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Maximum Principle of Stochastic Controlled Systems of Functional Type *)

Zhou Xunyu (周迅宇)
Institute of Mathematics, Fudan University

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Abstract. This paper studies the optimal controls of stochastic systems of functional type with end constraints. The systems considered may be degenerate and the control region may be nonconvex. A stochastic maximum principle is derived. The method is based on the idea that stochastic systems are essentially infinite dimensional systems.

§1. Introduction

A number of studies have been devoted to the necessary conditions of the optimal controls of stochastic systems. Many of these studies, however, are limited to the Markovian type, i.e., the systems are related only to the current state instead of the whole "past". For example, Kushner[3] has obtained a maximum principle of the systems with a finite numbers of inequality constraints by employing the variational theory of Neustadt. Bensoussan[1] has proved a maximum principle under the assumption that the control region is convex. But it seems more natural to study the stochastic systems of functional type, e.g., time delay system is an example of such systems. Works on the functional type are mainly those of Haussmann[2,3], but his method is based on the Girsanov transformation which is effective only for the nondegenerate systems.

The purpose of this paper is to derive a stochastic maximum principle for the systems of functional type. A rather general case is attacked: the systems may be degenerate; an end constraint is posed; the control region is arbitrary. Our method is different from those of the existing results in literature; it is based on the idea that a stochastic integral $\int \lambda(t, \omega)dt$ may be regarded as a Bochner integral $(B)\int \lambda(t, \cdot)dt$ valued at ω, which allows us to apply the theory developed by Li and Yao[6] for distributed parameter systems.

The paper is organized as follows: In Section 2 we formulate the problem and give some basic notations and assumptions. Section 3 is devoted to the study of the variation equation, which is a linear stochastic differential equation of functional type. A variation-of-constants formula is proved. In Section 4, we give the proof of the maximum principle.

§2. Preliminaries

Let $(\Omega, \mathscr{F}, P)$ be a standard probability space with a right-continuous increasing family $\{\mathscr{F}_t: 0 \leqslant t \leqslant 1\}$ of sub σ-fields of $\mathscr{F}$, each containing all P-null sets. Let $\{B(t): 0 \leqslant t \leqslant 1\}$ be an r-dimensional $\mathscr{F}_t$-Brownian motion. Consider the following stochastic controlled system:

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$$\begin{cases} dx(t) = \sigma(t, x)dB(t) + f(t, x, u(t))dt, & 0 \leqslant t \leqslant 1, \\ x(0) = x_0 \in R^d. \end{cases} \tag{2.1}$$

Let the cost functional be

$$J(u) := E\int_0^1 f^0(S, x^u, u(s))ds, \; u \in U_{ad}, \tag{2.2}$$

with the end constraint

$$x^u(1) \in Q \subset L^2(\Omega; R^d), \tag{2.3}$$

where the totality of admissible controls is defined as

$U_{ad} := \{u : u$ is a Γ-valued $\mathscr{F}_t$-adapted measurable process on $[0, 1]\}$,

here Γ being an arbitrarily prescribed subset in R^m.

The optimal control problem is to find such controls among U_{ad} satisfying (2.3) so as to achieve the minimum of the cost (2.2).

Here are some notations throughout this paper. Let X, Y be Banach spaces. $\mathscr{B}(X)$ denotes the topological σ-field in X; $L(X \to Y)$ denotes the totality of linear continuous mappings from X to Y. We denote by C the space of R^d-valued continuous functions defined on $[0, 1]$, under the sup norm; $\mathscr{B}_t(C)$ is the σ-field generated by $\{x : x \in C, x(t_1) \in G_1, \cdots, x(t_m) \in G_m\}$ where $0 \leqslant t_i \leqslant t$, G_i are Borel sets in R^d, $i = 1, 2, \cdots, m$ and $m = 1, 2, \cdots$; for $x \in C$, $t \in [0, 1]$, $x_t := x(t \wedge \cdot)$, $\|x\|_t := \|x_t\|$. Finally, we define

$A^{m,n} := \{a : a$ is a measurable mapping from $[0, 1] \times C$ to $R^{m\times n}$; and for each $t \in [0, 1]$, $a(t, \cdot)$ is $\mathscr{B}_t(C)/\mathscr{B}(R^{m\times n})$-measurable$\}$,

where $R^{m\times n} :=$ totality of $m \times n$ matrices.

Remark 2. 1. To simplify notations, we shall assume $r = 1$ (i. e., the Brownian motion is one-dimensional) in the sequel. There is no essential difficulty when $r > 1$.

Definition 2.1. Suppose T is a mapping from C to R^d. T is called Fréchet differentiable at $\hat{x} \in C$ if there exists $S \in L(C \to R^d)$ such that

$$T(y) = T(\hat{x}) + S(y - \hat{x}) + o(\|y - \hat{x}\|), \text{ for any } y \in C. \tag{2.4}$$

Furthermore, the operator S is called the Fréchet differential of T at $\hat{x}$. We denote $T_x(\hat{x}) := S$.

The following assumptions remains in force throughout this paper:

(A1) $\sigma \in A^{d,1}$; for any $u \in \Gamma$, $f(\cdot, \cdot, u) \in A^{d,1}$, $f^0(\cdot, \cdot, u) \in A^{1,1}$;

(A2) For any $(t, x) \in [0,1] \times C$, $f(t, x, \cdot)$ and $f^0(t, x, \cdot)$ are continuously

(A3) For any $(t, u) \in [0,1] \times \Gamma$, $\sigma(t, \cdot)$, $f(t, \cdot, u)$ and $f^0(t, \cdot, u)$ are continuous; Fréchet differentiable; and $\sigma_x(\cdot, \cdot)$, $f_x(\cdot, \cdot, \cdot)$ and $f^0_x(\cdot, \cdot, \cdot)$ are measurable;

(A4) There exists a finite nonnegative measure M on $[0, 1]$ and a positive constant K such that

$$|\sigma(t, x) - \sigma(t, y)|^2 + |f(t, x, u) - f(t, y, u)|^2 + |f^0(t, x, u) - f^0(t, y, u)|^2$$

$$\leqslant K\left(\int_0^t |x(s) - y(s)|^2 dM(s) + |x(t) - y(t)|^2\right), \text{ for any } u \in \Gamma, \; x, y \in C;$$

$$|\sigma(t, x)|^2 + |f(t, x, u)|^2 + |f^0(t, x, u)|^2 \leqslant K(1 + \|x\|_t^2), \text{ for any } u \in \Gamma, \; x \in C;$$

(A5) Q is a relatively convex body with finite condimension in $L^2(\Omega; R^d)$(for definition see [6]).

