

for a nonincreasing and left-continuous function g. Considering the upper quantiles of both sides in the above, it follows that

$$g(Q^-_{\tilde{\rho}}(1 - p)) = p$$

for all p, implying $g = 1 - F_{\tilde{\rho}}$. Thus,

$$F_{\tilde{e}_1}(\tilde{e}_1) = 1 - F_{\tilde{\rho}}(\tilde{\rho}).$$

Consequently,

$$\tilde{\rho} = Q^-_{\tilde{\rho}}(1 - F_{\tilde{e}_1}(\tilde{e}_1)) \quad \text{a.s.}$$

Then, by a substitution and by (5.1), we deduce

$$\tilde{\rho} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))u'_{1\lambda^*}(\tilde{e}_1) \quad \text{a.s.}$$

Finally, at equilibrium, by the market clearing condition, we have $c_0^* = e_0$, implying $u'_{0\lambda^*}(e_0) = u'_{0\lambda^*}(c_0^*) = 1$. Thus, we arrive at (5.2). □

This theorem asserts that, under its assumptions, the state-price density $\tilde{\rho}$ is a *weighted* marginal rate of substitution between initial and end-of-period consumption of the representative agent, where the (random) weight is $w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))$ depending on the rank $1 - F_{\tilde{e}_1}(\tilde{e}_1)$ of the realization of the aggregate future endowment $\tilde{e}_1$.[13] So, via (5.2), we have effectively established a rank-dependent CCAPM.

Now, suppose the probability weighting function w is convex (corresponding to risk aversion; see Yaari 1987). In states of nature in which future consumption is sufficiently high, marginal utility, $u'_{1\lambda^*}(\tilde{e}_1)$, is low, but $w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))$ is less than one, which further lowers the price of a future payoff. Conversely, an asset is even more desired in low consumption states compared to the classical setting. So, the effects of a convex probability weighting and a concave outcome utility are compounded in the same direction: they both highly value low consumption states and lowly value high consumption ones. By contrast, with a concave probability weighting all of the conclusions are reversed: the weighting offsets the effect of concave outcome utility in asset pricing. Finally, if the probability weighting is inverse-S shaped (namely $w(p)$ is concave close to $p = 0$ and convex close to $p = 1$), then the market offers a premium when evaluating assets in *both* very high and very low future consumption states.[14]

We provide further discussion of the implications, interpretation, and application of the rank-dependent CCAPM in the next two sections.

REMARK 5.3. The identity (5.1) reads

$$\frac{Q^-_{\tilde{\rho}}(p)}{w'(p)} = u'_{1\lambda^*}\left(Q^+_{\tilde{e}_1}(1-p)\right),$$

which implies that, *necessarily*, $\frac{Q^-_{\tilde{\rho}}(p)}{w'(p)}$ is monotonically increasing. This suggests that even though the monotone condition on $\frac{Q^-_{\tilde{\rho}}}{w'}$ introduced by Jin and Zhou (2008) is restrictive for an *individual* agent portfolio choice problem, it is automatically satisfied in equilibria provided that the state-price density has no atom and (3.11) is valid.

REMARK 5.4. We note that (5.2) is not an explicit formula for calculating the state-price density $\tilde{\rho}$, since λ^* implicitly depends on $\tilde{\rho}$ via (3.12). Instead, (5.2) can be considered as an (explicit) necessary condition for the existence of a state-price density (the same can be said about the EUT counterpart of the result). As discussed earlier, this condition has rich economic interpretations. Moreover, it can be used to derive the RDUT-based equity premium and risk-free rate formulae (see Section 6).

[13] In the classical EUT setting, the state-price density is simply the marginal rate of substitution between initial and end-of-period consumption.

[14] A similar phenomenon was qualitatively observed and discussed by Shefrin (2008, p. 471).

5.2. Existence of Equilibria

Theorem 5.2 requires the existence of an equilibrium of the economy $\mathcal{E}$ with the state-price density having a continuous CDF. We now derive conditions that ensure this existence. To this end, we introduce an auxiliary, "weighted" economy as follows:

- Each agent i has an endowment $(e_{0i}, \tilde{e}_{1i})$;
- The agents have homogeneous beliefs $\mathbb{P}^\diamond$ about the states of nature, which is given by[15]

$$\frac{d\mathbb{P}^\diamond}{d\mathbb{P}} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1));$$

- The preference of each agent i over consumption plans $(c_{0i}, \tilde{c}_{0i})$ is represented by

$$V_i^\diamond(c_{0i}, \tilde{c}_{1i}) = u_{0i}(c_{0i}) + \beta_i \mathbb{E}^\diamond[u_{1i}(\tilde{c}_{1i})],$$

  where $\mathbb{E}^\diamond[\cdot]$ stands for the expectation under $\mathbb{P}^\diamond$.

