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INVERSE S-SHAPED PROBABILITY WEIGHTING AND ITS IMPACT ON INVESTMENT

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Abstract. In this paper we analyze how changes in inverse S-shaped probability weighting influence optimal portfolio choice in a rank-dependent utility model. We derive sufficient conditions for the existence of an optimal solution of the investment problem, and then define the notion of a more inverse S-shaped probability weighting function. We show that an increase in inverse S-shaped weighting typically leads to a lower allocation to the risky asset, regardless of whether the return distribution is skewed left or right, as long as it offers a non-negligible risk premium. Only for lottery stocks with poor expected returns and extremely positive skewness does an increase in inverse S-shaped probability weighting lead to larger portfolio allocations.

1. **Introduction.** When making decisions under risk many people display a tendency to overweight both extremely positive and extremely negative events that occur with small probabilities. Overweighting of the probability of tail events can explain why some people buy both lottery tickets and insurance policies. This type of behaviour can be modelled by an inverse S-shaped probability weighting function, one of the central features of prospect theory [18]. In financial economics, probability weighting has been incorporated into portfolio choice models to explain portfolio underdiverification [24] and the demand for lottery stocks [3]. Inverse S-shaped

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weighting functions fit the aggregate experimental data well; see [13].[1] However, there is considerable heterogeneity in the degree of inverse S-shaped weighting at the individual level. Therefore, it is important to understand how changes in inverse S-shaped weighting influence optimal portfolio choice, an open question that we aim to address in this paper.

In this paper we study a general one-period portfolio choice problem with one risky asset (e.g., a stock) and one riskless asset, when the investor's preferences are modelled by rank-dependent utility (RDU). RDU is a parsimonious preference model that accommodates probability weighting, including overweighting of extreme events. The portfolio choice problem considered in this paper allows the return distribution to be very general including non-continuous distributions, and includes a constraint on short-selling. We first derive two sufficient conditions for the existence of an optimal solution of the investment problem, and then show that an investor with a more risk-averse RDU preference relation will invest less in the risky asset, generalizing an existing result of [9] to possibly non-continuous return distributions.

To study the impact of overweighting the probability of unlikely events, we first define the notion of a *more* inverse S-shaped probability weighting function, using a concave-convex transformation function. Intuitively, the impact of more inverse S-shaped probability weighting on the optimal asset allocation should depend on whether the risky asset's return distribution is skewed to the right or the left. When the risky asset return follows a skewed Bernoulli distribution, we prove analytically that the impact of a more inverse S-shaped weighting function on the stock allocation depends on the skewness parameter of the distribution.

For general return distributions, however, the effect of more inverse S-shaped weighting on the risky asset allocation is complicated, and will depend on the interplay between the weights given to the tails and the middle region of the return distribution, as well as the curvature of the utility function (risk aversion). We illustrate these effects numerically, using simulations of a skew-normal return distribution, and two different utility functions: power and exponential functions. We find that an increase in inverse S-shaped weighting typically leads to a lower allocation to the risky asset, regardless of whether the distribution is skewed left or right, as long as the asset offers a reasonable risk premium. An explanation is that an increase in inverse S-shaped weighting lowers the probability weight given to the positive returns around the median of the distribution, while also increasing the weight of negative extremes. The extra weight given to extremely positive returns cannot compensate for these two negative effects even if the return is skewed to the right. Only when the risk premium of the stock is close to zero or even negative (thereby offering a poor reward-to-risk ratio), and the return distribution is skewed to the right, then do we find a positive relation between the inverse S-shaped weighting and the optimal stock weight.

Finally, we use empirical stock market data to demonstrate the potential effects of probability weighting on investor portfolios in practice. We calibrate a skew-normal distribution using a time series of aggregate U.S. stock market returns, as well as the return distribution of a randomly sampled individual stock. As the returns of the aggregate U.S. stock market have a left-skewed distribution, a more

[1] Fehr-Duda and Epper [13] provide a review of the probability weighting literature, as well as empirical evidence in favor of inverse S-shaped probability weighting functions for a large representative sample from the Swiss population. Earlier experimental evidence supporting inverse S-shaped probability weighting can be found in [26] and [15].

inverse S-shaped weighting function leads to a lower optimal stock allocation. We find similar results if the return distribution has positive skewness and a relatively high expected return, using the historical return distribution of the listed company Apple as an example. Only for stocks offering relatively poor expected returns, or very extreme positive skewness, features often associated with so-called lottery stocks [20], is the relation between inverse S-shaped weighting and the optimal stock weight positive.

Related to our work are [9], [24], [3], [12], [17] and [16], amongst others, who study the influence of probability weighting on optimal portfolio choice and asset pricing, either using RDU or cumulative prospect theory. The contribution of our work to this literature is that we explicitly focus on the question whether an increase in inverse S-shaped probability weighting leads to a lower or higher allocation to stocks, and under what conditions. We discover that it is the combination of the agent preferences (utility function and probability weighting function) and the stock return characteristics (skewness and mean return) that dictates the demand for stocks, and there are no simple comparative statics.

In the following, Section 2 reviews the RDU theory and the characterization of a more risk-averse attitude in this framework. We then propose a single-period portfolio choice model in Section 3 and investigate the impact of changes in probability weighting on asset allocation theoretically in Section 4. Numerical experiments are presented in Section 5 and some technical results are placed in the Appendix.

2. **Rank-dependent utility.**

A preference relation $\succcurlyeq$ is a partial order on a set of random payoffs. A mapping V from the set of random payoffs to real numbers is said to be a representation of $\succcurlyeq$ if $X \succcurlyeq Y$ if and only if $V(X) \geq V(Y)$, in which case $V(X)$ is called the preference value of X. Rank-dependent utility (RDU) is a representation of preference relation defined as follows:

$$V(X) := \int_{\mathbb{R}} U(x) d[1 - w(1 - F_X(x))], \tag{1}$$

where $F_X(\cdot)$ is the cumulative distribution function (CDF) of X. The function U, which is increasing and continuous in its domain, is called a *utility function* and the function w, which is an increasing mapping from the unit interval onto itself, is called a *probability weighting function*.

A preference relation is called law-invariant if any two random payoffs sharing the same distribution are equivalent. Obviously, RDU represents a law-invariant preference relation.

Let $\succcurlyeq$ be a law-invariant preference relation, and denote its implied equivalence relation as $\sim$. A distribution F differs from another one F^* by a simple compensated spread from the point of view of $\succcurlyeq$ if $F \sim F^*$ and there exists $x_0 \in \mathbb{R}$ such that $F(x) \geq F^*(x)$ for all $x < x_0$ and $F(x) \leq F^*(x)$ for all $x \geq x_0$.