Remark 2.2. Under the above assumptions, equation (2.1) admits a unique solution x^u for any $u\in U_{ad}$ (see Appendix).

Remark 2.3. We have by (A3), (A4),

$$|\sigma_x(t,z)y|^2=\lim_{\lambda\downarrow 0}\left|\frac{\sigma(t,z+\lambda y)-\sigma(t,z)}{\lambda}\right|^2\leqslant K\left(\int_0^t|y(s)|^2dM(s)+|y(t)|^2\right). \quad (2.5)$$

The same conclusion holds for f_x and f_x^0.

§3. Variation Equation

In this section we will derive a variation-of-constants formula for the variation equation. Suppose $(\hat{x},\hat{u})$ is optimal for the problem (2.1)–(2.3), the variation equation is a linear stochastic differential equation of functional type as follows:

$$\begin{cases} d\delta x^v(t)=\sigma_x(t,\hat{x})\delta x^v dB(t)+f_x(t,\hat{x},\hat{u}(t))\delta x^v dt+(f(t,\hat{x},v(t)) \\ \qquad -f(t,\hat{x},\hat{u}(t)))dt, \\ \delta x^v(0)=0, \end{cases} \quad (3.1)$$

where $v\in U_{ad}$.

In general we consider the following equation:

$$\begin{cases} dy(t)=\sigma_x(t,\hat{x})ydB(t)+f_x(t,\hat{x},\hat{u}(t))ydt+h(t)dt, \\ y(0)=0, \end{cases} \quad (3.2)$$

where h is an R^d-valued $\mathscr{F}_t$-adapted measurable process on $[0,1]$.

It is easy to prove that for any $p\geqq 2$, there exists a constant $k=k(p)>0$ such that (cf. [2])

$$\sup_{u\in\Gamma}E\|x^u\|^p\leqslant k. \quad (3.3)$$

So we may assume that

$$E\int_0^1|h(t)|^2dt<+\infty. \quad (3.4)$$

We denote by $V_0[0,1]$ the set of all functions from $[0,1]$ to $R^{d\times d}$ such that they are left continuous, of bounded variation and vanish at 1. By Rietz's representation theorem, there exist $\eta_i(\cdot,t,\omega)\in V_0[0,1]$, $i=1,2$, such that

$$\sigma_x(t,\hat{x}(\omega))y=\int_0^1 d_\theta\eta_1(\theta,t,\omega)y(\theta),$$

$$f_x(t,\hat{x}(\omega),\hat{u}(t,\omega))y=\int_0^1 d_\theta\eta_2(\theta,t,\omega)y(\theta),\ \text{for any } y\in C.$$

Lemma 3.1. *For any $y\in C$, we have*

$$\sigma_x(t,\hat{x}(\omega))y=\int_0^t d_\theta\eta_1(\theta,t,\omega)y(\theta), \quad (3.5)$$

$$f_x(t,\hat{x}(\omega),\hat{u}(t,\omega))y=\int_0^t d_\theta\eta_2(\theta,t,\omega)y(\theta). \quad (3.6)$$

Proof. Since $\sigma \in A^{d,1}$, it is well known that for any $y, z \in C$, if $y_t = z_t$, then $\sigma(t, y) = \sigma(t, z)$. Hence for any $y \in C$,

$$\sigma_x(t, x(\omega))y = \lim_{\lambda \downarrow 0} \frac{\sigma(t, \hat{x}(\omega) + \lambda y) - \sigma(t, \hat{x}(\omega))}{\lambda}$$

$$= \lim_{\lambda \downarrow 0} \frac{\sigma(t, \hat{x}(\omega) + \lambda y_t) - \sigma(t, \hat{x}(\omega))}{\lambda} = \sigma_x(t, \hat{x}(\omega))y_t.$$

So for any $\varphi \in C([t, 1] \to R^d)$ we have $\int_t^1 d_\theta \eta_1(\theta, t, \omega)\varphi(\theta) = 0$, which leads to $\eta_1(\cdot, t, \omega)|_{(t,1)} = 0$. This concluds (3.5). Similarly for (3.6). □

Lemma 3.2. *η_i is a measurable mapping from $[0, 1] \times [0, 1] \times \Omega$ to $R^{d\times d}$. Moreover, for any $\theta \in [0, 1]$, $\eta_i(\theta, \cdot, \cdot)$ is an $R^{d\times d}$-valued $\mathscr{F}_t$-adapted process on $[0, 1]$ $(i = 1, 2)$.*