For simplicity, the above economy is denoted by

$$\mathcal{E}^\diamond \triangleq \left\{(\Omega, \mathcal{F}, \mathbb{P}^\diamond),\ (e_{0i}, \tilde{e}_{1i})_{i=1}^I,\ \mathcal{C},\ \left(V_i^\diamond(c_{0i}, \tilde{c}_{0i})\right)_{i=1}^I\right\}.$$

In this economy, the agents have homogeneous "weighted" beliefs $\mathbb{P}^\diamond$ and the preferences for tomorrow consumptions $\tilde{c}_{1i}$ are represented by expected utilities $\mathbb{E}^\diamond[u_{1i}(\tilde{c}_{1i})]$. Individuals' optimization on consumption in this "weighted economy" follow the classical expected utility maximization.

An Arrow–Debreu equilibrium of economy $\mathcal{E}^\diamond$ is a collection $\left\{\tilde{\rho}^\diamond,\ (c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I\right\}$ consisting of a state-price density $\tilde{\rho}^\diamond$ and a collection $(c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I$ of feasible consumption plans, such that the feasible consumption plan $(c_{0i}^*, \tilde{c}_{1i}^*)$ solves the following optimization problem:

$$\begin{aligned} V_i^\diamond(c_{0i}^*, \tilde{c}_{1i}^*) = \max_{(c_{0i}, \tilde{c}_{1i}) \in C}\ & V_i^\diamond(c_{0i}, \tilde{c}_{1i}) \\ \text{subject to}\quad & c_{0i} + \mathbb{E}^\diamond[\tilde{\rho}^\diamond \tilde{c}_{1i}] \leq e_{0i} + \mathbb{E}^\diamond[\tilde{\rho}^\diamond \tilde{e}_{1i}] \end{aligned} \tag{5.3}$$

for all i, and such that

$$c_0^* \triangleq \sum_{i=1}^I c_{0i}^* = e_0 \quad \text{and} \quad \tilde{c}_1^* \triangleq \sum_{i=1}^I \tilde{c}_{1i}^* = \tilde{e}_1. \tag{5.4}$$

THEOREM 5.5. *Let two state-pricing densities, $\tilde{\rho}$ and $\tilde{\rho}^\diamond$, be related by*

$$\tilde{\rho} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))\tilde{\rho}^\diamond. \tag{5.5}$$

Moreover, set $\lambda^* = (\lambda_1^*, \ldots, \lambda_I^*)$, where λ_i^* is the Lagrange multiplier for optimization problem (5.3) *$(i = 1, \ldots, I)$. Then, under Assumption 5.1, the following two statements are equivalent:*

(i) $\left\{\tilde{\rho}^\diamond,\ (c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I\right\}$ *is an Arrow–Debreu equilibrium of the economy $\mathcal{E}^\diamond$ and Ψ_{λ^*} is strictly increasing on* $[0, 1)$*;*

[15]Noting that $1 - F_{\tilde{e}_1}(\tilde{e}_1)$ is uniformly distributed on $(0, 1)$, we have $\mathbb{E}[w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))] = \int_0^1 w'(t)dt = w(1) - w(0) = 1$. So $\mathbb{P}^\diamond$ is a probability measure.

(ii) $(\tilde{\rho}, (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I)$ *is an Arrow–Debreu equilibrium of the economy* $\mathcal{E}$, $\tilde{\rho}$ *has a continuous CDF and condition* (3.11) *is fulfilled for all* i *and* $\mu > 0$.

Furthermore, (i) implies that

$$\mathbb{E}^{\diamond}[(u'_{1i})^{-1}(\mu\tilde{\rho}^{\diamond})\tilde{\rho}^{\diamond}] < \infty \tag{5.6}$$

for all i *and* $\mu > 0$.

Proof. We prove only (i) $\Rightarrow$ (ii)+(5.6), since (ii) $\Rightarrow$ (i) can be similarly proved (with the help of Theorem 5.2).

Assume (i) holds. Then, for every i, the consumption plan $(c^*_{0i}, \tilde{c}^*_{1i})$ solves problem (5.3) and is given by

$$\begin{cases} c^*_{0i} = (u'_{0i})^{-1}(\lambda^*_i) \\ \tilde{c}^*_{1i} = (u'_{1i})^{-1}\left(\frac{\lambda^*_i}{\beta_i}\tilde{\rho}^{\diamond}\right), \end{cases} \tag{5.7}$$

where the Lagrange multiplier λ^*_i satisfies

$$(u'_{0i})^{-1}(\lambda^*_i) + \mathbb{E}^{\diamond}\left[\tilde{\rho}^{\diamond}(u'_{1i})^{-1}\left(\frac{\lambda^*_i}{\beta_i}\tilde{\rho}^{\diamond}\right)\right] = e_{0i} + \mathbb{E}^{\diamond}[\tilde{\rho}^{\diamond}\tilde{e}_{1i}]. \tag{5.8}$$

Let $\lambda^* \triangleq (\lambda^*_1, \ldots, \lambda^*_I)$, and let $h_{1\lambda^*}$ and $u_{1\lambda^*}$ be defined as in Section 4. Then, by (5.4), we have

$$h_{1\lambda^*}(\tilde{\rho}^{\diamond}) = \sum_{i=1}^{I}\tilde{c}^*_{1i} = \tilde{c}^*_1 = \tilde{e}_1.$$