A law-invariant preference relation $\succcurlyeq^*$ is said to be *more risk averse* than another one $\succcurlyeq$ if $F^* \succcurlyeq^* F$ for any distributions F and F^* such that F differs from F^* by a simple compensated spread from the point view of $\succcurlyeq$.

The following theorem proved by [9] characterizes when one RDU preference relation is more risk averse than another.

Theorem 2.1. *Let $V_i(X)$ be the RDU preference measure with utility function U_i and probability weighting function w_i, $i = 1, 2$, and assume that w_1 and w_2 are differentiable on $[0, 1]$. Then, the preference relation represented by V_1 is more risk* *averse than the preference relation represented by* V_2 *if and only if* U_1 *is a concave transformation of* U_2 *and* w_1 *is a convex transformation of* w_2.

Proof. See Theorem 1 and Corollary 1 in [9]. □

The precise definition of concave and convex transformation and its connection with the Arrow-Pratt index are provided in Appendix A. Intuitively, Theorem 2.1 shows that making the utility function more concave, or the probability weighting function more convex, makes the agent's preference relation more risk averse.

3. **Portfolio choice under RDU in single period.** We consider a market in which two assets are tradeable: one is a risk-free asset and the other one is a risky stock. An agent, who is endowed with initial capital x_0, decides the allocation between these two assets in one period. Suppose the return of the risk-free asset is zero and the net (excess) return of the stock is R. In the following, for any random variable X, denote its distribution function and quantile function as $F_X(\cdot)$ and $G_X(\cdot)$, respectively. For simplicity, we assume in the following theoretical analysis that $F_R(\cdot)$ has a compact support, i.e., $G_R(\cdot)$ is bounded. In some of the numerical examples, we may consider distributions with noncompact support.

In the following, we denote $\underline{R} := \operatorname{essinf} R = G_R(0+)$ and $\overline{R} := \operatorname{esssup} R = G_R(1-)$. To avoid arbitrage, we always assume that $\underline{R} < 0$ and $\overline{R} > 0$.

Suppose the agent invests an amount θ in the risky stock and the rest in the risk-free asset. Then the terminal wealth becomes

$$X = x_0 + \theta R.$$

We assume that shorting is not allowed, namely, $\theta \geq 0$.

The agent's preference is represented by RDU with utility function $u(\cdot)$ which is strictly increasing in its domain and probability weighting function $w(\cdot)$ which is a strictly increasing function mapping $[0, 1]$ onto $[0, 1]$. As a result, the RDU value of the agent's terminal wealth, with allocation θ, is

$$f(\theta) := V(X) = \int_{\mathbb{R}} u(x) d[-w(1 - F_X(x))].$$

Denote $\underline{x} := \inf\{x | u(x) > -\infty\}$. Then, the interior of the domain of $u(\cdot)$ is $(\underline{x}, +\infty)$.

In the following, we assume that $w(\cdot)$ is absolutely continuous. Then, we have

$$f(\theta) = \int_0^1 u(G_X(z)) w'(1 - z) dz = \int_0^1 u(x_0 + \theta G_R(z)) w'(1 - z) dz.$$

Because we assume that $G_R(\cdot)$ is bounded, $u(\cdot)$ is increasing and $w(\cdot)$ is increasing, $f(\theta)$ is always a well-defined function whose value may possibly be $-\infty$, i.e., $f(\theta) \in [-\infty, +\infty)$.[2] Note that if $u(\cdot)$ is concave, then $f(\cdot)$ is concave as well.

The agent's portfolio choice problem is

$$\max_{\theta \geq 0} \quad f(\theta). \tag{2}$$

We will study the impact of the utility function and probability weighting function on the optimal allocation θ^*.

To exclude trivial cases, we always assume that x_0 is in the interior of the domain of $u(\cdot)$, which means that investing all the money in the risk-free asset leads to a finite preference value.

---

[2]If $G_R(\cdot)$ is unbounded, some technical conditions are needed to make sure $f(\cdot)$ is well-defined.

Lemma 3.1. *The interior of the domain of $f(\cdot)$ is $(0,\overline{\theta})$ where $\overline{\theta} := -\frac{x_0-\underline{x}}{\underline{R}}$. If $u(\cdot)$ is continuous in its domain, then $f(\cdot)$ is continuous in its domain. If $u(\cdot)$ is continuously differentiable in the interior of its domain, then $f(\cdot)$ is continuously differentiable in $[0,\overline{\theta})$. Here, the differentiability of $f(\cdot)$ at 0 is understood to be the right-differentiability.*

Proof. For any $\theta > \overline{\theta}$, there exists $\delta > 0$ and $z_0 > 0$ such that $x_0 + \theta G_R(z) \le \underline{x} - \delta$ for all $z \le z_0$. As a result, $f(\theta) = -\infty$. For any $\theta < \overline{\theta}$, we have $x_0 + \theta G_R(z) \ge x_0 + \theta \underline{R} > \underline{x}$ for any $z \in (0,1)$. Thus, $f(\theta) > -\infty$ in this case. Consequently, the interior of the domain of f is $(0,\overline{\theta})$.

When $u(\cdot)$ is continuous, the continuity of $f(\cdot)$ in $[0,\overline{\theta})$ is a result of the bounded dominance theorem. Next, we show that $f(\cdot)$ is continuous at $\overline{\theta}$ When $\overline{\theta} < +\infty$. It is obvious to see that $f(\overline{\theta}) > -\infty$ if and only if

$$\int_0^1 u(x_0 - \overline{\theta}\max(-G_R(z),0))w'(1-z)dz < \infty.$$

Because for any $\theta \in [0,\overline{\theta})$, we have $u(x_0 - \overline{\theta}\max(-G_R(z),0))w'(1-z) \le u(x_0 + \theta G_R(z))w'(1-z) \le u(x_0 + \overline{\theta}\max(G_R(z),0))w'(1-z) \le u(x_0 + \overline{\theta R})w'(1-z)$, the bounded dominance convergence theorem shows that $\lim_{\theta\uparrow\overline{\theta}} f(\theta) = f(\overline{\theta})$ when $f(\overline{\theta}) > -\infty$. On the other hand,

$$\begin{aligned}
&u(x_0 + \theta G_R(z))w'(1-z) \\
&= u(x_0 + \theta G_R(z))w'(1-z)\mathbf{1}_{G_R(z)\le 0} + u(x_0 + \theta G_R(z))w'(1-z)\mathbf{1}_{G_R(z)>0} \\
&\le u(x_0 + \theta G_R(z))w'(1-z)\mathbf{1}_{G_R(z)\le 0} + u(x_0 + \overline{\theta R})w'(1-z).
\end{aligned}$$

The monotone convergence theorem shows that $\lim_{\theta\uparrow\overline{\theta}} f(\theta) = -\infty = f(\overline{\theta})$ when $f(\overline{\theta}) = -\infty$.

