Portfolio Choice Under Cumulative Prospect Theory: An Analytical Treatment*

Xue Dong He† and Xun Yu Zhou‡

September 14, 2010

Abstract

We formulate and carry out an analytical treatment of a single-period portfolio choice model featuring a reference point in wealth, S-shaped utility (value) functions with loss aversion, and probability weighting under Kahneman and Tversky's cumulative prospect theory (CPT). We introduce a new measure of loss aversion for large payoffs, called the *large-loss aversion degree* (LLAD), and show that it is a critical determinant of the well-posedness of the model. The sensitivity of the CPT value function with respect to the stock allocation is then investigated, which, as a by-product, demonstrates that this function is neither concave nor convex. We finally derive optimal solutions explicitly for the cases when the reference point is the risk-free return and when it is not (while the utility function is piece-wise linear), and we employ these results to investigate comparative statics of optimal risky exposures with respect to the reference point, the LLAD, and the curvature of the probability weighting.

Key words: Portfolio choice, single period, cumulative prospect theory, reference point, loss aversion, S-shaped utility function, probability weighting, well-posedness

---

*We are grateful for comments from seminar and conference participants at Cambridge, London School of Economics, King's College, TU Berlin, Le séminaire Bachelier, National University of Singapore, Academia Sinica, Chinese Academy of Sciences, Nankai, Princeton, UT Austin, Man's Oxford Offsite Quant Forum, the 2007 Workshop on Mathematical Control Theory and Finance, the 2007 Financial Engineering and Risk Management Conference, the 2008 Oxford Behavioural Finance Conference, and the 4th Annual CARISMA Conference. He acknowledges a start-up fund of Columbia University. Zhou acknowledges financial support from the RGC Earmarked Grant CUHK418605, and a start-up fund of the University of Oxford. The usual disclaimer applies.

†Industrial Engineering and Operations Research Department, Columbia University, 500 West 120th Street New York, NY 10027. Email: <xh2140@columbia.edu>.

‡Nomura Centre for Mathematical Finance, The University of Oxford, 24–29 St Giles, Oxford OX1 3LB, UK, and Oxford–Man Institute of Quantitative Finance, The University of Oxford, and Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. Email: <zhouxy@maths.ox.ac.uk>.

1 Introduction

Expected utility maximization has been a predominant model for portfolio choice. The model is premised upon the assumption that people are rational. Its basic tenets are as follows: Investors evaluate wealth according to final asset positions; they are uniformly risk averse; and they are able or willing to evaluate probabilities objectively. However, substantial experimental evidence has suggested that human behaviors may significantly deviate from or simply contradict these classical principles when facing uncertainties. In the 1970s Kahneman and Tversky (1979) proposed *prospect theory* (PT) for decision-making under uncertainty. The theory was further developed by Tversky and Kahneman (1992) into *cumulative prospect theory* (CPT) in order to be consistent with first-order stochastic dominance. In the context of financial asset allocation, the key elements of CPT are

- People evaluate assets in comparison with certain benchmarks, rather than on final wealth positions;
- People behave differently on gains and on losses; they are not uniformly risk averse and are distinctively more sensitive to losses than to gains (the latter is a behavior called *loss aversion*); and
- People tend to overweight small probabilities and underweight large probabilities.

These elements translate respectively into the following technical features for the formulation of a portfolio choice model:

- A reference point (or neutral outcome/benchmark/breakeven point/status quo) in wealth that defines gains and losses[1];
- A value function (which replaces the notion of utility function), concave for gains and convex for losses (such a function is called *S-shaped*) and steeper for losses than for gains; and
- A probability weighting function that is a nonlinear transformation of probability measure, which inflates a small probability and deflates a large probability.

There has been burgeoning research interest in incorporating CPT into portfolio choice in a single-period setting; see, for example, Benartzi and Thaler (1995), Levy and Levy (2004), Gomes (2005), Barberis and Huang (2008), and Bernard and Ghossoub (2010). However, save for the last one, most of these works have focused on empirical/experimental

[1] Markowitz (1952) is probably the first to put forth the notion of a wealth reference point, termed *customary wealth*.

study and/or numerical solutions[2]. Moreover, the portfolio choice models therein are either rather special or exclude some of the key elements of CPT (e.g., probability weighting is absent, and/or the reference point coincides with the risk-free return). A sufficiently general model and its rigorous analytical treatment seem to be still lacking in the single-period setting[3].

In this paper we consider a single-period portfolio choice model with a CPT agent in a market consisting of one risky asset and one risk-free account. Since the risky return distribution is arbitrary in this paper, the risky asset could also be interpreted as the market portfolio or an index fund.

The paper aims to address three issues: 1) *Modeling*: to establish a behavioral portfolio selection model featuring all the three key elements of CPT (namely, a reference point, an S-shaped value function, and probability weighting) and to address the well-posedness of such a model; 2) *Solutions*: to carry out an analytical study of the general model and to derive explicit solutions for some important special cases; and 3) *Comparative statics analysis*: to investigate the sensitivity of the optimal risky allocation with respect to key exogenous variables such as the reference point, the level of agent loss aversion with respect to large payoffs, and the curvature of the probability weighting.

In the modelling part we highlight the ill-posedness issue. An ill-posed model is one whose optimal strategy is simply to take the greatest possible risky exposure. Such a situation arises when the utility associated with gains substantially outweighs the disutility associated with losses. The ill-posedness has been hardly an issue with classical portfolio choice models such as those of mean variance and expected utility. We shall show that this is no longer the case with a CPT model.

To certify whether a given model is well-posed or otherwise, we define a new measure of loss aversion. Different from the known indices of loss aversion, which are typically defined for small losses and gains, our measure is for *large* values. We show that this measure is the key to the issue of ill-posedness, and we derive explicitly a critical large-loss aversion level that divides between the well-posedness and ill-posedness of the underlying model. The result indicates that market investment opportunities such as stock return and risk-free return must be consistent with market participants' psychology (including their preferences), such as utility functions and probability weighting functions, before any reasonable model can be formulated. An important implication of this is that, in reaching an equilibrium, the stock return must be adjusted according to market participants' preferences

[2]The Bernard and Ghossoub (2010) paper came to our attention after we had completed the first version of our paper. An explanation of the differences between this paper and Bernard and Ghossoub (2010) is provided at the end of this section.

[3]Perversely, research on dynamic, continuous-time portfolio selection models, albeit only a little, has led to more general, analytical results; see Berkelaar et al. (2004) and Jin and Zhou (2008). One reason is that a continuous-time model renders a complete market under certain conditions, whereas in a single-period model the market is inherently incomplete.

to avoid the "ill-posedness" of the model. Therefore, the avoidance of "ill-posedness" will help us understand the market equilibrium prices better.

The solution part (for a well-posed model) poses the main technical challenges, especially in comparison to the classical utility or mean-variance model. The S-shaped utility functions are inherently non-concave and non-smooth and, moreover, the probability weighting generates nonlinear expectations. Consequently, the known, standard approaches in optimization, such as Lagrangian and convex duality, do not work (in particular we can no longer say anything about global optimality). Indeed, we prove in this paper that the CPT value function (as a function of the stock allocation) is in general non-convex and non-concave. Hence, this function may have many local maxima.

The problem of non-concavity and non-smoothness has been acknowledged in the literature, and the approaches to solving this type of problem have so far been largely limited to numerical schemes. In this paper we resort to an analytical approach attempting to obtain closed-form solutions. As with the classical portfolio models, closed-form solutions will provide insights in understanding the interrelationship between solutions and parameters, in carrying out a comparative statics analysis, and in testing and validating the model. In the present paper, while a solution in its greatest generality is yet to be derived, we examine two special cases: one in which the reference point is risk-free return and the other in which the utility function is piece-wise linear but the reference point may differ from risk-free return. These two cases are important. First, some, if not many, investors naturally choose risk-free return as a benchmark to evaluate their investment performances. Second, in many economic applications, a piece-wise linear utility function is convenient, yet at the same time it can still reveal many economic insights. For instance, Benartzi and Thaler (1995) use a piece-wise linear utility function to explain the equity premium puzzle. It is noteworthy that no portfolio choice problem was formulated in Benartzi and Thaler (1995). Furthermore, portfolio choice under CPT with a general reference point has not been thoroughly studied in the literature.

The explicit forms of optimal solutions derived in this paper make it possible to evaluate, analytically and numerically, the effects on equity allocation of various parameters, especially the reference point, the large-loss aversion degree (LLAD), the curvature of the probability weighting, and the planning horizon. In particular, it is shown that risky exposures monotonically decrease as the LLAD value increases. All these results reinforce the important role the LLAD plays in both modeling and solving CPT portfolio choice problems.

Before concluding this section, we comment on the differences between this paper and Bernard and Ghossoub (2010), the latter independently studying a similar CPT portfolio choice model with borrowing constraints and deriving results similar to some of the results here (e.g., Theorem 3 in Section 5.1). First, Bernard and Ghossoub (2010) do not formulate and address the general well-posedness issue, which is painstakingly dealt with in this paper[4]. Second, in deriving an optimal portfolio for their model, they do not consider the case in which the reference point is different from the risk-free return. Finally, they focus on a piece-wise power utility function where the power of the gain part (α) is no greater than its loss counterpart (β), whereas the case $\alpha > \beta$ is covered by this paper (see Theorem 3-(i)). On the other hand, Bernard and Ghossoub (2010) investigate some properties of the optimal portfolio and show, interestingly, that a CPT investor is very sensitive to the skewness of the stock excess return. This is not covered by our paper, although we do sensitivity analysis with respect to other parameters. In summary, there is some overlap between the two papers; yet there are sufficient differences in focus and scope.

The remainder of this paper is organized as follows. Section 2 formulates the CPT model. In Section 3 the model well-posedness is studied in great detail, while in Section 4 the sensitivity of the CPT value function with respect to stock allocation is investigated. Section 5 is devoted to the analytical solutions of the model for two important special cases. Finally, Section 6 concludes. All the proofs are relegated to an appendix.

2 Model

Consider a market consisting of one risky asset (stock) and one risk-free account and an agent with an investment planning horizon from date 0 to date T. The risk-free total return over this period is a deterministic quantity, $r(T)$ (i.e., \$1 invested in the risk-free account returns \$ $r(T)$ at T). The stock total excess return, $R(T) - r(T)$, is a random variable following a cumulative distribution function (CDF) $F_T(\cdot)$. Shorting in this market is allowed and there is no restriction on the levels of stock position and leverage[5]. It follows from the no-arbitrage rule that that

$$0 < F_T(0) \equiv P(R(T) \le r(T)) < 1. \tag{1}$$

There is an agent in the market with CPT preference. She has a reference point or benchmark in wealth denoted by B, which serves as a base point to distinguish gains from losses evaluated at the end of the investment horizon. Moreover, there are two utility functions[6], $u_+(\cdot)$ and $u_-(\cdot)$, both mapping from $\mathbb{R}_+$ to $\mathbb{R}_+$, that measure gains and losses respectively[7].

[4] Theorem 3.1 in Bernard and Ghossoub (2010) does present a case where an optimal solution is to take infinite risky exposure. But they neither formalize nor study ill-posedness in a general setting as we do in this paper.

[5] We think it is more sensible to let the final solution suggest what constraints should be in place rather than exogenously and arbitrarily imposing constraints on an asset allocation model. For example, later we will study the conditions under which an optimal solution of the model will endogenously avoid shorting.

[6] These are called *value functions* in the Kahneman–Tversky terminology. In this paper we still use the term *utility function* in order to distinguish it from the *CPT value function* defined below.

[7] Strictly speaking, $u_-(\cdot)$ is the disutility of losses. Note that in Kahneman and Tversky (1979) and Tversky

There are two additional functions, $w_+(\cdot)$ and $w_-(\cdot)$ from $[0, 1]$ to $[0, 1]$, representing the agent's weighting (sometimes also termed *distortion*) of probability for gains and losses respectively.

