Optimal consumption in a growth model with the Cobb-Douglas production function*

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Abstract

An optimal consumption problem is studied in a growth model for the Cobb-Douglas production function in a finite horizon. The problem is transfered into a stochastic Ramsey problem so as to reduce the dimension of the state space. The corresponding state equation is a stochastic differential

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equation with inherently non-Lipschitz coefficients, whose unique solvability is established. The unique existence of the classical solution of the Hamilton-Jacobi-Bellman equation associated with the original problem is proved, and a synthesis of the optimal consumption policy is presented in the feedback form.

Key words. Economic growth, Cobb-Douglas production function, Ramsey problem, Hamilton-Jacobi-Bellman equation, viscosity solutions.

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1 Introduction

We deal with the economic growth model originated by R.C. Merton [7] for the Cobb-Douglas production function in the finite horizon. Define the following quantities:

$T =$ finite horizon,

$y_t =$ labour supply at time $t \in [0, T]$,

$z_t =$ capital stock at time $t \in [0, T]$,

$\nu =$ the constant rate of depreciation, $\nu \geq 0$,

$c_t z_t =$ consumption rate at time $t \in [0, T]$, $0 \leq c(t) \leq 1$,

$c_t z_t / y_t =$ the totality of consumption rate per person,

$F(z, y) =$ the Cobb-Douglas production function $z^\alpha y^{1-\alpha}, 0 < \alpha < 1$,

producing the commodity for the capital stock $z > 0$ and the labour force $y > 0$,

$n, \sigma =$ nonzero constant coefficients,

$U(c) =$ the utility function for the consumption rate $c \geq 0$.

We assume that the labour supply y_t and the capital stock z_t are governed by the stochastic differential equation

$$dy_t = ny_t dt + \sigma y_t dB_t, \quad y_0 = y > 0, \tag{1.1}$$

$$\dot{z}_t = F(z_t, y_t) - \nu z_t - c_t z_t, \quad 0 < t \leq T, \quad z_0 = z > 0, \tag{1.2}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ carrying a standard Brownian motion $\{B_t\}$. Let $c = \{c_t\}$ be a consumption policies per capita such that

$$\begin{aligned} &c_t \text{ is progressively measurable w.r.t. the filtration } \mathcal{F}_t = \sigma(B_s, s \leq t), \\ &0 \leq c_t \leq 1, \quad 0 \leq t \leq T, \end{aligned} \tag{1.3}$$

and we denote by $\mathcal{A}$ the class of all consumption policies $\{c_t\}$ per capita.

The purpose of this paper is to present a synthesis of optimal consumption policy c^* so as to maximize the the expected utilities:

$$J(c) = E[\int_0^T U(c_t z_t / y_t)\ dt] \tag{1.4}$$

per person with finite horizon T over the class $\mathcal{A}$. The Hamilton-Jacobi-Bellman (for short, HJB) equation associated with this problem is given by

$$\begin{aligned} &V_t + \frac{1}{2}\sigma^2 y^2 V_{yy} + nyV_y + \{F(z,y) - \nu z\}V_z + \max_{0\leq c\leq 1}\{U(cz/y) - czV_z\} = 0,\ 0 \leq t < T, \\ &V(T,z,y) = 0, \qquad z > 0,\ y > 0, \end{aligned} \tag{1.5}$$

where the subscripts denote the partial derivatives and the utility function $U(c)$ is assumed to have the following properties:

$$\begin{aligned} &U \in C[0,\infty) \cap C^2(0,\infty), \quad U''(c) < 0 \text{ for } c > 0, \\ &U'(\infty) = U(0+) = 0, \quad U'(0+) = U(\infty) = \infty. \end{aligned} \tag{1.6}$$

The last two conditions constitute what is known as the Inada condition. Its economic interpretation is that, while the utility is very small (respectively very large) for a very small (respectively very large) consumption rate, the marginal utility diminishes as the consumption rate becomes extremely large.

Under (1.6), by the uniform continuity of U near 0, we have that

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0 :\ |U(c) - U(\bar{c})| \leq C_\varepsilon |c - \bar{c}| + \varepsilon \ \text{ for } \ c, \bar{c} > 0. \tag{1.7}$$

The technical difficulty in solving the problem lies in the fact that the HJB equation (1.5) is a parabolic PDE with two spatial variables y and z. The main approach to be employed is to reduce the dimension by turning the problem into a so-called Ramsey problem [7]. Through an analysis on the Ramsey problem together with the viscosity solution technique, we are able to show that (1.5) admits a smooth solution V and the optimal consumption policy c^* can be represented in a feedback form. A major technical hurdle to overcome is to prove the existence and uniqueness of solutions to the state equation of the Ramsey problem, whose drift coefficient is inherently non-Lipschitz. Moreover, we need to estimate the Hölder order of the solution in time. It should be noted that the stochastic Ramsey problem is analytically studied in [5], nevertheless in the *infinite* time horizon. The resulting HJB equation is an elliptic PDE, which is very different from the parabolic PDE dealt with in the present paper. We also refer to [6] for the growth model with the CES production function replacing $F(z, y)$ of (1.2).

This paper is organized as follows. In sections 2 and 3, we reduce (1.5) to the 2-dimensional HJB equation associated with the stochastic Ramsey problem, and we show the existence of viscosity solutions of the HJB equation. Sections 4 and 5 are respectively devoted to the C^2-regularity and the concavity of the viscosity solution. In section 6, we give a synthesis of the optimal consumption policy.

2 The Stochastic Ramsey Problem

We consider the HJB equation (1.5) and seek the solution $V(t, z, y)$ of (1.5) of the form

$$V(t, z, y) = v(t, x), \quad x = z/y. \tag{2.1}$$

Clearly,

$$yV_y = -xv_x, \quad yV_z = v_x, \quad y^2V_{yy} = x^2v_{xx} + 2xv_x. \tag{2.2}$$

Then, by (1.5), $v(t,x)$ solves the HJB equation

$$v_t(t,x) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) + (x^\alpha - \mu x)v_x(t,x) + \tilde{U}(x, v_x(t,x)) = 0,\ 0 \le t < T, \quad (2.3)$$
$$v(T,x) = 0, \quad x > 0,$$

where $\mu = n + \nu - \sigma^2$ and

$$\tilde{U}(x,p) = \max_{0 \le c \le 1}\{U(cx) - cxp\}, \quad p \in \mathbf{R}. \tag{2.4}$$

We observe that (2.3) is the HJB equation associated with the stochastic Ramsey problem so as to maximize

$$\bar{J}(c) = E[\int_0^T U(c_t R_t)dt], \tag{2.5}$$

over the class $\mathcal{A}$, subject to

$$dR_t = (R_t^\alpha - \mu R_t - c_t R_t)dt - \sigma R_t dB_t, \quad 0 < t \le T, \tag{2.6}$$
$$R_0 = x > 0.$$

The above SDE does not satisfy the Lipschitz condition as normally required for the existence and uniqueness. Moreover, we need to estimate the dependence of the solution, if any, on the time and the initial state. We solve these problem by an *ad hoc* technique.