Finally, the dominated convergence theorem can be applied to show that $f(\cdot)$ is continuously differentiable in $[0,\overline{\theta})$. ☐

Lemma 3.2. *Suppose $u(\cdot)$ is continuous in its domain. Then, an optimal solution to (2) exists if one of the following two conditions holds:*

1. *$\overline{\theta} < +\infty$ (which is equivalent to $\underline{x} > -\infty$).*
2. *$\overline{\theta} = +\infty$ (which is equivalent to $\underline{x} = -\infty$), $\lim_{x\downarrow-\infty} u(x) = -\infty$, and there exist $C \ge 0$, $0 \le \gamma_+ < \gamma_-$ such that for any $\lambda \ge 1$,*

$$\begin{aligned}
u(\lambda x) &\le \lambda^{\gamma_+}(u(x)+C)+C, \quad \forall x \ge 0, \\
u(\lambda x) &\le \lambda^{\gamma_-}(u(x)+C)+C, \quad \forall x \le 0.
\end{aligned} \tag{3}$$

Proof. If $\overline{\theta} < +\infty$, then from Lemma 3.1, it is either the case in which $f(\overline{\theta}) > -\infty$ and $f(\cdot)$ is continuous in $[0,\overline{\theta}]$ or the case in which $f(\overline{\theta}) = -\infty$, $f(\cdot)$ is continuous in $[0,\overline{\theta})$, and $\lim_{\theta\uparrow\overline{\theta}} f(\theta) = -\infty$. Thus, the optimal solution exists.

Next, consider the case in which $\overline{\theta} = +\infty$. In this case, the domain of $u(\cdot)$ is the whole real line. Denote $x^+ := \max(x,0)$ and fix $\delta \in (0,1)$ such that $\gamma_+ < \delta\gamma_-$. We have

$$\begin{aligned}
f(\theta) &= \int_0^1 u(x_0 + \theta G_R(z))w'(1-z)dz \\
&= \int_0^1 u(x_0 + \theta G_R(z))w'(1-z)\mathbf{1}_{G_R(z)\ge 0}dz
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^1 u(x_0 + \theta G_R(z)) w'(1-z) \mathbf{1}_{G_R(z)<0} dz \\
&\le \int_0^1 u(x_0^+ + \theta G_R(z)) w'(1-z) \mathbf{1}_{G_R(z)\ge 0} dz \\
&\quad + \int_0^1 u(x_0^+ + \theta G_R(z)) w'(1-z) \mathbf{1}_{G_R(z)<-\frac{x_0^+}{\theta}} dz \\
&\quad + \int_0^1 u(x_0^+ + \theta G_R(z)) w'(1-z) \mathbf{1}_{-\frac{x_0^+}{\theta}\le G_R(z)<0} dz \\
&\le C + \theta^{\gamma_+} \int_0^1 [u(\frac{x_0^+}{\theta} + G_R(z)) + C] w'(1-z) \mathbf{1}_{G_R(z)\ge 0} dz \\
&\quad + \theta^{\delta\gamma_-} \int_0^1 [u(\frac{x_0^+}{\theta^\delta} + \theta^{1-\delta} G_R(z)) + C] w'(1-z) \mathbf{1}_{G_R(z)<-\frac{x_0^+}{\theta}} dz dz \\
&\quad + \int_0^1 u(x_0^+) w'(1-z) \mathbf{1}_{-\frac{x_0^+}{\theta}\le G_R(z)<0} dz.
\end{aligned}
$$

Because $u(-\infty) = -\infty$, there exists $\theta_0 \ge 0$ and $\epsilon_0 > 0$ such that for any $\theta \ge \theta_0$,

$$\int_0^1 [u(\frac{x_0^+}{\theta^\delta} + \theta^{1-\delta} G_R(z)) + C] w'(1-z) \mathbf{1}_{G_R(z)<-\frac{x_0^+}{\theta}} dz \le -\epsilon_0.$$

In addition, $\int_0^1 u(x_0^+) w'(1-z) \mathbf{1}_{-\frac{x_0^+}{\theta}\le G_R(z)<0} dz$ goes to zero as θ goes to infinity. Consequently, $\lim_{\theta\to+\infty} f(\theta) = -\infty$ and the optimal solution exists. □

Lemma 3.2 provides two sufficient conditions for the existence of optimal solution to problem (2). The first condition yields that $u(x) = -\infty \ \forall x < \underline{x}$ for some $\underline{x} \in \mathbb{R}$, which can be interpreted as a no-bankruptcy constraint or a limited borrowing constraint once we set $\underline{x} > -\infty$. The second condition stipulates that the disutility of losing the same amount of money dominates the utility of wining a significant amount of money.[3] Either of the two conditions guarantees that the agent will not take infinite leverage, leading to the existence of optimal portfolios.

Note that condition (3) is preserved under affine transformation of $u(\cdot)$. This condition is related to the asymptotic elasticity of $u(\cdot)$ at infinity. In the following, we show that

$$\liminf_{x\to+\infty} \frac{xu'(x)}{u(x)} < \limsup_{x\to-\infty} \frac{xu'(x)}{u(x)},$$

implies condition (3). Indeed, suppose

$$\liminf_{x\to+\infty} \frac{xu'(x)}{u(x)} < \gamma_+ < \gamma_- < \limsup_{x\to-\infty} \frac{xu'(x)}{u(x)}$$

for some $\gamma_- > \gamma_+ \ge 0$. Then according to Lemma 6.3 of [19], there exist $x_2 \le 0 \le x_1$ such that

$$u(\lambda x) \le \lambda^{\gamma_+} u(x),\ \forall x \ge x_1, \quad u(\lambda x) \le \lambda^{\gamma_-} u(x),\ \forall x \le x_2.$$

---

[3] To see this, let us fix $x = 1$ in the first inequality in condition (3) and fix $x = -1$ in the second inequality. Then, for a sufficiently large λ, condition (3) implies that the utility of a gain of λ dollars, $u(\lambda)$, is approximately smaller than λ^{γ_+} and that the disutility of a loss of λ dollars, $-u(-\lambda)$, is approximately larger than λ^{γ_-} and thus dominates the utility of a gain of λ dollars.

On the other hand, for any $x \in [0, x_1]$, we have $u(\lambda x) \le u(\lambda x_1) \le \lambda^{\gamma_+} u(x_1)$. For any $x \in [x_2, 0]$, we have $\lambda^{\gamma_-} u(x) \ge \lambda^{\gamma_-} u(x_2)$ and $u(\lambda x) \le u(0)$. As a result, condition (3) is satisfied with $C := \max(0, u(0), u(x_1), -u(x_2))$.