The agent is initially endowed with an amount, W_0, with an objective to maximize his CPT preference value at $t = T$ by investing once at $t = 0$. Specifically, suppose an amount, θ, is invested in the stock and the remainder in the risk-free account, and set $x_0 = r(T)W_0 - B$. This quantity, x_0, is the deviation of the reference point from the risk-free payoff. Then the terminal wealth is

$$X(x_0, \theta, T) = x_0 + B + [R(T) - r(T)]\theta. \tag{2}$$

Let X be a random wealth and B the reference point. The *CPT (preference) value* of X is defined to be

$$\begin{aligned} V(X) = &\int_B^{+\infty} u_+(x - B)d[-w_+(1 - F(x))] \\ &- \int_{-\infty}^{B} u_-(B - x)d[w_-(F(x))], \end{aligned} \tag{3}$$

where $F(\cdot)$ is the CDF of X and the integral is in the Lebesgue–Stieltjes sense. Notice that in Tversky and Kahneman (1992) the corresponding CPT value is defined only for X with discrete values. It is easy to show that the preceding formula does coincide with the Tversky and Kahneman's definition, if X is purely discrete; hence (3) is a natural extension of the Tversky–Kahneman definition that applies to both discrete and continuous X.

Now, we evaluate the CPT value of (2) by applying (3), leading to a function of θ (called *CPT value function*), which is denoted by $U(\theta)$. When $\theta = 0$,

$$U(0) = \begin{cases} u_+(x_0), & \text{if } x_0 \geq 0 \\ -u_-(-x_0), & \text{if } x_0 < 0. \end{cases} \tag{4}$$

When $\theta > 0$, by changing variables, one obtains from (3) that

$$\begin{aligned} U(\theta) = &\int_{-x_0/\theta}^{+\infty} u_+(\theta t + x_0)d[-w_+ (1 - F_T(t))] \\ &- \int_{-\infty}^{-x_0/\theta} u_-(-\theta t - x_0)d[w_- (F_T(t))]. \end{aligned} \tag{5}$$

and Kahneman (1992) a utility function $u(\cdot)$ is given on the whole real line, which is convex on $\mathbb{R}_-$ and concave on $\mathbb{R}_+$ (corresponding to the observation that people tend to be risk-averse on gains and risk-seeking on losses), hence S-shaped. In our model we separate the utility on gains and losses by letting $u_+(x) := u(x)$ and $u_-(x) := -u(-x)$ whenever $x \geq 0$. Thus the concavity of $u_\pm(\cdot)$ corresponds to an overall S-shaped utility function.

Similarly, when $\theta < 0$, one has

$$
\begin{aligned}
U(\theta) = & \int_{-\infty}^{-x_0/\theta} u_+(\theta t + x_0) d[w_+ (F_T(t))] \\
- & \int_{-x_0/\theta}^{+\infty} u_-(-\theta t - x_0) d[-w_- (1 - F_T(t))].
\end{aligned} \tag{6}
$$

The CPT portfolio choice model is, therefore:

$$
\max_{\theta \in \mathbb{R}} \quad U(\theta). \tag{P}
$$

The following assumptions on the utility functions, $u_\pm(\cdot)$, and the weighting functions, $w_\pm(\cdot)$, are imposed throughout this paper.

Assumption 1. $u_\pm(\cdot)$*:* $\mathbb{R}_+ \to \mathbb{R}_+$ *are continuous, strictly increasing, strictly concave, and twice differentiable, with* $u_\pm(0) = 0$*.*

Assumption 2. $w_\pm(\cdot)$*:* $[0, 1] \to [0, 1]$ *are non-decreasing and differentiable, with* $w_\pm(0) = 0, w_\pm(1) = 1$*.*

Other than the differentiability condition, which is purely technical, these assumptions have well-established economic interpretations. Tversky and Kahneman (1992) use the following particular functional forms for the utility and weighting functions:

$$
u_+(x) = x^\alpha, \qquad u_-(x) = kx^\beta, \tag{7}
$$

$$
w_+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}, \qquad w_-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}, \tag{8}
$$

and they estimate the parameter values (from experimental data) as follows: $\alpha = \beta = 0.88$, $k = 2.25$, $\gamma = 0.61$, and $\delta = 0.69$. These functions satisfy Assumptions 1 and 2 with the specified parameters. Other representative weighting functions include the ones proposed by Tversky and Fox (1995):

$$
w_+(p) = \frac{\delta^+ p^{\gamma^+}}{\delta^+ p^{\gamma^+} + (1-p)^{\gamma^+}}, \quad w_-(p) = \frac{\delta^- p^{\gamma^-}}{\delta^- p^{\gamma^-} + (1-p)^{\gamma^-}} \tag{9}
$$

where $0 < \gamma^+, \gamma^- < 1$, and $\delta^+, \delta^- > 0$, and the ones by Prelec (1998):

$$
w_+(p) = e^{-\delta^+ (-\ln p)^\gamma}, \quad w_-(p) = e^{-\delta^- (-\ln p)^\gamma} \tag{10}
$$

for $0 < \gamma < 1$ and $\delta^+, \delta^- > 0$. Numerical estimates of all the parameters above are available in Abdellaoui (2000) and Wu and Gonzalez (1996).

Let us end this section by remarking that, when there is no probability weighting, $U(\theta)$ reduces to the normal expected utility (although the overall utility function is still S-shaped instead of globally concave). The combination of the probability weighting and the S-shaped utility function pose a major challenge in analyzing and solving our CPT model (P).

3 Well-posedness

We say that Problem (P) is *well-posed* if it admits a finite optimal solution $\theta^* \in \mathbb{R}$ with a finite CPT value; otherwise it is *ill-posed*. Well-posedness is more than a modeling issue; it also sheds light on the interplay between investors and markets and has important implications on market equilibrium, as will be discussed in detail in Section 3.3. It appears to us that well-posedness has not received adequate attention in the literature of portfolio choice. Well-posedness is usually imposed – explicitly or implicitly – as a standing assumption for modeling.

Since we do not impose constraints on portfolios in this paper, ill-posedness is equivalent to an *infinite* exposure to the risky asset under the assumption that the CPT value function is continuous[8]. Economically, an ill-posed model sets wrong incentives that (mis)lead the investor to take infinite leverage[9].

This section investigates the conditions under which our portfolio choice model (P) is well-posed. The key result is that the well-posedness is predominantly determined by a specific measure of loss aversion, a notion which will be introduced momentarily.

3.1 Infinite CPT value

To start, we examine whether $U(\theta)$ may have an infinite value at a certain finite θ (i.e., whether a particular portfolio will achieve an infinite CPT value). We give the following assumption.

Assumption 3. *$F_T(\cdot)$ has a probability density function $f_T(\cdot)$. Moreover, there exists $\epsilon_0 > 0$ such that $w'_{\pm}(1 - F_T(x))f_T(x) = O(|x|^{-2-\epsilon_0})$, $w'_{\pm}(F_T(x))f_T(x) = O(|x|^{-2-\epsilon_0})$ for $|x|$ sufficiently large and $0 < F_T(x) < 1$.*

Proposition 1. *Under Assumption 3, $U(\theta)$ has a finite value for any $\theta \in \mathbb{R}$, and $U(\cdot)$ is continuous on $\mathbb{R}$.*

[8] Mathematically, an infinite risky exposure can be formulated as: There exists a sequence θ_n with $|\theta_n| \to \infty$ such that $U(\theta_n) \to \sup_{\theta \in \mathbb{R}} U(\theta)$.

[9] In the presence of a leverage constraint, one can still discuss whether the model sets improper incentives by checking whether an extreme solution (boundary solution) is optimal.

Assumption 3 quantifies certain minimally needed coordination between the probability weighting and the market[10]. It is natural to ask whether this assumption is restrictive (if it were, then even the very definition of the CPT value would be questionable). The following result serves to clarify this point.

Proposition 2. *If the stock return $R(T)$ follows a lognormal or normal distribution, and $w'_{\pm}(x) = O(x^{-\alpha})$, $w'_{\pm}(1-x) = O(x^{-\alpha})$ for sufficiently small $x > 0$ with some $\alpha < 1$, then Assumption 3 holds for any $\epsilon_0 > 0$.*

It is straightforward to check that the Kahneman–Tversky weighting function (8) satisfies

$$
\begin{aligned}
w'_+(x) = O(x^{-(1-\gamma)}), \quad w'_+(1-x) = O(x^{-(1-\gamma)}); \\
w'_-(x) = O(x^{-(1-\delta)}), \quad w'_-(1-x) = O(x^{-(1-\delta)}).
\end{aligned}
$$

Similar estimates hold for the Tversky–Fox weighting (9) as well as for the Prelec weighting (10) if the return is normal or when $\gamma > \frac{1}{2}$ (γ is estimated to be 0.74 by Wu and Gonzalez 1996) if the return is lognormal. In other words, Assumption 3 holds for a number of interesting cases. That said, it is very wrong to think that the opposite would never occur. Indeed, it is not difficult to show that the CPT value is infinite for any stock allocation for the lognormal return and the Prelec weighting with $\gamma < \frac{1}{2}$. Hence, Assumption 3, which we shall assume to be in force hereafter, specifies some minimum requirement for a reasonable CPT model.[11].

3.2 Well-posedness and ill-posedness

We now address the well-posedness issue. In view of the continuity of $U(\cdot)$, investigating well-posedness boils down to investigating the asymptotic behavior of $U(\theta)$ when $|\theta| \to \infty$, namely, the CPT value when the stock is heavily invested (either long or short).

We define the following quantity

$$
k := \lim_{x \to +\infty} \frac{u_-(x)}{u_+(x)} \geq 0, \tag{11}
$$

assuming that the limit exists. This quantity measures the ratio between the pain of a substantial loss and the pleasure of a gain of the same magnitude; hence, it is a certain indication of the level of loss aversion. It is referred to hereafter as *large-loss aversion degree*

[10]It is noteworthy that the utility functions are not part of this required coordination.

[11]It has already been observed in the literature that CPT preferences might lead to infinite preference values of some prospects having finite expectations. Rieger and Wang (2006) give some results similar to Proposition 2. Barberis and Huang (2008) show that the preference value is finite if the variance of the prospect is finite and the CPT parameters are in reasonable ranges.

(LLAD)[12]. Notice that by its definition, k may take an infinite value. It turns out, as will be seen in the sequel, that the LLAD plays a central role in studying the CPT portfolio choice model (P).

Before we proceed, let us remark that this new index of loss aversion is markedly different from several existing indices. In CPT loss aversion is based on the experimental observation for small gains and losses. Therefore, the indices introduced in the literature, including those of Tversky and Kahneman (1992), Benartzi and Thaler (1995), and Köbberling and Wakker (2005), describe loss aversion in a neighborhood of zero instead of one of much greater magnitude, thus representing the kink of the utility function around the reference point. We argue that the LLAD defined here characterizes loss aversion from a different and important, yet hitherto largely overlooked angle for the following reasons. First of all, it is plausible that people are loss averse, both when the loss is small and when it is big. While there is very little experimental evidence of people's attitudes with respect to large payoffs, our analysis below shows that LLAD determines whether or not people will take infinite leverage, and hence LLAD affects market equilibrium. So it is important to consider loss aversion that refers to large payoffs. Second, for the Kahneman–Tversky utility functions (7) our definition of LLAD coincides with those of the aforementioned indices[13]. However, in general they do not have to be the same, because they are used to address different problems. In the context of portfolio choice, small-loss aversion determines whether one would be better off investing slightly in a risky asset when the reference point is the risk-free return, whereas LLAD determines whether one will take infinite leverage if allowed to do so. Last but not least, the results in this paper will justify the importance of this new notion at least on a theoretical (as opposed to experimental) ground: We will prove that LLAD is an exclusively critical determinant for the well-posedness of a CPT portfolio choice model and plays an important role in the final solution of the model. To conclude, although LLAD in general does not capture loss aversion for small gains and losses, it does provide an additional dimension in addressing the central issue of loss aversion in CPT.

Theorem 1. *We have the following conclusions:*

(i) *If* $k = +\infty$, *then* $\lim_{|\theta|\to+\infty} U(\theta) = -\infty$, *and consequently Problem* (P) *is well-posed.*

(ii) *If* $k = 0$, *then Problem* (P) *is ill-posed.*

This theorem suggests that, if the utility of a large gain increases much faster than that of a large loss (i.e., $k = 0$), then the agent will simply buy stock on maximum possible

[12]For the utilities suggested by Tversky and Kahneman (1992), the LLAD value is 2.25.

[13]Indeed, Köbberling and Wakker (2005), Section 7, point out that there is some technical complication with their definition when applied to power utilities; so they have to define the corresponding index by a convention.

margin since the pain incurred by a loss is overwhelmed by the happiness brought by a possible gain. This leads to a trivial (ill-posed, that is) problem. Conversely, if the investor is overwhelmingly loss averse ($k = +\infty$), then a large equity position, either long or short, is not preferable (its CPT value decreases to minus infinity as the risky position grows to infinity). This calls for a finite amount of money allocated to the stock, giving rise to a well-posed model.