Thus, $\tilde{\rho}^{\diamond} = u'_{1\lambda^*}(\tilde{e}_1)$, and hence by (5.5),

$$\tilde{\rho} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))u'_{1\lambda^*}(\tilde{e}_1) = \Psi_{\lambda^*}(1 - F_{\tilde{e}_1}(\tilde{e}_1)).$$

Since $\tilde{e}_1$ has a continuous CDF, $1 - F_{\tilde{e}_1}(\tilde{e}_1)$ is uniformly distributed on (0, 1). Moreover, Ψ_{λ^*} is strictly increasing, so we know that $\tilde{\rho}$ has a continuous CDF. By the left-continuity of Ψ_{λ^*}, we have (see Föllmer and Schied 2011, appendix A.3)

$$Q^-_{\tilde{\rho}}(p) = \Psi_{\lambda^*}(p).$$

By Remark 3.4 and by a substitution, we have

$$N(q) = -\int_q^{1-}\frac{\Psi_{\lambda^*}(w^{-1}(1-p))}{w'(w^{-1}(1-p))}\,dp = -\int_q^{1-} u'_{1\lambda^*}\left(Q^+_{\tilde{e}_1}(1 - w^{-1}(1-p))\right)dp.$$

It is easy to see that N is strictly concave, implying $\hat{N} = N$. Hence,

$$\hat{N}'(q) = N'(q) = u'_{1\lambda^*}\left(Q^+_{\tilde{e}_1}(1 - w^{-1}(1-q))\right).$$

Moreover,

$$F_{\tilde{\rho}}(\tilde{\rho}) = F_{\tilde{\rho}}\left(\Psi_{\lambda^*}(1 - F_{\tilde{e}_1}(\tilde{e}_1))\right) = F_{\tilde{\rho}}\left(Q^-_{\tilde{\rho}}(1 - F_{\tilde{e}_1}(\tilde{e}_1))\right) = 1 - F_{\tilde{e}_1}(\tilde{e}_1).$$

Thus, we have

$$N'\big(1 - w(F_{\tilde\rho}(\tilde\rho))\big) = N'\big(1 - w(1 - F_{\tilde e_1}(\tilde e_1))\big) = u'_{1\lambda^*}(\tilde e_1) = \tilde\rho^\diamond.$$

By (5.7) and by substitutions, we get

$$c^*_{0i} = (u'_{0i})^{-1}(\lambda^*_i)$$

and

$$\tilde c^*_{1i} = (u'_{1i})^{-1}\left(\frac{\lambda^*_i}{\beta_i}\tilde\rho^\diamond\right) = (u'_{1i})^{-1}\left(\frac{\lambda^*_i}{\beta_i}\hat N'\Big(1 - w(F_{\tilde\rho}(\tilde\rho))\Big)\right).$$

Moreover, for any nonnegative random variable $\tilde x$, we have

$$\mathbb{E}[\tilde\rho\tilde x] = \mathbb{E}[w'(1 - F_{\tilde e_1}(\tilde e_1))\tilde\rho^\diamond\tilde x] = \mathbb{E}^\diamond[\tilde\rho^\diamond\tilde x].$$

Then, (5.8) reads

$$(u'_{0i})^{-1}(\lambda^*_i) + \mathbb{E}\left[\tilde\rho(u'_{1i})^{-1}\left(\frac{\lambda^*_i}{\beta_i}\hat N'\Big(1 - w(F_{\tilde\rho}(\tilde\rho))\Big)\right)\right] = e_{0i} + \mathbb{E}[\tilde\rho\tilde e_{1i}].$$

Furthermore, recall that the asymptotic elasticity of each u_{1i} is assumed to be strictly less than one. Then, by Kramkov and Schachermayer (1999), theorems 2.2 and 3.2, for every $\mu > 0$, $(u'_{1i})^{-1}(\mu\tilde\rho^\diamond)$ solves the problem

$$\operatorname*{Maximize}_{\tilde x\geq 0}\ \mathbb{E}^\diamond[u_{1i}(\tilde x)] \quad \text{subject to } \mathbb{E}^\diamond[\tilde\rho^\diamond\tilde x] = a$$

for some $a > 0$. Thus (5.6) follows. Moreover,

$$\begin{aligned}\mathbb{E}^\diamond[(u'_{1i})^{-1}(\mu\tilde\rho^\diamond)\tilde\rho^\diamond] &= \mathbb{E}[(u'_{1i})^{-1}(\mu\tilde\rho^\diamond)\tilde\rho]\\ &= \mathbb{E}[(u'_{1i})^{-1}(\mu\hat N'(1 - w(F_{\tilde\rho}(\tilde\rho)))\tilde\rho]\\ &= \int_0^{1-}(u'_{1i})^{-1}(\mu\hat N'(\bar w(p)))Q^-_{\tilde\rho}(1-p)\,dp,\end{aligned}$$

implying (3.11). Therefore, it follows from Theorem 3.3 that $(c^*_{0i}, \tilde c^*_{1i})$ is the optimal consumption plan in economy $\mathcal{E}$ for agent i. Finally, (5.4) yields that the market clears in economy $\mathcal{E}$. Thus, (ii) follows. □

We now introduce the following condition, which is needed to ensure the existence of an equilibrium of economy $\mathcal{E}$.

