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Dynamic Portfolio Choice when Risk is Measured by Weighted VaR

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We aim to characterize the trading behavior of an agent, in the context of a continuous-time portfolio choice model, if she measures the risk by a so-called weighted value-at-risk (VaR), which is a generalization of both VaR and conditional VaR. We show that when bankruptcy is allowed the agent displays extremely risk-taking behaviors, unless the downside risk is significantly penalized, in which case an asymptotically optimal strategy is to invest a very small amount of money in an extremely risky but highly rewarded lottery and save the rest in the risk-free asset. When bankruptcy is prohibited, extremely risk-taking behaviors are prevented in most cases in which the asymptotically optimal strategy is to spend a very small amount of money in an extremely risky but highly rewarded lottery and put the rest in an asset with moderate risk. Finally we show that the trading behaviors remain qualitatively the same if the weighted VaR is replaced by a law-invariant coherent risk measure.

Key words: mean-risk portfolio choice; weighted value-at-risk; coherent risk measures; well-posedness; optimal investment strategies; binary and ternary payoffs
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1. Introduction Return and risk are two indispensable sides of investment activities. Both practitioners and researchers have tried for a long time to understand the tradeoff between them. On the one hand, the expected utility theory has been developed, in which the tradeoff between the return and risk of an investment is coded into a single number, the so-called expected utility. Abundant research has been devoted to optimal portfolio choice in this framework. On the other hand, Markowitz [35] proposes a portfolio choice problem in which the agent chooses among the portfolios that yield a pre-specified level of expected return while minimizing the variance of the portfolio's return. The pair of the expected return level and the corresponding minimal variance represent the best tradeoff between risk and return under this mean-variance portfolio theory, which traces out what is known as an *efficient frontier* on the return-risk plane.

Although the mean-variance theory has achieved a great success in portfolio choice and asset pricing, there have been criticisms about using variance as a risk measure, especially in terms of treating volatile positive returns as part of the risk. Some alternative risk measures have been put forth, examples including semi-variance and value-at-risk (VaR). On the other hand, some researchers argue that a good risk measure should satisfy a set of reasonable axioms, one example being the so-called coherent risk measures [5]. An interesting and important coherent risk measure is the conditional VaR, or CVaR. Numerous studies have been conducted to characterize and extend the notion of coherent risk measures; see also more discussions in Section 2.

Researchers have been applying VaR and CVaR to portfolio choice in the same way Markowitz developed mean-variance theory. For instance, Kast et al. [27], Alexander and Baptista [3, 4], Campbell et al. [12], and Vorst [43] study single-period mean-VaR portfolio choice problems in different settings; Krokhmal et al. [30], Acerbi and Simonetti [1], and Bassett et al. [9] consider single-period mean-CVaR portfolio choice models, in which the dual representation formula for CVaR established by Rockafellar and Uryasev [38, 39] plays a key role.

There have been also extensions of the Markowitz model from the single-period setting to the dynamic, continuous-time one in the past decade; see for instance Zhou and Li [45], Bielecki et al. [10]. In addition, Jin et al. [23] consider a continuous-time mean-risk portfolio choice problem where the risk is measured by a general convex function of the deviation of the random wealth from its mean. This risk measure includes variance and semi-variance as special cases. An intriguing result therein is that optimality of the corresponding mean-risk problem may not be achievable by any admissible portfolios.

The study of continuous-time mean-risk portfolio choice problems with VaR or a general coherent law-invariant risk measure is still lacking in the literature. Two relevant papers in this study are Basak and Shapiro [8] and Gabih et al. [21], where the authors consider continuous-time expected utility maximization problems with a risk constraint on the terminal wealth. The risk is quantified by different risk measures including VaR, limited expected loss and expected loss of the terminal wealth. However, in the problems considered in those two papers, it is the (strictly concave) expected utility of the terminal wealth, rather than the mean of the terminal wealth, that is maximized. As a result, the problems therein do not exactly belong to the class of mean-risk portfolio choice problems.

The present paper complements the portfolio selection literature by studying continuous-time mean-risk portfolio choice problems with general risk measures including VaR, CVaR and law-invariant coherent risk measures. We first consider the case in which the risk is quantified by the so-called weighted VaR, or WVaR in short, which includes both VaR and CVaR as special cases. Applying the so-called quantile formulation technique, which has been developed recently for solving mainly robust or behavioral portfolio selection problems (see Schied [41], Carlier and Dana [13], Jin and Zhou [24], and He and Zhou [22]), we solve the problem completely. Specifically, we provide sufficient and necessary conditions for both well-posedness (i.e., the optimal value is finite) and existence of optimal solutions.[1] These results indicate that the model is prone to be ill-posed especially when bankruptcy is allowed, leading to extremely risk-taking behaviors. Moreover, we find that the optimal value is independent of the expected terminal wealth target, which leads to a vertical efficient frontier on the mean-risk plane. Furthermore, the optimal terminal wealth, if it exists, must be a binary or digital option. When the problem is well-posed, we find an asymptotically optimal solution, which is also a digital option. This solution suggests the following strategy: bank most of the money and invest the rest to buy an extremely risky but highly rewarded lottery. The winning chance of this lottery is extremely small, but the winning payoff is sufficiently

[1] The well-posedness is equivalent to, in our setting, the absence of "nirvana strategies" defined by Kim and Omberg [28].

high to boost the expected terminal wealth to the desired target. A characterizing feature of this strategy is that it entails little downside risk since the agent saves most of his money in the risk-free asset. This type of strategy underlines the key investment methodology behind the so-called *principal guaranteed fund*.[2]

We then consider the case in which bankruptcy is not allowed.[3] Again, we are able to solve the problem thoroughly. A major difference, though, is that an optimal solution, when it exists, is no longer binary. It is ternary, i.e., takes only three values. Furthermore, an asymptotically optimal solution we derive is also three-valued, implying the following strategy: spend a very small amount of money buying an extremely risky but highly rewarded lottery to boost the expected return and invest the rest to an asset with medium risk. As a result, when the market turns out to be very good, the agent wins the lottery yielding an extremely high payoff. When the market is mediocre, the agent loses the lottery but wins the investment in the asset, which leads to a moderate level of terminal wealth. When the market is bad, the agent loses all his investment.

Finally, we change the mean-risk portfolio choice problem by replacing WVaR with a law-invariant coherent risk measure. While the latter is not a special case of the former, we are able to solve the latter based on the obtained results for the former along with a minimax theorem. We find that the optimal value of the problem is again independent of the expected terminal wealth target. Furthermore, when bankruptcy is allowed, the same strategy as in the case of WVaR is derived.

The main contributions of our paper are two-fold. First, by solving explicitly and completely the corresponding continuous-time portfolio choice problems, we are able to understand economically the dynamic portfolio behaviors when the risk is measured by VaR and its variants/generalizations. We find that the solutions have a significant qualitative change compared with those using variance as the risk measure. Moreover, the model is very likely to be ill-posed, indicating an improper modeling of the tradeoff between return and risk. As noted by Bassett et al. [9], portfolio choice is "... an acid test of any new criterion designed to evaluate risk alternatives". In this sense our results constitute a critique, rather than advocacy, of using WVaR (and its similar peers) to measure risk.

Secondly, the technical analysis performed in our paper contributes mathematically to the literature. Although one of the main techniques in the analysis, i.e., quantile formulation, has been employed in the literature [e.g., 41, 13, 24, 22], we develop new techniques in other parts of the analysis. For instance, after applying the quantile formulation, we need to solve an optimization problem in which the optimal quantile function is to be found. This problem is nontrivial and cannot be solved by directly applying the Lagrange dual method as in the literature [e.g., 22]. We design a new machinery that combines the dual method and a "corner-point searching" idea.

Finally, let us comment on the setting of our model. Following the classical mean-variance analysis, we measure investment risk by applying a risk measure on the terminal wealth. This setting is standard and dominant in the literature even for continuous-time models. There are papers, e.g., Cuoco et al. [16], Yiu [44] and Leippold et al. [33], that study investment problems with *dynamic* VaR constraint. More precisely, at each time t, the portfolio risk in a small time period $[t, t+\tau]$ (e.g., in every day) is evaluated as the VaR of profit in this period by assuming the portfolio is "constant". The agent then maximizes the expected utility of her terminal wealth or consumption while the portfolio risk at any time is controlled at certain level. In Section 7.2, we investigate this dynamic VaR version of our problem and discuss how the optimal trading strategies will change.

[2] A principal guaranteed fund is a mutual fund that ensures the return of the whole principal (or a large percentage of it) while providing potential additional return. The fund manager of such a fund typically adopts the strategies described above.

[3] Precisely, here and hereafter, by "bankruptcy is not allowed" we really mean "a strictly negative wealth is not allowed". A zero wealth is still allowed, in which case the agent is no longer able to trade, and therefore forced to exit the market.

Generally speaking, the mean-risk problem in this paper is not time consistent, i.e., the optimal dynamic trading strategy solved today is not necessarily optimal in the future if the agent has the same expected wealth target and risk attitude. In this paper, we focus on the so-called *pre-committed* strategies, which the agent commits herself to follow. Note that pre-committed strategies are important. Firstly, they are frequently applied in practice, sometimes with the help of certain commitment devices. For instance, Barberis [6] finds that the pre-committed strategy of a casino gambler is a stop-loss one (when the model parameters are within reasonable ranges). Many gamblers indeed follow this strategy by applying some commitment measures, such as leaving ATM cards at home or bringing little money to the casino; see Barberis [6] for a full discussion. The results in the present paper are useful to understand trading behaviors of a pre-committed mean-risk optimizer. Secondly, time-inconsistency arises in most mean-risk portfolio choice problems. Indeed, the dynamic Markowitz mean-variance portfolio choice problem itself is time-inconsistent [see e.g., 7, 11]. Further, Kupper and Schachermayer [31] show that the only dynamic risk measure that is law-invariant and time consistent is the entropic one. In this sense, time-consistent risk measures are too restrictive to accommodate many interesting cases such as VaR and CVaR.

The rest of the paper is organized as follows: In Section 2 we review different risk measures such as VaR, coherent risk measures, and convex risk measures, as well as a representation theorem for law-invariant convex risk measures. We then formulate the mean-WVaR portfolio choice problem in Section 3 and solve it completely. In Section 4 we revise the portfolio choice problem by adding a no-bankruptcy constraint and then provide the solution. Section 5 is devoted to solving the mean-risk portfolio choice problem in which the risk is measured by a law-invariant coherent risk measure. Several examples are provided in Section 6. Discussions and conclusions are presented in Section 7. Proofs are placed in an Appendix.

2. Risk Measures on the Space of Lower-Bounded Random Variables

Artzner et al. [5] argue that a good risk measure should satisfy several reasonable axioms and introduce the notion of *coherent* risk measures. Föllmer and Schied [17] and Frittelli and Rosazza Gianin [19] extend this notion to *convex* risk measures. It is notable that VaR, a risk measure commonly used in practice, is not a convex risk measure. Kou et al. [29], Cont et al. [15], and Cont et al. [14] note that when using historical data to estimate the risk of a portfolio, VaR leads to more robust estimates than convex risk measures do. In this section, we give a brief review of these risk measures for the purpose of our later study of mean-risk portfolio choice. Note that, unlike most existing works which define risk measures on the space of essentially bounded random variables, we define them on the space of lower-bounded random variables. We choose the latter space because in the portfolio choice problem we are going to consider, the payoffs are lower-bounded due to the classical *tame* portfolio setting [26].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the space of essentially bounded random variables, and $LB(\Omega, \mathcal{F}, \mathbb{P})$ be the space of lower-bounded finite-valued random variables (but different random variables in the space may have different lower bounds). Each element in $LB(\Omega, \mathcal{F}, \mathbb{P})$ represents the profit and loss (P&L) of a portfolio (in a fixed period). A mapping ρ from $LB(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R} \cup \{-\infty\}$ with $\rho(0) = 0$ is called a *risk measure* if it satisfies

1. *Monotonicity:* $\rho(X) \geq \rho(Y)$ for any $X, Y \in LB(\Omega, \mathcal{F}, \mathbb{P})$ such that $X \leq Y$;
2. *Cash Invariance:* $\rho(X + c) = \rho(X) - c$ for any $X \in LB(\Omega, \mathcal{F}, \mathbb{P})$ and $c \in \mathbb{R}$; and
3. *Truncation Continuity:* $\rho(X) = \lim_{n \to +\infty} \rho(X \wedge n)$ for any X in $LB(\Omega, \mathcal{F}, \mathbb{P})$.

The truncation continuity property, which is new compared to the risk measures defined on the space of bounded P&Ls, is imposed to guarantee that the risk of any unbounded P&L can be computed through its truncations. A simple example in which this property is violated is as follows: $\rho(X) = 0$ if X is bounded and $\rho(X) = -\infty$ otherwise. The truncation continuity property essentially excludes this type of risk measures.

A risk measure is *convex* if it satisfies

4 *Convexity:* $\rho(\alpha X+(1-\alpha)Y)\le \alpha\rho(X)+(1-\alpha)\rho(Y)$ for any $X,Y\in LB(\Omega,\mathcal{F},\mathbb{P})$ and $\alpha\in(0,1)$, and is *coherent* if it further satisfies

5 *Positive Homogeneity:* $\rho(\lambda X)=\lambda\rho(X)$ for any $X\in LB(\Omega,\mathcal{F},\mathbb{P})$ and $\lambda>0$.