Proof. Denote by $\mathscr{B}_\Omega$ the totality of measurable subsets of $[0, 1] \times \Omega$ whose t-sections belong to $\mathscr{F}_t$; by $\mathscr{B}_c$ the totality of measurable subsets of $[0, 1] \times C$ whose t-sections belong to $\mathscr{B}_t(C)$. $\mathscr{B}_\Omega$ and $\mathscr{B}_c$ are σ-fields on $[0, 1] \times \Omega$ and $[0, 1] \times C$ respectively. It is easy to see by assumptions (A1) and (A3) that $\sigma_x(\cdot, \cdot)$ is a $\mathscr{B}_c / \mathscr{B}(L(C \to R^d))$-measurable mapping from $[0, 1] \times C$ to $L(C \to R^d)$, but the mapping: $(t, \omega) \in [0, 1] \times \Omega \to (t, \hat{x}(\omega)) \in [0, 1] \times C$ is $\mathscr{B}_\Omega / \mathscr{B}_c$-measurable. All these facts together with the isomorphism of Riesz's representation imply that the mapping: $(t, \omega) \in [0, 1] \times \Omega \to \eta_1(\cdot, t, \omega) \in V_0[0, 1]$ is $\mathscr{B}_\Omega / \mathscr{B}(V_0[0, 1])$-measurable. Since $\eta_1(\cdot, t, \omega)$ is left continuous, therefore

$$\eta_1(\theta, t, \omega) = \lim_{n \to \infty} \eta_{1,n}(\theta, t, \omega),$$

where

$$\eta_{1,n}(\theta, t, \omega) := \sum_{i=1}^{n-1} \eta_1\left(\frac{i-1}{n}, t, \omega\right)\chi_{\left\{\frac{i-1}{n} \leq \theta < \frac{i}{n}\right\}}(\theta) + \eta_1\left(\frac{n-1}{n}, t, \omega\right)\chi_{\left\{\frac{n-1}{n} \leq \theta \leq 1\right\}}(\theta).$$

This yields the desired results for η_1. Similarly for η_2. □

For $\tau \in [0, 1]$, let $\varphi(\cdot, \tau)$ be a solution of the following matrix-valued equation:

$$\varphi(t, \tau) = I + \int_\tau^t \int_\tau^1 d_\theta \eta_1(\theta, s)\varphi(\theta, \tau)dB(s) + \int_\tau^t \int_\tau^1 d_\theta \eta_2(\theta, s)\varphi(\theta, \tau)ds, \quad t \in [\tau, 1]; \tag{3.7}$$

$$\varphi(t, \tau) = 0, \quad t \in [0, \tau). \tag{3.8}$$

where $I :=$ the $d \times d$ identity matrix.

(3.7) is a stochastic differential equation of functional type on $[\tau, 1]$. Since its coefficients satisfy Lipschitz condition, there exists uniquely on $[\tau, 1]$ an $R^{d\times d}$-valued $\mathscr{F}_t$-adapted continuous process satisfying (3.7) due to the following Lemma 3.3 (see Appendix).

Let $C[\tau, 1] := C([\tau, 1] \to R^d)$, and we can define $\mathscr{B}_t(C[\tau, 1])$ in a similar way to $\mathscr{B}_t(C)$ for $t \in [\tau, 1]$.

Lemma 3.3. *Suppose a_i is a mapping from $[\tau, 1]\times\Omega\times C[\tau, 1]$ to R^d:*

$$a_i(t, \omega, y) := \int_\tau^1 d_\theta \eta_i(\theta, t, \omega) y(\theta) \qquad (i=1, 2). \tag{3.9}$$

Then a_i is measurable. Moreover, $a_i(t, \cdot, y)$ is $\mathscr{F}_t$-measurable and $a_i(t, \omega, \cdot)$ is $\mathscr{B}_t(C[\tau, 1])$-measurable.

Proof. 1° When $\tau=0$, we have

$$\int_0^1 d_\theta \eta_1(\theta, t, \omega) y(\theta) = \sigma_x(t, \hat{x}(\omega)) y = \lim_{\lambda \downarrow 0} \frac{\sigma(t, \hat{x}(\omega)+\lambda y)-\sigma(t, \hat{x}(\omega))}{\lambda},$$

but it is easy to see that the mapping: $(t, \omega, y)\in[0, 1]\times\Omega\times C[0, 1] \to \sigma(t, \hat{x}(\omega) + \lambda y) \in R^d$ is a composition of some measurable mappings, therefore is measurable itself.

2° For a general $\tau\in(0, 1]$, if $y\in C[\tau, 1]$, define $\tilde{y}\in C[0, 1]$ as follows:

$$\tilde{y}(s) := \begin{cases} y(\tau), & \text{if } s\in[0, \tau], \\ y(s), & \text{if } s\in(\tau, 1]. \end{cases}$$

Then

$$a_1(t, \omega, y) = \int_0^1 d_\theta \eta_1(\theta, t, \omega)\tilde{y}(\theta) - (\eta_1(\tau+, t, \omega) - \eta_1(0, t, \omega)) y(\tau),$$

so we get the first two assertions of the lemma by 1° and Lemma 3.2. Further, note that if $y, z\in C[\tau, 1]$, and $y|_{[\tau, t]} = z|_{[\tau, t]}$, then by Lemma 3.1 we have $a_1(t, \omega, y) = a_1(t, \omega, z)$ which yields the last assertion of the lemma.

3° Similarly for a_2. □

The solution $\varphi(t, \tau, \omega)$ of (3.7) is not necessarily measurable in τ, which will cause some difficulty in the sequel. To solve this technical problem, we shall make a "good" modification of φ. Now let us introduce a result about the existence of a measurable modification.

Let S be a metric space. Denote by $L^0(\Omega; S)$ the totality of S-valued random variables on $(\Omega, \mathscr{F}, P)$. $L^0(\Omega; S)$ becomes a topological vector space with the following psedo-metric (cf. [7]):

$$d(X, Y) := \inf_{\varepsilon>0} [\varepsilon + P\{\omega: d'(X(\omega), Y(\omega))>\varepsilon\}],$$

where d' is the metric on S.

Remark 3.1. It is easy to see that $d(X_n, X)$ goes to zero whenever X_n goes to X in probability.