It is straightforward to verify that the exponential utility function $u(x) = 1 - e^{-\eta x}, x \in \mathbb{R}$, for some $\eta > 0$, satisfies condition (3) with $\gamma_+ = 0$ and any $\gamma_- > 0$. On the other hand, the following S-shaped utility function $u(x) = (x - B)^\alpha, x \ge B$, $u(x) = -k(B - x)^\beta, x \le B$ for some $k > 0, 0 < \alpha < \beta \le 1, B \in \mathbb{R}$, which appears in the cumulative prospect theory [26], satisfies condition (3) with $\gamma_+ = \alpha$ and $\gamma_- = \beta$.

A similar condition is used in [8] to prove the existence of the optimal solution in a multi-period expected utility portfolio choice problem. Condition (3) is also related to Theorem 2 of [17].

4. **Comparative statics.** Consider two agents whose preferences are represented by RDU. The first agent's utility function and probability weighting function are $u(\cdot)$ and $w(\cdot)$, respectively. The second agent's utility function and probability weighting function are $\tilde{u}(\cdot)$ and $\tilde{w}(\cdot)$, respectively. As usual, $u(\cdot)$ and $\tilde{u}(\cdot)$ are strictly increasing mappings from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty\}$ and $w(\cdot)$ and $\tilde{w}(\cdot)$ are strictly increasing and absolutely continuous mappings from $[0, 1]$ onto $[0, 1]$.

Suppose $\tilde{u}(x) = H(u(x))$ for some function $H(\cdot) : \mathbb{R} \cup \{-\infty\} \to \mathbb{R} \cup \{-\infty\}$ with $H(-\infty) = -\infty$ and $\tilde{w}(x) = T(w(x))$ for some increasing and absolutely continuous function $T(\cdot)$ mapping $[0, 1]$ onto $[0, 1]$. Denote the objective functions for the first and the second agents as $f(\cdot)$ and $\tilde{f}(\cdot)$, respectively. Define $\theta^* := \inf\{\theta_0 \mid f(\theta_0) = \sup_{\theta \ge 0} f(\theta)\}$ when the optimal solution exists and define $\tilde{\theta}^*$ similarly.

4.1. **Risk aversion and asset allocation.** We first investigate the impact of risk aversion on the optimal allocation. The following theorem is a (slight) generalization of Theorem 3 of [9] to allow non-continuous return distributions.

Theorem 4.1. *Assume both θ^* and $\tilde{\theta}^*$ exist and $u(\cdot)$ and $\tilde{u}(\cdot)$ are continuously differentiable. If $H(\cdot)$ is concave and $T(\cdot)$ is convex in their domains, respectively, then $\tilde{\theta}^* \le \theta^*$.*

Proof. We first conclude from Theorem A.1 in the Appendix that the domain of $\tilde{u}(\cdot)$ is contained in the domain of $u(\cdot)$. As a result, according to Lemma 3.1, the domain of $\tilde{f}(\cdot)$ is contained in the domain of $f(\cdot)$.

Let $z_0 := \sup\{z \in [0, 1] \mid G_R(t) \le 0\}$. Then, $G_R(z) \le 0, \forall z \le z_0$ and $G_R(z) > 0, \forall z > z_0$. Consider the functions

$$g(\theta) := \frac{1}{u'(x_0) w'(1 - z_0)} f(\theta), \quad \tilde{g}(\theta) := \frac{1}{\tilde{u}'(x_0) \tilde{w}'(1 - z_0)} \tilde{f}(\theta). \tag{4}$$

Then, θ^* and $\tilde{\theta}^*$ are the maximizers of $g(\theta)$ and $\tilde{g}(\theta)$, respectively.

We find that

$$\begin{aligned}
\tilde{g}'(\theta) - g'(\theta) &= \frac{1}{\tilde{u}'(x_0)\tilde{w}'(1 - z_0)} \int_0^1 G_R(z) \tilde{u}'(x_0 + \theta G_R(z)) \tilde{w}'(1 - z) dz \\
&\quad - \frac{1}{u'(x_0) w'(1 - z_0)} \int_0^1 G_R(z) u'(x_0 + \theta G_R(z)) w'(1 - z) dz \\
&= \int_0^1 G_R(z) \frac{T'(u(x_0 + \theta G_R(z))) u'(x_0 + \theta G_R(z)) H'(w(1 - z)) w'(1 - z)}{T'(u(x_0)) u'(x_0) H'(w(1 - z_0)) w'(1 - z_0)} dz
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^1 \frac{G_R(z)u'(x_0+\theta G_R(z))w'(1-z)}{u'(x_0)w'(1-z_0)}dz \\
&= \int_0^1 \frac{u'(x_0+\theta G_R(z))w'(1-z)}{u'(x_0)w'(1-z_0)}\left[\frac{T'(u(x_0+\theta G_R(z)))H'(w(1-z))}{T'(u(x_0))H'(w(1-z_0))}-1\right]G_R(z)dz.
\end{aligned}
$$

Because $T(\cdot)$ is concave and $H(\cdot)$ is convex, we conclude that

$$
\frac{T'(u(x_0+\theta G_R(z)))H'(w(1-z))}{T'(u(x_0))H'(w(1-z_0))}
$$

is larger than 1 when $z \le z_0$ and is less than 1 when $z > z_0$. Consequently, $\tilde{g}'(\theta) - g'(\theta) \le 0$ for any θ in the interior of the domain of $\tilde{f}$.

Next, we show that $\tilde{\theta}^* \le \theta^*$. Otherwise, $\tilde{\theta}^* > \theta^*$. Then, because $\tilde{g}'(\theta) - g'(\theta) \le 0$ for any θ in the interior of the domain of $\tilde{f}(\cdot)$, we conclude that

$$
\tilde{g}(\tilde{\theta}^*) - g(\tilde{\theta}^*) - (\tilde{g}(\theta^*) - g(\theta^*)) \le 0.
$$

As a result,

$$
g(\tilde{\theta}^*) \ge \tilde{g}(\tilde{\theta}^*) - \tilde{g}(\theta^*) + g(\theta^*) > g(\theta^*),
$$

which is a contradiction. □

Theorem 4.1 shows that an investor with a more risk-averse RDU preference relation will invest less in the risky asset, generalizing an existing result in [9] to non-continuous return distributions. Note that in Theorem 3 of [9], it is implicitly assumed that the distribution of the excess return R is continuous. Indeed, the authors wrote

$$
f(\theta) = \int_{\mathbb{R}} u(x_0+\theta x)w'(1-F_R(x))dF_R(x),
$$

which is valid only when $F_R(\cdot)$ is continuous. On the other hand, Theorem 4.1 does not impose any assumption on the distribution of R. In particular, Theorem 4.1 also holds when R follows a discrete distribution, such as a Bernoulli distribution. We refer to [16] for a similar result in a single-period complete market in which the RDU investor can trade a continuum of Arrow-Debreu securities.