Before we proceed, let us remark that in order to verify that a mapping from $LB(\Omega,\mathcal{F},\mathbb{P})$ to $\mathbb{R}\cup\{-\infty\}$ satisfies some of the aforementioned Properties 1–5, we only need to show that these properties are satisfied when the mapping is restricted on $L^\infty(\Omega,\mathcal{F},\mathbb{P})$ and the truncation continuity property is satisfied.

Two random variables X and Y on $(\Omega,\mathcal{F})$ are called *comonotone* if

$$(X(\omega)-X(\omega'))\,(Y(\omega)-Y(\omega'))\ge 0\quad\text{for all } (\omega,\omega')\in\Omega\times\Omega.$$

A risk measure ρ is called *comonotonic additive* if

$$\rho(X+Y)=\rho(X)+\rho(Y)$$

for any comonotone pair (X,Y) in $LB(\Omega,\mathcal{F},\mathbb{P})$. Convex risk measures that are comonotonic additive are of particular interest in the literature, and it is easy to see that they are actually coherent risk measures.

Throughout this paper we are interested in *law-invariant* risk measures. A risk measure ρ is law-invariant if $\rho(X)=\rho(Y)$ for any $X,Y\in LB(\Omega,\mathcal{F},\mathbb{P})$ that share the same probability distribution. There is a representation theorem in the literature for law-invariant convex risk measures. To state this theorem, let us first define two simple risk measures that are commonly used in practice.

For any random variable $X\in LB(\Omega,\mathcal{F},\mathbb{P})$, let G_X be its left-continuous quantile function. As a result, G_X is a left-continuous increasing function defined on $(0,1]$ with the possibility of taking $+\infty$ at 1. We extend G_X to the whole unit interval $[0,1]$ by defining $G_X(0):=\lim_{z\downarrow 0}G_X(z)>-\infty$, where the inequality is the case because X is lower-bounded. From the definition we see that G_X is continuous at 0. In the rest of the paper, denote by $\mathbb{G}$ the set of such quantile functions, i.e.,

$$\begin{aligned}\mathbb{G}:=\{G:[0,1]\to\mathbb{R}\cup\{+\infty\}\mid\ & G \text{ is increasing and left-continuous on } [0,1],\\ & \text{continues at 0, and finite-valued on } [0,1)\}.\end{aligned}\tag{1}$$

The α-VaR, denoted by $V@R_\alpha(X)$, of a P&L, X, is defined as

$$V@R_\alpha(X):=-G_X(\alpha)\tag{2}$$

where $\alpha\in[0,1]$ is the confidence level. The α conditional VaR or CVaR in short, denoted by $CV@R_\alpha(X)$, is defined as

$$CV@R_\alpha(X):=\frac{1}{\alpha}\int_0^\alpha V@R_s(X)ds\tag{3}$$

where $\alpha\in(0,1]$, with the convention

$$CV@R_0(X):=V@R_0(X)=-G_X(0).$$

Hence, $CV@R_\alpha(X)$ is decreasing and continuous with respect to α on $[0,1]$ and finite-valued on $[0,1)$.

It is easy to verify that both VaR and CVaR are comonotonic additive risk measures defined on $LB(\Omega,\mathcal{F},\mathbb{P})$. In addition, CVaR is convex, but VaR is not.[4] The following representation theorem shows that CVaR is the building block for law-invariant convex risk measures. To state the theorem, let us introduce the notation $\mathcal{P}([0,1])$ as the set of all probability measures on $[0,1]$. After equipped with the usual weak topology, $\mathcal{P}([0,1])$ becomes a compact space [36].

[4] Indeed, it is known that both VaR and CVaR satisfy the monotonicity and cash invariance properties on $L^\infty(\Omega,\mathcal{F},\mathbb{P})$, and CVaR is convex on $L^\infty(\Omega,\mathcal{F},\mathbb{P})$; see for instance Pflug [37]. On the other hand, using the monotone convergence theorem, it is easy to show that both VaR and CVaR satisfy the truncation continuity property. Thus, both VaR and CVaR are risk measures defined on $LB(\Omega,\mathcal{F},\mathbb{P})$ and CVaR is convex on this space.

THEOREM 1. *Assume that $(\Omega,\mathcal{F},\mathbb{P})$ is an atomless probability space. Then, ρ is a law-invariant convex risk measure on $LB(\Omega,\mathcal{F},\mathbb{P})$ if and only if there exists a lower-semi-continuous convex function $v:\mathcal{P}([0,1])\to[0,+\infty]$ satisfying $\inf_{\mu\in\mathcal{P}([0,1])} v(\mu)=0$ such that*

$$\rho(X)=\sup_{\mu\in\mathcal{P}([0,1])}\left\{\int_{[0,1]} CV@R_z(X)\mu(dz)-v(\mu)\right\}. \tag{4}$$

Furthermore, ρ is coherent if and only if

$$\rho(X)=\sup_{\mu\in\mathcal{A}}\left\{\int_{[0,1]} CV@R_z(X)\mu(dz)\right\} \tag{5}$$

for some closed convex set $\mathcal{A}\subset\mathcal{P}([0,1])$, and ρ is convex and comonotonic additive if and only if

$$\rho(X)=\int_{[0,1]} CV@R_z(X)\mu(dz) \tag{6}$$

for some $\mu\in\mathcal{P}([0,1])$.

This theorem has been proved in the literature when the risk measure is defined on $L^\infty(\Omega,\mathcal{F},\mathbb{P})$; see for instance Kusuoka [32] for the coherent case, Frittelli and Rosazza Gianin [20], Jouini et al. [25], and Rüschendorf [40] for the convex case, and Föllmer and Schied [18] for the convex and comonotonic additive case. We will supply a proof for the current setting of $LB(\Omega,\mathcal{F},\mathbb{P})$ in Appendix.

From Theorem 1, a law-invariant, convex, and comonotonic additive risk measure has representation (6). Using integration by parts, we obtain

$$\int_{[0,1]} CV@R_z(X)\mu(dz)=\mu(\{0\})V@R_0(X)+\int_{[0,1]} V@R_z(X)\int_{[z,1]}\frac{1}{s}\mu(ds)dz. \tag{7}$$

Thus, a law-invariant, convex, and comonotonic additive risk measure can be regarded as a weighted average of VaR at different confidence levels.

In general, we define the *weighted VaR* (or WVaR):

$$WV@R_m(X):=\int_{[0,1]} V@R_z(X)m(dz) \tag{8}$$

for some $m\in\mathcal{P}([0,1])$. It is easy to verify that WVaR is a law-invariant comonotonic additive risk measure on $LB(\Omega,\mathcal{F},\mathbb{P})$. [5] In addition, according to representation (6), WVaR is convex if and only if m on $(0,1]$ admits a *decreasing* density with respect to the Lebesgue measure.

WVaR generalizes many known risk measures, and meanwhile can generate other types of risk measures. If m is a Dirac measure, WVaR just becomes VaR. Acerbi [2] considers a family of risk measures defined as $\int_0^1 V@R_z(X)\phi(z)dz$ for some probability density ϕ on $(0,1)$. This class of risk measures are obviously special examples of WVaR. When ϕ is a decreasing function, Acerbi calls such a risk measure *spectral risk measure*, which is a law-invariant convex comonotonic additive risk measure according to Theorem 1 and equation (7).

Furthermore, let us define the mapping $\varphi:\mathcal{P}([0,1])\to\mathcal{P}([0,1])$ as

$$\varphi(\mu)(A)=\mu(\{0\})\mathbf{1}_{0\in A}+\int_{A\setminus\{0\}}\int_{[z,1]}\frac{1}{s}\mu(ds)dz, \tag{9}$$

[5] A law-invariant and comonotonic additive risk measure is not necessarily WVaR. For instance, for a fixed $\alpha\in(0,1)$, the risk measure $\lim_{z\downarrow\alpha} V@R_z(X)$ is comonotonic additive, but it is not a WVaR risk measure.

where A is any measurable subset of $[0,1]$. Using this notation, risk measure (7) can be written as $WV@R_{\varphi(\mu)}$. Moreover, any law-invariant convex risk measure on $LB(\Omega,\mathcal{F},\mathbb{P})$ can be written as $\sup_{\mu\in\mathcal{P}([0,1])}\{WV@R_{\varphi(\mu)} - v(\mu)\}$ for some lower-semi-continuous function v on $\mathcal{P}([0,1])$ with $\inf_{\mu\in\mathcal{P}([0,1])} v(\mu)=0$. For a law-invariant coherent risk measure, it has the representation $\sup_{\mu\in\mathcal{M}}\left\{WV@R_{\varphi(\mu)}\right\}$ for some closed convex set $\mathcal{M}\subset\mathcal{P}([0,1])$.

In the rest of the paper, we study a mean-risk portfolio choice problem and its variants in which the risk of a portfolio is evaluated by either WVaR or law-invariant coherent risk measures. The reason of using these two types of risk measures in the study of mean-risk portfolio choice is twofold. On the one hand, coherent risk measures has the desirable property that diversification reduces risk, as reflected by the convexity property. On the other hand, VaR, which is not a coherent risk measure, is popularly employed in practice. As discussed earlier, WVaR either covers or is building blocks to these two, and indeed many more, types of risk measures.

3. Mean-WVaR Portfolio Choice Model

In this section, we formulate our continuous-time mean-risk portfolio choice problem in which the risk of a portfolio is measured by WVaR.

3.1. A Continuous-Time Market

Let $T>0$ be a given terminal time and $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le T},\mathbb{P})$ be a filtered probability space on which is defined a standard $\mathcal{F}_t$-adapted n-dimensional Brownian motion $W(t)\equiv(W^1(t),\cdots,W^n(t))^\top$ with $W(0)=0$, and hence the probability space is atomless. It is assumed that $\mathcal{F}$ is $\mathbb{P}$-complete and $\mathcal{F}_t=\sigma\{W(s): 0\le s\le t\}$ augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Here and henceforth, $A^\top$ denotes the transpose of a matrix A and $x\vee y := \max(x,y)$.

We consider a continuous-time financial market where there are $m+1$ assets being traded continuously. One of the assets is a bank account with risk-free interest rate r. We assume without loss of generality that $r\equiv 0$. The other m assets are risky stocks whose price processes $S_i(t)$, $i=1,\cdots,m$, satisfy the following stochastic differential equation (SDE):

$$dS_i(t)=S_i(t)\left[b_i(t)dt+\sum_{j=1}^n\sigma_{ij}(t)dW^j(t)\right],\ t\in[0,T];\ \ S_i(0)=s_i>0, \tag{10}$$

where $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$, the appreciation and volatility rate of stock i, respectively, are scalar-valued, $\mathcal{F}_t$-progressively measurable stochastic processes with

$$\int_0^T\left[\sum_{i=1}^m|b_i(t)|+\sum_{i=1}^m\sum_{j=1}^n|\sigma_{ij}(t)|^2\right]dt<+\infty, \text{ a.s.}$$

Define the appreciation rate of return process

$$B(t):=(b_1(t),\cdots,b_m(t))^\top,$$

and the volatility matrix process $\sigma(t):=(\sigma_{ij}(t))_{m\times n}$. The following assumption, which leads to the market being arbitrage-free, is imposed throughout this paper:

Assumption 1. *There exists an $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^n$-valued process $\theta_0(\cdot)$, the so-called market price of risk, with $\mathbb{E}\left[e^{\frac{1}{2}\int_0^T|\theta_0(t)|^2dt}\right]<+\infty$ such that*

$$\sigma(t)\theta_0(t)=B(t),\quad a.s.,\ a.e.\ t\in[0,T].$$

Consider an agent who has an initial endowment $x>0$ and the investment period $[0,T]$. Assume that the trading of shares takes place continuously in a self-financing fashion and there are no transaction costs. Then, the agent's wealth process $X(\cdot)$ satisfies

$$dX(t)=\pi(t)^\top B(t)dt+\pi(t)^\top\sigma(t)dW(t),\ \ t\in[0,T];\ \ X(0)=x, \tag{11}$$

where $\pi_i(t)$ denotes the total market value of the agent's wealth in stock i at time t. The process $\pi(\cdot)\equiv(\pi_1(\cdot),\cdots,\pi_m(\cdot))^\top$ is called an *admissible portfolio* if it is $\mathcal{F}_t$-progressively measurable with

$$\int_0^T|\sigma(t)^\top\pi(t)|^2dt<+\infty,\ \text{ a.s.}$$

and is tame (i.e., the corresponding wealth process $X(\cdot)$ is almost surely bounded from below—although the bound may depend on $\pi(\cdot)$). It is standard in the continuous-time portfolio choice literature that a portfolio be required to be tame so as to, among other things, exclude the doubling strategy.

3.2. Mean-WVaR Portfolio Choice Problem We consider the following mean-risk portfolio choice problem: At time 0, the agent decides the dynamic portfolio in the period $[0,T]$ so as to minimize the risk of his portfolio given that some pre-specified expected return target is achieved at time T. The risk is evaluated by a WVaR risk measure on $LB(\Omega,\mathcal{F}_T,\mathbb{P})$: $WV@R_m$ for some given $m\in\mathcal{P}([0,1])$. Then, the portfolio choice problem can be formulated as

$$\begin{array}{ll}\underset{\pi}{\text{Min.}} & WV@R_m(X(T))\\ \text{subject to:} & \pi \text{ is admissible},\\ & dX(t)=\pi(t)^\top B(t)dt+\pi(t)^\top\sigma(t)dW(t),\ \ t\in[0,T],\ \ X(0)=x,\\ & \mathbb{E}[X^{x,\pi}(T)]\geq\Theta,\end{array} \tag{12}$$

where x is the initial wealth and Θ is the expected terminal wealth target set by the agent.[6] It is reasonable to assume that the expected terminal wealth target is larger than the initial wealth, i.e., $\Theta>x$.