What if, then, k is finitely positive, i.e., the two utilities increase at the same speed? The answer lies in some critical statistics, to be defined via the following lemma, which are related to both individual preferences and market opportunities.

Lemma 3. *Assume* $\lim_{x\to+\infty} \frac{u_+(tx)}{u_+(x)} = g_+(t)\ \forall t \geq 0$ *and* $\lim_{x\to+\infty} \frac{u_-(tx)}{u_-(x)} = g_-(t)\ \forall t \geq 0$*, then the following statistics*

$$a_1 := \int_0^{+\infty} g_+(t) d[-w_+ (1 - F_T(t))], \tag{12}$$

$$a_2 := \int_{-\infty}^0 g_-(-t) d[w_- (F_T(t))], \tag{13}$$

$$b_1 := \int_{-\infty}^0 g_+(-t) d[w_+ (F_T(t))], \tag{14}$$

$$b_2 := \int_0^{+\infty} g_-(t) d[-w_- (1 - F_T(t))], \tag{15}$$

are well-defined and strictly positive. Furthermore, we have

$$\begin{aligned}
&\lim_{\theta\to+\infty} \int_{-x_0/\theta}^{+\infty} [u_+(\theta t + x_0)/u_+(\theta)] d[-w_+ (1 - F_T(t))] = a_1,\\
&\lim_{\theta\to+\infty} \int_{-\infty}^{-x_0/\theta} [u_-(-\theta t - x_0)/u_-(\theta)] d[w_- (F_T(t))] dt = a_2,\\
&\lim_{\theta\to-\infty} \int_{-\infty}^{-x_0/\theta} [u_+(\theta t + x_0)/u_+(-\theta)] d[w_+ (F_T(t))] = b_1,\\
&\lim_{\theta\to-\infty} \int_{-x_0/\theta}^{+\infty} [u_-(-\theta t - x_0)/u_-(-\theta)] d[-w_- (1 - F_T(t))] dt = b_2.
\end{aligned}$$

If, in addition, $0 < k < +\infty$*, then* $g_+(t) \equiv g_-(t)$*.*

The following is a result of well-posedness when $u_+(x)$ and $u_-(x)$ increase at the same speed. It involves a critical value,

$$k_0 := \max\left(\frac{a_1}{a_2}, \frac{b_1}{b_2}\right). \tag{16}$$

Theorem 2. *Assume that* $0 < k < +\infty$, $\lim_{x\to+\infty} u_+(x) = +\infty$*, and* $\lim_{x\to+\infty} u_+(tx)/u_+(x) = g(t)\ \forall t \geq 0$*. We have the following conclusions:*

(i) *If* $k > k_0$, *then* $\lim_{|\theta|\to+\infty} U(\theta) = -\infty$, *and consequently Problem* (P) *is well-posed.*

(ii) *If* $k < k_0$, *then either* $\lim_{\theta\to+\infty} U(\theta) = +\infty$ *or* $\lim_{\theta\to-\infty} U(\theta) = +\infty$, *and consequently Problem* (P) *is ill-posed.*

To better understand these results, let us first explain the economic interpretations of the parameters a_1, a_2, b_1, b_2, and (therefore) k_0. Assume for the moment that there is no probability weighting, i.e., $w_\pm(p) = p$. If $u_+(x) = x^\alpha,\ \ u_-(x) = kx^\alpha$ with $k > 0,\ 0 < \alpha \le 1$, then $g_+(t) = g_-(t) = t^\alpha$. In this case $a_1 = E[(\tilde{R}^+)^\alpha]$, and $a_2 = E[(\tilde{R}^-)^\alpha]$, where $\tilde{R} := R(T) - r(T)$ is the stock total excess return. Thus the ratio

$$\frac{a_1}{a_2} = \frac{E[(\tilde{R}^+)^\alpha]}{E[(\tilde{R}^-)^\alpha]}.$$

Similarly,

$$\frac{b_1}{b_2} = \frac{E[(\tilde{R}^-)^\alpha]}{E[(\tilde{R}^+)^\alpha]}.$$

If we take exponential or logarithmic utility functions[14], namely,

$$u_+(x) = 1 - e^{-\alpha x}, \quad u_-(x) = k(1 - e^{-\alpha x}), \qquad \alpha > 0,\ k > 0,$$

or

$$u_+(x) = \log(1 + x), \quad u_-(x) = k\log(1 + x), \qquad k > 0.$$

In both cases $g_+(t) = g_-(t) \equiv 1$, so (assuming again that there is no probability weighting)

$$\frac{a_1}{a_2} = \frac{P(\tilde{R} > 0)}{P(\tilde{R} < 0)} \equiv \frac{E[1_{\{\tilde{R}>0\}}]}{E[1_{\{\tilde{R}<0\}}]}; \quad \frac{b_1}{b_2} = \frac{P(\tilde{R} < 0)}{P(\tilde{R} > 0)}.$$

Clearly, the ratio a_1/a_2 represents some investor-preference adjusted criterion of the upside potential of the stock relative to the downside potential or the attractiveness of the stock, *if* one takes a long position. Moreover, since in our model shorting is allowed, the ratio b_1/b_2 quantifies the attractiveness of shorting the stock. The quantity k_0, being the larger of the two ratios, represents the overall desirability of the investment in the stock, which is subjective and investor-specific.

Now, if probability weighting is present, the above intepretations are still valid. We only need to replace the expectation in the definition of the criterion with one under probability weighting – the latter is called the *Choquet expectation*, a nonlinear expectation[15].

---

[14]An exponential function $u_+(x) = 1 - e^{-\alpha x}$ does not satisfy $\lim_{x\to+\infty} u_+(x) = +\infty$ required by Theorem 2. However, k_0 is still well defined.

[15]See Denneberg (1994) for a detailed account on the Choquet expectation.

The parameter k_0 is related to, but not the same as, the CPT-ratio introduced by Bernard and Ghossoub (2010). It is an easy exercise to show that, if the utility function is a two-piece power function with the same power parameter for gains and losses, then k_0 coincides with the CPT ratio[16]. However, the two are different for most other utility functions such as exponential and logarithmic ones. Theorem 2 suggests that k_0 defined here is a more appropriate parameter in addressing well-posedness in a more general setting[17].

So, Theorems 1 and 2 indicate that the LLAD, k, is crucial in determining whether the model is well-posed or otherwise. If $k_0 > k$, namely the stock is so attractive that it overrides the large loss aversion, then the investor will take infinite leverage leading to an ill-posed model. If $k_0 < k$, the stock is only moderately attractive, then there is a trade-off between the stock desirability and avoidance of potential large losses, resulting in a well-posed model. In short, if the investor is not sufficiently loss averse with large payoffs, then the model will be ill-posed.

The boundary case when $k = k_0$, which is not covered in Theorem 2, may correspond to either well-posedness or ill-posedness and thus requires further investigation[18]. However the boundary case is of technical interest only, as practically all the parameters of the model – including utilities and probability weighting – are estimates and prone to errors. So we are not going to pursue this line of inquiry further.

Corollary 4. *If the utility functions are of the power ones:*

$$u_+(x) = x^\alpha, \;\; u_-(x) = kx^\beta, \qquad k > 0, \; 0 < \alpha, \beta \le 1, \tag{17}$$

then Problem (P) *is well-posed when $\alpha < \beta$, or $\alpha = \beta$ and $k > k_0$, and is ill-posed when $\alpha > \beta$, or $\alpha = \beta$ and $k < k_0$, where k_0 is defined by (16) (via Lemma 3) with $g_+(t) \equiv g_-(t) = t^\alpha$.*

Note that in the above it is a slight abuse of notation when we use the same letter k, reserved for LLAD, for the coefficient of $u_-(\cdot)$ in (17). However, the constant k in (17) is indeed the LLAD value when $\alpha = \beta$, which is the case of the Kahneman–Tversky utilities and the only case that is interesting in our investigations below.

[16] In Bernard and Ghossoub (2010) shorting is prohibited; so only a_1/a_2 is necessary in defining k_0.

[17] Both k_0 and the CPT-ratio are also related to the so-called Omega measure introduced by Keating and Shadwick (2002) and the gain–loss ratio of Bernardo and Ledoit (2000). A key difference though is that both the Omega measure and the gain–loss ratio depend only on market opportunities, whereas k_0 and the CPT-ratio involve investor preferences.

[18] For example, consider $u_+(x) = x + x^\alpha$ and $u_-(x) = k_0 x$ with $0 < \alpha < 1$ fixed and $k_0 = a_1/a_2$. Then $k := \lim_{x\to\infty} \frac{u_-(x)}{u_+(x)} = k_0$. For simplicity, let $x_0 = 0$. From the last equation in the proof of Theorem 2 (see Appendix), we see that $\lim_{\theta\to+\infty} U(\theta) = +\infty$, which implies ill-posedness. On the other hand, if $u_+(x) = x$ and $u_-(x) = k_0(x + x^\alpha)$ with $0 < \alpha < 1$ fixed and $k_0 = a_1/a_2$, a similar argument shows that $k := \lim_{x\to\infty} \frac{u_-(x)}{u_+(x)} = k_0$ and $\lim_{\theta\to+\infty} U(\theta) = -\infty$, leading to well-posedness.

3.3 Discussions

Let us discuss several important issues. First, in sharp contrast with a classical expected utility model, a CPT model could be easily ill-posed. The possible ill-posedness suggests the importance of the interplay between investors and markets. As shown by Lemma 3, the critical value k_0 in determining a model's well-posedness depends not only on the agent preference set (utilities, probability weighting, and investment horizon) but also on the investment opportunity set (asset return distributions). However, an infinite exposure to a risky portfolio is rarely observed in reality; hence, in response to the agent's preference set, the market is forced to price the assets in such a way that the agent's CPT model is well-posed. This, in turn, has potential implications for studying market equilibrium. For example, De Giorgi et al. (2004) and Barberis and Huang (2008) define a market equilibrium based on the avoidance of essentially that which we call ill-posedness here, assuming that the reference point is the risk-free return. In the next section we will also show that the equilibrium may exist under the same assumption. So, the possible ill-posedness is by no means a negative result in relation to CPT; rather, it helps in understanding the market equilibrium. More precisely, the market responds to the preferences of the market participants in such a way that the ill-posedness is avoided.

Secondly, Theorem 2 applies largely to utilities with constant relative risk aversion (CRRA) such as power functions, owing to the condition that the utilities have to be unbounded. This fails to hold with utilities that exhibit constant absolute risk aversion (CARA) – e.g., an exponential function. Although the piece-wise power utility function is used in CPT preference in a number of works (Barberis and Huang 2008, Bernard and Ghossoub 2010), it is also argued in some literature that this utility function may cause several problems and is less favorable than a piece-wise exponential utility function.

De Giorgi et al. (2004) show that under a multi-asset economy with the asset returns being jointly normally distributed, the equilibrium does not exist when the agents have (heterogeneous) piece-wise power utility functions and risk-free return reference points. Based on this, they declare that a piece-wise exponential utility function may perform better. However, it seems to us that there may be some problem in their argument[19]. On the other

[19] In De Giorgi et al. (2004) the optimal stock allocation of each agent, i, is determined by the sign of the function $f^i(q)$, defined in the last equation on p. 25. (Actually the CPT preference used in that paper is a special, rank-dependent one: From (1) in the paper, their weighting functions on gains and losses, $T_\pm(\cdot)$, satisfy $T_-(z) = T_+(1-z)$. However, most of their results can go through with a general CPT preference.) If $f^i(q) < 0$, agent i's optimal allocation is to invest only in the risk-free asset. If $f^i(q) > 0$, he will take an infinite leverage in the risky stocks. If $f^i(q) = 0$, then he is indifferent to the choices available among all possible allocations. See the discussion on pp. 28–29 in that paper. Then the authors argue that "as soon as the investors are a little heterogeneous (in preferences) no equilibrium exists because there is no common q at which all investors would be indifferent with respect to the degree of leverage" (p. 29). It is true that there does not exist a common q such that all the agents would be indifferent with respect to the level of leverage, i.e., $f^i(q) = 0, i = 1, \ldots, I$. However, the equilibrium still exists as long as there are some agents who are indifferent among all possible degrees of leverage and the rest of them optimally invest in the risk-free asset. In

hand, Barberis and Huang (2008) show that the equilibrium does exist when investors have homogeneous preferences – assuming the same asset returns and the same type of piece-wise power utility functions as in De Giorgi et al. (2004). Our result in the next section also shows that an equilibrium exists.