Proposition 2.1 *For each $c \in \mathcal{A}$, there exists a unique positive solution $\{R_t\} = \{R_t^x\}$ of (2.6), which satisfies*

$$E[\sup_{0 \le t \le T} R_t^2] \le C(1 + x^2), \tag{2.7}$$
$$E[|R_r - R_s|] \le C(1 + x)|r - s|^{1/2}, \quad 0 \le s \le r \le T, \tag{2.8}$$
$$E[|R_s^x - R_s^y|] \le C|x - y|^{1-\alpha}(1 + x^\alpha + y^\alpha), \quad x, y > 0, \quad 0 \le s \le T, \tag{2.9}$$

where the constant $C > 0$ depends only on α, T, μ, σ.

Proof. By Itô's formula

$$\begin{aligned} dR_t^{1-\alpha} &= (1-\alpha)R_t^{-\alpha}\{(R_t^\alpha - \mu R_t - c_t R_t)dt - \sigma R_t dB_t\} \\ &\quad + \frac{1}{2}\sigma^2(1-\alpha)(-\alpha)R_t^{-\alpha-1}R_t^2 dt. \end{aligned}$$

Hence, setting $x_t = R_t^{1-\alpha}$, we have

$$\begin{aligned} dx_t &= (1-\alpha)\{1-(\mu + c_t + \frac{1}{2}\sigma^2\alpha)x_t\}dt - (1-\alpha)\sigma x_t dB_t \qquad (2.10) \\ &= (1-\alpha)\{1-(c_t + \frac{1}{2}\sigma^2\alpha)x_t\}dt - x_t(1-\alpha)(\mu dt + \sigma dB_t), \quad x_0 = x^{1-\alpha}. \end{aligned}$$

By linearity, (2.10) admits a unique solution $\{x_t\}$. Also, we apply the comparison theorem to (2.10) and

$$d\bar{x}_t = (1-\alpha)\{-(\mu + c_t + \frac{1}{2}\sigma^2\alpha)\bar{x}_t\}dt - (1-\alpha)\sigma\bar{x}_t dB_t, \quad \bar{x}_0 = x_0.$$

Then

$$\begin{aligned} x_t &\geq \bar{x}_t \qquad (2.11) \\ &= x_0 \exp\{(1-\alpha)(-\mu t - \int_0^t c_s ds - \frac{1}{2}\sigma^2\alpha t) \\ &\quad -(1-\alpha)\sigma B_t - \frac{1}{2}(1-\alpha)^2\sigma^2 t\} > 0. \end{aligned}$$

Thus, we obtain a positive solution $\{R_t\}$ of (2.6). Let $\{\hat{x}_t\}$ be the solution of

$$d\hat{x}_t = -\hat{x}_t(1-\alpha)(\mu dt + \sigma dB_t), \quad \hat{x}_0 = x_0.$$

Setting $H_t = x_t/\hat{x}_t$ and $\bar{\alpha} = \sigma^2(1-\alpha)\alpha/2$, we have

$$\begin{aligned} dH_t &= (1-\alpha)\{\frac{1}{\hat{x}_t} - (c_t + \frac{1}{2}\sigma^2\alpha)H_t\}dt \\ &\leq (\frac{1-\alpha}{\hat{x}_t} - \bar{\alpha}H_t)dt, \quad H_0 = 1. \end{aligned}$$

Therefore

$$x_t \leq \hat{x}_t e^{-\bar{\alpha}t}\{1 + (1-\alpha)\int_0^t \frac{e^{\bar{\alpha}s}}{\hat{x}_s} ds\}, \tag{2.12}$$

which yields (2.7).

Now, let $\beta = 1/(1-\alpha) > 1$ and $M_t = \exp\{-(1-\alpha)\sigma B_t - \frac{1}{2}(1-\alpha)^2\sigma^2 t\}$. By (2.12) and Doob's maximal inequality, we have

$$\begin{aligned} E[\sup_{0\leq t\leq T} x_t^\beta] &\leq C(1 + x_0^\beta E[\sup_{0\leq t\leq T} M_t^\beta]) \\ &\leq C(1 + x_0^\beta (\frac{\beta}{\beta-1})^\beta E[M_T^\beta]) \\ &\leq C'(1+x), \end{aligned} \tag{2.13}$$

where the constant $C' > 0$ depends only on α, T, μ, σ. Hence, by (2.10), (2.13) and the moment inequality for martingales, we get

$$\begin{aligned} E[|x_r - x_s|^\beta] &\leq 2^\beta \Bigg(E[|\int_s^r (1-\alpha)\{1 - (\mu + c_t + \frac{1}{2}\sigma^2\alpha)x_t\}dt|^\beta] \\ &\quad + E[|\int_s^r (1-\alpha)\sigma x_t dB_t|^\beta] \Bigg) \\ &\leq C\Bigg(E[(\int_s^r (1 + x_t^2)dt)^{\beta/2}]|r-s|^{\beta/2} + E[(\int_s^r x_t^2 dt)^{\beta/2}] \Bigg) \\ &\leq C'(1+x)|r-s|^{\beta/2}, \qquad 0 \leq s \leq r \leq T. \end{aligned}$$

Since

$$|x^\beta - y^\beta| = |\int_y^x \beta t^{\beta-1} dt| \leq \beta|x-y|(|x|^{\beta-1} + |y|^{\beta-1}), \quad x, y \geq 0,$$

we observe by Hölder's inequality that

$$\begin{aligned} E[|R_r - R_s|] &= E[|x_r^\beta - x_s^\beta|] \\ &\leq \beta(E[|x_r - x_s|^\beta])^{1/\beta}(E[(|x_r|^{\beta-1} + |x_s|^{\beta-1})^{\beta/(\beta-1)}])^{1-1/\beta} \\ &\leq \beta(C'(1+x)|r-s|^{\beta/2})^{1/\beta}(E[2^{\beta/(\beta-1)} \sup_{0\leq t\leq T}|x_t|^\beta])^{1-1/\beta} \\ &\leq C(1+x)|r-s|^{1/2}, \end{aligned}$$

which implies (2.8).