In general, problem (12) is time-inconsistent (i.e., the class of problems in the future using the same objective function as in (12) are time-inconsistent). We aim only at solving (12) at $t=0$; namely, we are interested in pre-committed strategies only. Because of time-inconsistency, dynamic programming is not applicable and we use the martingale method instead. We assume

ASSUMPTION 2. *The market price of risk is unique, i.e., the market is complete.*

With the complete market assumption, we can define the pricing kernel

$$\xi:=\exp\left\{-\int_0^T\frac{1}{2}|\theta_0(s)|^2ds-\int_0^T\theta_0(s)^\top dW(s)\right\}, \tag{13}$$

where θ_0 is the unique market price of risk. Then, by the standard martingale approach [26], finding the optimal portfolio in the mean-risk portfolio choice problem is equivalent to finding the optimal terminal wealth in the following optimization problem:

$$\begin{array}{ll}\underset{X}{\text{Max.}} & -WV@R_m(X)\\ \text{subject to:} & X \text{ is } \mathcal{F}_T\text{-measurable and bounded from below},\\ & \mathbb{E}[X]\geq\Theta,\\ & \mathbb{E}[\xi X]\leq x,\end{array} \tag{14}$$

[6] Recall that in Section 2 risk measures are defined on P&Ls. Thus, the more appropriate objective function in the mean-risk portfolio choice problem should be $WV@R_m(X(T)-x)$. However, the cash-invariance property implies that $WV@R_m(X(T)-x)=WV@R_m(X(T))+x$, for which reason we simply use $WV@R_m(X(T))$ as the objective function in the portfolio choice problem.

where X represents the terminal (time T) payoff of certain portfolio. Here, we switch from minimization to maximization because doing so could simplify the notation.

3.3. Optimal Solutions In this section we present the solution to the mean-WVaR portfolio choice problem (14). Note that the objective function in (14) is neither convex nor concave in X; hence the normal optimization technique fails. To overcome this difficulty, we employ the *quantile formulation* technique. To this end, we impose the following assumption:

ASSUMPTION 3. *ξ has no atom, i.e., its distribution function is continuous.*

This assumption is satisfied when the investment opportunity set, i.e., the triplet $(r(\cdot), b(\cdot), \sigma(\cdot))$, is deterministic, in which case ξ is lognormally distributed (which is the case with the Black–Scholes market). This assumption, together with Assumptions 1 and 2 will be in force throughout of the paper.

The basic idea of the quantile formulation technique is to change the decision variable of a portfolio choice problem from terminal payoffs to their quantile functions so as to obtain some nice properties of the problem. The details of the quantile formulation are provided in Appendix A, and here we focus on presenting the solution to (14).

We will use the following terminology frequently. An optimization problem is *feasible* if it admits at least one feasible solution (i.e., a solution that satisfies all the constrained involved) and is *well-posed* if it has a finite optimal value. A feasible solution is *optimal* if it achieves the finite optimal value.

Before we present the optimal solution to (14), let us introduce the following assumption

ASSUMPTION 4. *$\operatorname{essinf} \xi = 0$, i.e., $\mathbb{P}(\xi \le x) > 0$ for any $x > 0$.*

This assumption stipulates that, for any given value, there exists a state of nature in which the market offers a return exceeding that value. In particular, when the investment opportunity set, i.e., the triplet $(r(\cdot), b(\cdot), \sigma(\cdot))$, is deterministic and ξ is lognormally distributed (which is the case with the Black–Scholes market), this assumption is satisfied.

In the following, we denote $F_\xi(\cdot)$ as the CDF of ξ and $F_\xi^{-1}(\cdot)$ as the quantile function of ξ, i.e., left-continuous inverse of $F_\xi(\cdot)$. It is straightforward to see that $\int_0^1 F^{-1}(1-z)dz = \mathbb{E}[\xi] = 1$.

We further introduce the function

$$\zeta(c) := \frac{m((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz}, \quad 0 < c < 1 \tag{15}$$

and define the quantity

$$\gamma^* := \sup_{0<c<1} \zeta(c). \tag{16}$$

This quantity plays an important role in determining the optimal solution.

THEOREM 2. *Let Assumption 4 holds. We have the following assertions:*
(i) If $\gamma^* > 1$, then $V(m) = +\infty$. Moreover, there exists $\beta^* > 0$ such that

$$X_n := x - n\mathbb{E}[\xi \mathbf{1}_{\xi \le \beta^*}] + n\mathbf{1}_{\xi \le \beta^*}$$

is feasible to problem (14) and its objective value goes to infinity as $n \to \infty$.

(ii) If $\gamma^* \le 1$, then $V(m) = x$. Furthermore, for any $\beta_n > 0$ such that $\lim_{n\to\infty} \beta_n = 0$, the following terminal wealth

$$X_n := \frac{x\mathbb{P}(\xi \le \beta_n) - \Theta\mathbb{E}[\xi \mathbf{1}_{\xi \le \beta_n}]}{\mathbb{P}(\xi \le \beta_n) - \mathbb{E}[\xi \mathbf{1}_{\xi \le \beta_n}]} + \frac{\Theta - x}{\mathbb{P}(\xi \le \beta_n) - \mathbb{E}[\xi \mathbf{1}_{\xi \le \beta_n}]}\mathbf{1}_{\xi \le \beta_n} \tag{17}$$

is feasible and asymptotically optimal to problem (14) as $n \to \infty$, i.e., its objective value converges to the optimal value of (14) as $n \to \infty$.

(iii) If $\gamma^* = 1$ and there exists $c^* \in (0,1)$ such that $\zeta(c^*) = \gamma^*$, then

$$X^* := \frac{x\mathbb{P}(\xi \le \beta^*) - \Theta \mathbb{E}[\xi \mathbf{1}_{\xi \le \beta^*}]}{\mathbb{P}(\xi \le \beta^*) - \mathbb{E}[\xi \mathbf{1}_{\xi \le \beta^*}]} + \frac{\Theta - x}{\mathbb{P}(\xi \le \beta^*) - \mathbb{E}[\xi \mathbf{1}_{\xi \le \beta^*}]} \mathbf{1}_{\xi \le \beta^*},$$

where $\beta^* := F_\xi^{-1}(1-c^*)$, is optimal to (14).

(iv) If $\zeta(c) < 1, 0 < c < 1$, then the optimal solution to (14) does not exist.

We observe from Theorem 2 that the well-posedness of mean-WVaR model depends only on the market parameter (represented by ξ) and the risk measure (represented by m). As discussed extensively in Jin and Zhou (2008) and He and Zhou (2011), an ill-posed model is one where the incentives implied by the specific model parameters are set wrongly, and hence the agent can push the objective value to be arbitrarily high. In the context of portfolio choice an ill-posed model usually leads to trading strategies that take the greatest possible risk exposure; see also Kim and Omberg [28], pp. 142-143, for a similar observation. For example, in the case of Theorem 2-(i) when the model is ill-posed, one can take terminal payoff $X_n := x - n\mathbb{E}[\xi \mathbf{1}_{\xi \le \beta^*}] + n\mathbf{1}_{\xi \le \beta^*}$ to achieve as large objective value, or as small risk value, as possible. Note that this payoff may place the investor in a deep loss position (with loss amount, $n\mathbb{E}[\xi \mathbf{1}_{\xi \le \beta^*}] - x$, arbitrarily large as n goes to infinity). Hence, in the region of the parameter space specified by $\gamma^* > 1$, the chosen risk measure – the WVaR – will lead (or indeed *mis*lead) the agent to take extremely risky positions.

Theorem 2-(i) gives a qualitative characterization of the ill-posedness (or the presence of nirvana strategies): the less risk measure penalizes the downside outcomes, or the higher the market return, the more likely the model becomes ill-posed (and hence the more risk the agent is tempted to take). On the other hand, when the problem is well-posed, the optimal value is x, which is irrelevant to the expected terminal wealth target Θ and is even irrelevant to the risk measure (represented by m)! In addition, (17) provides an asymptotically optimal solution when the problem is well-posed. Write this solution as $X_n = a_n + b_n \mathbf{1}_{\xi \le \beta_n}$. It is easy to show that $a_n \to x$ and $b_n \to +\infty$ as $n \to +\infty$. Thus, the approximating solution implies the following strategy: banking most of you money (a_n) and using the rest ($x - a_n$) to buy a lottery ($b_n \mathbf{1}_{\xi \le \beta_n}$). The probability of winning the lottery, $\mathbb{P}(\xi \le c_n)$, is small, but the winning payoff, b_n, is high. By implementing this strategy, the agent bears little risk because this strategy barely entails losses. Meanwhile, the lottery could boost the expected terminal wealth to the desired level. In particular, since $x > 0$, we have $a_n > 0$ and hence $X_n > 0$ a.s. when n is sufficiently large, implying a no-bankruptcy strategy.

We plot the efficient frontier for the mean-WVaR portfolio choice problem (12). Because the optimal value does not depend on the expected terminal wealth target Θ, the efficient frontier is a vertical line on the return-risk plane, as shown by Figure 3.3.

4. Mean-WVaR Portfolio Choice with Bankruptcy Prohibition

Following the classical setting for mean-variance portfolio choice problems like in Markowitz [35] and Zhou and Li [45], we have assumed in the mean-WVaR portfolio choice problem (14) that the agent can continue trading even when she is bankrupt. In some circumstances, however, bankruptcy may not be allowed (i.e. the agent is no longer able to trade when her wealth reaches zero). Thus, in the following we consider the mean-WVaR portfolio choice problem in the presence of the no-bankruptcy constraint.

Consider the following portfolio selection problem

$$\begin{array}{ll} \underset{X}{\text{Max.}} & -WV@R_m(X) \\ \text{subject to:} & X \text{ is } \mathcal{F}_T\text{-measurable and bounded from below,} \\ & \mathbb{E}[X] \ge \Theta, \\ & \mathbb{E}[\xi X] \le x, \\ & X \ge 0. \end{array} \tag{18}$$

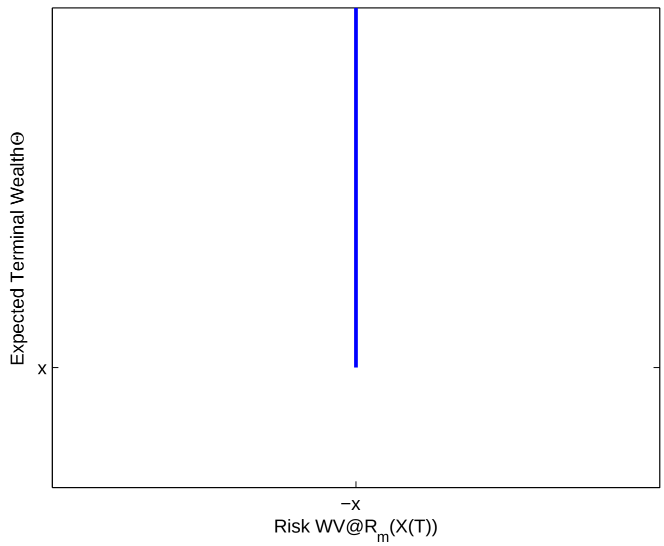

FIGURE 1. Efficient frontier of problem (12) on return-risk panel.

Compared to problem (14), we have an additional constraint $X \geq 0$ preventing the wealth from going strictly negative. It is easy to see that replacing the constraint $X \geq 0$ with $X \geq b$ for some $b < x$ does not essentially change the problem because we could do the change-of-variable $\tilde{X} = X - b$. The constraint $X \geq b$ can be understood as a stop-loss bound preset by the agent or a limited borrowing constraint.

For the same reason as in problem (14), we assume that $\Theta > x > 0$. Denote the optimal value of problem (18) by $\bar{V}(m)$. Recall $\zeta(\cdot)$ defined by (15) and γ^* defined by (16). The following theorem provides the solutions to (18) completely.

THEOREM 3. *Let Assumption 4 hold. Then, we have the following assertions:*

(i) If $\gamma^* = +\infty$, then $\bar{V}(m) = +\infty$. Moreover, there exists β_n, β_n', b_n, and b_n' with $\beta_n \to 0$ and $b_n \to \infty$ as $n \to \infty$ so that $X_n := b_n' \mathbf{1}_{\xi \leq \beta_n'} + b_n \mathbf{1}_{\xi \leq \beta_n}$ is feasible to problem (18) and its objective value goes to infinity.

(ii) If $\gamma^* \leq 1$, then $\bar{V}(m) = x$. Furthermore, the terminal payoff (17) is asymptotically optimal to problem (18).

(iii) If $1 < \gamma^* < +\infty$, then $\bar{V}(m) = \gamma^* x$. Furthermore, there exists β_n, β_n', b_n, and b_n' with $\beta_n \to 0$ and $b_n \to \infty$ as $n \to \infty$ so that

$$X_n := b_n' \mathbf{1}_{\xi \leq \beta_n'} + b_n \mathbf{1}_{\xi \leq \beta_n} \tag{19}$$

is feasible and asymptotically optimal to problem (18).