Rieger and Wang (2006) prefer a piece-wise exponential utility function to a piece-wise power one as the CPT preference, since the latter may lead to infinite preference values of some prospects, in a spirit similar to that of Proposition 1 in our paper. This, however, is not a convincing reason to reject the piece-wise power utility function. Actually, Proposition 1 shows that the preference value is finite as long as the tails of the asset return prospects are properly controlled, *independent* of the utility functions being used! As discussed above, the condition of Proposition 1 infers a range of market returns in an equilibrium.

Rieger (2007) further argues that the piece-wise power utility function cannot capture very high degrees of risk-aversion in simple lotteries. This is certainly true, since in CPT the power index $\alpha \geq 0$ and hence the relative Arrow–Pratt index $(1-\alpha)$ does not exceed 1. However, this fact alone is not sufficient to reject the piece-wise power utility function. First, Rieger (2007) found that the CPT preference with piece-wise power utility functions can still capture the levels of risk-aversion of about 70% of the respondents in the Tversky and Kahneman (1992) experiment. Second, by incorporating the probability weighting functions, the CPT preference with piece-wise power utility functions can still describe higher levels of risk-averse behavior, while in Rieger (2007) the weighting functions are fixed.

Köbberling and Wakker (2005) reject the piece-wise utility function because it makes CPT inconsistent when describing the loss aversion referring to small payoffs and large payoffs. We have offered an extensive discussion of this issue in Section 3.2. Here let us add that there is no inconsistency when the power indices of the utility functions in gains and in losses are the same (such as the one proposed by Kahneman and Tversky). Furthermore, the inconsistency in describing loss aversion follows from the particular form of the piece-wise power utility function, and it may disappear with other utility functions that are unbounded.

To take our argument further, we now show that the piece-wise exponential utility cannot rule out ill-posedness at all. First, recall that a behavioral model is well-posed if infinite exposure to the risky asset is not optimal. Assuming the piece-wise exponential utility,

$$u_+(x) = 1 - e^{-\alpha x}, \quad u_-(x) = \lambda(1 - e^{-\beta x}), \qquad \alpha, \beta > 0,\ \lambda > 0,$$

other words, once there exists some appropriate $q \geq 0$ such that $\max_{i \in I} f^i(q) = 0$, then an equilibrium exists. Such a q can indeed be found because one can show that each $f^i(q)$ is continuous in q, $\lim_{q\to\infty} f^i(q) = \infty$, and $f^i(0) < 0$ under some mild conditions, in the same spirit of Lemma 2 in De Giorgi et al. (2004).

we can compute that

$$\lim_{\theta\to\infty}[\theta^2 U'(\theta)] = f_T(0)\left\{\frac{w'_+(1-F_T(0))}{\alpha} - \frac{\lambda w'_-(F_T(0))}{\beta} - x_0\left[w'_+(1-F_T(0)) + \lambda w'_-(F_T(0))\right]\right\}.$$

When the reference point coincides with the risk-free return (i.e. $x_0 = 0$) and there is no probability weighting, then $\lim_{\theta\to\infty}[\theta^2 U'(\theta)] < 0$ if $\lambda > \frac{\beta}{\alpha}$. In this case the behavioral model is well-posed. However, a general reference point and the presence of weighting functions would make things very different. We have carried out some numerical computation which shows that a high (though reasonable) reference point, i.e., a sufficiently negative x_0, can make the problem ill-posed even if there is no probability weighting and the loss aversion $\lambda > \frac{\beta}{\alpha}$. This result can be explained intuitively as follows. Consider the case in which $x_0 < 0$. Noting $\lim_{x\to\infty} u_+(x) = 1$, $\lim_{x\to-\infty}[-u_-(-x)] = -\lambda$, the maximum possible increase in CPT value is $1 + u_-(-x_0)$; and the maximum possible decrease is $\lambda - u_-(-x_0)$, if the agent starts with the prospective value $-u_-(-x_0)$ and then takes infinite leverage on stocks. Clearly, for a sufficiently negative x_0, the maximum possible happiness dominates the maximum possible pain, if the agent takes infinite risky exposure. This causes the ill-posedness. Our discussion also shows that the well-posedness with piece-wise exponential utilities depends not only on the LLAD (λ in this case) but also on the reference point. In contrast, for unbounded utilities only the LLAD plays a role. It is worth mentioning that piece-wise exponential utilities have been applied to portfolio choice in the literature. However, most of these works focus on the risk-free reference point and thus overlook the problems that reference points may cause.

4 Sensitivity

We have shown that the CPT value function $U(\theta)$ depends continuously on the amount allocated to equity, θ. In this section, we examine the sensitivity of this dependence, especially around $\theta = 0$. This sensitivity, in turn, will tell whether it is optimal to invest in equity at all. As a by-product, we will show that $U(\theta)$ is generally neither concave nor convex.

We first introduce the following statistics:

$$\begin{aligned} &\lambda_1^+ := \textstyle\int_{-\infty}^{+\infty} td[w_+(F_T(t))],\ \lambda_2^+ := \int_{-\infty}^{+\infty} td[-w_+(1-F_T(t))], \\ &\lambda_1^- := \textstyle\int_{-\infty}^{+\infty} td[w_-(F_T(t))],\ \lambda_2^- := \int_{-\infty}^{+\infty} td[-w_-(1-F_T(t))]. \end{aligned} \tag{18}$$

Proposition 5. *Suppose $x_0 \neq 0$. Then $U(\theta)$ is continuously differentiable on $[0, +\infty)$ and*

$(-\infty, 0]$, *where the derivative at 0 is the right one and left one respectively. In particular,*

$$U'(0+) = \begin{cases} u'_+(x_0)\lambda_2^+, & x_0 > 0, \\ u'_-(-x_0)\lambda_1^-, & x_0 < 0 \end{cases}, \quad U'(0-) = \begin{cases} u'_+(x_0)\lambda_1^+, & x_0 > 0, \\ u'_-(-x_0)\lambda_2^-, & x_0 < 0 \end{cases}. \tag{19}$$

Let us look more closely at the signs of $U'(0+)$ and $U'(0-)$ by examining the statistics $\lambda_i^{\pm}$, $i = 1, 2$. Indeed, integrating by parts and noting Assumption 3, we derive the following alternative definitions of these statistics:

$$\begin{aligned} \lambda_1^{\pm} &= \int_0^{+\infty} [1 - w_{\pm}(F_T(t))]dt - \int_{-\infty}^0 w_{\pm}(F_T(t))dt, \\ \lambda_2^{\pm} &= \int_0^{+\infty} w_{\pm}(1 - F_T(t))dt - \int_{-\infty}^0 [1 - w_{\pm}(1 - F_T(t))]dt. \end{aligned} \tag{20}$$

These formulae show that $\lambda_1^{\pm}, \lambda_2^{\pm}$ are various generalized versions of the expected stock excess return. One crucial point is that these weighted expectations are investor-specific, namely, they all depend on the specific investor probability weighting functions. If there were no probability weighting, then they would all reduce to the usual expectations[20]. Thus, if these weighted expectations are larger than 0, then the derivative of $U(\theta)$ near 0 is larger than 0. This suggests that, when the reference point is different from the risk-free return and the investor's expectations on the stock excess return are positive, then to him investing some amount in the stock is better than not holding the stock at all.

Notice that the case when $x_0 = 0$ is excluded from consideration in Proposition 5, as it will be investigated separately in the next section.

We have so far investigated the sensitivity of $U(\theta)$ near $\theta = 0$. Next we study its asymptotic property at $\theta = +\infty$.

Proposition 6. *Assume* $\lim_{x\to+\infty} u'_{\pm}(x) = 0$. *Then*

$$\lim_{|\theta|\to+\infty} U'(\theta) = 0. \tag{21}$$

So, the CPT value function has a diminishing marginal value if the utility function does.

Corollary 7. *If* $\lim_{|\theta|\to+\infty} U(\theta) = -\infty$ *and* $\lim_{x\to+\infty} u'_{\pm}(x) = 0$, *then* $U(\cdot)$ *is non-concave on either* $\mathbb{R}_-$ *or* $\mathbb{R}_+$. *If in addition* $\lambda_1^-, \lambda_2^+ > 0$, *then* $U(\cdot)$ *is non-convex on* $\mathbb{R}_+$.

5 Two Explicitly Solvable Cases

The non-concavity of the CPT value function $U(\cdot)$ stated in Corollary 7 imposes a major difficulty in solving the model (P). Concavity/convexity, which renders powerful techniques

[20] In fact, $\lambda_2^{\pm}$ are precisely the Choquet expectations.

such as Lagrangian and duality workable, is crucial in solving optimization problems. The rich theories on mean-variance and classical expected utility models – including the related asset pricing – have been exclusively built on the premise that the respective value functions to be maximized are concave. Now, the non-concavity is inherent in a CPT model, and hence its analysis and solutions call for new techniques and approaches. In this section we will study two economically interesting and explicitly solvable cases, while comparing our approaches to those in the literature.

5.1 Case 1: Reference point coincides with the risk-free return

It is plausible that many investors – especially ordinary households – tend to compare their investment performance with that in fixed income securities, especially when the equity market is bearish. So the risk-free return serves as a natural (if pessimistic) psychological reference point. For our behavioral model this case corresponds to $x_0 = 0$, which simplifies the problem greatly. Specifically, the value function in this case reduces to

$$U(\theta) = \int_0^{+\infty} u_+(\theta t) d[-w_+ (1 - F_T(t))] - \int_{-\infty}^{0} u_-(-\theta t) d[w_- (F_T(t))], \quad \theta \geq 0,$$
$$U(\theta) = \int_{-\infty}^{0} u_+(\theta t) d[w_+ (F_T(t))] - \int_0^{+\infty} u_-(-\theta t) d[-w_- (1 - F_T(t))], \quad \theta < 0.$$

If the utility functions are of the power ones, as in (17), then we can explicitly solve Problem (P) by investigating the local maxima of $U(\cdot)$.

Theorem 3. *Assume $x_0 = 0$ and that the utility functions and k_0 are as in Corollary 4. We have the following conclusions:*

(i) *If $\alpha > \beta$, or $\alpha = \beta$ and $k < k_0$, then* (P) *is ill-posed.*

(ii) *If $\alpha = \beta$ and $k > k_0$, then the only optimal solution to* (P) *is $\theta^* = 0$.*

(iii) *If $\alpha = \beta$ and $k = k_0 = a_1/a_2$, then any $\theta^* \geq 0$ is optimal to* (P).

(iv) *If $\alpha = \beta$ and $k = k_0 = b_1/b_2$, then any $\theta^* \leq 0$ is optimal to* (P).

(v) *If $\alpha < \beta$, then the only optimal solution to* (P) *is*

$$\theta^* = \left[\frac{1}{k}\frac{\alpha}{\beta}\frac{a_1}{a_2}\right]^{\frac{1}{\beta-\alpha}} \tag{22}$$

if $a_1^\beta/a_2^\alpha \geq b_1^\beta/b_2^\alpha$, and it is

$$\theta^* = -\left[\frac{1}{k}\frac{\alpha}{\beta}\frac{b_1}{b_2}\right]^{\frac{1}{\beta-\alpha}} \tag{23}$$

if $a_1^\beta / a_2^\alpha < b_1^\beta / b_2^\alpha$.

The results in this theorem suggest that a CPT investor with a risk-free reference point and power utilities will invest in stocks as much as he can if he is not sufficiently large-loss averse (quantified by the conditions in Theorem 3-(i)). On the contrary, if he is overwhelmingly large-loss averse (corresponding to Theorem 3-(v)[21]), then the optimal solution would be to take a fixed equity position regardless of his initial endowment (although the level of the position does depend on the other model parameters). This investment behavior resembles that of a classical utility maximizer with an exponential utility. Moreover, $|\theta|$ monotonically decreases as k increases. This implies that the higher large-loss aversion is, the less risky an exposure (either long or short) becomes with an optimal strategy. On the other hand, when $\alpha = \beta$ (the Kahneman–Tversky utilities) with a sufficiently large LLAD (Theorem 3-(ii)), not holding stocks is the only optimal solution. This, however, explains why many households did not, historically, invest in equities at all[22]. The contrapositive conclusion is that, if a CPT investor (with the Kahneman–Tversky utilities) does indeed invest in stocks, then his reference point must be different from the risk-free return.