Next, we set $r_t = (R^y_t)^{1-\alpha}$. Then, by (2.10)

$$d(x_t - r_t) = (1-\alpha)(-\mu - c_t - \frac{1}{2}\sigma^2\alpha)(x_t - r_t)dt - (1-\alpha)\sigma(x_t - r_t)dB_t,$$

or equivalently

$$x_s - r_s = (x_0 - r_0)\exp\{(1-\alpha)(-\mu t - \int_0^s c_t dt - \frac{1}{2}\sigma^2\alpha s) - (1-\alpha)\sigma B_s - \frac{1}{2}(1-\alpha)^2\sigma^2 s\}.$$

Hence

$$E[|x_s - r_s|^\beta] \le C|x_0 - r_0|^\beta.$$

By Hölder's inequality, we deduce

$$\begin{aligned} E[|R^x_s - R^y_s|] &= E[|x^\beta_s - r^\beta_s|] \\ &\le \beta(E[|x_s - r_s|^\beta])^{1/\beta}(E[(|x_s|^{\beta-1} + |r_s|^{\beta-1})^{\beta/(\beta-1)}])^{1-1/\beta} \\ &\le C|x_0 - r_0|(1 + x^\alpha + y^\alpha), \end{aligned}$$

which implies (2.9).

3 Viscosity solutions

We study the viscosity solution v of the HJB equation (2.3), i.e.,

$$v_t + \frac{1}{2}\sigma^2 x^2 v_{xx} + (x^\alpha - \mu x)v_x + \tilde{U}(x, v_x) = 0 \quad \text{in } Q := [0,T) \times (0,\infty), \tag{3.1}$$

$$v(T,x) = 0, \qquad x > 0. \tag{3.2}$$

Definition 3.1 *Let $v \in C([0,T] \times (0,\infty))$ satisfy (3.2). Then v is called a viscosity solution of (3.1) if the following assertions are satisfied:*

$$a + \frac{1}{2}\sigma^2 x^2 X + (x^\alpha - \mu x)\lambda + \tilde{U}(x,\lambda) \ge 0, \quad \forall (a,\lambda,X) \in \mathcal{P}^{2,+}v(s,x), \quad \forall (s,x) \in Q,$$

$$a+\frac{1}{2}\sigma^2 x^2 X+(x^\alpha-\mu x)\lambda+\tilde{U}(x,\lambda) \le 0, \quad \forall (a,\lambda,X) \in \mathcal{P}^{2,-}v(s,x), \quad \forall (s,x) \in Q,$$

where $\mathcal{P}^{2,+}$ and $\mathcal{P}^{2,-}$ are the second parabolic superdifferentials and subdifferentials [1] defined by

$$\begin{aligned}
\mathcal{P}^{2,+}v(s,x) &= \{(a,\lambda,X) \in \mathbf{R}^3 : \\
&\limsup_{(t,y)\in Q \to (s,x)} \frac{v(t,y)-v(s,x)-a(t-s)-\lambda(y-x)-\frac{1}{2}X(y-x)^2}{|t-s|+|y-x|^2} \le 0\}, \\
\mathcal{P}^{2,-}v(s,x) &= \{(a,\lambda,X) \in \mathbf{R}^3 : \\
&\liminf_{(t,y)\in Q \to (s,x)} \frac{v(t,y)-v(s,x)-a(t-s)-\lambda(y-x)-\frac{1}{2}X(y-x)^2}{|t-s|+|y-x|^2} \ge 0\}.
\end{aligned}$$

Define

$$v(s,x) = \sup_{c\in\mathcal{A}} E[\int_s^T U(c_t X_t)\ dt], \tag{3.3}$$

where $\{X_t\}$ is the solution of (2.6) for $t \in (s,T]$ with $X_s = x$, that is,

$$dX_t = (X_t^\alpha - \mu X_t - c_t X_t)dt - \sigma X_t dB_t, \quad s < t \le T, \quad X_s = x > 0, \tag{3.4}$$

and the supremum is taken over all systems $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; \{B_t\}, \{c_t\})$. We choose $b_1 > 0$ such that $x^\alpha - \mu x \le b_1$. Taking sufficiently large $b_0 > b_1$, we observe that $\zeta(t,x) := e^{T-t}(x+b_0)$ fulfils

$$\begin{aligned}
\zeta_t + \frac{1}{2}\sigma^2 x^2 \zeta_{xx} + (x^\alpha - \mu x)\zeta_x + \tilde{U}(x,\zeta_x) &\le e^{T-t}\{-b_0 + (x^\alpha - \mu x)\} + \tilde{U}(x, e^{T-t}) \\
&\le e^{T-t}(-b_0+b_1) + U\circ(U')^{-1}(e^{T-t}) \qquad (3.5) \\
&\le -b_0 + b_1 + U\circ(U')^{-1}(1) < 0, \quad (t,x) \in [0,T) \times (0,\infty).
\end{aligned}$$

Lemma 3.2 *We assume (1.6). Then the following assertions are valid:*

$$0 \le v(s,x) \le \zeta(s,x). \tag{3.6}$$

For any $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that*

$$|v(s,x) - v(r,y)| \le C_\varepsilon\{|s-r|^{1/2}(1+x+y) + |x-y|\} + \varepsilon(1+x+y),$$
$$x,\ y > 0, \quad 0 \le r \le s \le T. \tag{3.7}$$

Proof. By Itô's formula and (3.5), we have

$$\begin{aligned} 0 &\le \zeta(T, X_T) \\ &= \zeta(s,x) + \int_s^T \{\zeta_t(t,X_t) + [(X_t)^\alpha - \mu X_t - c_t X_t]\zeta_x(t,X_t) \\ &+ \frac{1}{2}\sigma^2 X_t^2 \zeta_{xx}(t,X_t)\}dt - \int_s^T \sigma X_t \zeta_x(t,X_t) dB_t \\ &\le \zeta(s,x) - \int_s^T U(c_t X_t)dt - \int_s^T \sigma X_t e^{T-t} dB_t, \quad a.s. \end{aligned} \tag{3.8}$$

By (2.7), we note that $\{\int_s^t \sigma X_r e^{-r} dB_r\}$ is a martingale. Therefore, we deduce (3.6).