An interesting problem is if the converse of Theorem 4.1 is true. Precisely, if the agent with utility function $\tilde{u}(\cdot)$ and probability weighting function $\tilde{w}(\cdot)$ invests less in the risky stock than the agent with utility function $u(\cdot)$ and probability weighting function $w(\cdot)$, can we conclude that $\tilde{u}(\cdot)$ is a concave transformation of $u(\cdot)$ and $\tilde{w}(\cdot)$ is a convex transformation of $w(\cdot)$? To our best knowledge this problem is still open.

4.2. **A sufficient and necessary condition.** In general, if $\tilde{u}(\cdot)$ is not a concave transform of $u(\cdot)$ or $\tilde{w}(\cdot)$ is not a convex transform of $w(\cdot)$, it is unclear whether $\tilde{\theta}^* \le \theta^*$. The following theorem provides a sufficient and necessary condition under which $\tilde{\theta}^* \le \theta^*$.

Theorem 4.2. *Assume that both $u(\cdot)$ and $\tilde{u}(\cdot)$ are continuously differentiable with $\tilde{u}(x) = H(u(x))$ and $u(\cdot)$ is concave. Suppose that both $\tilde{\theta}^*$ and θ^* exists and $\tilde{\theta}^*$ is in the interior of the domains of $f(\cdot)$ and $\tilde{f}(\cdot)$. Then, $\tilde{\theta}^* \le \theta^*$ if and only if*

$$
\begin{aligned}
\int_0^1 \frac{u'(x_0+\tilde{\theta}^* G_R(z))w'(1-z)}{u'(x_0)w'(1-z_0)}&\left[\frac{T'(u(x_0+\tilde{\theta}^* G_R(z)))H'(w(1-z))}{T'(u(x_0))H'(w(1-z_0))}-1\right] \\
&\times G_R(z)dz \le 0. \qquad (5)
\end{aligned}
$$

Proof. Because $\tilde{\theta}^*$ is optimizer of $\tilde{f}(\cdot)$ and is in the interior of the domain of $\tilde{f}(\cdot)$, we immediately conclude that $\tilde{f}'(\tilde{\theta}^*) = 0$. On the other hand, because $u(\cdot)$ is concave, $f(\cdot)$ is also concave. Because $\tilde{\theta}^*$ is in the interior of the domain of $f(\cdot)$ and θ^* is the maximizer of f in its domain, we conclude that $\theta^* \geq \tilde{\theta}^*$ if and only if $f'(\tilde{\theta}^*) \geq 0 = \tilde{f}'(\tilde{\theta}^*)$, which is the case if and only if $\tilde{g}'(\tilde{\theta}^*) - g'(\tilde{\theta}^*) \leq 0$ by the definition of $\tilde{g}$ and g in (4). Finally, one can see from the calculation in the proof of Theorem 2 that $\tilde{g}'(\tilde{\theta}^*) - g'(\tilde{\theta}^*) \leq 0$ if and only if (5) holds. □

Suppose the interior of the domain of $f(\cdot)$ and $\tilde{f}(\cdot)$ are $(0, \overline{\theta})$ and $(0, \overline{\tilde{\theta}})$, respectively. Lemma 3.1 shows that under certain conditions, $\lim_{\theta \to \overline{\theta}} f(\theta) = -\infty$ and $\lim_{\theta \to \overline{\tilde{\theta}}} \tilde{f}(\theta) = -\infty$. In this case, $\tilde{\theta}^*$ is an interior point of the domain of $\tilde{f}(\cdot)$ and $f(\cdot)$ if and only if $\tilde{\theta}^* > 0$. A sufficient condition for $\tilde{\theta}^* > 0$ is that $\tilde{f}'(0) > 0$, which is equivalent to

$$\int_0^1 G_R(z)\tilde{w}'(1-z)dz > 0.$$

This condition states that the expected excess return adjusted for probability weighting is strictly positive.

4.3. **More inverse S-shaped probability weighting.** Our aim is to study the effect of changes in the degree of inverse S-shaped probability weighting on optimal portfolio choice. We define a probability weighting function $\tilde{w}(\cdot)$ to be *more inverse S-shaped* than another one $w(\cdot)$ if there exists $z_0 \in (0,1)$, a concave function $H(\cdot)$, and a convex function $\bar{H}(\cdot)$ such that $H(0) = 0, H(w(z_0)) = w(z_0) = \bar{H}(w(z_0)), \bar{H}(1) = 1$ and

$$\tilde{w}(z) = \begin{cases} H(w(z)) & z \in [0, z_0], \\ \bar{H}(w(z)) & z \in [z_0, 1]. \end{cases}$$

We call this z_0 the *reflection point*. It is easy to check that

$$\frac{d}{dz}(\tilde{w}(z) - w(z)) = (H'(w(z)) - 1)w'(z)$$

is first positive and then negative on $[0, z_0]$ because $H(\cdot)$ is concave. Thus, we have $\tilde{w}(z) \geq w(z)$ on $[0, z_0]$. A similar argument shows that $\tilde{w}(z) \leq w(z)$ on $[z_0, 1]$. As a result, for any $p \in (0, z_0)$ and any $q \in (z_0, 1)$, we have

$$\begin{aligned} \tilde{w}(p) - \tilde{w}(0) &\geq w(p) - w(0), \quad \tilde{w}(1) - \tilde{w}(q) \geq w(1) - \tilde{w}(q), \\ \tilde{w}(z_0) - \tilde{w}(p) &\leq w(z_0) - w(p), \quad \tilde{w}(q) - \tilde{w}(z_0) \leq w(q) - w(z_0) \end{aligned} \tag{6}$$

Thus, the decision weight for the worst outcome of a random payoff that occurs with probability p is higher under $\tilde{w}$ than under w. Similarly, the decision weight for the best outcome of a random payoff that occurs with probability $1-q$ is higher under $\tilde{w}$ than under w. Consequently, the total decision weights for the outcomes other than the worst and the best ones is lower under $\tilde{w}$ than under w. Letting p go to 0 and z_0, respectively, and letting q go to 1 and z_0, respectively, we can also conclude that

$$\tilde{w}'(0+) \geq \tilde{w}'(0+),\ \tilde{w}'(z_0-) \leq w'(z_0-),\ \tilde{w}'(z_0+) \leq w'(z_0+),\ \tilde{w}'(1-) \geq w'(1-). \tag{7}$$

To further illustrate probability weighting functions with different degrees of inverse S-shape, we plotted a family of probability weighting functions, $w(z) = az^2 + (1 - a/2)z, z \in [0, 0.5]$, $w(z) = 1 - w(1-z), z \in (0.5, 1]$, for three values of a,