Lemma 3.4 ([7, p. 16]). *Suppose T and S are complete separable metric spaces. λ is a mapping from $T\times\Omega$ to S; and for any $t\in T$, $\lambda(t, \cdot)\in L^0(\Omega; S)$. Suppose λ is measurable when regarded as a mapping from T to $L^0(\Omega; S)$. Then there exists a measurable mapping $\tilde{\lambda}$ from $T\times\Omega$ to S such that for any $t\in T$, $P\{\omega: \tilde{\lambda}(t, \omega) = \lambda(t, \omega)\}=1$.*

Lemma 3.5. *There exists a measurable function $\tilde{\varphi}(t, \tau, \omega)$: $[0, 1]\times[0, 1]\times\Omega \to R^{d\times d}$ satisfying (3.7) and (3.8).*

Proof. For each fixed $\tau\in[0, 1]$, there exists $\varphi(t, \tau)$ $(t\geqslant\tau)$ satisfying (3.7). φ can be regarded as a mapping from Δ to $L^0(\Omega; R^{d\times d})$ where $\Delta := \{(t, \tau): 0\leqslant\tau\leqslant t\leqslant 1\}$. We want to show that

$$\lim_{(t_n,\tau_n)\uparrow(t,\tau)} E|\varphi(t_n, \tau_n)-\varphi(t, \tau)|^2=0, \tag{3.10}$$

where $(t_n,\tau_n)\uparrow(t,\tau)$ means: $(t_n,\tau_n)\in\Delta$, $(t,\tau)\in\Delta$, $t_n\uparrow t$, $\tau_n\uparrow\tau$. We assume $\eta_2=0$ for simplicity. First we claim that

$$\lim_{n\to\infty} E \sup_{\tau\leqslant\theta\leqslant 1} |\varphi(\theta, \tau_n)-\varphi(\theta,\tau)|^2=0. \tag{3.11}$$

Indeed, when $\theta\geqslant\tau$, we can write

$$\varphi(\theta,\tau_n)-\varphi(\theta,\tau)=\int_{\tau_n}^{\tau}\int_{\tau_n}^{1} d_\beta\eta_1(\beta,s)\varphi(\beta,\tau_n)dB(s)+\int_{\tau}^{\theta}\int_{\tau_n}^{\tau} d_\beta\eta_1(\beta,s)\varphi(\beta,\tau_n)dB(s)$$
$$+\int_{\tau}^{\theta}\int_{\tau}^{s} d_\beta\eta_1(\beta,s)(\varphi(\beta,\tau_n)-\varphi(\beta,\tau))dB(s). \tag{3.12}$$

Denote by $\bigvee_a^b \eta_1$ the total variation of η_1 on $[a, b]$. Then (3.12) yields

$$E\sup_{\tau\leqq\theta\leqq h}|\varphi(\theta,\tau_n)-\varphi(\theta,\tau)|^2\leqslant\text{const.}\left(\int_{\tau_n}^{\tau} E\left|\bigvee_{\tau_n}^{1}\eta_1(\cdot,s)\cdot\sup_{\tau_n\leqq\beta\leqq 1}|\varphi(\beta,\tau_n)|\right|^2 ds\right.$$
$$+\int_{\tau}^{h} E\left|\bigvee_{\tau_n}^{\tau}\eta_1(\cdot,s)\sup_{\tau_n\leqslant\beta\leqslant 1}|\varphi(\beta,\tau_n)|\right|^2 ds+\int_{\tau}^{h} E\left|\bigvee_{\tau}^{1}\eta_1(\cdot,s)\sup_{\tau\leqslant\beta\leqslant s}|\varphi(\beta,\tau_n)\right.$$
$$\left.\left.-\varphi(\beta,\tau)|\right|^2 ds\right)$$

$$\leqslant\text{const.}\left(|\tau_n-\tau|+E\int_{\tau}^{1}\left|\bigvee_{\tau_n}^{\tau}\eta_1(\cdot,s)\right|^2\cdot\sup_{\tau_n\leqslant\beta\leqslant 1}|\varphi(\beta,\tau_n)|^2 ds\right.$$
$$\left.+\int_{\tau}^{h} E\sup_{\tau\leqslant\beta\leqslant s}|\varphi(\beta,\tau_n)-\varphi(\beta,\tau)|^2 ds\right).$$

Similar to (3.3) we can verify that $\sup_{0\leqslant\tau\leqslant 1} E\sup_{\tau\leqslant\theta\leqslant 1}|\varphi(\theta,\tau)|^p\leqslant k(p)$, and by (2.5) we know that $\bigvee_0^1\eta_1(\cdot, t,\omega)\leqslant K(\int_0^1 dM(\theta)+1)$. Moreover, $\bigvee_{\tau_n}^{\tau}\eta_1(\cdot,s)\to 0$ as $\tau_n\uparrow\tau$ due to the left continuity of $\eta_1(\cdot, s)$. Hence (3.11) follows from the Gronwall's inequality.

Now suppose $(t_n,\tau_n)\uparrow(t,\tau)$. We may assume that $\tau_n\leqslant\tau\leqslant t_n\leqslant t$. Then we have

$$\varphi(t_n,\tau_n)-\varphi(t,\tau)=\int_{\tau_n}^{\tau}\int_{\tau_n}^{1} d_\theta\eta_1(\theta, s)\varphi(\theta,\tau_n)dB(s)$$
$$+\int_{\tau}^{t_n}\int_{\tau_n}^{\tau} d_\theta\eta_1(\theta,s)\varphi(\theta,\tau_n)dB(s)+\int_{\tau}^{t_n}\int_{\tau}^{1} d_\theta\eta_1(\theta,s)(\varphi(\theta,\tau_n)$$
$$-\varphi(\theta,\tau))dB(s)-\int_{t_n}^{t}\int_{\tau}^{1} d_\theta\eta_1(\theta,s)\varphi(\theta,\tau)dB(s).$$

Appealing to (3.11), and estimating the above term by term, we get (3.10).