ASSUMPTION 5.6. *For any λ, function Ψ_λ is strictly increasing on* $[0, 1)$.

THEOREM 5.7. *Under Assumptions 5.1 and 5.6, assume further that*

$$\begin{cases}\mathbb{E}[w'(1 - F_{\tilde e_1}(\tilde e_1))u_{1i}(\tilde e_1)] < \infty\\ \mathbb{E}\left[w'(1 - F_{\tilde e_1}(\tilde e_1))u'_{1i}\left(\frac{\tilde e_1}{I}\right)\right] < \infty\end{cases} \tag{5.9}$$

for all $i = 1, \ldots, I$. Then, there exists an Arrow–Debreu equilibrium of economy $\mathcal{E}$ where the state-price density has a continuous CDF and (3.11) *is satisfied for all i and $\mu > 0$. If,* *in addition, the utility functions u_{1i} satisfy*

$$-\frac{cu''_{1i}(c)}{u'_{1i}(c)} \leq 1 \textit{ for all } i = 1, \ldots, I \textit{ and } c > 0,$$

then the equilibrium is unique.

Proof. In view of Theorem 5.5, we obtain the existence of an equilibrium from the standard existence result for an EUT economy; see, e.g., Dana (1993a, b) and Föllmer and Schied (2011, section 3.6). The uniqueness follows from Dana (1993a, b). □

Let us now discuss the key condition for the existence of equilibria stipulated in Assumption 5.6. This condition has a prominent economic interpretation, which will be elaborated in Section 7 after we have introduced the notion of "implied relative risk aversion." Mathematically, the condition is *necessary*, at least for $\lambda = \lambda^*$, for the existence of an equilibrium, as stipulated in Theorem 5.2. It is automatically satisfied, for example, when w is strictly convex, since function $p \mapsto u'_{1\lambda}(Q^+_{\tilde{e}_1}(1-p))$ is strictly increasing. Note that the convexity of the probability weighting function underlines risk aversion in the Yaari sense.

Nevertheless, Assumption 5.6 may hold even when w is concave or inverse-S shaped. Here are examples:

EXAMPLE 5.8. Assume that the aggregate future endowment $\tilde{e}_1$ follows the Pareto distribution, namely,

$$F_{\tilde{e}_1}(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^{\gamma}, & x \geq x_m, \\ 0, & x < x_m, \end{cases}$$

where $x_m > 0$ is the scale parameter and $\gamma > 0$ the Pareto index. Obviously, $Q^+_{\tilde{e}_1}(p) = x_m(1-p)^{-\frac{1}{\gamma}}$.

(i) **A concave weighting function.** Take the power weighting function $w(p) = p^{1-\delta}$, where $\delta \in (0, 1)$ measures the degree of concavity of w (and hence the level of risk-loving associated with the probability weighting). Clearly, $w'(p) = (1-\delta)p^{-\delta}$.

(i.a) Consider the utility functions in Example 4.2-(a). In this case,

$$\Psi_\lambda(p) = w'(p)u'_{1\lambda}\left(Q^+_{\tilde{e}_1}(1-p)\right) = \left[\sum_{i=1}^{I}\left(\frac{\beta_i}{\lambda_i}\right)^{\frac{1}{\alpha}}\right]^{\alpha}(1-\delta)x_m^{-\alpha}p^{\frac{\alpha}{\gamma}-\delta}.$$

This is a strictly increasing function if and only if $\delta < \frac{\alpha}{\gamma}$. The latter condition is more likely to be satisfied with a less concave weighting function, a more concave outcome utility function, or a less positively skewed future total consumption distribution.

(i.b) Consider the utility functions in Example 4.2-(b). In this case,

$$\Psi_\lambda(p) = \frac{1-\delta}{\sqrt{2}}\sqrt{\sqrt{4\left(\frac{\beta_2}{\lambda_2}\right)^4\frac{1}{x_m}p^{\frac{1}{\gamma}-4\delta} + \left(\frac{\beta_1}{\lambda_1}\right)^4\frac{1}{x_m^2}p^{\frac{2}{\gamma}-4\delta}} + \left(\frac{\beta_1}{\lambda_1}\right)^2\frac{1}{x_m}p^{\frac{1}{\gamma}-2\delta}}.$$

This is a strictly increasing function if $\delta < \frac{1}{4\gamma} = \frac{\min\{\alpha_1,\alpha_2\}}{\gamma}$.