Now, by (3.3), we have

$$\begin{aligned} |v(s,x) - v(r,y)| &\le \sup_{c\in\mathcal{A}} E[|\int_s^T U(c_t X_t)dt - \int_r^T U(c_t Y_t)dt|] \qquad (3.9) \\ &\le \sup_{c\in\mathcal{A}} E[\int_s^T |U(c_t X_t) - U(c_t Y_t)|dt] + \sup_{c\in\mathcal{A}} E[\int_r^s U(c_t Y_t)dt] \\ &\equiv J_1 + J_2, \end{aligned}$$

where $\{Y_t\}$ denotes the solution of (3.4) with $Y_r = y$. By (2.9) and Young's inequality, choosing a suitable constant $\delta > 0$ for any $\varepsilon' > 0$, we note that

$$\begin{aligned} E[|X_t - Y_t|] &\le C\{\frac{1-\alpha}{\delta}(|x-y|^{1-\alpha})^{1/(1-\alpha)} + \alpha\delta(1 + x^\alpha + y^\alpha)^{1/\alpha}\} \\ &= C_{\varepsilon'}|x-y| + \varepsilon'(1+x+y). \end{aligned}$$

Also, by (2.7)

$$E[X_t] \le C(1+x).$$

Hence, by (1.7)

$$\begin{aligned} J_1 &\le \sup_{c\in\mathcal{A}} E[\int_0^T \{C_\varepsilon |X_t - Y_t| + \varepsilon\} dt] \\ &\le C_\varepsilon T\{C_{\varepsilon'}|x-y| + \varepsilon'(1+x+y)\} + \varepsilon T. \end{aligned}$$

By the same calculation as (3.8), taking into account (2.7) and (2.8), we get

$$\begin{aligned} J_2 &\le E[\zeta(r,Y_r) - \zeta(s,Y_s)] \\ &\le E[|\zeta(r,Y_r) - \zeta(r,Y_s)|] + E[|\zeta(r,Y_s) - \zeta(s,Y_s)|] \\ &\le e^T\{E[|Y_s - Y_r|] + E[|s-r||Y_s + b_0|]\} \\ &\le C|s-r|^{1/2}(1+y). \end{aligned}$$

Therefore, we deduce (3.7).

Theorem 3.3 *We assume (1.6). Then the value function v of (3.3) is a viscosity solution of (3.1).*

Proof. By Lemma 3.2, we see that $v \in C([0,T]\times(0,\infty))$, and by (3.3), $v(T,x) = 0$. According to [2], the viscosity property of v follows from the dynamic programming principle for v, that is,

$$v(s,x) = \sup_{c\in\mathcal{A}} E[\int_s^\tau U(c_t X_t)dt + v(\tau, X_\tau)], \quad \forall (s,x) \in [0,T)\times(0,\infty) \qquad (3.10)$$

for any $\tau \in [s,T)$, where the supremum is taken over all systems $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}; \{B_t\}, \{c_t\})$.

Let $\bar{v}$ be the right-hand side of (3.10) and we set $J_{(s,x)}(c) = E[\int_s^T U(c_t X_t)dt]$. For each $c = \{c_t\} \in \mathcal{A}$, let $\tilde{X}_t = X_{t+\tau}$ and $\tilde{B}_t = B_{t+\tau} - B_\tau$. Then we have

$$d\tilde{X}_t = (\tilde{X}_t^\alpha - \mu\tilde{X}_t - \tilde{c}_t\tilde{X}_t)dt - \sigma\tilde{X}_t d\tilde{B}_t, \quad t \in (0, T-\tau], \quad \tilde{X}_0 = X_\tau,$$

where $\tilde{c} = \{\tilde{c}_t\}$ is the shifted process of c by τ, i.e., $\tilde{c}_t = c_{t+\tau}$. By (3.4), we see that

$$E[\int_\tau^T U(c_t X_t)dt|\mathcal{F}_\tau] = E[\int_0^{T-\tau} U(\tilde{c}_t \tilde{X}_t)dt|\mathcal{F}_\tau] = J_{(\tau,X_\tau)}(c), \quad a.s.,$$

with respect to the conditional probability measure $P(\cdot|\mathcal{F}_\tau)$. Hence

$$\begin{aligned} J_{(s,x)}(c) &= E[\int_s^\tau U(c_t X_t)dt + \int_\tau^T U(c_t X_t)dt] \\ &\leq E[\int_s^\tau U(c_t X_t)dt + v(\tau, X_\tau)]. \end{aligned}$$

Taking the supremum, we deduce $v \leq \bar{v}$.

Conversely, let $\{S_j : j = 1, \ldots, n+1\}$ be a sequence of disjoint subsets of $(0, \infty)$ such that

$$diam(S_j) < \delta, \quad \bigcup_{j=1}^n S_j = (0, R) \quad \text{and} \quad S_{n+1} = [R, \infty)$$

for $\delta, R > 0$ chosen later. For any $\varepsilon > 0$, we take $x_j \in S_j$ and $c^{(j)} \in \mathcal{A}$ such that

$$v(\tau, x_j) - \varepsilon \leq J_{(\tau,x_j)}(c^{(j)}), \quad j = 1, \ldots, n+1. \tag{3.11}$$

By the same argument as (3.7), we note that

$$|J_{(\tau,x)}(c) - J_{(\tau,y)}(c)| + |v(\tau,x) - v(\tau,y)| \leq C_\varepsilon|x-y| + \frac{\varepsilon}{4}(1+x+y), \quad x, y > 0, \ c \in \mathcal{A}$$

for some constant $C_\varepsilon > 0$. We choose $0 < \delta < 1$ such that $C_\varepsilon \delta < \varepsilon/2$. Then, we have

$$|J_{(\tau,x)}(c^{(j)}) - J_{(\tau,y)}(c^{(j)})| + |v(\tau,x) - v(\tau,y)| \leq \varepsilon(1+x), \quad x, y \in S_j,$$

from which

$$J_{(\tau,X_\tau)}(c^{(j)}) \geq J_{(\tau,x_j)}(c^{(j)}) - \varepsilon(1 + X_\tau) \quad \text{if} \quad X_\tau \in S_j, \ j = 1, \ldots, n.$$