Finally, Theorem 3-(iii) has an important implication in the existence of a market equilibrium. In fact, if there are some agents in the market who follow CPT in the Tversky and Kahneman (1992) setting (i.e., (7) and (8), where $\alpha = \beta$) and whose reference point coincides with the risk-free return, then, whatever their LLAD value k is, there is always some expected return of the stock (the one that makes $k = k_0 = a_1/a_2$) for which these agents are willing to hold any positive amount of the stock. As long as the total shares of the stock optimally held by all the other investors do not exceed the aggregate number of the circulating shares of the stock, these CPT investors also achieve the optimality by holding the remaining shares of the stock. This is consistent with the notion of equilibrium where there is some expected return that clears the market.

It is interesting to compare our findings here with those of Barberis and Huang (2008), who consider CPT portfolio selection with multiple stocks[23]. Under the assumption that the asset prices are normally distributed, they derive analytically the optimal solution which is quite close to the one presented in Theorem 3. They have a numerical solution when the normality assumption is removed.[24] Moreover, assuming that all the investors in the market

[21]It is estimated in Abdellaoui (2000) that the medians of α and β are 0.89 and 0.92 respectively, where the difference is rather small, corresponding to this case.

[22]This phenomenon has been noted for a long time; see Mankiw and Zeldes (1991) for example. A similar result to the one presented here is derived in Gomes (2005) for his portfolio selection model with loss averse investors, albeit in the absence of probability weighting.

[23]The main focus of Barberis and Huang (2008) is the asset pricing implications of the CPT preferences (specifically how positive skewness is priced by CPT investors). For that they need to solve a CPT portfolio selection problem.

[24]To be specific, in studying the skewness, Barberis and Huang (2008) introduce a skewed security in addition to J risky assets with joint normal distributed return. This skewed security is independent of the J risky assets.

follow the CPT preference in Tversky and Kahneman (1992) with $\alpha = \beta$ and that their reference point is the same as the risk-free return, they obtain the equilibrium, which is exactly the case when $k = k_0$.[25] However, the case $\alpha \neq \beta$ or the one in which the reference point is different from the risk-free return is not treated in Barberis and Huang (2008).

5.2 Case 2: Linear utilities

The second case is the one with linear utility functions, namely,

$$u_+(x) = x, \;\; u_-(x) = kx, \qquad k \geq k_0, \tag{24}$$

yet with a general reference point (i.e., x_0 is arbitrary). We are interested in this case because for many applications the concavity/convexity is insignificant and hence can be ignored. An analytical solution to this case, according to our best knowledge, is unavailable in the literature.

In the present case, we have for $\theta > 0$:

$$U'(\theta) = \int_{-x_0/\theta}^{+\infty} t w_+'(1 - F_T(t)) f_T(t) dt + k \int_{-\infty}^{-x_0/\theta} t w_-'(F_T(t)) f_T(t) dt, \tag{25}$$

and

$$U''(\theta) = \frac{x_0^2}{\theta^3} f_T(-\frac{x_0}{\theta}) \left[w_+' \left(1 - F_T(-\frac{x_0}{\theta})\right) - k w_-' \left(F_T(-\frac{x_0}{\theta})\right)\right]. \tag{26}$$

To find the conditions leading to the solution of (P), we need to introduce the following function:

$$\varphi(p) := \frac{w_+'(1-p)}{w_-'(p)}, \;\; 0 < p < 1. \tag{27}$$

Here we set $\varphi(p) = +\infty$ whenever $w_-'(p) = 0$. The following assumptions will be in force throughout this subsection.

Assumption 4. *There exists* $0 \leq p_1 < p_2 \leq 1$ *such that*

$$\varphi(p) \begin{cases} \geq k, & \textit{if } 0 < p \leq p_1, \\ < k, & \textit{if } p_1 < p < p_2, \\ \geq k, & \textit{if } p_2 \leq p < 1. \end{cases}$$

Assumption 5. $U(\theta) \leq U(0)$ *for* $\theta < 0$.

We defer the discussions of these two assumptions to the end of this section.

By assuming a binomial distribution for this skewed security, they compute the equilibrium numerically.

[25] It is a simple exercise to show that $k = k_0$ reduces to equation (19), p. 12 in Barberis and Huang (2008) as a special case.

Theorem 4. *If $x_0 \neq 0$ and $\lambda_1^- > 0$, $\lambda_2^+ > 0$, then* (25) *has a unique root $\theta^* > 0$ which is an optimal solution to* (P)*.*

Corollary 8. *Under the same assumptions of Theorem 4, an optimal solution to* (P) *with parameters x_0, k, and T is given by*

$$\theta^*(x_0, k, T) = \begin{cases} \frac{-x_0}{v_+^*(k,T)}, & \textit{if } x_0 \leq 0, \\ \frac{-x_0}{v_-^*(k,T)}, & \textit{if } x_0 > 0, \end{cases} \tag{28}$$

where $v_+^*(k, T)$ and $v_-^*(k, T)$ are the unique roots of

$$h(v) := \int_v^{+\infty} tw'_+(1 - F_T(t))f_T(t)dt + k \int_{-\infty}^{v} tw'_-(F_T(t))f_T(t)dt \tag{29}$$

on $(0, +\infty)$ and $(-\infty, 0)$ respectively.

Based on (28) we have the following monotonicity result of the optimal risky exposure.

Theorem 5. *Under the same assumptions of Theorem 4, the stock allocation $\theta^*(x_0, k, T)$ strictly increases in $|x_0|$ and strictly decreases in k.*

This result shows that the higher the large-loss aversion is, the less the investment in stocks is. Hence, it confirms decisively, via an analytical argument, that investors tend to allocate relatively less to stocks due to the aversion to significant losses. The monotonicity (indeed the proportionality; see (28)) of the equity allocation with respect to $|x_0|$ is equally intriguing[26], as it reveals the important role the reference point plays in asset allocation. When $x_0 > 0$, i.e., the reference point is smaller than the risk-free return[27], the larger the gap (between the reference point and risk-free return) is, the greater the investment in stocks is. This can be explained as follows. A greater x_0 yields more room before a loss would be triggered; hence, the investor feels safer and hence becomes more aggressive. When $x_0 < 0$, i.e., the reference point is larger than the risk-free return[28], again the larger the gap is, the more weight is given to stocks. The economic intuition is that the investor in this case starts off in a loss position compared to the higher reference wealth set for the final date; hence, his behavior is risk seeking, trying to get out of the hole as soon as possible.

Next, we investigate the comparative statics in terms of the *curvature* of the probability weighting.

[26] Actually this monotonicity holds for more general power utilities: $u_+(x) = x^\alpha, u_-(x) = kx^\alpha$. In this case it is easy to see that $U(\theta; x_0) = x_0^\alpha U(\theta; 1)$, where $U(\cdot; x_0)$ denotes the CPT value function given x_0. However, for this general case we have yet to obtain a result as complete as Corollary 8.

[27] This corresponds to the type of investors who would feel the pains of losses only when the investment returns fall substantially below those of the fixed income. This may well describe the investment behaviors of those with higher tolerance for losses, say, some very wealthy people.

[28] Most equity and hedge fund managers should belong to this category.

Theorem 6. *Under the same assumptions of Theorem 4, define $z_+ := 1 - F_T(v_+^*(k,T))$ and $z_- := 1 - F_T(v_-^*(k,T))$.*

(i) *Let $\bar{w}_+(\cdot)$ be a different probability weighting function on gains while keeping the weighting on losses unchanged, and let $\bar{\theta}(x_0, k, T)$ be the corresponding optimal solution to (P). If*

$$\bar{w}_+'(z) \geq w_+'(z), \quad \forall z \in (0, z_+),$$

then $\bar{\theta}(x_0, k, T) \geq \theta^*(x_0, k, T)$, $\forall x_0 < 0$.

(ii) *Let $\tilde{w}_-(\cdot)$ be a different weighting function on losses while keeping the weighting on gains unchanged, and let $\tilde{\theta}(x_0, k, T)$ be the corresponding optimal solution to (P). If*

$$\tilde{w}_-'(z) \leq w_-'(z), \quad \forall z \in (0, 1 - z_-),$$

then $\tilde{\theta}(x_0, k, T) \geq \theta^*(x_0, k, T)$, $\forall x_0 > 0$.

Theorem 6-(i) shows that if the agent starts in the loss domain, then the greater $w_+'(z)$ around $z = 0$ is, the more risky the allocation becomes. Recall (see (3)) that the derivatives of the probability weighting on gains around 0 are the weights to significant gains when evaluating risky prospects, thereby indicating the level of exaggerating small probabilities associated with huge gains. In other words, $w_+'(\cdot)$ near 0 quantifies the agent's hope of very good outcomes. Likewise, Theorem 6-(ii) stipulates that in the gain domain, the smaller $w_-'(\cdot)$ around 0 is, the more risky is the exposure. The derivatives of the probability weighting on losses near 0 capture the agent's fear of very bad scenarios. Note that, since the utility function is linear, the behavior of risk-seeking on losses and risk-aversion on gains is solely reflected by the probability weighting. This theorem is consistent with the intuition that a more hopeful or less fearful agent will invest more heavily in risky stocks.

The remainder of this subsection examines the interpretations and validity of Assumptions 4 and 5.

First of all, Assumption 4 ensures, via (26), that $U''(\theta)$ changes its sign at most twice. In other words, although $U(\cdot)$ is in general not concave, the assumption prohibits it from switching too often between being convex and being concave. Moreover, Assumption 4 accommodates the case when $p_1 = 0$ and/or $p_2 = 1$; so φ satisfies the assumption if it is strictly bounded by k, in which case $U(\cdot)$ is concave on $\mathbb{R}_+$. In particular, the case without weighting, i.e., $w_\pm(p) = p$, and the symmetric case as in the so-called rank-dependent models (see Tversky and Kahneman 1992, p. 302), i.e., $1 - w_+(1-p) = w_-(p)$, do satisfy Assumption 4 for any LLAD value $k > 1$ (recall that the Kahneman–Tversky LLAD value is $k = 2.25$).

Next, let us check the Kahneman–Tversky type weighting with a same power parameter:

$$w_+(p) = w_-(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}, \qquad 1/2 \le \gamma < 1. \tag{30}$$

Proposition 9. *Suppose $w_\pm(p)$ are in the form* (30). *Then*

$$\varphi(p) \le \lambda_0(\gamma) := \left(\gamma - \frac{1-\gamma}{[1/p_1(\gamma) - 1]^\gamma}\right)^{-1} \quad \forall p \in (0,1),$$

where $p_1(\gamma)$ is the unique root of the following function on $(0, 1/2]$:

$$K(p) := [p^\gamma + (1-p)^\gamma] - [p^{\gamma-1} - (1-p)^{\gamma-1}].$$

Now, if we take $\gamma = 0.61$, then $\lambda_0(0.61) \approx 2.0345$; and if we take $\gamma = 0.69$, then $\lambda_0(0.69) \approx 1.5871$. Hence, it follows from Proposition 9 that Assumption 4 is valid for the Kahneman–Tversky LLAD value $k = 2.25 > 2.0345$ if we take the same power parameter in the weighting functions.

We have shown analytically that, for the Kahneman–Tversky weighting functions with the same parameter (either $\gamma = 0.61$ or $\gamma = 0.69$), $\varphi(p)$ is indeed strictly bounded by the Kahneman–Tversky LLAD value $k = 2.25$, and hence, as pointed out earlier, the corresponding value function $U(\cdot)$ is uniformly concave. Now, if the weighting functions for gains and losses have slightly different power parameters as in (8) with $\gamma = 0.61$ and $\delta = 0.69$, we no longer have an analytical bound. However, one can depict the corresponding function, $\varphi(p)$, as Figure 1. By inspection we see that there is no uniform boundedness in this case ($\varphi(p)$ goes to infinity near 0 and 1); yet Assumption 4 is indeed satisfied with $k = 2.25$.

It is also worth mentioning that Assumption 4 holds true with the Tversky–Fox weighting (9) and the Prelec weighting (10), if we take the specific parameter values estimated by Abdellaoui (2000) and Wu and Gonzalez (1996).

Next, let us turn to Assumption 5, which essentially excludes short-selling from a good investment strategy.