(ii) **An inverse-S shaped weighting function.** Take the Prelec weighting function $w(p) = e^{-\delta(-\ln p)^{\sigma}}$ where $\delta > 0$ and $\sigma \in (0,1)$, which is inverse-S shaped (see Prelec 1998). Obviously, $w'(p) = \sigma\delta e^{-\delta(-\ln p)^{\sigma}}(-\ln p)^{\sigma-1}p^{-1}$.

(ii.a) Consider the utility functions in Example 4.2-(a). In this case,

$$\Psi_{\lambda}(p) = \left[\sum_{i=1}^{I}\left(\frac{\beta_i}{\lambda_i}\right)^{\frac{1}{\alpha}}\right]^{\alpha} \sigma\delta x_m^{-\alpha} p^{\frac{\alpha}{\gamma}-1} e^{-\delta(-\ln p)^{\sigma}}(-\ln p)^{\sigma-1}.$$

This is a strictly increasing function if $\frac{\alpha}{\gamma} > 1$.

(ii.b) Consider the utility functions in Example 4.2-(b). In this case,

$$\Psi_{\lambda}(p) = \frac{1}{\sqrt{2}}\sigma\delta e^{-\delta(-\ln p)^{\sigma}}(-\ln p)^{\sigma-1} \times\sqrt{\sqrt{4\left(\frac{\beta_2}{\lambda_2}\right)^4 \frac{1}{x_m} p^{\frac{1}{\gamma}-4} + \left(\frac{\beta_1}{\lambda_1}\right)^4 \frac{1}{x_m^2} p^{\frac{2}{\gamma}-4}} + \left(\frac{\beta_1}{\lambda_1}\right)^2 \frac{1}{x_m} p^{\frac{1}{\gamma}-2}}.$$

This is a strictly increasing function if $\gamma < \frac{1}{4}$, or, equivalently, $\frac{\min\{\alpha_1,\alpha_2\}}{\gamma} > 1$.

6. EQUITY PREMIUM

In this section, we first derive some approximation forms of the equity premium and risk-free rate based on the rank-dependent CCAPM (5.2), and then suggest a possible direction of thinking about the equity premium puzzle.

Consider a security whose payoff at $t = 1$ is a random variable $\tilde{x}$. Its price is given by $s = \mathbb{E}[\tilde{\rho}\tilde{x}]$. In particular, for a riskless security whose payoff $x_f = 1$, its price is $s_f = \mathbb{E}[\tilde{\rho}]$. The rate of return of the risky security is $\tilde{r} = \frac{\tilde{x}}{s} - 1$ and the risk-free rate is $r_f = \frac{1}{s_f} - 1$. It is easy to see that $\mathbb{E}[\tilde{\rho}\tilde{r}] = \mathbb{E}[\tilde{\rho}r_f]$, or

$$-\mathbf{Cov}(\tilde{\rho},\tilde{r}) = \mathbb{E}[\tilde{\rho}](\bar{r} - r_f),$$

where $\bar{r} = \mathbb{E}[\tilde{r}]$. Thus, we arrive at

$$\bar{r} - r_f = -\frac{\mathbf{Cov}(\tilde{\rho},\tilde{r})}{\mathbb{E}[\tilde{\rho}]} = -(1 + r_f)\mathbf{Cov}(\tilde{\rho},\tilde{r}). \tag{6.1}$$

In the following, we write

$$u_{1\lambda^*}(c) = \beta u_{0\lambda^*}(c), \quad \beta \in (0,1]$$

in order to highlight the discount factor β for the representative agent.[16] This is necessary since we are going to derive the risk-free rate, which is partly related to the time impatience of the agent.

[16]For the benchmark utility given by Example 4.2-(a), we have $u_{1\lambda^*}(c) = \beta u_{0\lambda^*}(c) = \beta\frac{c^{1-\alpha}}{1-\alpha}$ for some $\beta \in (0,1]$.

Assume that the absolute value of the growth rate $\tilde{g} \triangleq \frac{\tilde{e}_1}{e_0} - 1$ of aggregate endowment is very small. Then, by (5.2), we have

$$\tilde{\rho} = \beta w'(1 - F_{\tilde{e}_1}(e_0(1+\tilde{g})) \frac{u'_{1\lambda^*}(e_0(1+\tilde{g}))}{u'_{1\lambda^*}(e_0)}$$

and, therefore, by an expansion,

$$\tilde{\rho} = \beta w'(1 - F_{\tilde{e}_1}(e_0)) \left[1 - \alpha \tilde{g} - \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \tilde{g} \right] + o(|\tilde{g}|), \tag{6.2}$$

where

$$\alpha \triangleq - \frac{e_0 u''_{1\lambda^*}(e_0)}{u'_{1\lambda^*}(e_0)}$$

is the relative risk aversion index of $u_{1\lambda^*}$. Setting $\bar{g} = \mathbb{E}[\tilde{g}]$, we can obtain

$$1 + r_f = \frac{1}{\mathbb{E}[\tilde{\rho}]} \approx \frac{1}{\beta w'(1 - F_{\tilde{e}_1}(e_0)) \left[1 - \alpha \bar{g} - \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \bar{g} \right]}$$