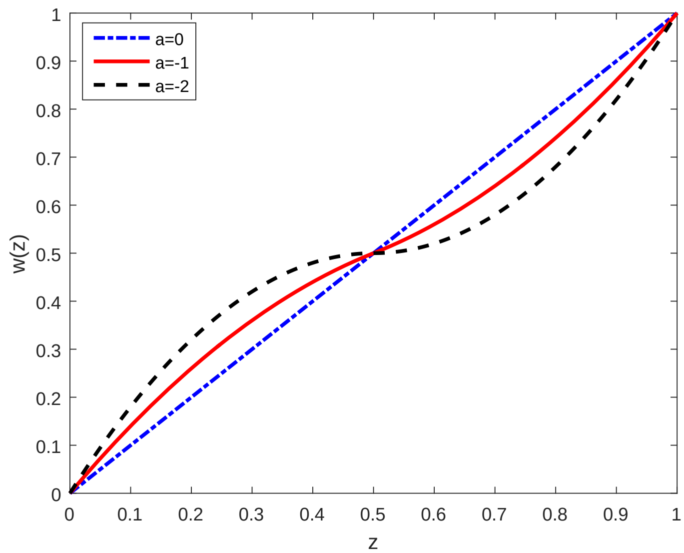

FIGURE 1. Comparative inverse S-shape. A family of probability weighting functions, $w(z) = az^2 + \big(1 - a/2\big)z, z \in [0, 0.5]$, $w(z) = 1 - w(1 - z), z \in (0.5, 1]$, are plotted for three values of a, 0, -1, and -2, in dash-dotted, solid, and dashed lines, respectively. As a becomes more negative, the probability weighting function becomes more inverse S-shaped.

0, -1, and -2, in dash-dotted, solid, and dashed lines, respectively, in Figure 1. As a becomes more negative, the probability weighting function becomes more inverse S-shaped.

4.4. **Explicit result for skewed Bernoulli distributions.** Inverse S-shaped probability weighting has two conflicting effects as it exaggerates the small probabilities of *both* good and bad extremes. Hence, intuitively, its *overall* impact on optimal portfolio choice should depend on the skewness of the stock return distribution. We examine this net impact via a family of Bernoulli distributions with the same mean and variance but different skewness. Fix $\sigma > 0$ and $s \in \mathbb{R}$, which stand for the standard deviation and Sharpe ratio of the excess return R, respectively. Let μ be the expected excess return, then $\mu = s\sigma$. We consider the following distribution of R, parameterized by p:

$$\mathbb{P}\left(R = \sigma\left(s + \left(\frac{p}{1-p}\right)^{1/2}\right)\right) = 1 - p,\ \mathbb{P}\left(R = \sigma\left(s - \left(\frac{1-p}{p}\right)^{1/2}\right)\right) = p. \quad (8)$$

It is easy to check that $\mathbb{E}(R) = s\sigma = \mu$ and $\mathrm{var}(R) = \sigma^2$. In addition, straightforward calculation shows that the skewness of the distribution, denoted as $\mathrm{Skew}(R)$ is

$$\frac{2p-1}{(p(1-p))^{1/2}}.$$

It is easy to see that $\mathrm{Skew}(R)$ is strictly increasing in p, $\mathrm{Skew}(R) = 0$ at $p = 1/2$, and

$$\lim_{p\to 1} \mathrm{Skew}(R) = +\infty, \quad \lim_{p\to 0} \mathrm{Skew}(R) = -\infty.$$

There is a restriction on p to ensure that $\overline{R} > 0$ and $\underline{R} < 0$, making p lie in the range $(0, 1/(1+s^2))$. Because typical values of the Sharpe ratio (s) are less than 60%, p can still take values in a fairly large interval. As a result, the family of Bernoulli distributions (8) is flexible enough to generate a fairly wide range of levels of skewness with the mean and standard deviation being fixed. The family of distributions (8) is also employed by [22] to study skewness-award asset allocation problems.

We now prove that the skewness (parameter p) of the Bernoulli distribution determines if an investor with a more inverse S-shaped probability weighting function invests more or less in the risky asset.

Theorem 4.3. *Suppose $\tilde{u}(\cdot) = u(\cdot)$ and $\tilde{w}(\cdot)$ is more inverse S-shaped than $w(\cdot)$ with the reflection point z_0. Assume the return rate R follows the distribution (8). Suppose that both θ^* and $\tilde{\theta}^*$ exist. Then, if $p \geq 1 - z_0$, $\tilde{\theta}^* \geq \theta^*$. If $p \leq 1 - z_0$, $\tilde{\theta}^* \leq \theta^*$.*

Proof. Denote $b = \sigma\left(s + \left(\frac{p}{1-p}\right)^{1/2}\right) > 0$ and $a = -\sigma\left(s - \left(\frac{1-p}{p}\right)^{1/2}\right) > 0$. Then,

$$G_R(z) = \begin{cases} -a & 0 < z \leq p, \\ b & p < z < 1. \end{cases}$$

As a result,

$$f(\theta) = u(x_0 - \theta a)(1 - w(1-p)) + u(x_0 + \theta b)w(1-p)$$

and

$$f'(\theta) = -au'(x_0 - a\theta)(1 - w(1-p)) + bu'(x_0 + \theta b)w(1-p).$$

Similarly,

$$\tilde{f}'(\theta) = -au'(x_0 - a\theta)(1 - \tilde{w}(1-p)) + bu'(x_0 + \theta b)\tilde{w}(1-p).$$

Because $\tilde{w}(z) \geq w(z)$ on $[0, z_0]$ and $\tilde{w}(z) \leq w(z)$ on $[z_0, 1]$, we immediately conclude that

$$\frac{d}{d\theta}(\tilde{f}(\theta) - f(\theta)) = [bu'(x_0 + \theta b) - (-au'(x_0 - a\theta))]\,[\tilde{w}(1-p) - w(1-p)]$$

is negative if $p \leq 1 - z_0$ and is positive otherwise. The conclusion follows. $\square$

The above theorem shows that when the probability weighting function of the agent becomes more inverse S-shaped and the excess return follows a Bernoulli distribution, the allocation to negatively skewed assets (corresponding to $p \leq \min(1 - z_0, 1/2)$) becomes less and the allocation to positively skewed assets (corresponding to $p \geq \max(1 - z_0, 1/2)$) becomes more. This result is intuitive. Suppose the excess return R follows a negatively skewed Bernoulli distribution, with sufficiently low p such that an increase in inverse S-shaped weighting will amplify the weight of the bad outcome: $p \leq \min(1 - z_0, 1/2)$. A more inverse S-shaped weighting function will then increase the bad outcome's weight, $w(p)$, while simultaneously decreasing the weight of the good outcome, $1 - w(p)$. As a result, the risky stock becomes less attractive, and the investor reduces his optimal allocation.