(iv) Problem (18) admits an optimal solution if and only if $1 \leq \gamma^* < +\infty$ and there exists $c^* \geq c_0$ such that $\gamma^* = \zeta(c^*)$ where c_0 is the unique solution in $(0,1)$ to the equation $\frac{1-c}{\int_c^1 F_\xi^{-1}(1-z)dz} = \frac{\Theta}{x}$. In this case,

$$X^* := \frac{x}{\mathbb{E}\left[\xi 1_{\xi \leq \beta^*}\right]} \mathbf{1}_{\xi \leq \beta^*},$$

where $\beta^* := F_\xi^{-1}(1 - c^*)$, is optimal to problem (18).

Theorem 3 shows, again, that the optimal value of (18) is independent of the expected terminal wealth target Θ, as with the case in which bankruptcy is allowed. Thus, if we draw the efficient frontier on the mean-risk plane, we will also obtain a vertical line as shown in Figure 3.3. However, unlike problem (12) whose optimal value, if finite, does not depend on the choice of risk measure (represented by m), the optimal value of (18) depends on the risk measure (through γ^*).

We have noted in the previous section that when bankruptcy is allowed, one can always construct a series of asymptotically optimal strategies that *automatically* avoid bankruptcy, *so long as* the problem is well-posed. At a first glance, this result seems to defeat the need of introducing the model with bankruptcy prohibition. However, whether the bankruptcy constraint is in place or not significantly changes the likelihood of the problem being ill-posed (and, correspondingly, the risk-taking degree of the agent). Indeed, when bankruptcy is not allowed, the problem is ill-posed only when $\gamma^* = +\infty$, while when bankruptcy is allowed, the model is much more likely to be ill-posed—as long as the same $\gamma^* > 1$! This certainly makes perfect sense economically: other things being equal, no restriction on bankruptcy leads the agent to be exposed to more risk in order to increase her objective value.

When the bankruptcy constraint is absent, we have shown in Theorem 2 that the set of binary payoffs is sufficiently large to achieve the optimal value. The resulting asymptotically optimal strategy (in Theorem 2-(ii)) is to invest a very small amount of money in an extremely risky but highly rewarded lottery and the rest in the bank account. The agent's final payoff is binary, depending on whether the eventual market outcome is extremely favorable or otherwise. However, in the presence of the bankruptcy constraint, the asymptotically optimal solution (in Theorem 3-(iii)) is ternary, which suggests the following strategy: invest little in an extremely risky but highly rewarded lottery and the rest in an asset with medium risk. In good states of world, the agent wins the lottery and his payoff is boosted. In medium states, the agent loses the lottery but wins the investment in the asset, ending up with a moderate level of payoff. In bad states, the agent simply goes bankrupt and exits the market.

Indeed the following proposition stipulates that it is just impossible to use only binary payoffs to achieve the optimal value when bankruptcy is not allowed.

PROPOSITION 1. *Let Assumption 4 hold. Let $\hat{V}(m)$ be the optimal value of problem (18) restricted to binary payoffs, i.e., the optimal value of the following problem*

$$\begin{array}{ll}\underset{X}{\mathit{Max.}} & -WV@R_m(X)\\ \textit{subject to:} & X \textit{ is } \mathcal{F}_T\textit{-measurable and bounded from below},\\ & \mathbb{E}[X] \geq \Theta,\\ & \mathbb{E}[\xi X] \leq x,\\ & X \geq 0,\\ & X \textit{ is binary.}\end{array}$$

If $\gamma^* > 1$ and $\sup_{c_0 \leq c < 1} \zeta(c) < \gamma^*$, then $\hat{V}(m) < \bar{V}(m)$.

To conclude, the presence of the no-bankruptcy constraint markedly alters the model and the resulting (asymptotically) optimal investment behaviors.

5. Mean-Risk Portfolio Choice with Law-Invariant Coherent Risk Measures

In this section, we consider a mean-risk portfolio choice problem where the risk is evaluated by a law-invariant coherent risk measure, with both bankruptcy allowed and prohibited. The market setting is the same as in Sections 3 and 4.

Because the probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$ is atomless, any law-invariant coherent risk measure can be represented as in (5). Therefore, we may just consider the following law-invariant coherent risk measure in our portfolio choice problem:

$$\sup_{\mu \in \mathcal{M}} WV@R_{\varphi(\mu)}(X),$$

where $\mathcal{M}$ is a given closed convex set in $\mathcal{P}([0,1])$ and φ is defined as in (9). Given this risk measure, the mean-risk portfolio choice problem with bankruptcy allowed can be formulated as

$$\begin{array}{ll} \underset{X}{\text{Max.}} & \inf_{\mu\in\mathcal{M}} -WV@R_{\varphi(\mu)}(X) \\ \text{subject to:} & X \text{ is } \mathcal{F}_T\text{-measurable and bounded from below,} \\ & \mathbb{E}[X] \geq \Theta, \\ & \mathbb{E}[\xi X] \leq x. \end{array} \tag{20}$$

When bankruptcy is not allowed, the portfolio choice problem is

$$\begin{array}{ll} \underset{X}{\text{Max.}} & \inf_{\mu\in\mathcal{M}} -WV@R_{\varphi(\mu)}(X) \\ \text{subject to:} & X \text{ is } \mathcal{F}_T\text{-measurable and bounded from below,} \\ & \mathbb{E}[X] \geq \Theta, \\ & \mathbb{E}[\xi X] \leq x, \\ & X \geq 0. \end{array} \tag{21}$$

THEOREM 4. *Let Assumption 4 hold.*
(i) Problem (20) is well-posed if and only if

$$\inf_{\mu\in\mathcal{M}} \sup_{0<c<1} \frac{\varphi(\mu)((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz} \leq 1. \tag{22}$$

Furthermore, when problem (20) is well-posed, its optimal value is x, and the terminal wealth (17) is asymptotically optimal as $n \to \infty$.
(ii) Problem (21) is well-posed if and only if

$$\inf_{\mu\in\mathcal{M}} \sup_{0<c<1} \frac{\varphi(\mu)((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz} < +\infty. \tag{23}$$

Furthermore, when problem (21) is well-posed, its optimal value is

$$\max\left\{\inf_{\mu\in\mathcal{M}} \sup_{0<c<1} \frac{\varphi(\mu)((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz}, 1\right\} x. \tag{24}$$

So, the efficient frontiers here inherit the same property of those with the WVaR case: they are vertical lines on the corresponding mean-risk planes. Furthermore, in the absence of the bankruptcy constraint, the terminal wealth (17) that is asymptotically optimal in the WVaR case is also asymptotically optimal in the case with law-invariant coherent risk measures.

6. Examples In this section we present two examples to illustrate the general results obtained in the previous sections. For simplicity, we assume that essinf $\xi = 0$ and that $F_\xi^{-1}(\cdot)$ is continuous.

EXAMPLE 1. Let $m = \varphi(\mu)$ where $\mu = (1-w)\delta_0 + w\delta_{z_1}$, φ is defined as in (9), $z_1 \in (0,1)$, and $w \in [0,1]$. In this case,

$$WV@R_m(X) = (1-w)CV@R_0(X) + wCV@R_{z_1}(X),$$

which is a weighted average of two CVaR risk measures at two different confidence levels (hence a law-invariant, convex, and comonotonic additive risk measure). In particular, when $w = 0$ or 1, it degenerates into a single CVaR.

By straightforward calculation, we have

$$\zeta(c) = \begin{cases} \frac{w - \frac{w}{z_1}c}{\int_c^1 F_\xi^{-1}(1-z)dz}, & 0 < c \leq z_1, \\ 0, & z_1 < c < 1. \end{cases}$$

Taking the derivative with respect to c, we have, for any $c\in(0,z_1)$, that

$$\zeta'(c)=\frac{\frac{w}{z_1}h(c)}{\left(\int_c^1 F_\xi^{-1}(1-z)dz\right)^2},$$

where

$$h(c)=-\int_c^1 F_\xi^{-1}(1-z)dz+(-c+z_1)F_\xi^{-1}(1-c).$$

Observe that

$$\begin{aligned}h(c)&=\int_c^{z_1}[F_\xi^{-1}(1-c)-F_\xi^{-1}(1-z)]dz-\int_{z_1}^1 F_\xi^{-1}(1-z)dz\\&=\int_0^{z_1}\max\{F_\xi^{-1}(1-c)-F_\xi^{-1}(1-z),0\}dz-\int_{z_1}^1 F_\xi^{-1}(1-z)dz.\end{aligned}$$

Because $\max\{F_\xi^{-1}(1-c)-F_\xi^{-1}(1-z),0\}$ is decreasing with respect to c, we have that $h(\cdot)$ is strictly decreasing in $(0,z_1)$. In addition, it is easy to observe that $h(z_1)<0$.

If $F_\xi^{-1}(1-)\le\frac{1}{z_1}$, then $h(c)<h(0+)\le 0$ for any $c\in(0,z_1)$. Thus, $\zeta(c)$ is strictly decreasing in $(0,z_1)$. As a result, $\gamma^*=\zeta(0+)=w\le 1$ and cannot be achieved by any $c\in(0,1)$. By Theorems 2 and 3, the optimal values for both problem (14) and problem (18) are x and the optimal solutions do not exist.

If $F_\xi^{-1}(1-)>\frac{1}{z_1}$, which is the case for a lognormally distributed ξ, then $h(0+)>0$. Hence $\zeta(\cdot)$ is first strictly increasing and then strictly decreasing in $(0,z_1)$; so there exists a unique $c^*\in(0,z_1)$ such that $\gamma^*=\zeta(c^*)$. Furthermore, c^* is the unique root of $h(\cdot)$ in $(0,z_1)$, leading to

$$\gamma^*=\zeta(c^*)=\frac{w}{z_1F_\xi^{-1}(1-c^*)}.$$

Clearly, c^* does not depend on w; hence we can denote $c^*=c^*(z_1)$, which is strictly increasing in z_1. On the other hand,

$$\begin{aligned}z_1F_\xi^{-1}(1-c^*)&=\int_{c^*}^1 F_\xi^{-1}(1-z)dz+c^*F_\xi^{-1}(1-c^*)\\&=\int_0^1\min\{F_\xi^{-1}(1-z),F_\xi^{-1}(1-c^*)\}dz,\end{aligned}$$

showing that $z_1F_\xi^{-1}(1-c^*)$ is strictly decreasing in z_1. In conclusion, γ^* is strictly increasing in z_1 and strictly increasing in w. Therefore, by Theorem 2, when $w>z_1F_\xi^{-1}(1-c^*)$, problem (14) is ill-posed. When $w=z_1F_\xi^{-1}(1-c^*)$, the optimal value of (14) is x and the optimal solution exists. When $w<z_1F_\xi^{-1}(1-c^*)$, the optimal value of (14) is x and the optimal solution does not exist.

Interestingly, in the special case in which $w=1$ and thus $WV@R_m$ becomes $CV@R_{z_1}$, problem (14) is ill-posed. This is because

$$z_1F_\xi^{-1}(1-c^*)=\int_{c^*}^1 F_\xi^{-1}(1-z)dz+c^*F_\xi^{-1}(1-c^*)<\int_0^1 F_\xi^{-1}(1-z)dz=1.$$

As a result, $\gamma^*=\frac{1}{z_1F_\xi^{-1}(1-c^*)}>1$. By Theorem 2, problem (14) is ill-posed.

Next, consider problem (18), in which bankruptcy is not allowed, under the condition that $F_\xi^{-1}(1-)>\frac{1}{z_1}$. It is *always* well-posed because $\gamma^*<+\infty$, and the optimal value is γ^*x. So the corresponding CVaR problem is also well-posed, as opposed to the case when bankruptcy is allowed. On the other hand, the optimal solution exists if and only if $c^* \geq c_0$ where c_0 is defined as in Theorem 3. Notice that the existence of optimal solution does not depend on w. Moreover, by Proposition 1, if $w > z_1 F_\xi^{-1}(1-c^*)$ and $c^* < c_0$, then $\hat{V}(\mu) < \bar{V}(\mu)$, and consequently the set of two-valued quantile functions cannot achieve the optimal value.

EXAMPLE 2. Consider

$$m = \sum_{i=0}^{n} w_i \delta_{z_i}$$

where $w_i \geq 0$, $i = 0, \ldots, n$, $\sum_{i=0}^n w_i = 1$, and $0 = z_0 < z_1 < \cdots < z_n = 1$. This problem covers VaR as a special case. It is easy to compute that

$$\zeta(c) = \frac{\sum_{j=i}^n w_j}{\int_c^1 F_\xi^{-1}(1-z)dz}, \quad z_{i-1} < c \leq z_i,\ i = 1, \ldots, n.$$

As a result,

$$\gamma^* = \sup_{0<c<1} \zeta(c) = \begin{cases} \max_{1\leq i\leq n-1} \frac{\sum_{j=i}^n w_j}{\int_{z_i}^1 F_\xi^{-1}(1-z)dz}, & w_n = 0, \\ +\infty, & w_n > 0. \end{cases}$$

Then, by Theorem 2, we know problem (14) is ill-posed if and only if $w_n > 0$ or $\max_{1\leq i\leq n-1} \frac{\sum_{j=i}^n w_j}{\int_{z_i}^1 F_\xi^{-1}(1-z)dz} > 1$, and is well-posed and admits an optimal solution if and only if $w_n = 0$ and $\max_{1\leq i\leq n-1} \frac{\sum_{j=i}^n w_j}{\int_{z_i}^1 F_\xi^{-1}(1-z)dz} = 1$, and is well-posed but does not admit a solution in other cases.