and, therefore,

$$1 + r_f \approx \frac{1}{\beta w'(1 - F_{\tilde{e}_1}(e_0))} \left[1 + \alpha \bar{g} + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \bar{g} \right], \tag{6.3}$$

provided that

$$\alpha \bar{g} + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \bar{g}$$

is small. Substituting (6.2) and (6.3) into (6.1) leads to

$$\begin{aligned} \bar{r} - r_f &\approx \left[\alpha + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \right] \left[1 + \alpha \bar{g} + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \bar{g} \right] \mathbf{Cov}(\tilde{g}, \tilde{r}) \\ &\approx \left[\alpha + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \right] \mathbf{Cov}(\tilde{g}, \tilde{r}). \end{aligned}$$

The first term above, $\alpha \mathbf{Cov}(\tilde{g}, \tilde{r})$, is provided by the standard CCAPM under EUT. The second term,

$$\frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \mathbf{Cov}(\tilde{g}, \tilde{r}),$$

depends on the weighted probability of the future endowment exceeding the current endowment, namely, the rank of the current endowment over all possible realizations of the future endowment. Thus, we have established an approximation of the equity premium:

$$\bar{r} - r_f \approx \alpha \mathbf{Cov}(\tilde{g}, \tilde{r}) + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \mathbf{Cov}(\tilde{g}, \tilde{r}). \tag{6.4}$$

We may write (6.4) in a more compact way. Recall that $\tilde{g} = \frac{\tilde{e}_1 - e_0}{e_0}$; thus,

$$F_{\tilde{e}_1}(x) = F_{\tilde{g}}\left(\frac{x - e_0}{e_0}\right), \quad f_{\tilde{e}_1}(x) = f_{\tilde{g}}\left(\frac{x - e_0}{e_0}\right)\frac{1}{e_0}.$$

Therefore,

$$\frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0)e_0 = \frac{w''(1 - F_{\tilde{g}}(0))}{w'(1 - F_{\tilde{g}}(0))} f_{\tilde{g}}(0).$$

This leads to the following alternative form of the approximated equity premium:

$$\bar{r} - r_f \approx \left[\alpha + \frac{w''(1 - F_{\tilde{g}}(0))}{w'(1 - F_{\tilde{g}}(0))} f_{\tilde{g}}(0)\right] \mathbf{Cov}(\tilde{g}, \tilde{r}). \tag{6.5}$$

Similarly, we may rewrite (6.3) as

$$1 + r_f \approx \frac{1}{\beta w'(1 - F_{\tilde{g}}(0))}\left[1 + \alpha\bar{g} + \frac{w''(1 - F_{\tilde{g}}(0))}{w'(1 - F_{\tilde{g}}(0))} f_{\tilde{g}}(0)\bar{g}\right]. \tag{6.6}$$

The corresponding approximations under the classical (EUT) CCAPM can be reproduced by letting $w'(\cdot) = 1$ in (6.5) and (6.6); that is,

$$\bar{r} - r_f \approx \alpha \mathbf{Cov}(\tilde{g}, \tilde{r}), \tag{6.7}$$

$$1 + r_f \approx \frac{1 + \alpha\bar{g}}{\beta} \tag{6.8}$$

provided that $\alpha\bar{g}$ is small.

The classical formula (6.7) stipulates that the equity premium depends *only* on the relative risk aversion level and on the "beta" (i.e., correlation between the equity return and the overall economy); nevertheless, the rank-dependent formula (6.5) suggests that it depends additionally on agents' subjective views of the overall consumption growth, represented by $1 - F_{\tilde{g}}(0) \equiv \mathbb{P}(\tilde{g} > 0)$. This, in turn, provides a new perspective from which to consider the classical equity premium puzzle.

Briefly put, the equity premium puzzle (Mehra and Prescott 1985) states that the observed equity premium is too high to be explainable by the classical CCAPM. In particular, Mehra and Prescott (1985) found the historical average annual equity premium of the S&P 500 index for the period 1889–1978 to be 6.18%, implying an implausibly high relative risk aversion based on the classical CCAPM. Subsequent empirical studies have confirmed that the equity premium puzzle is robust across different time periods and different countries.

The rank-dependent CCAPM, in particular the presence of the probability weighting function, may offer a way, at least at the qualitative level, to explain the puzzle.