Hence

$$\begin{aligned} J_{(\tau,X_\tau)}(c^{(j)}) &= J_{(\tau,X_\tau)}(c^{(j)}) - J_{(\tau,x_j)}(c^{(j)}) + J_{(\tau,x_j)}(c^{(j)}) \\ &\geq -\varepsilon(1+X_\tau) + v(\tau, x_j) - \varepsilon \qquad (3.12) \\ &\geq -2\varepsilon(1+X_\tau) + v(\tau, X_\tau) - \varepsilon \quad \text{if} \quad X_\tau \in S_j,\ j=1,\ldots,n. \end{aligned}$$

By definition, we find $c \in \mathcal{A}$ such that

$$\bar{v}(s,x) - \varepsilon \leq E[\int_s^\tau U(c_t X_t)dt + v(\tau, X_\tau)].$$

As in the proof of Theorem IV-1.1 [3], we can take $c, c^{(j)}$ on the same probability space. Define

$$c_t^\tau = c_t 1_{\{t<\tau\}} + c_t^{(j)} 1_{\{\tau \leq t \leq T\}} \quad \text{if} \quad X_\tau \in S_j,\ j=1,\ldots,n+1.$$

It is easy to see that $\{c_t^\tau\}$ belongs to $\mathcal{A}$. Let $\{X_t^\tau\}$ be the solution of

$$dX_t^\tau = [(X_t^\tau)^\alpha - \mu X_t^\tau - c_t^\tau X_t^\tau]dt - \sigma X_t^\tau dB_t, \quad s < t \leq T,\ X_s^\tau = x > 0.$$

Clearly, $X_t^\tau = X_t$ a.s. if $s \leq t < \tau$. Further, for each $j = 1, \ldots, n+1$, we have on $\{X_\tau \in S_j\}$

$$X_r^\tau = X_\tau + \int_\tau^r [(X_t^\tau)^\alpha - \mu X_t^\tau - c_t^\tau X_t^\tau]dt - \int_\tau^r \sigma X_t^\tau dB_t, \quad \tau < r \leq T, \quad a.s.$$

Hence $X_t^\tau = X_t^{(j)}$ for all $t \in [\tau, T]$ a.s. on $\{X_\tau \in S_j\}$, where $\{X_t^{(j)}\}$ denotes the solution of

$$dX_t^{(j)} = [(X_t^{(j)})^\alpha - \mu X_t^{(j)} - c_t^{(j)} X_t^{(j)}]dt - \sigma X_t^{(j)} dB_t, \quad \tau < t \leq T, \quad X_\tau^{(j)} = X_\tau.$$

Thus, we get

$$\begin{aligned} J_{(\tau,X_\tau)}(c^\tau) &= E[\int_\tau^T U(c_t^\tau X_t^\tau)dt | \mathcal{F}_\tau] \\ &= E[\int_\tau^T U(c_t^{(j)} X_t^{(j)})dt | \mathcal{F}_\tau] \qquad (3.13) \\ &= J_{(\tau,X_\tau)}(c^{(j)}) \quad a.s. \ \text{on} \quad \{X_\tau \in S_j\}. \end{aligned}$$

Next, taking into account (3.6) and (2.7), we choose $R > 0$ such that

$$\begin{aligned}
\sup_{c\in\mathcal{A}} E[v(\tau, X_\tau)1_{\{X_\tau \geq R\}}] &\leq \sup_{c\in\mathcal{A}} e^T E[(X_\tau + b_0)1_{\{X_\tau \geq R\}}] \\
&\leq \sup_{c\in\mathcal{A}} \frac{e^T}{R} E[X_\tau^2 + b_0 X_\tau] \qquad (3.14)\\
&\leq \frac{e^T}{R}\{(1+b_0)C(1+x^2) + b_0\} < \varepsilon.
\end{aligned}$$

By (3.11)-(3.14) and (2.7), we have

$$\begin{aligned}
E[\int_\tau^T U(c_t^\tau X_t^\tau)dt] &= E[E[\int_\tau^T U(c_t^\tau X_t^\tau)dt|\mathcal{F}_\tau]] \\
&= E[J_{(\tau,X_\tau)}(c^\tau)] \\
&= E[\sum_{j=1}^{n+1} J_{(\tau,X_\tau)}(c^{(j)})1_{\{X_\tau\in S_j\}}] \\
&\geq E[\sum_{j=1}^{n} \{v(\tau, X_\tau)) - 3\varepsilon(1+X_\tau)\}1_{\{X_\tau\in S_j\}}] \\
&\geq E[\{v(\tau, X_\tau)) - v(\tau, X_\tau)1_{\{X_\tau\geq R\}}\}] - 3\varepsilon E[1+X_\tau] \\
&\geq E[v(\tau, X_\tau)] - \varepsilon - 3\varepsilon\{2 + C(1+x^2)\}.
\end{aligned}$$

Thus

$$\begin{aligned}
v(s,x) &\geq E[\int_s^\tau U(c_t^\tau X_t^\tau)dt + \int_\tau^T U(c_t^\tau X_t^\tau)dt] \\
&\geq E[\int_s^\tau U(c_t X_t)dt + v(\tau, X_\tau)] - 4\varepsilon\{2 + C(1+x^2)\} \\
&\geq \bar{v}(s,x) - \varepsilon - 4\varepsilon\{2 + C(1+x^2)\}.
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we obtain $\bar{v} \leq v$. The proof is complete.