If we consider the special case in which $m = \delta_{z_1}$ for some $z_1 \in (0,1)$, corresponding to the VaR, then $\gamma^* = \frac{1}{\int_{z_1}^1 F_\xi^{-1}(1-z)dz} > 1$. As a result, problem (14) is ill-posed.

Next, we consider problem (18). By Theorem 3, this problem is well-posed if and only if $w_n = 0$. Moreover, it admits an optimal solution if and only if $w_n = 0$ and there exists i^* such that $z_{i^*} \geq c_0$ and

$$\max_{1\leq i\leq n-1} \frac{\sum_{j=i}^n w_j}{\int_{z_i}^1 F_\xi^{-1}(1-z)dz} = \frac{\sum_{j=i^*}^n w_j}{\int_{z_{i^*}}^1 F_\xi^{-1}(1-z)dz}.$$

The preceding two examples indicate that the mean-risk model with VaR or CVaR as risk measure is ill-posed if bankruptcy is allowed, and well-posed if otherwise. Note that even in the well-posed case, an optimal solution may not exist in general. Basak and Shapiro [8] and Gabih et al. [21] also consider the VaR constraint on the terminal wealth in a dynamic, complete-market setting. Their problems admit (unique) optimal solutions due to the strong concavity and the Inada condition of their utility functions, while our problem may not because our objective function is linear.

7. Discussions and Conclusions

7.1. Comparison with mean-variance problems We have shown in the previous sections that optimal portfolios of the continuous-time mean-risk portfolio choice problems where the risk is measured by WVaR or law-invariant coherent risk measures exhibit markedly different qualitative behaviors than their mean-variance counterpart.

First, the mean-WVaR portfolio choice problem is prone to be ill-posed, especially when bankruptcy is allowed. The mean-variance problem, however, is *always* well-posed in the same market setting [45]. This difference is one of the most significant: it indicates that if one uses VaR, CVaR, a coherent risk measure, or a generalized version of these as a risk measure in a dynamic portfolio choice model, then one will be likely to be attracted to take excessive risk in order to achieve "optimality" or "nirvana". One important implication of our results is that imposing a no-bankruptcy constraint (or generally an upper bound constraint for losses) will greatly curtail, but not completely eliminate, the aforementioned risk-taking behavior.

Secondly, no matter whether or not bankruptcy is allowed, the optimal value of the mean-risk portfolio choice problem with WVaR or law-invariant coherent risk measures is independent of the expected terminal wealth target; in other words, the efficient frontier on the mean-risk plane is a vertical line. By contrast, the minimal variance in the mean-variance model depends on the expected terminal wealth target, yielding a nontrivial efficient frontier curve on the mean-variance plane; see Zhou and Li [45] and Bielecki et al. [10]. This suggests one of the disadvantages of the mean-WVaR model, that is, there is no explicit tradeoff between risk and return if the risk is evaluated by WVaR. This feature itself leads to some of the unusual trading behaviors to be discussed in details below.

Thirdly, according to Theorem 2, the optimal terminal payoff of the mean-WVaR model (when bankruptcy is allowed), if it exists, is a digital option: it has two deterministic values, depending on the market condition (whether $\xi \le \beta^*$ or otherwise). Even when an optimal solution does not exist, we have found that the two-valued terminal wealth (17) to be asymptotically optimal, as shown by Theorems 2 and 4. When bankruptcy is prohibited, from the proof of Theorem 3, asymptotically optimal terminal wealth can be three-valued. By contrast, the optimal terminal wealth in the mean-variance model changes continuously with respect to the market condition (represented by the pricing kernel ξ); see for instance Bielecki et al. [10].

Finally, the asymptotically optimal solution (17) in the mean-risk problems (14) and (20) represents the following strategy: bank most of the capital and invest the rest to an extremely risky by highly rewarded lottery. The chance of winning the lottery is slight, and consequently the probability of the terminal wealth exceeding the expected payoff target is tiny. A similar observation can be made for the asymptotically optimal solutions to problem (18) in which bankruptcy is not allowed. By contrast, the probability of the optimal terminal wealth exceeding the expected terminal wealth target in the mean-variance portfolio choice problem is high, at least theoretically. Indeed, Li and Zhou [34] show that this probability is at least 80% under all parameter values, assuming a deterministic investment opportunity set.

For classical continuous-time models such as the mean-variance or expected utility maximization, the optimal trading strategies are to continuously re-balance in such a way that the proportion among risky allocations is kept in a pre-determined pattern. While one cannot claim that such a trading behavior captures most of what we see in practice, it does describe what would be the best in an ideal world. In comparison, the strategies derived from the mean-WVaR model and its variants display a very specific pattern of trading behavior, one that is typically taken by the so-called principal guaranteed funds. This type of strategies is not commonly seen in real investment practice, nor does it provide an investment guide suitable for a typical investor.

The essential reason behind our resulting strategies can be seen from the quantile formulation of the mean-WVaR model (27): it is a linear program in $\mathbb{G}$ due to the linearity of $U(G(\cdot), m)$ in $G(\cdot)$. This linearity is intrinsic to the comonotonic additivity of WVaR. For law-invariant coherent risk measures, the objective function becomes sublinear in $G(\cdot)$, and this sublinearity is the result of the positive homogeneity property. This sublinearity in turn leads to the optimal quantile functions being "corner points", or step functions. The corresponding terminal random payoffs are therefore like digital options, leading to the class of "gambling" trading strategies discussed above. For expected utility or mean-variance models, however, the objective functions involved are not sublinear.

7.2. Comparison with Cuoco et al. [16] In this paper, we evaluate portfolio risk by applying a risk measure on the *terminal* wealth only. In real world, however, people may monitor the risk of investment dynamically as time goes by. Cuoco et al. [16] study a model in which the VaR of the portfolio's profit is constrained to be no higher than some specified level at any time t. To compare their results and ours, we consider a variant of our mean-risk portfolio selection problem (12) with a dynamic risk constraint as in Cuoco et al. [16].

To match the setting in Cuoco et al. [16], we switch the roles of the return and risk in the mean-risk problem by fixing a risk level and optimizing the return. As in Cuoco et al. [16], we prohibit bankruptcy. At each time t, the VaR of a portfolio in the period $[t, t+\tau]$ is evaluated by assuming the portfolio in this period is static, where $\tau > 0$ is a given horizon. For simplicity, we assume there is only one risky stock in the market and the appreciation rate $B(\cdot)$ and volatility $\sigma(\cdot)$ of the stock are positive constants. In addition, we assume that at each time t the limit imposed on the VaR of the portfolio in the period $[t, t+\tau]$ is a constant proportion β of the portfolio's value at time t.

According to Cuoco et al. [16, Remark 2], the VaR constraint can be translated into the following portfolio constraint

$$a \le \frac{\pi(t)}{X(t)} \le b, \quad 0 \le t \le T$$

for some constants $a \in \mathbb{R}$ and $b > 0$ that are dependent on the appreciation and volatility of the stock, the confidence level of VaR and the duration τ.[7] Consequently, the revised portfolio choice problem can be formulated as

$$\begin{array}{ll} \underset{\pi(\cdot)}{\text{Max}} & \mathbb{E}[X(T)] \\ \text{subject to} & dX(t) = \pi(t)Bdt + \pi(t)\sigma dW(t), \quad 0 \le t \le T, \quad X(0) = x, \\ & a \le \frac{\pi(t)}{X(t)} \le b, \quad X(t) \ge 0, \quad 0 \le t \le T. \end{array} \tag{25}$$

One can apply the same dynamic programming method as in Cuoco et al. [16] to obtain the optimal percentage allocation to the stock: $\pi^*(t)/X^*(t) = b, 0 \le t \le T$.[8] In other words, the optimal portfolio must bind the VaR constraint. Note that the optimal portfolio in Cuoco et al. [16] does not necessarily bind the VaR constraint because the expected utility rather than the mean of the terminal wealth is maximized therein.

Let us return to the mean-risk problem (18) (having no-bankruptcy constraint). When the risk is measured by VaR, Example 2 illustrates that problem (18) is well-posed, terminal wealth (19) is asymptotically optimal, and the optimal value is independent of the expected return target, i.e., the mean-risk efficient frontier is vertical. This efficient frontier in turn suggests that if an agent restricts the VaR of her terminal wealth to be lower than a fixed risk level and then maximizes the mean of the terminal wealth, the optimal value is infinity (i.e., the problem is ill-posed) and a sequence of terminal wealths in form (19) with $\beta_n \to 0$ and $b_n \to +\infty$ can be constructed to achieve the infinity.[9] Furthermore, the trading strategies that replicate this sequence of terminal wealths

[7] Note that $\pi(t)$ stands for the percentage allocation to the stock in Cuoco et al. [16] while it is used as the dollar amount allocation in the present paper.

[8] Indeed, one can check that the value function corresponding to this portfolio satisfies the HJB equation (15) in Cuoco et al. [16] when the utility function $u(\cdot)$ is the identity function.

[9] The optimal value of problem (18) is $\gamma^* x$ (when $\gamma^* > 1$), which shows that with any expected terminal wealth target (higher than the risk-free payoff), the minimum risk level one can achieve is $-\gamma^* x$. Consequently, in order to make the mean-VaR problem, in which the VaR of the terminal wealth is restricted to be lower than a fixed risk level and the expected terminal wealth is maximized, meaningful, one needs to set this fixed risk level to be strictly higher than $-\gamma^* x$.

must involve asymptotically infinite leverage on the stock. Indeed, as problem (25) shows, when the percentage allocation to the stock is bounded at any time (e.g., $a \le \pi(t)/X(t) \le b, t \ge 0$), the maximum expected terminal wealth one can attain is finite. Hence infinite leverage must be taken to achieve infinitely high expected terminal wealth, which is the case when the portfolio risk is evaluated as the VaR of the terminal wealth.

To summarize, when portfolio risk is evaluated only on the terminal wealth, investors tend to take infinite leverage on risky stocks in order to achieve mean-risk optimality. When portfolio risk is evaluated dynamically as in Cuoco et al. [16], investors are prevented from taking excess leverage.

7.3. Conclusions To conclude, when the risk criterion is changed from variance to WVaR or a law-invariant coherent risk measure, the solutions in the mean-risk portfolio choice problem have a significant qualitative change. Although the resulting optimal trading strategies can explain or capture the behavior of a specific class of agents, they do not seem to be appropriate for many other investors. Moreover, the model is very likely to be ill-posed, indicating an improper modeling of the tradeoff between return and risk. So, while mathematically this paper has solved this class of portfolio selection problems thoroughly and explicitly, economically it is more a critique, than advocacy, of using WVaR or law-invariant coherent measures (including VaR and CVaR) on terminal wealth only to model risk for portfolio choice.

Appendix A: Quantile Formulation Quantile formulation, developed in a series of papers including Schied [41], Carlier and Dana [13], Jin and Zhou [24], and He and Zhou [22], is a technique of solving optimal terminal payoff in portfolio choice problems. This technique can be applied once the objective function and constraints, except for the initial budget constraint, in a portfolio choice problem are law-invariant and the objective function is improved with a higher level of the terminal wealth (i.e., the more the better). The basic idea of quantile formulation is to choose quantile functions as the decision variable.

The mean-risk problem (14) satisfies the aforementioned assumptions, so we can apply quantile formulation here. Recall that $\mathbb{G}$ is the quantile function set as defined in (1). Define

$$U(G(\cdot),m) := \int_{[0,1]} G(z)m(dz), \quad G(\cdot) \in \mathbb{G}. \tag{26}$$

By this definition, $WV@R_m(X) = -U(G_X,m)$. Then, the quantile formulation of (14) is the following optimization problem

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max.}} & U(G(\cdot),m) \\ \text{subject to:} & \int_0^1 G(z)dz \ge \Theta, \\ & \int_0^1 F_\xi^{-1}(1-z)G(z)dz \le x, \\ & G(\cdot) \in \mathbb{G}. \end{array} \tag{27}$$

The following theorem verifies the equivalence of the portfolio selection problem (14) and the quantile formulation (27) in terms of feasibility, well-posedness, and existence and uniqueness of the optimal solution.

Theorem 5. *We have the following assertions.*

(i) Problem (14) is feasible (well-posed) if and only if problem (27) is feasible (well-posed). Furthermore, they have the same optimal value.

(ii) The existence (uniqueness) of optimal solutions to (14) is equivalent to the existence (uniqueness) of optimal solutions to (27).

(iii) If X^* is optimal to (14), then $G_{X^*}(\cdot)$ is optimal to (27). If $G^*(\cdot)$ is optimal to (27), then $G^*(1-F_\xi(\xi))$ is optimal to (14).

Proof. This can be proved in exactly the same way as in He and Zhou [22]. □

Theorem 5 stipulates that in order to study the mean-WVaR problem (14), we only need to study its quantile formulation (27).