Applying Lemma 3.4 and Remark 3.1 ($T:=\Delta$, $S:=R^{d\times d}$), we get that there exists a measurable $\hat{\varphi}:\Delta\times\Omega\rightarrow R^{d\times d}$ such that for any $(t,\tau)\in\Delta$, $P\{\omega: \hat{\varphi}(t,\tau,\omega)=\varphi(t,\tau,\omega)\}=1$. But when τ is fixed, it is easy to see that
$E|\hat{\varphi}(t_1,\tau)-\hat{\varphi}(t_2,\tau)|^4=E|\varphi(t_1,\tau)-\varphi(t_2,\tau)|^4\leqslant \text{const.}|t_1-t_2|^2 \quad (t_1, t_2\geqslant\tau)$.
By the proof of Komogorov's theorem (cf. [4, Th. 1.4.3.]), $\lim\limits_{\substack{r\rightarrow t+\\ r\in D}}\hat{\varphi}(r,\tau,\omega)$ exists with probability 1, where D is the set of all binary rationals on $[0, 1]$. Hence it follows by Fubini's theorem that there exists on $[0,1]\times\Omega$ a null set A_0 such that $\lim\limits_{\substack{r\rightarrow t+\\ r\in D}}\hat{\varphi}(r,\tau,\omega)$ exists when $(\tau,\omega)\overline{\in}A_0$. Set

$$\widetilde{\varphi}(t,\tau,\omega)=\begin{cases}\lim\limits_{\substack{r\rightarrow t+\\ r\in D}}\hat{\varphi}(r,\tau,\omega), & \text{if } (t,\tau,\omega)\in D^c\times A_v^c,\\ \hat{\varphi}(t,\tau,\omega), & \text{otherwise}.\end{cases}$$

It is easy to show that for any $\tau\in[0,1]$, $P\{\omega:\widetilde{\varphi}(t,\tau,\omega)=\varphi(t,\tau,\omega)$, for any $t\in[\tau,1]\}=1$. Therefore $\widetilde{\varphi}$ satisfies (3.7) which completes the proof. □

Remark 3.2. By virtue of the above lemma, we can assume that φ in (3.7) and (3.8) is measurable in (t,τ,ω).

Lemma 3.6 (*Stochastic Fubini's theorem*). *Suppose* λ *is a mapping from* $[0,1]\times[0,1]\times\Omega$ *to* R^d *satisfying*:

1° λ *is measurable*;

2° *For any* $\tau\in[0,1]$, $\lambda(\cdot,\tau,\cdot)$ *is an* $\mathscr{F}_t$*-adapted process*;

3° $E\int_0^1\int_0^1|\lambda(s,\tau)|^2\,dsd\tau<+\infty$.

Then for any $[a,b]\subset[0,1]$, *we have*

$$\int_a^b\int_a^s\lambda(s,\tau)d\tau dB(s)=\int_a^b\int_\tau^b\lambda(s,\tau)dB(s)d\tau,\quad \text{P-a.s.}$$

Proof. Using a standard argument (see for example [4, Lemma 2.1.1]), we can show that there exists a sequence of step function λ_n such that $E\int_0^1\int_0^1|\lambda_n(s,\tau)-\lambda(s,\tau)|^2\,dsd\tau\rightarrow 0$ as $n\rightarrow\infty$. But the lemma is true for step functions.

Theorem 3.1. *The solution of* (3.2) *can be represented by*

$$y(t,\omega)=\int_0^t\varphi(t,\tau,\omega)h(\tau,\omega)d\tau. \tag{3.13}$$

Proof. The Lebesque integral in (3.13) is well defined owing to the measurability of φ and (3.4). Define $\widetilde{y}(t):=\int_0^t\varphi(t,\tau)h(\tau)d\tau$. Then

$$\begin{aligned}\int_0^t\int_0^s d_\theta\eta_1(\theta,s)\widetilde{y}(\theta)dB(s)&=\int_0^t\int_0^s d_\theta\eta_1(\theta,s)\int_0^\theta\varphi(\theta,\tau)h(\tau)d\tau dB(s)\\ &=\int_0^t\int_0^s d_\theta\eta_1(\theta,s)\int_0^s\varphi(\theta,\tau)h(\tau)d\tau dB(s)\\ &=\int_0^t\int_0^s\int_0^s(d_\theta\eta_1(\theta,s)\varphi(\theta,\tau))h(\tau)d\tau dB(s)\\ &=\int_0^t\left(\int_\tau^t\int_0^s d_\theta\eta_1(\theta,s)\varphi(\theta,\tau)dB(s)\right)h(\tau)d\tau\\ &=\int_0^t\left(\int_\tau^t\int_\tau^1 d_\theta\eta_1(\theta,s)\varphi(\theta,\tau)dB(s)\right)h(\tau)d\tau,\end{aligned}$$

where the exchangeability of integrations holds due to Lemmas 3.2, 3.5 and 3.6. Similarly, we have

$$\int_0^t\int_0^s d_\theta\eta_2(\theta,s)\tilde{y}(\theta)ds=\int_0^t\left(\int_\tau^t\int_\tau^1 d_\theta\eta_2(\theta,s)\varphi(\theta,\tau)ds\right)h(\tau)d\tau,$$

hence $\tilde{y}$ is a continuous process satisfying (3.2). Now the desired result follows from the uniqueness of solutions of (3.2). □

§4. Maximum Principle

In this section we will derive the maximum principle.

The following lemma will play an essential role in this paper.