4 Classical solutions

In this section, we study the classical solutions of the HJB equation (3.1) with the terminal condition (3.2). First, for any interval $[\xi_1, \xi_2]$ with $\xi_1 > 0$, we consider the parabolic equation

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + (x^\alpha - \mu x)u_x + \tilde{U}(x, u_x) = 0, \quad 0 \le t < T, \quad \xi_1 < x < \xi_2, \tag{4.1}$$

with the (parabolic) boundary condition

$$u(T, x) = v(T, x) = 0, \quad x \in [\xi_1, \xi_2], \tag{4.2}$$

$$u(t, \xi_1) = v(t, \xi_1), \quad u(t, \xi_2) = v(t, \xi_2), \quad t \in [0, T). \tag{4.3}$$

Theorem 4.1 *Let $u_i \in C([0,T] \times [\xi_1, \xi_2]), i = 1, 2,$ be two viscosity solutions of (4.1)-(4.3). Then, under (1.6), we have $u_1 = u_2$.*

Proof. It is sufficient to show that $u_1 \le u_2$. Suppose there exists $(t_0, x_0) \in [0, T) \times (\xi_1, \xi_2)$ such that $u_1(t_0, x_0) - u_2(t_0, x_0) > 0$. Then we find $\eta > 0$ such that

$$\varrho := \sup_{(t,x)\in(0,T)\times(\xi_1,\xi_2)} \{u_1(t,x) - u_2(t,x) - 2\eta\frac{1}{t}\} > 0.$$

By boundedness, we have

$$u_1(t,x) - u_2(t,x) - 2\eta\frac{1}{t} \quad \to \quad -\infty, \quad \text{uniformly in } x \text{ as } t \downarrow 0. \tag{4.4}$$

Thus, by (4.2) and (4.3), there exists $(\bar{t}, \bar{x}) \in (0, T) \times (\xi_1, \xi_2)$ such that

$$\varrho = u_1(\bar{t}, \bar{x}) - u_2(\bar{t}, \bar{x}) - 2\eta\frac{1}{\bar{t}}.$$

Define

$$\Psi_k(t, x, y) = u_1(t, x) - u_2(t, y) - \frac{k}{2}|x - y|^2 - 2\eta\frac{1}{t}$$

for $k > 0$. By (4.2) and (4.4), there exists $(t_k, x_k, y_k) \in (0, T) \times [\xi_1, \xi_2]^2$ such that

$$\Psi_k(t_k, x_k, y_k) = \sup \Psi_k(t, x, y) \ge \Psi_k(\bar{t}, \bar{x}, \bar{x}) = \varrho, \tag{4.5}$$

from which

$$\frac{k}{2}|x_k - y_k|^2 < u_1(t_k, x_k) - u_2(t_k, y_k) - 2\eta\frac{1}{t_k}.$$

Thus

$$|x_k - y_k| \to \quad 0 \quad \text{as} \quad k \to \infty. \tag{4.6}$$

By the definition of (t_k, x_k, y_k), we have

$$\Psi_k(t_k, x_k, y_k) \geq \Psi_k(t_k, x_k, x_k).$$

Hence, by uniform continuity

$$\frac{k}{2}|x_k - y_k|^2 \leq u_2(t_k, x_k) - u_2(t_k, y_k) \quad \to \quad 0 \quad \text{as} \quad k \to \infty. \tag{4.7}$$

By (4.6), (4.5) and (4.3), extracting a subsequence, we have

$$t_k \to \tilde{t} \in (0, T), \quad (x_k, y_k) \to (\tilde{x}, \tilde{x}) \in (\xi_1, \xi_2)^2 \quad \text{as} \quad k \to \infty.$$

Now, we may consider that $(t_k, x_k, y_k) \in (0, T) \times (\xi_1, \xi_2)^2$. Applying Ishii's lemma [1, Thm. 8.3] to

$$\Psi_k(t, x, y) = w_1(t, x) - w_2(t, y) - \frac{k}{2}|x - y|^2,$$

we obtain $a, b \in \mathbf{R}$ and $X, Y \in \mathbf{R}$ such that

$$\begin{aligned} &(a, k(x_k - y_k), X) \in \bar{\mathcal{P}}^{2,+} w_1(t_k, x_k), \\ &(b, k(x_k - y_k), Y) \in \bar{\mathcal{P}}^{2,-} w_2(t_k, y_k), \\ &a - b = 0, \quad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \end{aligned} \tag{4.8}$$

where $w_1(t, x) = u_1(t, x) - \eta/t$ and $w_2(t, y) = u_2(t, y) + \eta/t$. From the definition of $\mathcal{P}^{2,+}u_1(t_k, x_k), \mathcal{P}^{2,-}u_2(t_k, y_k)$, it follows that

$$\mathcal{P}^{2,+}u_1(t, x) = \{(\hat{a}, \lambda, \hat{X}) + \eta(\frac{-1}{t^2}, 0, 0) : (\hat{a}, \lambda, \hat{X}) \in \mathcal{P}^{2,+}w_1(t, x)\},$$

$$\mathcal{P}^{2,-}u_2(t,x) = \{(\hat{a},\lambda,\hat{X}) - \eta(\frac{-1}{t^2},0,0) : (\hat{a},\lambda,\hat{X}) \in \mathcal{P}^{2,-}w_2(t,x)\}.$$

Hence

$$(\bar{a},\lambda_1,\bar{X}) := (a,k(x_k-y_k),X) + \eta(\frac{-1}{t_k^2},0,0) \in \bar{\mathcal{P}}^{2,+}u_1(t_k,x_k),$$
$$(\bar{b},\lambda_2,\bar{Y}) := (b,k(x_k-y_k),Y) - \eta(\frac{-1}{t_k^2},0,0) \in \bar{\mathcal{P}}^{2,-}u_2(t_k,y_k).$$

By Definition 3.1

$$\bar{a} + \frac{1}{2}\sigma^2 x_k^2 \bar{X} + (x_k^\alpha - \mu x_k)\lambda_1 + \tilde{U}(x_k,\lambda_1) \geq 0,$$
$$\bar{b} + \frac{1}{2}\sigma^2 y_k^2 \bar{Y} + (y_k^\alpha - \mu y_k)\lambda_2 + \tilde{U}(y_k,\lambda_2) \leq 0.$$

Putting these inequalities together, we get

$$\begin{aligned} 2\eta\frac{1}{t_k^2} &\leq \frac{1}{2}\sigma^2(x_k^2\bar{X} - y_k^2\bar{Y}) + \{(x_k^\alpha - \mu x_k)\lambda_1 - (y_k^\alpha - \mu y_k)\lambda_2\} \\ &+ |\tilde{U}(x_k,\lambda_1) - \tilde{U}(y_k,\lambda_2)| \\ &\equiv I_1 + I_2 + I_3, \quad \text{say.} \end{aligned}$$