Proposition 10. *If $x_0 \ne 0$, $k > k_0$ and $\lambda_1^+ > 0$, $\lambda_2^- > 0$, then $U(\theta) \le U(0)$ $\forall \theta < 0$ if either*

$$\psi(p) := w_+'(p)/w_-'(1-p) \le k \;\; \forall p \in (0,1), \tag{31}$$

or

$$k \ge \max\left\{ \frac{\int_{-\infty}^0 w_+(F_T(t))dt}{\int_0^{+\infty} w_-(1-F_T(t))dt}, \zeta_1, \zeta_2 \right\}, \tag{32}$$

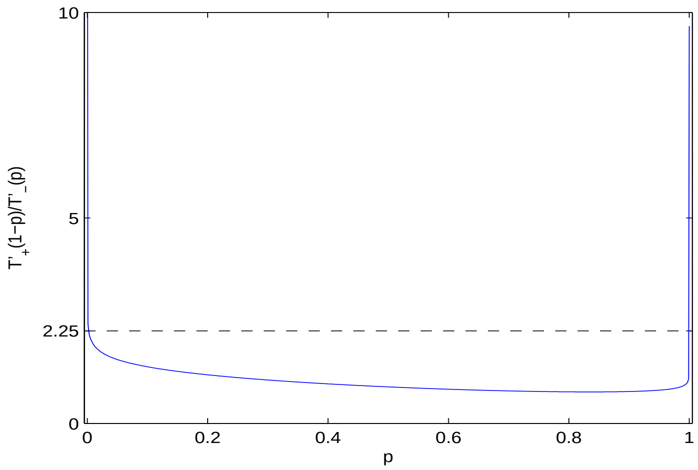

Figure 1: $\varphi(p)$ for Kahneman–Tversky's distortions with $\gamma = 0.61$ and $\delta = 0.69$

where $\zeta_1 = \sup_{0<p<1} \frac{1-w_+(p)}{w_-(1-p)}$ *and* $\zeta_2 = \sup_{0<p<1} \frac{w_+(p)}{1-w_-(1-p)}$, *both assumed to be finite*[29].

This result shows that prohibition of short-selling is endogenous when the LLAD exceeds certain critical levels, as determined by (32). This certainly makes perfect sense as short-selling involves tremendous risk and thus is not preferred by a sufficiently large-loss averse investor.

If we take the Kahneman–Tversky weighting functions with a same power parameter $\gamma = \delta = 0.61$ or 0.69, then the condition (31) is satisfied for any $k > 2.0345$ as shown earlier. If the two parameters are slightly different as in Tversky and Kahneman (1992), i.e., $\gamma = 0.61, \ \delta = 0.69$, then both conditions of Proposition 10 fail. However we can modify the conditions above to ones that are more complicated and heavily dependent on the probability distribution of the stock return. Since they are unduly technical, we choose not to pursue any further investigations along that line.

6 Conclusions

This paper formulates and develops a CPT portfolio selection model in a single period setting, where three key elements of CPT, namely the reference point, the S-shaped utilities, and the probability weighting, are all taken into consideration. We have introduced a new measure of loss aversion called LLAD, which is relevant only to large (instead of small) gains and losses. This measure is markedly different from the ones commonly employed in the literature; yet we have shown that it plays a prominent role in portfolio choice. The model is ill-posed if the investor is not sufficiently loss averse in the sense that LLAD is be-

[29]Here we set $\psi(p) = +\infty$ if $w'_-(1-p) = 0$.

low a certain critical level. For a well-posed model, deriving its solution analytically poses a great challenge due to the non-convexity nature of the underlying optimization problem. We have solved two special but important cases completely with explicit solutions obtained by exploiting carefully the special structures of the corresponding CPT value functions.

There are certainly many questions yet to be answered. An immediate challenge is to analyze the model beyond the two special cases. Another fundamental challenge is building an equilibrium model or capital asset pricing model built upon our CPT portfolio selection theory for an economy where some agents have CPT preferences while the rest are rational.

Appendix: Proofs

Proof of Proposition 1. Fix an arbitrary $\theta > 0$. Since $u_+(\cdot)$ is concave, we have $u_+(\theta t + x_0) \leq C(1 + \theta|t|)\ \forall t \in \mathbb{R}$ for some constant, C. By Assumption 3, the Lebesgue integrand of the first integral in (5) is of an order of at most $t^{-1-\epsilon_0}$ when t is large and hence integrable. Similarly, all the other integrals defining $U(\theta)$ have finite values for any $\theta \in \mathbb{R}$.

Next, we show the continuity of $U(\theta)$ at $\theta = 0$. Consider only $x_0 < 0$ and $\theta \downarrow 0$, the other cases being similar. The first integral of $U(\theta)$ in (5) goes to 0 when $\theta \downarrow 0$ by monotone convergence theorem. Denoting the second integral by $A(\theta)$ (excluding the negative sign), we have

$$
\begin{aligned}
A(\theta) = &\int_{-\infty}^{0} u_-(-\theta t - x_0) d[w_-\left(F_T(t)\right)] \\
&+ \int_{0}^{-x_0/\theta} u_-(-\theta t - x_0) d[w_-\left(F_T(t)\right)] \\
:= &A_1(\theta) + A_2(\theta).
\end{aligned}
$$

Clearly, $\lim_{\theta\downarrow 0} A_1(\theta) = u_-(-x_0)w_-[F_T(0)]$, whereas the dominated convergence theorem yields $\lim_{\theta\downarrow 0} A_2(\theta) = u_-(-x_0)(1 - w_-[F_T(0)])$. Thus we have $\lim_{\theta\downarrow 0} U(\theta) = U(0)$.

Finally, the continuity at any other point can be proved similarly. $\square$

Proof of Proposition 2. Suppose $R(T)$ follows a lognormal distribution, say, $\ln R(T) \sim N(\mu_T, \sigma_T)$. Then

$$
\begin{aligned}
F_T(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln[r(T)+x]-\mu_T}{\sigma_T}} e^{-t^2/2} dt, \\
f_T(x) &= \frac{1}{\sqrt{2\pi}\sigma_T} \frac{1}{r(T)+x} e^{-\frac{1}{2}\left[\frac{\ln(r(T)+x)-\mu_T}{\sigma_T}\right]^2}.
\end{aligned}
$$

Using integration by parts, we can easily show that

$$(x^{-1}-x^{-3})\exp(-x^2/2)\leq\sqrt{2\pi}[1-N(x)]\leq x^{-1}\exp(-x^2/2)\ \ \forall x>0,$$

where $N(x)$ is the CDF of the standard normal distribution. Thus, we have

$$w'_{\pm}(1-F_T(x))f_T(x)=O\left(e^{-\frac{1-\alpha}{2}\left[\frac{\ln(r(T)+x)-\mu_T}{\sigma_T}\right]^2}\right)$$

for all large enough $x>0$. Notice that $e^{-\left[\frac{\ln[r(T)+x]-\mu_T}{\sigma_T}\right]^2}=O(x^{-p})$ for all $p>0$ and hence the conclusion holds. The case of normal distribution can be verified in the same way. $\square$

Proof of Theorem 1. (i) First, we show that $\lim_{x\to+\infty}\frac{u_-(tx)}{u_+(x)}=+\infty$ for any $t>0$. Indeed, by monotonicity it holds for $t\geq 1$. For $0<t<1$, the conclusion follows from $u_-(tx)\geq tu_-(x)$ which is due to the concavity of $u_-(x)$ and the fact that $u_-(0)=0$. On the other hand, observe that for fixed $y_0\geq 0$,

$$0\leftarrow\frac{y_0u'_+(y+y_0)}{u_+(y)}\leq\frac{u_+(y+y_0)-u_+(y)}{u_+(y)}\leq\frac{y_0u'_+(y)}{u_+(y)}\to 0$$

as $y\to+\infty$. This is because either $u_+(y)\to+\infty$ or $u'_+(y)\to 0$, as $y\to+\infty$ (recall that $u_+(\cdot)$ is non-decreasing and concave). Thus $\lim_{y\to+\infty}u_+(y+y_0)/u_+(y)=1$. As a consequence, we have

$$\lim_{x\to+\infty}\frac{u_-(tx+x_1)}{u_+(x+x_2)}=\lim_{x\to+\infty}\frac{u_-(tx+x_1)}{u_+(x+x_1/t)}\lim_{x\to+\infty}\frac{u_+(x+x_1/t)}{u_+(x+x_2)}=+\infty\tag{33}$$

for any fixed x_1, x_2, and $t>0$.

Now, for $\theta>0$ it follows from (5) that

$$\begin{aligned}U(\theta)=&\int_{-x_0/\theta}^{+\infty}u_+(\theta t+x_0)d[-w_+\left(1-F_T(t)\right)]\\&-\int_{-\infty}^{-x_0/\theta}u_-(-\theta t-x_0)d[w_-\left(F_T(t)\right)]:=I_1-I_2\end{aligned}$$

Due to the concavity of $u_+(\cdot)$, for fixed $t_0>0$ we have

$$\begin{aligned}I_1\leq\ &u_+(\theta t_0+x_0)\int_{-x_0/\theta}^{+\infty}d[-w_+\left(1-F_T(t)\right)]\\&+u'_+(\theta t_0+x_0)\theta\int_{t_0}^{+\infty}(t-t_0)d[-w_+\left(1-F_T(t)\right)]\end{aligned}$$

Thus,

$$U(\theta) \leq u_+(\theta t_0 + x_0)\Big[w_+[1 - F_T(-x_0/\theta)]$$
$$+\frac{u'_+(\theta t_0 + x_0)\theta}{u_+(\theta t_0 + x_0)} \int_{t_0}^{+\infty} (t - t_0) d[-w_+ (1 - F_T(t))]$$
$$- \int_{-\infty}^{-x_0/\theta} h(\theta, t) d[w_- (F_T(t))]\Big],$$

where

$$h(\theta, t) := \frac{u_-(-\theta t - x_0)}{u_+(\theta t_0 + x_0)}.$$

Due to (1) we can find $b < a < 0$ such that $w_-[F_T(a)] - w_-[F_T(b)] > 0$. Let θ be large enough such that $-|x_0|/\theta > a$. Then for any fixed $t \in [b, a]$, it follows from (33) that $\lim_{\theta \to +\infty} h(\theta, t) = +\infty$. Consequently,

$$\liminf_{\theta \to +\infty} \int_{-\infty}^{-x_0/\theta} h(\theta, t) d[w_- (F_T(t))]$$
$$\geq \liminf_{\theta \to +\infty} \int_{b}^{a} h(\theta, t) d[w_- (F_T(t))] \geq +\infty,$$

where the last inequality is due to Fatou's lemma. On the other hand, the concavity of $u_+(\cdot)$ implies that $\frac{u'_+(\theta t_0 + x_0)\theta}{u_+(\theta t_0 + x_0)}$ is bounded in θ. Thus, we have $\lim_{\theta \to +\infty} U(\theta) = -\infty$. Similarly, $\lim_{\theta \to -\infty} U(\theta) = -\infty$. So Problem (P) is well-posed[30]. The conclusion of (ii) can be obtained similarly. □

Proof of Lemma 3. We prove only for a_1 and the first limit, the others being similar. Because $u_+(\cdot)$ is increasing and concave, we have $\min\{t, 1\} \leq g_+(t) \leq \max\{1, t\}$. Thus, by Assumption 3, a_1 is well-defined and strictly positive. Now for any fixed x_0, $\lim_{\theta \to +\infty} u_+(\theta t + x_0)/u_+(\theta) = g_+(t)$ $\forall t \geq 0$. Again from the concavity and monotonicity of $u_+(\cdot)$, $u_+(\theta t + x_0)/u_+(\theta) \leq \max\{1, t + x_0/\theta\}$. By Assumption 3 and the dominated convergence theorem, the limit exists and equals a_1.

Finally, if $0 < k < \infty$, we have

$$g_-(t) = \lim_{x \to +\infty} \frac{u_-(tx)}{u_-(x)} = \lim_{x \to +\infty} \left(\frac{u_-(tx)}{u_+(tx)} \cdot \frac{u_+(tx)}{u_+(x)} \cdot \frac{u_+(x)}{u_-(x)} \right) = g_+(t).$$

□

Proof of Theorem 2. First notice by Lemma 3-(ii), $g_+(t) \equiv g_-(t) \equiv g(t)$. For $\theta > 0$, we

[30]In this case $U(\cdot)$ is called *coercive*. Its (finite) maximum is achieved in a certain bounded interval due to the continuity of $U(\cdot)$.

have

$$U(\theta) = u_+(\theta)\Big(\int_{-x_0/\theta}^{+\infty}[u_+(\theta t + x_0)/u_+(\theta)]d[-w_+\,(1 - F_T(t))]$$
$$-[u_-(\theta)/u_+(\theta)]\int_{-\infty}^{-x_0/\theta}[u_-(-\theta t - x_0)/u_-(\theta)]d[w_-\,(F_T(t))]\Big).$$

Then, it follows from Lemma 3 that $\lim_{\theta\to+\infty} U(\theta) = -\infty$ if $k > a_1/a_2$, and $\lim_{\theta\to+\infty} U(\theta) = +\infty$ if $k < a_1/a_2$. The situation when $\theta \to -\infty$ is completely symmetric. Hence, the conclusions (i) and (ii) are evident.