Lemma 4.1. *Suppose λ is a mapping from $[0,1]\times\Omega$ to R^d such that it is an $\mathscr{F}_t$-adapted measurable process and $E\int_0^1|\lambda(t)|^2dt<+\infty$. Then when regarded as a mapping from $[0,1]$ to $L^2(\Omega;R^d)$, λ is Bochner integrable. Moreover,*

$$\left((B)\int_0^1\lambda(t,\cdot)dt\right)(\omega)=\int_0^1\lambda(t,\omega)dt,\quad \text{P-a.s.}\omega. \tag{4.1}$$

Proof. It is a known result that there exists a sequence of step functions λ_n (here "step function" is in the sense of an $L^2(\Omega;R^d)$-valued function) such that

$$E\int_0^1|\lambda_n(t)-\lambda(t)|^2dt\to 0,\quad \text{as } n\to\infty. \tag{4.2}$$

Hence we may choose a subsequence $\lambda_{n'}$ satisfying

$$E|\lambda_{n'}(t)-\lambda(t)|^2\to 0,\quad \text{as } n'\to\infty,\ \text{for a.e. } t\in[0,1],$$

which implies λ is a strongly measurable $L^2(\Omega;R^d)$-valued function, hence is Bochner integrable. Moreover it is clear that (4.1) is true for step functions, so is for λ by (4.2). □

Suppose (x,u) is optimal for the problem (2.1)–(2.3). For any $v\in U_{ad}$, set

$$x_0^v(t):=E\int_0^t f^0(s,x^v,v(s))ds, \tag{4.3}$$

$$\delta x_0^v(t):=E\left\{\int_0^t f_x^0(s,\hat{x},\hat{u}(s))\delta x^v ds+\int_0^t(f^0(s,\hat{x},v(s))-f^0(s,\hat{x},\hat{u}(s)))ds\right\}, \tag{4.4}$$

where δx^v is determined by (3.1). Now we define a convex cone in $R^1\times L^2(\Omega;R^d)$ as follows:

$$\Xi:=\left\{\sum_{i=1}^n\lambda_i(\delta x_0^{v_i}(1),\ \delta x^{v_i}(1))\,|\,\lambda=(\lambda_1\cdots,\lambda_n)\geqslant 0,\ \sum_{i=1}^n\lambda_i=1,\ v_i\in U_{ad},\ i=1,\cdots,n;\ n=1,2,\cdots\right\}.$$

Theorem 4.1. *For any $\xi\in\Xi$, there exists an $\varepsilon_0>0$ such that whenever $\varepsilon\in[0,\varepsilon_0]$ there is a $u^\varepsilon\in U_{ad}$ satisfying*

$$(x_0^\varepsilon(1),\ x^\varepsilon(1))=(\hat{x}_0(1),\ \hat{x}(1))+\varepsilon\xi+o(\varepsilon), \tag{4.5}$$

where $x_0^\varepsilon := x_0^{u^\varepsilon}$, $x^\varepsilon := x^{u^\varepsilon}$, $\hat{x}_0 := x_0^{\hat{u}}$, *and* $o(\varepsilon)$ *is concerned with the strong topology of* $R^1 \times L^2(\Omega;\ R^d)$.

Proof. For any $\xi \in \Xi$, write $\xi := \sum_{i=1}^{n} \lambda_i(\delta x_0^{v_i}(1), \delta x^{v_i}(1))$. Thanks to Lemma 4.1, we can apply the vector-valued measure theorem [6, Lemma 1] which yields that when ε is sufficiently small, there exists a family of Lebesque measurable sets E_i^ε in $[0, 1]$ $(i=1, \cdots, n]$ with $E_i^\varepsilon \cap E_j^\varepsilon = \varphi\ (i \neq j)$ and $\sum_{i=1}^{n} \mu(E_i^\varepsilon) = \varepsilon$, where μ is the Lebesque measure, such that

$$\varepsilon \sum_{i=1}^{n} \lambda_i \int_0^t \begin{pmatrix} E[f^0(s, \hat{x}, v_i(s)) - f^0(s, \hat{x}, \hat{u}(s))] \\ f(s, \hat{x}, v_i(s)) - f(s, \hat{x}, \hat{u}(s)) \end{pmatrix} ds$$

$$= \sum_{i=1}^{n} \int_{[0,t] \cap E_i^\varepsilon} \begin{pmatrix} E[f^0(s, \hat{x}, v_i(s)) - f^0(s, \hat{x}, \hat{u}(s))] \\ f(s, \hat{x}, v_i(s)) - f(s, \hat{x}, \hat{u}(s)) \end{pmatrix} ds + \begin{pmatrix} r_0(t, \varepsilon) \\ r_1(t, \varepsilon) \end{pmatrix}, \tag{4.6}$$

$$|r_0(t, \varepsilon)| + \|r_1(t, \varepsilon)\|_{L^2(\Omega; R^d)} \leqq \theta(\varepsilon) = o(\varepsilon). \tag{4.7}$$

Define

$$u^\varepsilon(t) := \begin{cases} v_i(t), & \text{if } t \in E_i^\varepsilon, \\ \hat{u}(t), & \text{if } t \in [0, 1] \setminus \bigcup_{i=1}^{n} E_i^\varepsilon. \end{cases}$$

Then $u^\varepsilon \in U_{ad}$, and we can write

$$x^\varepsilon(t) - \hat{x}(t) = \int_0^t (\sigma(s, x^\varepsilon) - \sigma(s, \hat{x}))dB(s) + \int_0^t (f(s, x^\varepsilon, u^\varepsilon(s)) - f(s, \hat{x}, u^\varepsilon(s)))ds + \sum_{i=1}^{n} \int_{[0,t] \cap E_i^\varepsilon} (f(s, \hat{x}, v_i(s)) - f(s, \hat{x}, \hat{u}(s)))ds. \tag{4.8}$$

It follows by Gronwall's inequality,

$$E\|x^\varepsilon - \hat{x}\|^2 \to 0, \quad \text{as } \varepsilon \to 0. \tag{4.9}$$

Set $\delta x_\lambda := \sum_{i=1}^{n} \lambda_i \delta x^{v_i}$. We want to show that

$$x^\varepsilon(t) = \hat{x}(t) + \varepsilon \delta x_\lambda(t) + r(t, \varepsilon), \tag{4.10}$$

where

$$\|r(t, \varepsilon)\|_{L^2(\Omega; R^d)} \leqslant \theta(\varepsilon) = o(\varepsilon).$$