By (4.8) and (4.7), it is clear that

$$I_1 = \frac{\sigma^2}{2}(x_k^2 X - y_k^2 Y) \leq \frac{\sigma^2}{2}3k|x_k - y_k|^2 \quad \to \quad 0 \quad \text{as} \quad k \to \infty.$$

Since x^α is Lipschitz on $[\xi_1,\xi_2]$, we see by (4.7) that

$$I_2 = k\{(x_k^\alpha - y_k^\alpha) - \mu(x_k - y_k)\}(x_k - y_k) \quad \to \quad 0 \quad \text{as} \quad k \to \infty.$$

By (1.7), (4.6) and (4.7), we have

$$\begin{aligned} I_3 &\leq \max_{0\leq c\leq 1}|U(cx_k) - U(cy_k)| + |x_k\lambda_1 - y_k\lambda_2| \\ &\leq C_\varepsilon|x_k - y_k| + \varepsilon + k|x_k - y_k|^2 \\ &\to \quad 0 \quad \text{as} \quad k \to \infty \quad \text{and} \quad \varepsilon \to 0. \end{aligned}$$

Consequently, we deduce

$$2\eta \frac{1}{T^2} \leq 2\eta \frac{1}{\tilde{t}^2} \leq 0,$$

which is a contradiction.

Theorem 4.2 *We assume (1.6). Then there exists a solution* $v \in C^{1,2}([0,T) \times (0,\infty)) \cap C([0,T] \times (0,\infty))$ *of (3.1), (3.2).*

Proof. By (1.6), we have

$$U(0) \leq U'(c)(-c) + U(c), \quad \forall c > 0.$$

Then for any $\xi_1 > 0$,

$$C_0 := \sup_{0<c\leq \xi_1} cU'(c) \leq U(\xi_1) < \infty.$$

Hence, for $x_1, x_2 \in [\xi_1, \xi_2]$ where $\xi_2 > \xi_1$,

$$|U(cx_1) - U(cx_2)| \leq cU'(c\xi_1)|x_1 - x_2| \leq \frac{C_0}{\xi_1}|x_1 - x_2|, \quad 0 \leq c \leq 1.$$

Thus, for $p_1, p_2 \in \mathbf{R}$

$$\begin{aligned} |\tilde{U}(x_1, p_1) - \tilde{U}(x_2, p_2)| &\leq \max_{0\leq c\leq 1} |U(cx_1) - U(cx_2)| + |x_1 p_1 - x_2 p_2| \\ &\leq (\frac{C_0}{\xi_1} + |p_1|)|x_1 - x_2| + \xi_2 |p_1 - p_2|. \end{aligned}$$

According to [4], by uniform ellipticity, there exists a unique solution $u \in C([0,T] \times [\xi_1, \xi_2]) \cap C^{1,2}([0,T) \times (\xi_1, \xi_2))$ of (4.1)-(4.3). Clearly, v is a viscosity solution of (4.1)-(4.3). By Theorem 4.1, we have $u = v$ and v is smooth. Since ξ_1, ξ_2 are arbitrary, we obtain the assertion.

Corollary 4.3 *We make the assumption of Theorem 4.2. Then there exists a solution* $V \in C^{1,2}([0,T) \times (0,\infty)^2)$ *of (1.5).*

Proof. The proof follows from Theorem 4.2 and (2.1).

5 Concavity

In this section, we study the concavity of the solution v to (3.1), (3.2).

Theorem 5.1 *We assume (1.6). Then $v(s,x)$ is concave in $x \in (0,\infty)$ for each $s \in [0,T]$. In addition, we have*

$$v_x(s,x) > 0 \quad \textit{for} \quad x > 0. \tag{5.1}$$

Proof. Let $x_i > 0, i = 1, 2$, and $0 \le \theta \le 1$. For any $\varepsilon > 0$, there exists $c^{(i)} \in \mathcal{A}$ such that

$$v(s,x_i) - \varepsilon < E[\int_s^T U(c_t^{(i)} X_t^{(i)}) dt],$$

where $\{X_t^{(i)}\}$ denotes the solution of (3.4) corresponding to $c^{(i)}$ with $X_s^{(i)} = x_i$ on the same probability space. We set

$$\bar{c}_t = \frac{\theta c_t^{(1)} X_t^{(1)} + (1-\theta) c_t^{(2)} X_t^{(2)}}{\theta X_t^{(1)} + (1-\theta) X_t^{(2)}},$$

which belongs to $\mathcal{A}$. Define $\{\bar{X}_t\}$ and $\{\tilde{X}_t\}$ by

$$\begin{aligned} d\bar{X}_t &= [(\bar{X}_t)^\alpha - \mu \bar{X}_t - \bar{c}_t \bar{X}_t] dt - \sigma \bar{X}_t dB_t, \quad s < t \le T, \quad \bar{X}_s = \theta x_1 + (1-\theta) x_2, \\ \tilde{X}_t &= \theta X_t^{(1)} + (1-\theta) X_t^{(2)}. \end{aligned}$$

By concavity

$$\tilde{X}_r \le \theta x_1 + (1-\theta) x_2 + \int_s^r [(\tilde{X}_t)^\alpha - \mu \tilde{X}_t - \bar{c}_t \tilde{X}_t] dt - \int_s^r \sigma \tilde{X}_t dB_t, \quad a.s.$$

By the comparison theorem, we have

$$\tilde{X}_t \le \bar{X}_t, \quad t \in (s,T], \quad a.s.$$

Thus

$$\begin{aligned}
v(s, \theta x_1 + (1-\theta)x_2) &\geq E[\int_s^T U(\bar{c}_t \bar{X}_t)dt] \geq E[\int_s^T U(\bar{c}_t \tilde{X}_t)dt] \\
&= E[\int_s^T U(\theta c_t^{(1)} X_t^{(1)} + (1-\theta) c_t^{(2)} X_t^{(2)})dt] \\
&\geq \theta E[\int_s^T U(c_t^{(1)} X_t^{(1)})dt] + (1-\theta) E[\int_s^T U(c_t^{(2)} X_t^{(2)})dt] \\
&> \theta v(s, x_1) + (1-\theta) v(s, x_2) - \varepsilon.
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$, we obtain the concavity of v.