Indeed, assume that the probability weighting function w is inverse-S shaped, as supported by many experimental/empirical studies. Due to the high probability of positive economic growth (in particular on average over periods of approximately 100 years, as in the study of Mehra and Prescott 1985), it is plausible to assume that $\mathbb{P}(\tilde{g} > 0)$ is large. Hence, $1 - F_{\tilde{g}}(0) = \mathbb{P}(\tilde{g} > 0)$ lies in the convex domain of w and $\frac{w''(1-F_{\tilde{g}}(0))}{w'(1-F_{\tilde{g}}(0))} f_{\tilde{g}}(0)$ is positive. Therefore, the expected rate of return provided by our model (6.5) is indeed larger than that yielded by the standard model (6.7), a fact that contributes positively to the theoretical value of the equity premium. The economic reason is that the representative agent is effectively making decisions based primarily on the convex part of w. The convexity of probability weighting corresponds to risk aversion and therefore leads to both an enhancement of the overall risk aversion level and a higher rate of return than that encountered under the standard CCAPM. As a result, an extremely high α is no longer required to account for the observed equity premium. Moreover, the explicit expression (6.5) provides qualitative information on the premium. For example, it is seen that this "theoretical enhancement," $\frac{w''(1-F_{\tilde{g}}(0))}{w'(1-F_{\tilde{g}}(0))} f_{\tilde{g}}(0)\mathbf{Cov}(\tilde{g}, \tilde{r})$, increases with the level of convexity, measured by $\frac{w''}{w'}$.

We stress, however, that the above discussion is qualitative only. To investigate whether our theoretical model matches the observed market data will require a careful empirical study, including calibration of the weighting function, which goes beyond the scope and objectives of this paper. Nonetheless, our study shows that the rank-dependent pricing model provides a potentially fruitful path in explaining some of the intriguing puzzles in the literature.

7. THE RDUT ECONOMY VERSUS THE EUT ECONOMY

At the individual consumption choice level, RDUT agents exhibit remarkably different behavior than their EUT counterparts (see Remark 3.5). In terms of equilibrium and asset pricing, the RDUT model also displays distinctly different properties than the EUT one. However, in this section we will argue that at equilibrium an RDUT economy is "equivalent" to an EUT one with proper modification on some primitives. In fact, the proofs of the main results in this paper have been based on the idea of making a connection between the RDUT and EUT models and then utilizing the existing results that pertain to the latter. Let us examine this connection in two different directions.

Implied Relative Risk Aversion. Risk preferences are captured in an RDUT model via two components: outcome utility and probability weighting. The outcome utility function is assumed to be concave, representing risk aversion in the classical sense. If the probability weighting is convex, implying risk aversion in the Yaari sense, then the *overall, aggregate* risk taste of the agent is clearly risk aversion. If the probability weighting is concave or inverse-S shaped, then there are two conflicting preferences in force, and the overall attitude toward risk depends on a quantitative assessment of the two. Intriguingly, the pricing formula (5.2) can shed light on how this may be done. The relevant analysis, to which we now turn, leads to the conclusion of an implied relative risk aversion on the part of an RDUT agent.

The idea is to interpret (5.2) within the classical EUT setting where the representative agent has a *modified* outcome utility function. Indeed, under the assumptions of Theorem 5.2, we know that function

$$x \mapsto w'(1 - F_{\tilde{e}_1}(x))u'_{1\lambda^*}(x)$$

is strictly decreasing. Consider a function u_w such that

$$u'_w(x) = w'(1 - F_{\tilde{e}_1}(x))u'_{1\lambda^*}(x).$$

Then, u_w is strictly increasing and concave. The marginal utility $u'_{1\lambda^*}(x)$ weighted by $w'(1 - F_{\tilde{e}_1}(x))$ yields the marginal utility $u'_w(x)$. Now, we can rewrite (5.2) as

$$\tilde{\rho} = \frac{u'_w(\tilde{e}_1)}{u'_{0\lambda^*}(e_0)}.$$

This formulation can be regarded as representing a state-price density for a fictitious EUT economy (under the original probability $\mathbb{P}$ *without* weighting), where u_w is the outcome utility function of a "weighted" representative agent. Moreover, assuming $\tilde{e}_1$ has a probability density function $f_{\tilde{e}_1}$ and w is twice continuously differentiable, we have

$$R^w(x) \triangleq -\frac{xu''_w(x)}{u'_w(x)} = -\frac{xu''_{1\lambda^*}(x)}{u'_{1\lambda^*}(x)} + \frac{xw''(1 - F_{\tilde{e}_1}(x))}{w'(1 - F_{\tilde{e}_1}(x))} f_{\tilde{e}_1}(x). \tag{7.1}$$

This representation gives the rank-dependent *implied relative index of risk aversion*, which can be used to assess the *overall* degree of risk aversion (or risk-loving) of the RDUT agent. Specifically, the rank-dependent implied utility u_w is, in the Arrow–Pratt sense, more (less) risk averse than $u_{1\lambda^*}$ in the domain where w is convex (concave). In particular, if w is inverse-S shaped, then the probability weighting effectively increases the level of risk aversion in lower consumption states and enhances that of risk-seeking in higher consumption states.

The preceding analysis also explains and justifies the sufficient condition for the existence of equilibria, Assumption 5.6. The function Ψ_λ being strictly increasing is equivalent to the function

$$x \mapsto w'(1 - F_{\tilde{e}_1}(x))u'_{1\lambda}(x)$$

being strictly decreasing, and the arbitrariness of λ corresponds to that of the initial endowments. Therefore, in economics terms, Assumption 5.6 requires an implied overall positive index of relative risk aversion for any initial distribution of the wealth.