The quantile formulation technique also applies to the mean-WVaR problem with bankruptcy prohibition (18), mean-risk problem (20) when risk is measured by law-invariant coherent risk measures, and mean-risk problem with bankruptcy prohibition (21) when risk is measured by law-invariant coherent risk measures, and their quantile formulations are presented, respectively, as follows:

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max.}} & U(G(\cdot), m) \\ \text{subject to:} & \int_0^1 G(z)dz \geq \Theta, \\ & \int_0^1 F_\xi^{-1}(1-z)G(z)dz \leq x, \\ & G(0) \geq 0,\, G(\cdot) \in \mathbb{G}, \end{array} \tag{28}$$

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max.}} & \inf_{\mu \in \mathcal{M}} U(G(\cdot), \varphi(\mu)) \\ \text{subject to:} & \int_0^1 G(z)dz \geq \Theta, \\ & \int_0^1 F_\xi^{-1}(1-z)G(z)dz \leq x, \\ & G(\cdot) \in \mathbb{G}, \end{array} \tag{29}$$

and

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max.}} & \inf_{\mu \in \mathcal{M}} U(G(\cdot), \varphi(\mu)) \\ \text{subject to:} & \int_0^1 G(z)dz \geq \Theta, \\ & \int_0^1 F_\xi^{-1}(1-z)G(z)dz \leq x, \\ & G(0) \geq 0,\, G(\cdot) \in \mathbb{G}. \end{array} \tag{30}$$

Theorem 5 applies to these problems as well.

Finally, let us comment that although we use the complete market setting here, i.e., Assumption 2 is in force, quantile formulation can also be applied in the incomplete markets in which the investment opportunity set, $(r(\cdot), b(\cdot), \sigma(\cdot))$, is deterministic and conic constraints are imposed on the portfolios. In this case, the pricing kernel is not unique, and we need to choose the so-called *minimal* pricing kernel; see He and Zhou [22] for details.

Appendix B: Proofs

B.1. Proof of Theorem 1 We only consider the convex case here, the other two cases being similar.

When restricted on $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, ρ is still law-invariant and convex. According to Jouini et al. [25, Theorem 1.1], there exists $v : \mathcal{P}([0,1]) \to [0, +\infty]$ such that

$$\rho(X) = \sup_{\mu \in \mathcal{P}([0,1])} \left\{ \int_{[0,1]} CV@R_z(X)\mu(dz) - v(\mu) \right\} =: \tilde{\rho}(X)$$

for all $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$. Because we normalize the risk measure ρ so that $\rho(0) = 0$, we must have $\inf_{\mu \in \mathcal{P}([0,1])} v(\mu) = 0$. On the other hand, because $\int_{[0,1]} CV@R_z(X)\mu(dz)$ is continuous in μ for each fixed $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, replacing v with its lower-semi-continuous envelop does not change $\tilde{\rho}(X)$. Thus, we can assume that v is lower-semi-continous in $\mathcal{P}([0,1])$.

Clearly, $\tilde{\rho}(X)$ is well-defined for any $X \in LB(\Omega, \mathcal{F}, \mathbb{P})$. In order to prove the representation (4), we only need to prove that $\tilde{\rho}$ is truncation continuous on $LB(\Omega, \mathcal{F}, \mathbb{P})$.

With the notation $f(X,\mu) := \int_{[0,1]} -CV@R_z(X)\mu(dz)$, we have

$$\tilde{\rho}(X) = -\inf_{\mu\in\mathcal{P}([0,1])} \{f(X,\mu) + v(\mu)\}.$$

Because $-CV@R_z(X), z \in [0,1]$, is increasing and continuous on $[0,1]$ and is finite-valued on $[0,1)$, there exists a sequence of bounded continuous functions converging point-wisely to $-CV@R_z(X)$ from below. Consequently, $f(X,\mu)$ is lower-semi-continuous in μ. On the other hand, it is obvious that $f(X,\mu) \le f(Y,\mu)$ for any $X \le Y$.

Now, consider a fixed $X \in LB(\Omega,\mathcal{F},\mathbb{P})$. For any $n \ge 1$, due to the compactness of $\mathcal{P}([0,1])$ and the lower-semi-continuity of f and v in μ, there exists $\mu_n \in \mathcal{P}([0,1])$ such that $\tilde{\rho}(X\wedge n) = -[f(X\wedge n,\mu_n) + v(\mu_n)]$. Thanks again to the compactness of $\mathcal{P}([0,1])$, we can assume that μ_n converges to $\bar{\mu}$. Then, for any $m \ge 1$, we have

$$\begin{aligned}
\limsup_{n\to+\infty} \tilde{\rho}(X\wedge n) &= \limsup_{n\to+\infty} \{-[f(X\wedge n,\mu_n) + v(\mu_n)]\} \\
&= -\liminf_{n\to+\infty} [f(X\wedge n,\mu_n) + v(\mu_n)] \\
&\le -\liminf_{n\to+\infty} [f(X\wedge m,\mu_n) + v(\mu_n)] \\
&\le -[f(X\wedge m,\bar{\mu}) + v(\bar{\mu})]
\end{aligned}$$

where the first inequality is due to the monotonicity of f in its first argument and the second one is due to the lower-semi-continuity of f and v in μ. Sending m to infinity, by the monotone convergence theorem, we conclude that

$$\limsup_{n\to+\infty} \tilde{\rho}(X\wedge n) \le -[f(X,\bar{\mu}) + v(\bar{\mu})] \le -\inf_{\mu\in\mathcal{P}([0,1])} [f(X,\mu) + v(\mu)] = \tilde{\rho}(X).$$

On the other hand, it is easy to check that $\liminf_{n\to+\infty} \tilde{\rho}(X\wedge n) \ge \tilde{\rho}(X)$. Therefore, we conclude that $\tilde{\rho}$ satisfies the truncation property.

B.2. Proof of Theorem 2 The proof of Theorem 2 is decomposed into two steps: first solve (27) by Lagrange dual method; secondly, solve (14) by recalling Theorem 5.

We apply the Lagrange dual method to solve (27). However, we apply the multiplier only to the expected return constraint and keep the initial budget constraint unchanged. For any $\lambda \ge 0$, consider the following problem

$$\begin{aligned}
&\operatorname*{Max.}_{G(\cdot)} \quad U_\lambda(G(\cdot),m) := \left[\int_{[0,1]} G(z)m(dz)\right] + \lambda\int_0^1 G(z)dz - \lambda\Theta \\
&\text{Subject to } \int_0^1 F_\xi^{-1}(1-z)G(z) \le x, \\
&\qquad\qquad G(\cdot) \in \mathbb{G}
\end{aligned} \tag{31}$$

and denote its optimal value by $V_\lambda(m)$. Recall that $V(m)$ is the optimal value of problem (14), thus is also the optimal value of problem (27). Then, the following weak duality must hold:

$$V(m) \le \inf_{\lambda\ge 0} V_\lambda(m).$$

In the following, we solve (31) and show that the strong duality holds.

Let $\mathbb{S}$ be the set of two-valued quantile functions, i.e.,

$$\mathbb{S} := \{G(\cdot) \in \mathbb{G} \mid G(z) = a + b\mathbf{1}_{c<z\le 1}, a \in \mathbb{R}, b \in \mathbb{R}_+, c \in (0,1)\} \tag{32}$$

and consider

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max.}} & U_\lambda(G(\cdot),m) := \left[\int_{[0,1]} G(z)m(dz)\right] + \lambda\int_0^1 G(z)dz - \lambda\Theta \\ \text{Subject to} & \int_0^1 F_\xi^{-1}(1-z)G(z) \le x, \\ & G(\cdot)\in\mathbb{S}. \end{array} \tag{33}$$

Denote its optimal value by $\tilde{V}_\lambda(m)$.

Proposition 2. *Problems (31) and (33) have the same optimal value, i.e., $V_\lambda(m) = \tilde{V}_\lambda(m)$.*

Proof. The proof is similar to that of Jin and Zhou [24, Proposition D.3]. Because the proof is short, we supply it here for completeness. Clearly, we have $V_\lambda(m) \ge \tilde{V}_\lambda(m)$. If $V_\lambda(m) > \tilde{V}_\lambda(m)$, by the monotone convergence theorem, we can find a bounded feasible $G(\cdot)$ such that $\tilde{V}_\lambda(m) < U_\lambda(G(\cdot),m) < +\infty$. Then, by the dominated convergence theorem, we can find a feasible step function

$$\tilde{G}(\cdot) := a_0 + \sum_{i=1}^n b_i \mathbf{1}_{(t_i,1]}, \quad 0 < t_1 < \cdots < t_n < 1, b_1, \cdots, b_n > 0$$

such that $U(\tilde{G}(\cdot),m) > \tilde{V}_\lambda(m)$. Given any $\{\alpha_i\}_{i=1}^n$ such that $\alpha_i > 0, i = 1,\cdots,n$ and $\sum_{i=1}^n \alpha_i = 1$, let

$$d_i = \frac{b_i}{\alpha_i}, \quad a_i = \int_0^1 F_\xi^{-1}(1-z)\tilde{G}(z)dz - d_i\int_{t_i}^1 F_\xi^{-1}(1-z)dz.$$

Then, we can check that $G_i(\cdot) := a_i + d_i\mathbf{1}_{(t_i,1]}, i = 1,\cdots,n$ are feasible to problem (31) and $\tilde{G}(\cdot) = \sum_{i=1}^n \alpha_i G_i(\cdot)$. By the linearity of U_λ in G, there exists i_0 such that $U_\lambda(\tilde{G}(\cdot),m) \le U_\lambda(G_{i_0}(\cdot),m) \le \tilde{V}_\lambda(m)$, which is a contradiction. □

Next, we solve (33). Thanks to the monotonicity property of WVaR, we only need to consider those feasible solutions that bind the initial budget constraint. Thus, for any feasible solution $a + b\mathbf{1}_{(c,1]}$, we may set

$$a = a(b,c) := x - b\int_c^1 F_\xi^{-1}(1-z)dz. \tag{34}$$

Straightforward calculation shows that

$$U_\lambda(a(b,c) + b\mathbf{1}_{(c,1]}, m) = (k_1(c)\lambda - k_2(c))b + x - (\Theta - x)\lambda,$$

where

$$k_1(c) := \int_c^1 \left[1 - F_\xi^{-1}(1-z)\right]dz, \quad k_2(c) := \int_c^1 F_\xi^{-1}(1-z)dz - m((c,1]).$$

Therefore, (33) is equivalent to

$$\sup_{c\in(0,1),b\ge 0}\ (k_1(c)\lambda - k_2(c))b + x - (\Theta - x)\lambda. \tag{35}$$

Because ξ is atomless, $F_\xi^{-1}(\cdot)$ is strictly increasing on $(0,1)$; hence $k_1(\cdot)$ is strictly concave and $k_1(0+) = k_1(1-) = 0$. Consequently, $k_1(c) > 0$ for any $0 < c < 1$.

Define

$$\lambda^* := \inf_{0<c<1}\frac{k_2(c)}{k_1(c)} = \inf_{0<c<1}\frac{\int_c^1 F_\xi^{-1}(1-z)dz - m((c,1])}{\int_c^1\left[1 - F_\xi^{-1}(1-z)\right]dz}. \tag{36}$$

For any $\lambda \le \lambda^*$ and $c \in (0,1)$, we have $k_1(c)\lambda - k_2(c) \le 0$; hence the optimal value of (35) is $x-(\Theta-x)\lambda$. For any $\lambda > \lambda^*$, there exists $c \in (0,1)$ such that $k_1(c)\lambda - k_2(c) > 0$; hence the optimal value of (35) is $+\infty$. Thus, we have

$$V_\lambda(m) = \tilde{V}_\lambda(m) = \begin{cases} x-(\Theta-x)\lambda, & \lambda \le \lambda^*, \\ +\infty, & \lambda > \lambda^*, \end{cases} \tag{37}$$

Therefore, λ^* is the possible multiplier that closes the duality gap. The next proposition makes precise this observation.

PROPOSITION 3. *We have the following assertions:*
(i) If $\lambda^* < 0$, then $V(m) = \inf_{\lambda \ge 0} V_\lambda(m) = V_0(m) = +\infty$.
(ii) If $\lambda^* \ge 0$, then $V(m) = \inf_{\lambda \ge 0} V_\lambda(m) = V_{\lambda^*}(m) = x-(\Theta-x)\lambda^*$.
(iii) If $\lambda^* \ge 0$ and there exists $c^* \in (0,1)$ such that $\lambda^* = \frac{k_2(c^*)}{k_1(c^*)}$, then problem (27) has an optimal solution given in the form $a + b\mathbf{1}_{(c^*,1]}$ with $a \in \mathbb{R}$ and $b > 0$.

Proof. We first prove assertion (i). In this case, there exists $c^* \in (0,1)$ such that $k_2(c^*) < 0$. Consider $G_b(\cdot) := a(b,c^*) + b\mathbf{1}_{(c^*,1]}$ where $a(b,c^*)$ is given as in (34). Then, we have

$$\int_0^1 G_b(z)dz = a(b,c^*) + b(1-c^*) = x + k_1(c^*)b =: g(b).$$

Clearly, $\lim_{b\to+\infty} g(b) = +\infty$. Consequently, when b is large enough, the expected return constraint is satisfied. On the other hand, we have

$$U(G_b(\cdot), m) = -k_2(c^*)b + x,$$

which goes to infinity as b goes to infinity. Therefore, $V(m) = +\infty$.