In fact, (4.8) and (4.6) imply

$$x^\varepsilon(t) - x(t) = \int_0^t \int_0^1 \sigma_x(s, \hat{x} + \theta(x^\varepsilon - \hat{x}))(x^\varepsilon - \hat{x}) d\theta dB(s) + \int_0^t \int_0^1 f_x(s, \hat{x} + \theta(x^\varepsilon - \hat{x}), u^\varepsilon(s))(x^\varepsilon - \hat{x}) d\theta ds + \varepsilon \sum_{i=1}^{n} \lambda_i \int_0^t (f(s, \hat{x}, v_i(s)) - f(s, \hat{x}, \hat{u}(s)))ds - r_1(t, \varepsilon). \tag{4.11}$$

Denote $y^\varepsilon := \dfrac{x^\varepsilon - \hat{x}}{\varepsilon} - \delta x_\lambda$. Then noting (3.1), we have

$$y^\varepsilon(t) = \int_0^t \int_0^1 \sigma_x(s, \hat{x} + \theta(x^\varepsilon - \hat{x}))y^\varepsilon d\theta dB(s) + \int_0^t \int_0^1 f_x(s, \hat{x} + \theta(x^\varepsilon - \hat{x}), u^\varepsilon(s))y^\varepsilon d\theta ds$$

$$+ r_2(t, \varepsilon) - \frac{r_1(t, \varepsilon)}{\varepsilon}, \tag{4.12}$$

where

$$r_2(t, \varepsilon) := \int_0^t \int_0^1 (\sigma_x(s, \hat{x} + \theta(x^\varepsilon - \hat{x})) - \sigma_x(s, \hat{x}))\delta x_\lambda d\theta dB(s)$$

$$+ \int_0^t \int_0^1 (f_x(s, \hat{x} + \theta(x^\varepsilon - \hat{x}), u^\varepsilon(s)) - f_x(s, \hat{x}, \hat{u}(s)))\delta x_\lambda d\theta ds.$$

By assumption (A3), (4.9) and the dominated convergence theorem, we have easily

$$E|r_2(t, \varepsilon)|^2 \leqslant r(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0.$$

Applying (2.5) to (4.12), we have

$$E|y^\varepsilon(t)|^2 \leqslant \text{const. } E\left(\int_0^t \int_0^s |y^\varepsilon(\theta)|^2 dM(\theta) ds + \int_0^t |y^\varepsilon(s)|^2 ds \right.$$

$$\left. + \left| r_2(t, \varepsilon) + \frac{r_1(t, \varepsilon)}{\varepsilon} \right|^2 \right),$$

hence again by Gronwall's inequality, we arrive at

$$E|y^\varepsilon(t)|^2 \leqslant r(\varepsilon) \to 0, \quad \text{as } \varepsilon \to 0,$$

which proves (4.10). A similar argument yields

$$x_0^\varepsilon(t) = \hat{x}_0(t) + \varepsilon \sum_{i=1}^n \lambda_i \delta x_0^{v_i}(t) + r_3(t, \varepsilon), \tag{4.13}$$

where

$$|r_3(t, \varepsilon)| \leqslant \theta(\varepsilon) = o(\varepsilon).$$

Now (4.5) follows by combining (4.10) and (4.13). □

Theorem 4.2 (*Maximum principle*). *Suppose $(\hat{x}, \hat{u})$ is optimal. Then there exist $\psi_0 \leqslant 0$, $\psi \in L^2(\Omega; R^d)$ and an R^d-valued $\mathscr{F}_t$-adapted process $\{p(t): 0 \leqq t \leqq 1\}$ such that*

1° $H(t, \hat{x}, \hat{u}(t), p(t)) = \max_{u \in \Gamma} H(t, \hat{x}, u, p(t))$, P-a.s., a.e. $t \in [0, 1]$;

2° $E(\psi, x - \hat{x}(1)) \geqslant 0$, *for any* $x \in Q$, *where the Hamiltonian* $H(t, x, u, p) := (p, f(t, x, u)) + \psi_0 f^0(t, x, u)$, *for* $(t, x, u, p) \in [0, 1] \times C \times \Gamma \times R^d$.

Proof. Define $\Lambda := (-\infty, 0] \times (Q - \hat{x}(1))$. Due to assumption (A5) and Theorem 4.1, it follows exactly as in [6] that Λ and Ξ can be separated. So there exist $\psi_0 \in R^1$ and $\psi \in L^2(\Omega; R^d)$ satisfying

$$|\psi_0|^2 + E|\psi|^2 \neq 0, \tag{4.14}$$

$$\psi_0 E\int_0^1 f_x^0(t, x, \hat{u}(t))\delta x^v dt+\psi_0 E\int_0^1 (f^0(t, \hat{x}, v(t))-f^0(t, \hat{x}, \hat{u}(t)))dt$$

$$+E(\psi, \delta x^v(1))\leqslant 0, \qquad \text{for any } v\in U_{ad}, \tag{4.15}$$

$$\psi_0 x_0+E(\psi, x-\hat{x}(1))\geqslant 0, \qquad \text{for any } (x_0, x-\hat{x}(1))\in\Lambda. \tag{4.16}$$

Assume (by Riesz's representation theorem)

$$f_x^0(t, \hat{x}, \hat{u}(t))y=\int_0^1 (d_\theta\eta_0(\theta, t), y(\theta))\equiv\int_0^t (d_\theta\eta_0(\theta, t), y(\theta)), \quad \text{for any } y\in C.$$

Furthermore, using Theorem 3.1 to represent δx^v, (4.15) can be rewritten as

$$\int_0^1 E\{(\tilde{p}(t), f(t, \hat{x}, v(t))-f(t, \hat{x}, \hat{u}(t)))+\psi_0(f^0(t, \hat{x}, v(t))$$
$$-f^0(t, \hat{x}, \hat{u}(t)))\}dt\leqslant 0, \tag{4.17}$$

where $\tilde{p}(t):=\psi_0\int_t^1\int_0^\tau\varphi^T(\theta, t)d_\theta\eta_0(\theta, \tau)d\tau+\varphi^T(1, t)\psi$. Define $p(t):=E(\tilde{p}(t)|\mathscr{F}_t)$. Then $p(t)$ is an $\mathscr{F}_t$-adapted process, and (4.17) still holds with $p(t)$ replaced by $p(t)$, which yields

$$EH(t, \hat{x}, \hat{u}(t), p(t))\geqslant EH(t, \hat{x}, u, p(t)), \text{ for any } u\in\Gamma, \text{ a.e. } t\in[0, 1].$$

By a standard trick as in Kushner[5], we can conclude 1° of the theorem. On the other hand, (4.16) yields 2° and $\psi_0\leqslant 0$.