To prove (5.1), by Theorem 4.2, we note that v is smooth. By non-negativity and concavity, we see that

$$v_x(s, x) \geq 0, \quad x > 0$$

for every $s \in [0, T)$. Suppose that $v_x(s, x_0) = 0$ for some $x_0 > 0$. Then, $v_x(s, x) = 0$ for all $x \geq x_0$. Hence $v(s, x)$ can be written as $v(s, x) = h(s)$ for $x \geq x_0$. By (3.1), we have

$$U(x) = -v_t(s, x) = -h'(s), \quad x \geq x_0.$$

This is contrary with (1.6). Therefore we obtain (5.1).

6 Optimal policies

We give a synthesis of the optimal policy $c^* = \{c_t^*\}$ for the optimization problem (1.4) subject to (1.1) - (1.3). We consider the stochastic differential equation

$$dX_t^* = [(X_t^*)^\alpha - \mu X_t^* - \gamma(t, X_t^*) X_t^*]dt - \sigma X_t^* dB_t, \quad 0 < t \leq T, \;\; X_0^* = x > 0, \quad (6.1)$$

where $\gamma(t,x) = I(x, v_x(t,x))$ and $I(x,p)$ denotes the maximizer of (2.4) for $x, p > 0$, i.e.,

$$I(x,p) = \begin{cases} (U')^{-1}(p)/x & \text{if} \quad U'(x) \le p, \\ 1 & \text{otherwise.} \end{cases} \tag{6.2}$$

Lemma 6.1 *Under (1.6), there exists a unique positive solution $\{X_t^*\}$ of (6.1).*

Proof. By (5.1), we notice that $\gamma(t,x)$ is well defined. Let $\{N_t\}$ be the solution of (2.6) corresponding to $c_t = 0$. Define the probability measure $\hat{P}$ on $(\Omega, \mathcal{F}_T, P)$ by

$$d\hat{P}/dP = \exp\{\int_0^T \gamma(s, N_s)/\sigma \; dB_s - \frac{1}{2}\int_0^T (\gamma(s, N_s)/\sigma)^2 ds\}.$$

By the very definition (6.2) we have $0 \le \gamma(t,x) \le 1$; so Girsanov's theorem yields that

$$\hat{B}_t := B_t - \int_0^t \gamma(s, N_s)/\sigma \; ds \quad \text{is a Brownian motion on } (\Omega, \mathcal{F}_T, \hat{P}).$$

Hence

$$dN_t = [(N_t)^\alpha - \mu N_t - \gamma(t, N_t) N_t]dt - \sigma N_t d\hat{B}_t, \quad \text{under } \hat{P}.$$

Thus, (6.1) admits a positive weak solution.

Now, by (6.2), we have

$$\gamma(t,x)x = \min\{(U')^{-1} \circ v_x(t,x), x\}.$$

Also, by (1.6) and concavity

$$\frac{\partial}{\partial x}(U')^{-1} \circ v_x(t,x) = \frac{v_{xx}(t,x)}{U'' \circ (U')^{-1} \circ v_x(t,x)} \ge 0.$$

Thus, $\gamma(t,x)x$ is nondecreasing on $(0,\infty)$ for each t. We rewrite (6.1) as the form of (2.10). Then, we see that the pathwise uniqueness holds for (6.1). Therefore, by the Yamada-Watanabe theorem [3], we deduce that (6.1) admits a unique strong solution $\{X_t^*\}$.

Theorem 6.2 *We assume (1.6). Then the optimal consumption policy* $\{c_t^*\}$ *is given by the feedback form*

$$c_t^* = c^\star(t, z_t^*, y_t), \tag{6.3}$$

where $c^\star(t,z,y) = I(z/y, yV_z(t,z,y))$ *and* $\{z_t^*\}$ *is the unique solution of*

$$\dot{z}_t^* = F(z_t^*, y_t) - \nu z_t^* - c_t^* z_t^*, \quad 0 < t \leq T, \quad z_0^* = z > 0. \tag{6.4}$$

Proof. We set $X_t^\star = z_t^*/y_t$. By Itô's formula and (2.2), we see that $X_t^\star$ solves (6.1). Therefore, by Lemma 6.1, there exists a unique positive solution $\{z_t^*\}$ of (6.4).

By Theorems 4.2 and 5.1, we note that

$$0 < v_x(t,x)x \leq v(t,x) - v(t,0+) \leq v(t,x), \quad x > 0.$$

Hence, by (3.6) and (2.7)

$$\begin{aligned} E[\int_0^T \{v_x(s, X_s^\star)X_s^\star\}^2 ds] &\leq E[\int_0^T \{v(s, X_s^\star)\}^2 ds] \\ &\leq E[\int_0^T \zeta(s, X_s^\star)^2 ds] < \infty. \end{aligned}$$

By (2.2), this yields that $\{\int_0^t \sigma y_s V_y(s, z_s^*, y_s) dB_s\}$ is a martingale. By (1.5) and (6.2), $c^\star$ satisfies

$$V_t + \frac{1}{2}\sigma^2 y^2 V_{yy} + nyV_y + \{F(z,y) - \lambda z\}V_z + \{U(c^\star z/y) - c^\star z V_z\} = 0.$$

Applying Itô's formula to (1.1) and (6.4), we get

$$E[V(T, z_T^*, y_T)] = V(0,z,y) - E[\int_0^T U(c_t^* z_t^*/y_t)dt],$$

which implies

$$E[\int_0^T U(c_t^* z_t^*/y_t)dt] = V(0,z,y).$$

By the same calculation as above, we obtain

$$E[\int_0^T U(c_t z_t/y_t)dt] \le V(0,z,y)$$

for any $c \in \mathcal{A}$. The proof is complete.

Remark 6.3 *From the proof of Theorem 6.2, it follows that*

$$\inf_{c\in\mathcal{A}} E[\int_s^T U(c_t z_t/y_t)dt] = V(s,z,y).$$

Thus, under (1.6), we see that the smooth solution V of the HJB equation (1.5) is unique. Furthermore, let u be the solution of (4.1) on the entire domain $[0,T) \times (0,\infty)$ with $u(T,x) = 0,\ x > 0$. Setting $x = z/y$ and $V(t,z,y) = u(t,z/y)$ for $z, y > 0$, by (2.2), we have that V satisfies (1.5). Therefore, we obtain the uniqueness of u.

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