Next, consider assertion (ii). By the weak duality, we have $V(m) \le \inf_{\lambda\ge 0} V_\lambda(m) = V_{\lambda^*}(m) = x-(\Theta-x)\lambda^*$. On the other hand, let $c_n \in (0,1)$, $n \ge 1$, be a sequence such that $k_2(c_n)/k_1(c_n) \downarrow \lambda^*$. Let $G_n(\cdot) := a(b_n, c_n) + b_n\mathbf{1}_{(c_n,1]}$ and $b_n \in \mathbb{R}_+$ be the one binding the expected return constraint, i.e.,

$$b_n = \frac{\Theta - x}{\int_{c_n}^1 \left[1 - F_\xi^{-1}(1-z)\right] dz}.$$

Then, we have

$$\begin{aligned} U(G_n(\cdot), m) &= -k_2(c_n)b_n + x \\ &= -(\Theta-x)\frac{k_2(c_n)}{k_1(c_n)} + x, \end{aligned}$$

which converges to $x-(\Theta-x)\lambda^*$ as n goes to infinity. Therefore, we have $V(m) \ge x-(\Theta-x)\lambda^* = V_{\lambda^*}(m)$.

Finally, we prove assertion (iii). From (35), it follows that for any $b \in \mathbb{R}_+$, $G_b(\cdot) := a(b,c^*) + b\mathbf{1}_{(c^*,1]}$ is optimal to (33) and therefore optimal to (31). On the other hand

$$g(b) := \int_0^1 G_b(z)dz = a(b,c^*) + b(1-c^*) = x + k_1(c^*)b$$

is strictly increasing and $g(0) = x$, $\lim_{b\uparrow\infty} g(b) = +\infty$. Therefore, there exists a unique $b^* > 0$ such that $g(b^*) = \Theta$. Consequently, $G_{b^*}(\cdot)$ is optimal to (27). $\square$

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. Because $F_\xi^{-1}(1-z)$ is decreasing in z, we have

$$\limsup_{c\uparrow 1}\frac{k_2(c)}{k_1(c)}\le \limsup_{c\uparrow 1}\frac{\int_c^1 F_\xi^{-1}(1-z)dz}{\int_c^1[1-F_\xi^{-1}(1-z)]dz}\le \limsup_{c\uparrow 1}\frac{(1-c)F_\xi^{-1}(1-c)}{(1-c)(1-F_\xi^{-1}(1-c))}=0,$$

where the equality is due to $F_\xi^{-1}(0+)=0$, which follows from Assumption 4. Thus, we obtain that λ^*, as defined in (36), is larger than or equal to zero, and is equal to zero if and only if $\sup_{0<c<1}\zeta(c)\le 1$, i.e., $\gamma^*\le 1$. Therefore, recalling the equivalence between problems (14) and (27), we conclude $V(m)=+\infty$ in case (i) and $V(m)=x$ in case (ii) from Proposition 3.

From the proof of Proposition 3, we find that when $\gamma^*>1$, there exists $c^*\in(0,1)$ such that $G_n(\cdot):=a(n,c^*)+n\mathbf{1}_{(c^*,1]}$ is feasible and its objective value goes to infinity as $n\to\infty$. Letting $\beta^*:=F_\xi^{-1}(1-c^*)$ and recalling Theorem 5-(iii), we conclude that $X_n:=x-n\mathbb{E}\left[\xi\mathbf{1}_{\xi\le\beta^*}\right]+n\mathbf{1}_{\xi\le\beta^*}$ is feasible and its optimal value goes to infinity.

Next, we prove that the terminal wealth in (17) is asymptotically optimal in case (ii). It is easy to verify that $\mathbb{E}[\xi X_n]=x$ and $\mathbb{E}[X_n]=\Theta$. Thus, X_n is feasible to problem (14). On the other hand, we have

$$V(m)\ge -WV@R_m(X_n)\ge \frac{x\mathbb{P}(\xi\le\beta_n)-\Theta\mathbb{E}\left[\xi\mathbf{1}_{\xi\le\beta_n}\right]}{\mathbb{P}(\xi\le\beta_n)-\mathbb{E}\left[\xi\mathbf{1}_{\xi\le\beta_n}\right]}\to x=V(m).$$

Therefore, the objective value of X_n converges to the optimal value as $n\to\infty$.

For case (iii), we have $k_1(c^*)/k_2(c^*)=0=\lambda^*$. Recalling Proposition 3-(iii), $G^*=a(b^*,c^*)+b^*\mathbf{1}_{(c^*,1]}$ is optimal to (27), where b^* is the positive number so that $\int_0^1 G^*(z)dz=\Theta$. According to Theorem 5-(iii),

$$X^*=G^*(1-F_\xi(\xi))=\frac{x\mathbb{P}(\xi\le\beta^*)-\Theta\mathbb{E}\left[\xi\mathbf{1}_{\xi\le\beta^*}\right]}{\mathbb{P}(\xi\le\beta^*)-\mathbb{E}\left[\xi\mathbf{1}_{\xi\le\beta^*}\right]}+\frac{\Theta-x}{\mathbb{P}(\xi\le\beta^*)-\mathbb{E}\left[\xi\mathbf{1}_{\xi\le\beta^*}\right]}\mathbf{1}_{\xi\le\beta^*}$$

is optimal to (14).

Finally, we prove (iv). Suppose problem (14) admits an optimal solution, so does problem (27), and we denote the solution as $G^*(\cdot)$. Then, $G^*(\cdot)$ can not be constant on $(0,1)$ because it must satisfy the initial budget constraint and expected terminal wealth constraint at the same time and $\Theta>x$. Let $dG^*(s+)$ be the measure on $[0,1)$ induced by G^*, i.e.,

$$G^*(z)-G^*(0)=\int_{[0,z)}dG^*(s+),\quad z\in[0,1).$$

The above equality also holds true for $z=1$ because $G^*(1)=\lim_{z\uparrow 1}G^*(z)$. Then, we have

$$\begin{aligned}\int_{[0,1]}G^*(z)m(dz)&=\int_{[0,1]}\int_{[0,z)}dG^*(s+)m(dz)+G^*(0)\\&=\int_{[0,1)}m((s,1])dG^*(s+)+G^*(0).\end{aligned}$$

Similarly, we have

$$\int_0^1 F_\xi^{-1}(1-z)G^*(z)dz=\int_{[0,1)}\left(\int_s^1 F_\xi^{-1}(1-z)dz\right)dG^*(s+)+G^*(0).$$

Because G^* is nonconstant on $[0,1)$, $dG^*(s+)$ must be a strictly positive measure. Then, because $\zeta(c)<1$ for all $c\in(0,1)$, we conclude that

$$\int_{[0,1]}G^*(z)m(dz)<\int_0^1 F_\xi^{-1}(1-z)G^*(z)dz\le x,$$

which contradicts the optimality of $G^*(\cdot)$. $\square$

B.3. Proof Theorem 3 We only need to solve (28) and then apply Theorem 5.

For any $\gamma \geq 1$, we apply the multiplier $\gamma - 1$ to the last constraint in (28), i.e., consider the following problem

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max.}} & U(G(\cdot),m)+(\gamma-1)G(0)=\gamma\left[U(G(\cdot),m^{\gamma})\right] \\ \text{subject to:} & \int_0^1 G(z)dz \geq \Theta, \\ & \int_0^1 F_\xi^{-1}(1-z)G(z)dz \leq x, \\ & G(\cdot)\in\mathbb{G} \end{array} \tag{38}$$

where $m^\gamma := \frac{1}{\gamma}m + \frac{\gamma-1}{\gamma}\delta_0$ and δ_0 is the Dirac measure at 0. Denote the optimal value of (38) by $\bar{V}_\gamma(m)$. Obviously, we have $\bar{V}_\gamma(m) = \gamma V(m^\gamma)$. On the other hand,

$$\sup_{c\in(0,1)} \frac{m^\gamma((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz} = \frac{1}{\gamma} \sup_{c\in(0,1)} \frac{m((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz} = \frac{\gamma^*}{\gamma}.$$

Thus, it follows from Theorem 2 that

$$\bar{V}_\gamma(m) = \gamma V(m^\gamma) = \begin{cases} \gamma x & \gamma \geq \gamma^* \\ +\infty & \gamma < \gamma^*. \end{cases}$$

Then, by the weak duality, we have

$$\bar{V}(m) \leq \inf_{\gamma\geq 1} \bar{V}_\gamma(m) = (\gamma^* \vee 1)x.$$

Next, we prove that actually the strong duality holds.

First, consider the case in which $\gamma^* \leq 1$. Choosing $\gamma = 1$, then problem (38) becomes problem (27). By Theorem 2, the optimal value of problem (27) is x, and we can find a sequence of feasible quantiles G_n such that $U(G_n, m) \to x$ and $G_n(0) \to x$ as $n \to \infty$. As a result, G_n's are also feasible to problem (28), which implies that $\bar{V}(m) \geq U(G_n, m) \to x$. Combining this fact with the weak duality, we conclude that $\bar{V}(m) = x$. In addition, it is easy to see that the terminal wealth (17) is feasible and asymptotically optimal to problem (18).

Next, we consider the case in which $1 < \gamma^* < +\infty$. We can find a sequence $\{c_n'\}_{n\geq 1}$ in $(0,1)$ such that $\zeta(c_n')$ goes to γ^*. Because $F_\xi^{-1}(0+) = 0$, we have

$$\liminf_{c\uparrow 1} \frac{1-c}{\int_c^1 F_\xi^{-1}(1-z)dz} \geq \liminf_{c\uparrow 1} \frac{1-c}{(1-c)F_\xi^{-1}(1-c)} = +\infty.$$

Consequently, we can find a sequence $\{c_n\}$ in $(0,1)$ increasing to 1 and a sequence $\{w_n\}$ in $[0,1]$ converging to 0 such that

$$w_n \frac{1-c_n}{\int_{c_n}^1 F_\xi^{-1}(1-z)dz} \to +\infty.$$

Let

$$G_n(\cdot) := (1-w_n)\frac{x}{\int_{c_n'}^1 F_\xi^{-1}(1-z)dz}\mathbf{1}_{(c_n',1]} + w_n \frac{x}{\int_{c_n}^1 F_\xi^{-1}(1-z)dz}\mathbf{1}_{(c_n,1]}. \tag{39}$$

Then, it is easy to see that

$$\int_0^1 G_n(z)dz \geq w_n x \frac{1-c_n}{\int_{c_n}^1 F_\xi^{-1}(1-z)dz} \to +\infty.$$

Therefore, $G_n(\cdot)$ is feasible when n is sufficiently large. On the other hand, we have

$$U(G_n(\cdot),m) = \left[(1-w_n)\frac{m((c_n',1])}{\int_{c_n'}^1 F_\xi^{-1}(1-z)dz} + w_n\frac{m((c_n,1])}{\int_{c_n}^1 F_\xi^{-1}(1-z)dz}\right]x$$
$$\geq \left[(1-w_n)\frac{m((c_n',1])}{\int_{c_n'}^1 F_\xi^{-1}(1-z)dz}\right]x \to \gamma^* x.$$

Thus, we have $\bar{V}(m) \geq \gamma^* x$. Together with the weak duality, we have $\bar{V}(m) = \gamma^* x$. As a consequence, $X_n := G_n(1-F_\xi(\xi))$ is feasible and asymptotically optimal to problem (18). With $\beta_n := F_\xi^{-1}(1-c_n)$, $\beta_n' := F_\xi^{-1}(1-c_n')$, $b_n := w_n\frac{x}{\int_{c_n}^1 F_\xi^{-1}(1-z)dz}$, and $b_n' := (1-w_n)\frac{x}{\int_{c_n'}^1 F_\xi^{-1}(1-z)dz}$, X_n coincides with the one in (19).

Assertion (i) can be proved similarly.

Finally, we prove (iv). We only need show that (30) admits optimal solutions if and only if there exists $c^* \geq c_0$ such that $\gamma^* = \zeta(c^*)$.

Because $\frac{1-c}{\int_c^1 F_\xi^{-1}(1-z)dz}$ is continuous and strictly increasing in c on $(0,1)$ with

$$\lim_{c\downarrow 0}\frac{1-c}{\int_c^1 F_\xi^{-1}(1-z)dz} = 1, \quad \lim_{c\uparrow 1}\frac{1-c}{\int_c^1 F_\xi^{-1}(1-z)dz} = +\infty,$$

the existence and uniqueness of $c_0 \in (0,1)$ are justified.

First, we prove the sufficiency. Let $c^* \geq c_0$ such that $\gamma^* = \zeta(c^*) \geq 0$. Let

$$G^*(z) := \frac{x}{\int_{c^*}^1 F_\xi^{-1}(1-z)dz}\mathbf{1}_{c^*<z\leq 1}, \quad 0<z<1.$$

Then,

$$\int_0^1 G^*(z)dz = \frac{1-c^*}{\int_{c^*}^1 F_\xi^{-1}(1-z)dz}x \geq \Theta$$

where the last inequality is due to $c^* \geq c_0$. Therefore, $G^*(\cdot)$ is feasible. On the other hand,

$$U(G^*(z),m) = \frac{m((c^*,1])}{\int_{c^*}^1 F_\xi^{-1}(1-z)dz}x = \gamma^* x.$$

Thus, $G^*(\cdot)$ is an optimal solution to (30). As a result, $G^*(1-F_\xi(\xi))$ is optimal to (18)

Next, we prove the necessity. Suppose problem (18) admits an optimal solution, then so does problem (28). Denote $G^*(\cdot)$ as the optimal solution to problem (28). Then, we must have $\gamma^* < +\infty$. By the strong duality we just proved, $G^*(\cdot)$ is also optimal to (38) with $\gamma = \gamma^* \vee 1$. If $\gamma^* < 1$, problem (38) with $\gamma = \gamma^* \vee 1 = 1$ just becomes problem (27). Consequently, $G^*(\cdot)$ is optimal to problem (27), which, however, contradicts the result in Theorem 2-(iv) that the optimal solution to problem (27) does not exist when $\gamma^* < 1$. Thus, we must have $\gamma^* \geq 1$.