Remark 4.1. ψ_0 and p will not be identical to zero at the same time. Indeed, if $\psi_0=0$, then $E|\psi|^2>0$ by (4.14). Hence

$$E|p(t)|^2=E|\varphi^T(1, t)\psi|^2\geqslant\text{const.}\ (E|\psi|^2-E|\psi|^2E|\varphi(1, t)-I|^2).$$

But $E|\varphi(1, t)-I|^2\to 0$ as $t\to 1$ (see 3.10)); it follows that when t is sufficiently near 1, $E|\tilde{p}(t)|^2>0$, hence $E|p(t)|^2>0$.

Appendix. Stochastic Differential Equations with Random Coefficients (SDERC).

In many cases, we will encounter stochastic differential equations of functional type whose coefficients are given also at random, e.g. open-loop control problem. Such equations can be dealt with similarly to those in Ito's sense, but there are still some differences. The most important difference is that SDERC has no conceptions of weak and strong solution (cf.[4, Ch. 4]) because SDERC is on a given probability space. In this appendix, we shall give the definition of solutions of SDERC, and then give an existence and uniqueness theorem, the proof of which is omitted since it can be supplied by a standard approach (cf. [4, Th. 4.3.1]).

Given a standard probability space $(\Omega, \mathscr{F}, P)$ with a filtration $\{\mathscr{F}_t: 0\leqslant t\leqslant 1\}$ and an r-dimensional $\mathscr{F}_t$-Brownian motion $\{B(t): 0\leqslant t\leqslant 1\}$. σ and b are mappings from $[0, 1]\times C\times\Omega$ to $R^{d\times r}$ and R^d, respectively.

Definition. By a solution of the following SDERC

$$\begin{cases} dx(t, \omega)=\sigma(t, x, \omega)dB(t)+b(t, x, \omega)dt, & t\in[0, 1], \\ x(0, \omega)=x_0\in R^d, \end{cases} \tag{$*$}$$

we mean a d-dimensional continuous stochastic process $x=\{x(t): 0\leqslant t\leqslant 1\}$ on $(\Omega, \mathscr{F}, P)$ such that

1° x is $\mathscr{F}_t$-adapted;

2° If we denote $\varphi(t,\omega):=\sigma(t,x(\omega),\omega)$, $\psi(t,\omega):=b(t,x(\omega),\omega)$, then $\varphi\in\mathscr{L}_2^{\text{loc}}$ and $\psi\in\mathscr{L}_1^{\text{loc}}$ (cf [4, p. 16]);

3° With probability one,

$$x(t,\omega)=x_0+\int_0^t\sigma(s,x(\omega),\omega)dB(s,\omega)+\int_0^t b(s,x(\omega),\omega))ds, \text{ for any } t\in[0,1].$$

Definition. We say that the uniqueness of solutions of (*) holds if whenever x and x' are two solutions, then $P\{\omega: x(t,\omega)=x'(t,\omega),$ for any $t\in[0,1]\}=1$.

Theorem. *Under the following conditions, the SDERC (*) admits of a unique solution*:

(C1) *σ and b are measurable mappings*:

(C2) *For fixed $(t,x)\in[0,1]\times C$, $\sigma(t,x,\cdot)$ and $b(t,x,\cdot)$ are $\mathscr{F}_t$-measurable; for fixed $(t,\omega)\in[0,1]\times\Omega$, $\sigma(t,\cdot,\omega)$ and $b(t,\cdot,\omega)$ are $\mathscr{B}_t(C)$-measurable*;

(C3) *There exists a positive constant K such that*

$|\sigma(t,x,\omega)-\sigma(t,y,\omega)|+|b(t,x,\omega)-b(t,y,\omega)|\leqq K\|x-y\|_t$, *for any* $x,y\in C$;

$|\sigma(t,x,\omega)|^2+|b(t,x,\omega)|^2\leqq K(1+\|x\|_t^2)$, *for any* $x\in C$.

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References

[1] Bensoussan, A., Lectures on stochastic control, part I, Lecture Notes in Math. 972, Springer-Verlag, Berlin, Heidelberg, 1983, 1–39.

[2] Haussmann, U.G., General necessary conditions for optimal control of stochastic systems, *Math. Progr. Study*, **9** (1976), 30–48.

[3] ———— , On the stochastic maximum principle, *SIAM J. Control Optim.*, **16** (1978), 236–251.

[4] Ikeda, N. and Watanabe, S., Stochastic Differential Equations and Diffusion Processes, Kodansha Ltd, Tokyo, 1981.

[5] Kushner, H.J., Necessary conditions for continuous parameter stochastic optimization problems, *SIAM J. Control*, **10** (1972), 550–565.

[6] Li, X.J. and Yao, Y.L., Maximum principle of distributed parameter systems with time lags, Proc. Conference on Control Theory of Distributed Parameter Systems and Applications, Vorau, Austria (1984), Edited by F.Kappel, K.Kunisch and W.Schappacher, Springer-Verlag, New York, 1985, 410–427.

[7] Mohammed, S-E. A., Stochastic Functional Differential Equations, Res. Notes in Math. 99, Pitman, Boston, London, Melbourne, 1984.