□

Proof of Corollary 4. It is a direct consequence of Theorem 2. □

Proof of Proposition 5. If the change of differential and integral is valid, then we have

$$\begin{aligned} U'(\theta) = &\int_{-x_0/\theta}^{+\infty} t u'_+(\theta t + x_0) d[-w_+\,(1 - F_T(t))] \\ &+ \int_{-\infty}^{-x_0/\theta} t u'_-(-\theta t - x_0) d[w_-(F_T(t))], \quad \theta > 0, \end{aligned} \tag{34}$$

and

$$\begin{aligned} U'(\theta) = &\int_{-\infty}^{-x_0/\theta} t u'_+(\theta t + x_0) d[w_+\,(F_T(t))] \\ &+ \int_{-x_0/\theta}^{+\infty} t u'_-(-\theta t - x_0) d[-w_-\,(1 - F_T(t))], \quad \theta < 0. \end{aligned} \tag{35}$$

Now, we verify the validity of this change of order. We show this for the case in which $x_0 < 0$ and $\theta = 0$. Let $\theta > 0$. Then,

$$\begin{aligned} \frac{U(\theta) - U(0)}{\theta} = &\int_{-x_0/\theta}^{+\infty} \frac{u_+(\theta t + x_0)}{\theta} d[-w_+\,(1 - F_T(t))] \\ &+ (1/\theta)\int_{-x_0/\theta}^{+\infty} u_-(-x_0) d[w_-\,(F_T(t))] \\ &+ (1/\theta)\int_{-\infty}^{0}[u_-(-x_0) - u_-(-\theta t - x_0)]d[w_-\,(F_T(t))] \\ &+ (1/\theta)\int_{0}^{-x_0/\theta}[u_-(-x_0) - u_-(-\theta t - x_0)]d[w_-\,(F_T(t))] \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Since $u_+(\cdot)$ is concave, we have $u_+(\theta t + x_0) \le C(\theta t + 1)$ for some $C > 0$. Hence, it follows from Assumption 3 that $I_1, I_2 \to 0$ as $\theta \downarrow 0$. In addition, the dominated convergence theorem yields

$$\lim_{\theta\downarrow 0} I_3 = u'_-(-x_0)\int_{-\infty}^{0} td[w_-(F_T(t))].$$

Next, by the concavity of $u_-(\cdot)$, we have

$$\begin{aligned} I_4 &\geq \int_0^{-x_0/\theta} u'_-(-x_0)td[w_-(F_T(t))] \\ &\to u'_-(-x_0)\int_0^{+\infty} td[w_-(F_T(t))] \text{ as } \theta\downarrow 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} I_4 &\leq \int_0^{-x_0/\theta} u'_-(-\theta t - x_0)tw'_-[F_T(t)]f_T(t)ds \\ &= -\int_0^{-x_0}[(s+x_0)/\theta^2]u'_-(s)w'_-[F_T((-s-x_0)/\theta)]f_T((-s-x_0)/\theta)ds \\ &= -\int_0^{-x_0-\epsilon}[(s+x_0)/\theta^2]u'_-(s)w'_-[F_T((-s-x_0)/\theta)]f_T((-s-x_0)/\theta)ds \\ &\quad -\int_{-x_0-\epsilon}^{-x_0}[(s+x_0)/\theta^2]u'_-(s)w'_-[F_T((-s-x_0)/\theta)]f_T((-s-x_0)/\theta)ds \\ &:= I_5 + I_6, \end{aligned}$$

where $0<\epsilon<-x_0$ is a positive number. Fixing such an ϵ, we have

$$\begin{aligned} |I_5| &= \int_0^{-x_0-\epsilon}[(-s-x_0)/\theta^2]u'_-(s)O([(-s-x_0)/\theta)]^{-2-\epsilon_0})ds \\ &= O(\theta^{\epsilon_0}) \to 0 \text{ as } \theta\downarrow 0 \end{aligned}$$

by Assumption 3. For I_6 we have

$$\begin{aligned} I_6 &\leq -u'_-(-x_0-\epsilon)\int_{-x_0-\epsilon}^{-x_0}[(s+x_0)/\theta^2]w'_-[F_T((-s-x_0)/\theta)]f_T((-s-x_0)/\theta)ds \\ &= u'_-(-x_0-\epsilon)\int_0^{\epsilon/\theta} tw'_-[F_T(t)]f_T(t)dt \\ &\to u'_-(-x_0-\epsilon)\int_0^{+\infty} td[w_-(F_T(t))] \text{ as } \theta\downarrow 0. \end{aligned}$$

Now letting $\epsilon \downarrow 0$, we conclude that for $x_0 < 0$,

$$\begin{aligned} U'(0+) :&= \lim_{\theta \downarrow 0} \frac{U(\theta) - U(0)}{\theta} \\ &= u'_-(-x_0) \int_{-\infty}^{+\infty} t d[w_-\left(F_T(t)\right)] = u'_-(-x_0)\lambda_1^-. \end{aligned} \tag{36}$$

Similarly, we can derive the other identities in (19). Similar analysis can be employed to show that $U(\theta)$ is continuously differentiable at $\theta \neq 0$ and that

$$\lim_{\theta \downarrow 0} U'(\theta) = U'(0+), \qquad \lim_{\theta \uparrow 0} U'(\theta) = U'(0-),$$

so long as $x_0 \neq 0$. □

Proof of Proposition 6. Consider $x_0 < 0$ and $\theta > 0$:

$$\begin{aligned} U'(\theta) = & \int_{-x_0/\theta}^{+\infty} t u'_+(\theta t + x_0) w'_+(1 - F_T(t)) f_T(t) dt \\ & + \int_{-\infty}^{0} t u'_-(-\theta t - x_0) w'_-(F_T(t)) f_T(t) dt \\ & - \int_{0}^{-x_0} [(s + x_0)/\theta^2] u'_-(s) w'_-\left(F_T((-s - x_0)/\theta)\right) f_T((-s - x_0)/\theta) dt \\ := & I_1 + I_2 + I_3. \end{aligned}$$

By the monotone convergence theorem $I_1, I_2 \to 0$ as $\theta \to +\infty$. By the dominated convergence theorem $I_3 \to 0$ as $\theta \to +\infty$. Thus, we have $\lim_{\theta \to +\infty} U'(\theta) = 0$. The other cases can be proved in exactly the same way. □

Proof of Corollary 7. The non-concavity is evident by combining $\lim_{|\theta| \to +\infty} U(\theta) = -\infty$ and $\lim_{|\theta| \to +\infty} U'(\theta) = 0$. If $\lambda_1^-, \lambda_2^+ > 0$, then $U'(0+) > 0$. Hence, $U(\theta)$ cannot be convex on $\mathbb{R}_+$. □

Proof of Theorem 3. Case (i) is clear by Corollary 4. Next, we calculate the following derivatives:

$$\begin{aligned} U'(\theta) &= \theta^{\alpha-1}[\alpha a_1 - k\theta^{\beta-\alpha}\beta a_2], \quad \theta > 0, \\ U'(\theta) &= -(-\theta)^{\alpha-1}[\alpha b_1 - k(-\theta)^{\beta-\alpha}\beta b_2], \quad \theta < 0. \end{aligned}$$

Hence, in the case of (ii), we have $U'(\theta) < 0$ for $\theta > 0$ and $U'(\theta) > 0$ for $\theta < 0$. Thus, $\theta^* = 0$ is the unique optimal solution. (iii) and (iv) can be proved similarly.

Now, we turn to (v). Clearly,

$$\theta_1 = \left[\frac{1}{k}\frac{\alpha}{\beta}\frac{a_1}{a_2}\right]^{\frac{1}{\beta-\alpha}}, \quad \theta_2 = -\left[\frac{1}{k}\frac{\alpha}{\beta}\frac{b_1}{b_2}\right]^{\frac{1}{\beta-\alpha}}$$

are the unique roots of $U'(\theta) = 0$ on $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively. Notice that $U'(\theta) > 0$ at $\theta > 0$ near 0, and $U'(\theta) < 0$ at $\theta < 0$ near 0. Thus, θ_1 and θ_2 are the only two local maximums of $U(\cdot)$. To find which one is better, we need only to compare the corresponding CPT values. Straightforward calculation yields

$$U(\theta_1) = k^{-\frac{\alpha}{\beta-\alpha}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}} - \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}}\right]\left[\frac{a_1^\beta}{a_2^\alpha}\right]^{\frac{1}{\beta-\alpha}},$$
$$U(\theta_1) = k^{-\frac{\alpha}{\beta-\alpha}}\left[\left(\frac{\alpha}{\beta}\right)^{\frac{\alpha}{\beta-\alpha}} - \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\beta-\alpha}}\right]\left[\frac{b_1^\beta}{b_2^\alpha}\right]^{\frac{1}{\beta-\alpha}}.$$

The conclusion follows immediately. □

Proof of Theorem 4. Notice that $p(\theta) := F_T(-x_0/\theta)$ is monotone on $\theta \in \mathbb{R}_+$ and that $F_T(\cdot)$ has no atom due to assumption 3. Thus, under Assumption 4 $U''(\theta)$ is nonnegative on $(0, \theta_1)$, negative on (θ_1, θ_2), and nonnegative again on $(\theta_2, +\infty)$ for some $0 \le \theta_1 < \theta_2 \le +\infty$. Since $U'(0+) > 0$ due to $\lambda_1^- > 0$, $\lambda_2^+ > 0$, $U'(\theta) > 0$ on $[0, \theta_1]$. Also $U'(\theta) < 0$ on $[\theta_2, +\infty)$, otherwise $U(\theta)$ will increase from some point $\theta_3 > \theta_2$ due to $U''(\theta) \ge 0$ when $\theta > \theta_2$, and consequently this contradicts the fact that $U(+\infty) = -\infty$. Therefore, there must exist a root of (25), θ^*, that must lie on (θ_1, θ_2). Since $U''(\cdot)$ is strictly negative on this interval, such a root is unique. □

Proof of Corollary 8. The conclusion is clear via a change of variables $v = -x_0/\theta$. □

Proof of Theorem 5. Consider the case in which $x_0 < 0$. Since $v_+^*(k, T) > 0$, it follows from Corollary 8 that $\theta^*(x_0, k, T)$ strictly increases in $-x_0$. To see the monotonicity in k, let $k_2 > k_1$. Rewrite (29) as $h(v, k) \equiv h(v) = h_1(v) + kh_2(v)$. Since $v_+^*(k_1, T) > 0$ solves $h(v, k_1) = 0$ and $h_1(v) > 0\ \forall v > 0$, we must have $h_2(v_+^*(k_1, T)) < 0$. Thus, $h(v_+^*(k_1, T), k_2) = h(v_+^*(k_1, T), k_1) + (k_2 - k_1)h_2(v_+^*(k_1, T)) < 0$. However, $h(v, k)$ is strictly negative on $v \in (0, v_+^*(k, T))$ and strictly positive on $v \in (v_+^*(k, T), +\infty)$; hence, $v_+^*(k_2, T) > v_+^*(k_1, T)$, which means $\theta(x_0, k, T)$ strictly decreases in k.