In a similar spirit, we can recover (6.5) from (6.7) by replacing the relative risk aversion index α with the rank-dependent implied relative risk aversion index

$$\alpha + \frac{w''(1 - F_{\tilde{g}}(0))}{w'(1 - F_{\tilde{g}}(0))} f_{\tilde{g}}(0).$$

We can also recover (6.6) from (6.8) in the same way, while noticing that the former has one more term $w'(1 - F_{\tilde{g}}(0))$. This finding is interesting, since it suggests that the discount factor of the weighted agent should be multiplied by the factor $w'(1 - F_{\tilde{g}}(0))$ in addition to the adjustment of the degree of risk aversion. This factor is larger than one if the agents have an inverse-S shaped weighting function and envisage positive consumption growth. So, RDUT agents place more weight on future consumption—a finding that is, after all, consistent with the empirically observed low interest rate.

Instead of revising the outcome utility function of the *representative* agent, we can also revise those of the *individual* agents in the following way:

$$u_{wi}(x) = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))u_{1i}(x), \quad i = 1, 2, \cdots, I.$$

Each u_{wi} is a *state-dependent* utility function, inheriting all of the properties of u_{1i} at each state of nature. With this revision of individual utility functions, the original RDUT economy is equivalent to the resulting EUT economy in terms of equilibrium allocation and pricing.

The above discussion leads to the conclusion that, at equilibrium, one cannot actually distinguish an RDUT economy from an EUT one. However, the "reasonable" or "plausible" range of relative coefficients of risk aversion under the latter may be significantly shifted from that which pertains under the former. This is the case because under RDUT, the relative risk aversion consists of contributions from both the outcome utility and the probability weighting. For instance, Mehra and Prescott (1985) found that the observed equity premium corresponded to a relative index of risk aversion over 30. A measure of 30 on this index means that investors would have to be indifferent between a gamble equally likely to pay \$50,000 or \$100,000 and a certain payoff of \$51,209 (Mankiw and Zeldes 1991). Such a case, however, is highly implausible because few individuals are likely to be that risk averse. Nonetheless, once we understand that the probability weighting, in addition to the outcome utility, also contributes to this total measure of 30, then the number may no longer be implausible and the equity premium puzzle may be less—indeed, perhaps not at all—puzzling.

Rank-Neutral Probability. There is another way of reconciling our rank-dependent CCAPM with the standard one. The idea has actually been revealed in the proof of Theorem 5.5. Let

$$\tilde{\rho}^{\diamond} = \frac{u'_{1\lambda^*}(\tilde{c}_1^*)}{u'_{0\lambda^*}(c_0^*)} \tag{7.2}$$

and $\tilde{\rho}$ be determined by (5.5). In Section 5.2 of this paper, we obtained the result that $\tilde{\rho}^{\diamond}$ is the equilibrium state-price density in an EUT economy $\mathcal{E}^{\diamond}$ where the agents have homogeneous beliefs $\mathbb{P}^{\diamond}$, and that $\tilde{\rho}$ is the equilibrium state-price density in an RDUT economy $\mathcal{E}$ where the agents have homogeneous beliefs $\mathbb{P}$. The relation (5.5) between $\tilde{\rho}$ and $\tilde{\rho}^{\diamond}$ can be reformulated as follows:

$$\tilde{\rho} = \tilde{\rho}^{\diamond} \frac{d\mathbb{P}^{\diamond}}{d\mathbb{P}}.$$

Consequently, for any claim $\tilde{x}$ we have

$$\mathbb{E}[\tilde{\rho}\tilde{x}] = \mathbb{E}^{\diamond}[\tilde{\rho}^{\diamond}\tilde{x}].$$

Moreover, the two economies have the same equilibrium allocation.

Therefore, by analogy to the risk-neutral probability in option pricing, the RDUT economy $\mathcal{E}$ is equivalent, as far as asset pricing is concerned, to the EUT economy $\mathcal{E}^{\diamond}$ where a "rank-neutral" probability measure is used instead of the original measure.

8. CONCLUSION

We provide conditions on an RDUT economy under which the Arrow–Debreu equilibrium exists uniquely. We reveal, in an explicit way, how probability weighting will affect this economy. A key step in our derivation, which is also a major technical contribution of the paper, is to obtain an analytical solution to the individual consumption problem that involves the concave envelope of a certain nonconcave function. We find that asset prices depend not only upon level of risk aversion and beta, but also upon agents' subjective views regarding overall consumption growth. This suggests that probability weighting may help to explain a number of intriguing economic phenomena.

The setting of our economy is fairly general, with general future endowment distributions, general shape of probability weighting functions, and heterogeneous outcome utilities. However, we realize that one of the key assumptions in this paper is the homogeneity of the agents' probability weightings. Without this condition our approach in proving the existence of a representative agent fails. Economically, with both heterogeneous outcome utilities and probability weightings, the agents' preferences might be too diverse to permit the existence of an RDUT representative agent. This remains a significant research problem that we endeavor to attack in our future studies.

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