Because $1 \leq \gamma^* < +\infty$, again by the strong duality, we conclude that $G^*(\cdot)$ is optimal to problem (38) with γ chosen to be $\gamma^* \vee 1 = \gamma^*$. Note that in this case problem (38) is equivalent to problem (27) with m replaced by m^{γ^*}. Then, by Theorem 2, there must exist $c^* \in (0,1)$ such that

$$\sup_{c\in(0,1)}\frac{\frac{1}{\gamma^*}m((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz} = \frac{\frac{1}{\gamma^*}m((c^*,1])}{\int_{c^*}^1 F_\xi^{-1}(1-z)dz},$$

i.e., $\gamma^* = \zeta(c^*)$. Next, we show that we can actually find $c^* \geq c_0$ so that $\gamma^* = \zeta(c^*)$.

If it is not the case, we must have $\zeta(c) < \gamma^*$ for all $c \in [c_0, 1)$. Because $G^*(\cdot)$ is optimal to problem (28), by the strong duality, $G^*(\cdot)$ is also optimal to problem (38) with γ chosen to be $\gamma^* \vee 1 = \gamma^*$. Then, by the strong duality in Proposition 3, $G^*(\cdot)$ is optimal to problem (31) with λ chosen to be 0 and m replaced by m^{γ^*}. Define

$$\bar{G}(z) := G^*(z \wedge c_0) + \int_{c_0}^1 F_\xi^{-1}(1-s)(G^*(s) - G^*(c_0))ds, \quad 0 < z < 1.$$

Then, $\bar{G}(\cdot)$ is feasible to (31). On the other hand,

$$\begin{aligned}
&U(\bar{G}(z), m^{\gamma^*})\\
=&U(G^*(\cdot \wedge c_0), m^{\gamma^*}) + \int_{c_0}^1 F_\xi^{-1}(1-z)(G^*(z) - G^*(c_0))dz\\
=&U(G^*(\cdot), m^{\gamma^*}) - \frac{1}{\gamma^*}\int_{(c_0,1]}(G^*(z) - G^*(c_0))m(dz) + \int_{c_0}^1 F_\xi^{-1}(1-z)(G^*(z) - G^*(c_0))dz\\
=&U(G^*(\cdot), m^{\gamma^*}) - \frac{1}{\gamma^*}\int_{(c_0,1]}\int_{[c_0,z)} dG^*(s+)m(dz) + \int_{c_0}^1 F_\xi^{-1}(1-z)\int_{[c_0,z)} dG^*(s+)dz\\
=&U(G^*(\cdot), m^{\gamma^*}) + \int_{[c_0,1)}\left[\int_s^1 F_\xi^{-1}(1-z)(z)dz - \frac{1}{\gamma^*}m((s,1])\right]dG^*(s+),
\end{aligned}$$

where $dG^*(s+)$ is the measure on $[0,1)$ induced by G^*. Because

$$\int_c^1 F_\xi^{-1}(1-z)dz > \gamma^* m((c,1]), \quad c \in [c_0, 1),$$

we conclude that

$$U(\bar{G}(z), m^{\gamma^*}) \geq U(G^*(z), m^{\gamma^*})$$

and the inequality becomes equality if and only if $G^*(z) = G^*(c_0), z \geq c_0$. Therefore, by the optimality of $G^*(\cdot)$, we must have $G^*(\cdot) = G^*(\cdot \wedge c_0)$. Consequently

$$\begin{aligned}
\int_0^1 G^*(z)dz &= G^*(0) + \int_{[0,1)}(1-s)dG^*(s+)\\
&= G^*(0) + \int_{[0,c_0)}(1-s)dG^*(s+)\\
&\leq G^*(0) + \frac{\Theta}{x}\int_{[0,c_0)}\int_s^1 F_\xi^{-1}(1-z)dzdG^*(s+)\\
&= G^*(0) + \frac{\Theta}{x}\int_{[0,1)}\int_s^1 F_\xi^{-1}(1-z)dzdG^*(s+)\\
&= -\left(\frac{\Theta}{x} - 1\right)G^*(0) + \frac{\Theta}{x}\int_0^1 F_\xi^{-1}(1-z)G^*(z)dz\\
&\leq \Theta,
\end{aligned}$$

where the first inequality is due to the definition of c_0. Furthermore, both inequalities become equalities if and only if $G^*(\cdot) \equiv 0$, which is impossible because $G^*(\cdot)$ satisfies the initial budget constraint. Therefore, we must have

$$\int_0^1 G^*(z)dz < \Theta,$$

which contradicts the feasibility of $G^*(\cdot)$.

B.4. Proof of Proposition 1 According to Theorem 5, $\hat{V}(m)$ is the optimal value of the following problem

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max.}} & U(G(\cdot),m) \\ \text{Subject to} & \int_0^1 G(z)dz \geq \Theta, \\ & \int_0^1 F_\xi^{-1}(1-z)G(z)dz \leq x, \\ & G(\cdot) \in \mathbb{S},\ G(0) \geq 0, \end{array}$$

where $\mathbb{S}$ is the set of two-valued quantile functions as defined in (32).

Because of the monotonicity property of the WVaR risk measure, we only need to consider the feasible solutions in the following form

$$a(b,c) + b\mathbf{1}_{(c,1]}$$

where $b \in \mathbb{R}_+$, $c \in (0,1)$, and $a(b,c)$ is defined as in (34). The feasibility implies that

$$a(b,c) \geq 0, \quad a(b,c) + b(1-c) \geq \Theta,$$

which is equivalent to

$$c \geq c_0, \quad \frac{\Theta - x}{1 - c - \int_c^1 F_\xi^{-1}(1-z)dz} \leq b \leq \frac{x}{\int_c^1 F_\xi^{-1}(1-z)dz}.$$

On the other hand,

$$U(a(b,c) + b\mathbf{1}_{(c,1]}, m) = \left[m((c,1]) - \int_c^1 F_\xi^{-1}(1-z)dz \right] b + x.$$

Therefore, we have

$$\hat{V}(m) \leq \sup_{c_0 \leq c < 1} \{\max\{\zeta(c) - 1, 0\}x + x\} = \max\{\sup_{c_0 \leq c < 1} \zeta(c), 1\}x < \gamma^* x.$$

B.5. Proof Theorem 4 The following minimax result is useful in the proof of Theorem 4.

LEMMA 1. *Let $\mathbb{H} \subset \mathbb{G}$ be a nonempty convex set. Let v be a lower-semi-continuous convex function on $\mathcal{P}([0,1])$ and assume that the domain of v is closed. Then, we have*

$$\sup_{G(\cdot)\in\mathbb{H}} \inf_{\mu\in\mathcal{P}([0,1])} \{U(G(\cdot), \varphi(\mu)) + v(\mu)\} = \inf_{\mu\in\mathcal{P}([0,1])} \sup_{G(\cdot)\in\mathbb{H}} \{U(G(\cdot), \varphi(\mu)) + v(\mu)\}.$$

Proof. Let $\mathcal{M}$ be the domain of v. Then, we only need to prove

$$\sup_{G(\cdot)\in\mathbb{H}} \inf_{\mu\in\mathcal{M}} \{U(G(\cdot), \varphi(\mu)) + v(\mu)\} = \inf_{\mu\in\mathcal{M}} \sup_{G(\cdot)\in\mathbb{H}} \{U(G(\cdot), \mu) + v(\mu)\}.$$

Because $\mathcal{M}$ is a closed set in $\mathcal{P}([0,1])$, which is a compact space, $\mathcal{M}$ is compact. Furthermore, because v is convex, $\mathcal{M}$ is convex. On the other hand, using the same argument as in the proof of Theorem 1, we can show that U is lower-semi-continuous with respect to μ. Thus, if we write

$$f(G,\mu) := U(G(\cdot), \varphi(\mu)) + v(\mu),$$

then f is lower-semi-continuous in μ on $\mathcal{P}([0,1])$ and thus lower-semi-continuous on $\mathcal{M}$. Moreover, it is easy to see that f is convex in μ and linear (thus concave) in G.

Let $0 < t_n < 1, n \geq 1$ be a sequence of real numbers increasing to 1, and define

$$\mathbb{H}^n := \{G(\cdot \wedge t_n) \mid G(\cdot) \in \mathbb{H}\}.$$

Then, $\mathbb{H}^n$ is a nonempty convex subset of $\mathbb{G}$. Furthermore, f takes finite values on $\mathbb{H}^n \times \mathcal{M}$ because any $G(\cdot) \in \mathbb{H}^n$ is bounded. Thus, by Sion [42, Theorem 4.2'], we obtain that

$$\sup_{G(\cdot)\in\mathbb{H}^n} \inf_{\mu\in\mathcal{M}} f(G(\cdot),\mu) = \inf_{\mu\in\mathcal{M}} \sup_{G(\cdot)\in\mathbb{H}^n} f(G(\cdot),\mu).$$

Because f is lower-semi-continuous in μ and $\mathcal{M}$ is compact, there exists $\mu_n \in \mathcal{M}$ such that

$$\sup_{G(\cdot)\in\mathbb{H}^n} f(G(\cdot),\mu_n) = \inf_{\mu\in\mathcal{M}} \sup_{G(\cdot)\in\mathbb{H}^n} f(G(\cdot),\mu).$$

Again thanks to the compactness of $\mathcal{M}$, we can assume that $\mu_n \to \bar{\mu}$ without loss of generality. Now for any fixed $\bar{G}(\cdot) \in \mathbb{H}$ and $m \geq 1$, we have

$$\begin{aligned}
\sup_{G(\cdot)\in\mathbb{H}} \inf_{\mu\in\mathcal{M}} f(G(\cdot),\mu) &\geq \liminf_{n\to+\infty} \sup_{G(\cdot)\in\mathbb{H}^n} \inf_{\mu\in\mathcal{M}} f(G(\cdot),\mu) \\
&= \liminf_{n\to+\infty} \sup_{G(\cdot)\in\mathbb{H}^n} f(G(\cdot),\mu_n) \\
&\geq \liminf_{n\to+\infty} f(\bar{G}(\cdot\wedge t_n),\mu_n) \\
&\geq \liminf_{n\to+\infty} f(\bar{G}(\cdot\wedge t_m),\mu_n) \\
&\geq f(\bar{G}(\cdot\wedge t_m),\bar{\mu})
\end{aligned}$$

where the third inequality is due to the monotonicity of f with respect to G and the last inequality is due to the low-semi-continuity of f in μ. Letting m go to infinity, by the monotone convergence theorem, we have

$$\sup_{G(\cdot)\in\mathbb{H}} \inf_{\mu\in\mathcal{M}} f(G(\cdot),\mu) \geq f(\bar{G}(\cdot),\bar{\mu})$$

for any $\bar{G}(\cdot) \in \mathbb{H}$. Consequently,

$$\sup_{G(\cdot)\in\mathbb{H}} \inf_{\mu\in\mathcal{M}} f(G(\cdot),\mu) \geq \sup_{G(\cdot)\in\mathbb{H}} f(G(\cdot),\bar{\mu}) \geq \inf_{\mu\in\mathcal{M}} \sup_{G(\cdot)\in\mathbb{H}} f(G(\cdot),\mu) \geq \sup_{G(\cdot)\in\mathbb{H}} \inf_{\mu\in\mathcal{M}} f(G(\cdot),\mu). \quad \square$$

Proof of Theorem 4. First, we prove case (i). Recall that problem (20) is equivalent to its quantile formulation (29), so we only need to investigate the latter.

Let $\mathbb{H}$ be the set of quantiles that are feasible to problem (29). It is obvious that $\mathbb{H}$ is a nonempty convex set. Now, applying Lemma 1, we obtain that the optimal value of problem (29) is $\inf_{\mu\in\mathcal{M}} V(\varphi(\mu))$. As a result, problem (29) is well-posed if and only if $V(\varphi(\mu)) < \infty$ for some $\mu \in \mathcal{M}$. Recall that $V(m) < \infty$ if and only if $\sup_{0<c<1} \frac{m((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz} \leq 1$. Thus, problem (29) is well-posed if and only if $\sup_{0<c<1} \frac{\varphi(\mu)((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz} \leq 1$ for some $\mu \in \mathcal{M}$, which is equivalent to (22) because $\mathcal{M}$ is closed and

$$\varphi(\mu)((c,1]) = \int_{(c,1]} \frac{s-c}{s} \mu(ds)$$

is lower-semi-continuous in μ for each fixed $c \in (0,1)$.

In addition, from Theorem 2, if $V(m) < \infty$, then $V(m) = x$. Thus, if problem (29), thus problem (20), is well-posed, then its optimal value must be x. Finally, it is easy to show that X_n's are feasible to problem (20) and

$$\inf_{\mu\in\mathcal{M}} \left(-WV@R_{\varphi(\mu)}(X_n)\right) \geq \frac{x\mathbb{P}(\xi \leq c_n) - \Theta\mathbb{E}[\xi \mathbf{1}_{\xi\leq c_n}]}{\mathbb{P}(\xi \leq c_n) - \mathbb{E}[\xi \mathbf{1}_{\xi\leq c_n}]} \to x$$

as $n \to \infty$. Thus, the payoffs X_n's approximate the optimal value as $n \to \infty$.

Case (ii) can be proved similarly. $\square$

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