For general distributions with more than two outcomes, a more inverse S-shaped weighting function will increase the weight given to both positive *and* negative extreme outcomes with small probabilities, while reducing the weight of intermediate outcomes that have larger probabilities; see (6). The implication is that not only the skewness of the return distribution matters, but also its location parameter (the weights given to intermediate outcomes around the mean), as well as the investor's level of risk aversion. This makes it difficult to derive general analytical results. In the following section, we investigate the impact of inverse S-shaped probability weighting on asset allocation through numerical experiments.

5. **Numerical experiments.** In our numerical experiments we will use the following probability weighting function

$$w(p) = \frac{\delta p^{\gamma}}{\delta p^{\gamma} + (1-p)^{\gamma}}, \tag{9}$$

with $\gamma > 0$ and $\delta > 0$. This two-parameter weighting function was introduced by [14], and subsequently applied in influential papers by [21] and [25]. For $\gamma < 1$ the probability weighting is inverse S-shaped, while it is S-shaped for $\gamma > 1$. The parameter δ captures general overweighting of probabilities, with $\delta < 1$ corresponding to underweighting (pessimism) and $\delta > 1$ to overweighting (optimism). Indeed, we can calculate that

$$w'(p) = \delta\gamma p^{\gamma-1}(1-p)^{\gamma-1}(\delta p^{\gamma} + (1-p)^{\gamma})^{-2}$$

and

$$\begin{aligned}\frac{w''(p)}{w'(p)} &= -\frac{\delta(1+\gamma-2p)p^{\gamma} + (1-\gamma-2p)(1-p)^{\gamma}}{p(1-p)(\delta p^{\gamma}+(1-p)^{\gamma})}\\ &= -\frac{1+\gamma-2p}{p(1-p)} + \frac{2\gamma(1-p)^{\gamma}}{p(1-p)(\delta p^{\gamma}+(1-p)^{\gamma})}.\end{aligned}$$

Obviously, $w''(p)/w'(p)$ is strictly decreasing w.r.t. δ for each fixed $p \in (0,1)$ and fixed $\gamma > 0$. Thus, the larger value δ takes, the less risk averse the preference relation represented by RDU with the probability weighting function (9) is.

On the other hand, we can see that

$$\lim_{p\downarrow 0}\frac{pw''(p)}{w'(p)} = -(1-\gamma), \quad \lim_{p\uparrow 1}\frac{pw''(p)}{w'(p)} = 1-\gamma.$$

As a result, when γ takes a smaller value, $w''(p)/w'(p)$ becomes more negative for p in a neighbourhood of 0 and becomes more positive for p in a neighbourhood of 1, and, consequently, Theorem A.1-(v) in Appendix A shows that w becomes more concave in the neighbourhood of 0 and becomes more convex in the neighbourhood of 1.

We note that expected utility, or no probability weighting, is a special case for $\gamma = 1$ and $\delta = 1$. The case with $\delta = 1$ is also of importance; in this case, we have

$$\frac{w''(p)}{w'(p)} = -\frac{1-2p}{p(1-p)} + \frac{\gamma\big((1-p)^{\gamma} - p^{\gamma}\big)}{p(1-p)(p^{\gamma}+(1-p)^{\gamma})}.$$

Straightforward calculation yields

$$\frac{d}{d\gamma}\left(\frac{w''(p)}{w'(p)}\right) = \frac{p^{\gamma-1}(1-p)^{\gamma-1}}{(p^{\gamma}+(1-p)^{\gamma})^2}\cdot\left[\left(\frac{1-p}{p}\right)^{\gamma} - \left(\frac{1-p}{p}\right)^{-\gamma} + 2\gamma\ln\left(\frac{1-p}{p}\right)\right].$$

Because $f(x) := x^{\gamma} - x^{-\gamma} + 2\gamma \ln x$ is strictly increasing in x with $f(1/2) = 0$, we immediately conclude that $\frac{d}{d\gamma}\left(\frac{w''(p)}{w'(p)}\right)$ is strictly increasing in γ when $p < 1/2$ and strictly decreasing in γ when $p > 1/2$. Consequently, when $\delta = 1$, a decrease in γ makes the weighting function *more inverse S-shaped* according to our definition. As we will be focusing on the effect of being more inverse S-shaped, in our numerical examples we will use the probability weighting function in (9) with $\delta = 1$, while varying the parameter γ to illustrate its impact on optimal portfolio choice.

5.1. **Numerical examples.**

5.1.1. *Skew-normal return distribution.* We assume an exponential utility function, that is,

$$u(x) = 1 - e^{-\beta x}, \quad x \in \mathbb{R} \tag{10}$$

for some $\beta > 0$. We use the Goldstein-Einhorn probability weighting function (9) with $\delta = 1$. We assume that the excess return R follows a skew-normal distribution.

Denote by $\Phi(\cdot)$ the CDF of the standard normal distribution. The skew-normal distribution is defined by the following probability density function; see [1]:

$$f_{\xi,\omega,\alpha}(x) = \frac{2}{\omega}\Phi'\left(\frac{x-\xi}{\omega}\right)\Phi\left(\alpha\left(\frac{x-\xi}{\omega}\right)\right), \quad x \in \mathbb{R}, \tag{11}$$

where ξ is the location parameter, ω is the scale parameter, α is the shape parameter. The mean, variance, and skewness of this distribution are $\xi + \omega\kappa$, $\omega^2\left(1-\kappa^2\right)$, and $\frac{4-\pi}{2}\frac{\kappa^3}{(1-\kappa^2)^{3/2}}$, respectively, where $\kappa := \frac{\alpha}{\sqrt{1+\alpha^2}}\sqrt{\frac{2}{\pi}}$.

Suppose that the mean and standard deviation of the distribution are fixed at μ and σ, respectively. Then, we choose different values for κ from $-\sqrt{2/\pi}$ to $\sqrt{2/\pi}$ (i.e., choose different values of α from $-\infty$ to ∞) to model skewness, and set

$$\omega = \frac{\sigma}{\sqrt{1-\kappa^2}}, \quad \xi = \mu - \frac{\sigma\kappa}{\sqrt{1-\kappa^2}}$$

to match the mean and standard deviation.

In the following simulation study, we set $\mu = 6\%$ and $\sigma = 20\%$, representing the typical expected return and volatility of the aggregate U.S. stock market portfolio in one year. Because the exponential utility function has the property of constant absolute risk aversion, the optimal allocation does not depend on the initial wealth x_0, and hence we assume $x_0 = 1$.[4] Setting $\delta = 1$, we depict in Figure 3 the optimal allocation to the risky asset with different values of γ (recall the smaller γ is, the more inverse S-shaped the probability weighting function is). Three values of the skewness of R are chosen: -0.5, 0, and 0.5, representing the cases of a negatively skewed distribution, a symmetric distribution, and a positively skewed distribution, respectively; see Figure 2 for the probability density function of R in these three cases.