For the case in which $x_0 > 0$, the argument is similar. □

Proof of Theorem 6. We only prove (i), the other one being similar. Let $\bar{h}(\cdot)$ be defined by (29), where $w_+(\cdot)$ is replaced by $\bar{w}_+(\cdot)$, and let $\bar{v}_+(k, T)$ be the unique root of $\bar{h}(\cdot)$ on $(0, +\infty)$. It follows from $\bar{w}'_+(z) \ge w'_+(z)\ \forall z \in (0, z_+)$ that $\bar{h}(v_+^*(k, T)) \ge h(v_+^*(k, T)) =$

0. However, the proof of Theorem 4 and the change of variable $v = -x_0/\theta$ yield $\bar{h}(\cdot) > 0$ on $(\bar{v}_+(k,T), \infty)$ and $\bar{h}(\cdot) < 0$ on $(0, \bar{v}_+(k,T))$. As a result, $\bar{v}_+(k,T) \le v_+^*(k,T)$, and consequently $\bar{\theta}(x_0,k,T) \ge \theta^*(x_0,k,T)$ for $x_0 < 0$. $\square$

Proof of Proposition 9. Write $w(p) := w_+(p) = w_-(p)$. We have

$$w'(p) = a(p)^{-1/\gamma-1}[\gamma p^{\gamma-1}a(p) - p^\gamma(p^{\gamma-1} - (1-p)^{\gamma-1})], \quad 0 < p < 1,$$

where $a(p) := p^\gamma + (1-p)^\gamma$. We now introduce a parameter $\lambda > 1/\gamma$ (> 1), along with a function

$$\begin{aligned}\ell(p) :&= [w_+'(1-p) - \lambda w_-'(p)]a(p)^{1/\gamma+1} \\ &= \gamma[(1-p)^{\gamma-1} - \lambda p^{\gamma-1}][p^\gamma + (1-p)^\gamma] - [(1-p)^\gamma + \lambda p^\gamma][(1-p)^{\gamma-1} - p^{\gamma-1}].\end{aligned}$$

The rest of the proof is devoted to showing that $\ell(p) < 0$, $0 < p < 1$, for some carefully chosen λ. Indeed, when $1/2 \le p < 1$, we have $(1-p)^{\gamma-1} \ge p^{\gamma-1}$. Noting that $\lambda > 1$, we have

$$\ell(p) \le [p^\gamma + (1-p)^\gamma][(1-\lambda\gamma)p^{\gamma-1} + (\gamma-1)(1-p)^{\gamma-1}] < 0.$$

When $0 < p < 1/2$, we rewrite $\ell(p)$:

$$\begin{aligned}\ell(p) = &-(\lambda\gamma - 1)[p^{\gamma-1} - (1-p)^{\gamma-1}][p^\gamma + (1-p)^\gamma] \\ &- (\lambda-1)\gamma(1-p)^{\gamma-1}[p^\gamma + (1-p)^\gamma] \\ &+ (\lambda-1)p^\gamma[p^{\gamma-1} - (1-p)^{\gamma-1}].\end{aligned}$$

Since $0 < p < 1/2$, we have $p^{\gamma-1} > (1-p)^{\gamma-1}$. Thus, $\ell(p) < 0$, if

$$(\lambda\gamma - 1)[p^\gamma + (1-p)^\gamma] \ge (\lambda-1)p^\gamma,$$

which is equivalent to

$$p \le p_0 := \min\left\{\frac{1}{2}, \frac{1}{1 + \left[\frac{\lambda(1-\gamma)}{\lambda\gamma-1}\right]^{1/\gamma}}\right\}.$$

If $p_0 = 1/2$, then we are done. So consider the case in which $p_0 < p < 1/2$. In this case, from

$$\ell(p) < -(\lambda-1)\gamma(1-p)^{\gamma-1}[p^\gamma + (1-p)^\gamma] + (\lambda-1)p^\gamma[p^{\gamma-1} - (1-p)^{\gamma-1}]$$

it follows that $\ell(p) < 0$ provided both inequalities below hold:

$$(\lambda - 1)\gamma(1-p)^{\gamma-1} \geq (\lambda - 1)p^{\gamma}, \;\; [p^{\gamma} + (1-p)^{\gamma}] \geq [p^{\gamma-1} - (1-p)^{\gamma-1}]. \tag{37}$$

The first inequality of (37) is equivalent to

$$J(p) := \ln \gamma + (\gamma - 1)\ln(1-p) - \gamma \ln p \geq 0.$$

However, the above is seen from the facts that $J'(p) < 0$ on $p \in (0, 1/2)$ (since $p < \gamma$) and that $J(1/2) \geq 0$ (due to $\gamma \geq 1/2$). So it suffices to prove the second inequality of (37), which is equivalent to $K(p) \geq 0$ for $p_0 < p < 1/2$. It is easy to verify that $K(p)$ is strictly increasing on $p \in (0, 1/2]$ and $K(1/2) > 0, K(0+) = -\infty$. Thus $K(\cdot)$ has a unique root on $(0, 1/2)$, denoted by $p_1(\gamma)$. Now, by a direct calculation it can be shown that $p_0 \geq p_1(\gamma)$ as long as

$$\lambda \geq \lambda_0(\gamma) = \left(\gamma - \frac{1-\gamma}{[1/p_1(\gamma) - 1]^{\gamma}}\right)^{-1} \;\; (> 1/\gamma). \tag{38}$$

Consequently, when a λ is chosen satisfying the above, it must hold that $1/2 > p > p_0 \geq p_1(\gamma)$ and hence $K(p) \geq 0$ or the second inequality of (37) holds. □

Proof of Proposition 10. For $\theta < 0$, it can be calculated that

$$U''(\theta) = -\frac{x_0^2}{\theta^3} f_T(-\frac{x_0}{\theta}) \left[w_+' \left(F_T(-\frac{x_0}{\theta}) \right) - k w_-' \left(1 - F_T(-\frac{x_0}{\theta}) \right) \right]. \tag{39}$$

Thus, under condition (31), we have $U''(\theta) \leq 0$. Since $\lambda_1^+ > 0$, $\lambda_2^- > 0$, we have $U'(0-) > 0$, and therefore, $U'(\theta) > 0, \forall \theta < 0$. This leads to $U(\theta) \leq U(0)$ on $\mathbb{R}_-$.

Next, suppose (32) is valid, in which case we focus on $U(\cdot)$ itself. Denoting $v = -x_0/\theta$, we have

$$\begin{aligned} g(v) &:= \frac{1}{\theta}[U(\theta) - U(0)] \\ &= -\int_{-\infty}^{v} w_+(F_T(t))dt + k\int_{v}^{+\infty} w_-(1 - F_T(t))dt + \begin{cases} v, & v \geq 0 \\ kv, & v < 0 \end{cases}. \end{aligned}$$

It suffices to show that

$$g(v) \geq 0, \quad \forall v \in \mathbb{R}. \tag{40}$$

For $v \geq 0$, (40) reduces to

$$k \geq \frac{\int_{-\infty}^{v} w_{+}(F_T(t))dt - v}{\int_{v}^{+\infty} w_{-}(1 - F_T(t))dt} =: \ell_1(v). \tag{41}$$

To prove the above we are to find $\sup_{v \geq 0} \ell_1(v)$. Note that

$$h(v) := \int_{-\infty}^{v} w_{+}(F_T(t))dt - v = \int_{-\infty}^{0} w_{+}(F_T(t))dt - \int_{0}^{v} \left(1 - w_{+}(F_T(t))\right) dt;$$

hence, $h(+\infty) = -\lambda_1^{+} < 0$, and $\ell_1(v) < 0$ for all sufficiently large $v \geq 0$. However, $\ell_1(0) > 0$; so the maximum point v^* of $\ell_2(\cdot)$ on $\mathbb{R}_+$ exists, which satisfies either $v^* = 0$ or the following first-order condition:

$$\begin{aligned} &[w_{+}(F_T(v^*)) - 1] \int_{v^*}^{+\infty} w_{-}(1 - F_T(t))dt \\ &\quad + w_{-}(1 - F_T(v^*)) \left(\int_{-\infty}^{v^*} w_{+}(F_T(t))dt - v^* \right) = 0. \end{aligned}$$

Consequently,

$$\ell_1(v^*) = \frac{1 - w_{+}(F_T(v^*))}{w_{-}(1 - F_T(v^*))}.$$

If we have

$$\zeta_1 = \sup_{0<p<1} \frac{1 - w_{+}(p)}{w_{-}(1-p)} < +\infty,$$

then

$$\sup_{v \geq 0} \ell_1(v) \leq \max\{\ell_1(0), \zeta_1\}.$$

Let us now turn to the case in which $v < 0$. First of all, $g(v) \geq 0,\ \forall v < 0$ if and only if $j(v) := \int_{v}^{+\infty} w_{-}(1 - F_T(t))dt + v > 0,\ \forall v < 0$ and

$$k \geq \frac{\int_{-\infty}^{v} w_{+}(F_T(t))dt}{\int_{v}^{+\infty} w_{-}(1 - F_T(t))dt + v} =: \ell_2(v), \quad \forall v < 0. \tag{42}$$

Notice that

$$j(v) = \int_{0}^{+\infty} w_{-}(1 - F_T(t))dt - \int_{v}^{0} \left(1 - w_{-}(1 - F_T(t))\right) dt.$$

Thus, $j(\cdot)$ is non-decreasing in $v < 0$. However, $j(-\infty) = \lambda_2^- > 0$; so it holds automatically that $j(v) > 0\ \forall v < 0$. To handle (42), a similar argument as above yields

$$\sup_{v \le 0} \ell_2(v) \le \max\{\ell_2(0), \zeta_2\},$$

where

$$\zeta_2 = \sup_{0<p<1} \frac{w_+(p)}{1 - w_-(1-p)}.$$

Noting $\ell_1(0) = \ell_2(0) = \frac{\int_{-\infty}^{0} w_+[F_T(t)]dt}{\int_0^{+\infty} w_-[1-F_T(t)]dt}$, we complete our proof. $\square$

Acknowledgement. The authors are very grateful to the three anonymous referees and the Department Editor for their constructive comments on earlier versions of the manuscript, which have led to significant improvement of the paper.

References

Mohammed Abdellaoui. Parameter-free elicitation of utility and probability weighting functions. *Management Science*, 46(11):1497–1512, Nov. 2000.

Nicholas Barberis and Ming Huang. Stocks as lotteries: The implications of probability weighting for security prices. *The American Economic Review*, 98(5):2066–2100, Aug. 2008.

Shlomo Benartzi and Richard H. Thaler. Myopic loss aversion and the equity premium puzzle. *The Quarterly Journal of Economics*, 110(1):73–92, Feb. 1995.

Arjan Berkelaar, Roy Kouwenberg, and Thierry Post. Optimal portfolio choice under loss aversion. *The Review of Economics and Statistics*, 86(4):973–987, 2004.

Carole Bernard and Mario Ghossoub. Static portfolio choice under cumulative prospect theory. *Mathematics and Financial Economics*, 2(4):277 – 306, 2010.

Antonio E. Bernardo and Olivier Ledoit. Gain, loss and asset pricing. *Journal of Political Economy*, 108(1):144–172, 2000.

Enrico De Giorgi, Thorsten Hens, and Haim Levy. Existence of CAPM equilibria with prospect theory preferences. working paper, *Available at http://ssrn.com/abstract=420184*, 2004.

Dieter Denneberg. *Non-Additive Measure and Integral*. Kluwer Academic, 1994.

Francisco J. Gomes. Portfolio choice and trading volume with loss-averse investors. *Journal of Business*, 78(2):675–706, 2005.

Hanqing Jin and Xun Yu Zhou. Behavioral portfolio selection in continuous time. *Mathematical Finance*, 18:385–426, 2008.

Daniel Kahneman and Amos Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 47:263–291, 1979.

Con Keating and William F. Shadwick. A universal performance measure. *Journal of Performance Measurement*, 6(3):59–84, 2002.

Veronika Köbberling and Peter P. Wakker. An index of loss aversion. *Journal of Economic Theory*, 122(1):119–131, May 2005.

Haim Levy and Moshe Levy. Prospect theory and mean-variance analysis. *Review of Financial Studies*, 17(4):1015–1041, 2004.

Nicholas G. Mankiw and Stephen P. Zeldes. The consumption of stockholders and non-stockholders. *Journal of Financial Economics*, 29(1):94–112, Mar. 1991.

Harry Markowitz. The utility of wealth. *The Journal of Political Economy*, 60(2):151–158, Apr. 1952.

Drazen Prelec. The probability weighting function. *Econometrica*, 66(3):497–527, May 1998.

Marc Oliver Rieger. Too risk averse for prospect theory? Working paper, University of Zurich, 2007.

Marc Oliver Rieger and Mei Wang. Cumulative prospect theory and the St. Petersburg paradox. *Economic Theory*, 28(3):665–679, Aug. 2006.

Amos Tversky and Craig R. Fox. Weighing risk and uncertainty. *Psychological Review*, 102(2):269–283, 1995.

Amos Tversky and Daniel Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5:297–323, 1992.

George Wu and Richard Gonzalez. Curvature of the probability weighting function. *Management Science*, 42(12):1676–1690, Dec. 1996.