We observe from Figure 3 that when the distribution is negatively skewed, the optimal allocation decreases with respect to $1 - \gamma$. This is consistent with the intuition. Indeed, a more inverse S-shaped probability weighting function implies that the agent put higher weights on both the best and the worst outcomes that occur with small probability. For a negatively skewed distribution the magnitude of the best outcomes is relatively smaller than the magnitude of the worst outcomes;

[4]We refer to Appendix B for comparable results with a power utility function.

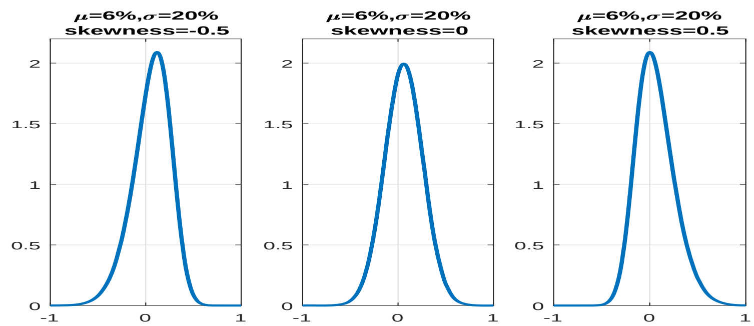

FIGURE 2. Probability density function of the excess return R of the risky asset when R follows a skew-normal distribution. The mean and standard deviation of R are set to be $\mu = 6\%$ and $\sigma = 20\%$, respectively, and the skewness of R takes three values: -0.5, 0, and 0.5, corresponding to the probability density functions in the left, middle, and right panes, respectively.

so the second impact dominates the first one, making the agent invest less in the stock market.

However, it is surprising that when the distribution is positively skewed or symmetric (dashed-dotted line and dashed line, respectively), the optimal allocation still decreases with respect to $1 - \gamma$. This can be explained by the following two effects: decreasing marginal utility and lower weights given to the middle region of the distribution. If a return distribution is symmetric, a more inverse S-shaped probability distribution makes the agent put higher weights on both the worst and the best outcomes (both of small probability). However, assuming the investor has a concave (i.e., risk averse) utility function, the marginal loss is higher than the marginal gain of the same magnitude. Hence, when the return distribution is symmetric, other things being equal a risk averse investor with a more inverse S-shaped weighting function will find the stock less attractive and reduce the allocation.

There is however a more important reason. Apart from increasing the weights given to the tails, a more inverse S-shaped function also lowers the weights given to the middle part of the return distribution. Typically, with a mean annual return rate of 6%, in the middle part of the return distribution the investor is more likely to earn positive returns than negative ones, and hence lowering the weights here makes the stock less attractive. We can observe in Figure 3 that positive skewness of 0.5 is not sufficient to overcome the two negative effects of more inverse S-shaped weighting: large losses hurting more than large gains for concave utility, and lower weights assigned to the middle region of the distribution. The highest level of positive skewness we can numerically assign to the skew-normal distribution is 0.99, and even for this value we get the same result: an increase in inverse S-shaped weighting (higher $1 - \gamma$) leads to a lower optimal stock weight.

To illustrate the effect of the middle part of the distribution on the optimal allocation, we change the expected excess stock return to $\mu = 1\%$. The stock investment is now overall less attractive, offering a low ratio of expected return to risk. Figure 4 shows that in this case when skewness is 0.5, the optimal allocation now does increase with respect to $1 - \gamma$ for most values of γ. The density functions of the return distributions in Figure 5 help us see why: the positively skewed

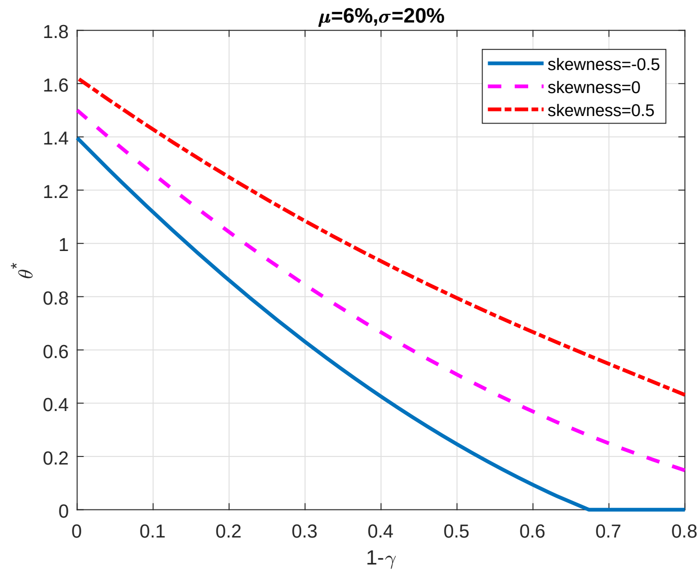

FIGURE 3. Optimal dollar amount θ^* invested in the risky asset with respect to different degrees of inverse S-shape of the probability weighting function. The utility function is the exponential one in (10) with $\beta = 1$. The probability weighting function is given by (9), so $1 - \gamma$ represents the degree of inverse S-shape of the probability weighting function. We set $\delta = 1$. The excess return of the risky asset R follows a skew-normal distribution with mean $\mu = 6\%$ and standard deviation $\sigma = 20\%$. The skewness takes three values -0.5, 0, and 0.5, corresponding to the solid line, dashed line, and dash-dotted line, respectively.

distribution ($skewness = 0.5$) has a negative mode and median, indicating that negative returns are more frequent than positive returns in the middle region of the distribution. An increase in inverse S-shaped weighting will also lower the weight assigned to the middle region, and hence make the stock relatively more attractive in this case. In Appendix B we show similar results for a power utility function, using a log-transformed skew-normal distribution.

5.2. **Empirical return distributions.** The simulation study above shows that the impact of a more inverse S-shaped probability weighting function on the allocation to risky assets not only depends on the skewness of the asset return distribution, but also on its location parameter. One may wonder what the overall impact is for the *empirical return distributions* of the aggregate stock market and the return of typical individual, exchange-listed stocks. In this section, we use historical data to address this question.

We use data on the annual excess returns of the aggregate U.S. stock market portfolio from the data library of Professor Kenneth French.[5] In the period 1962–2016, the estimated average excess return of the U.S. market is $\mu = 6.5\%$, with standard

[5] The link to the library is http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

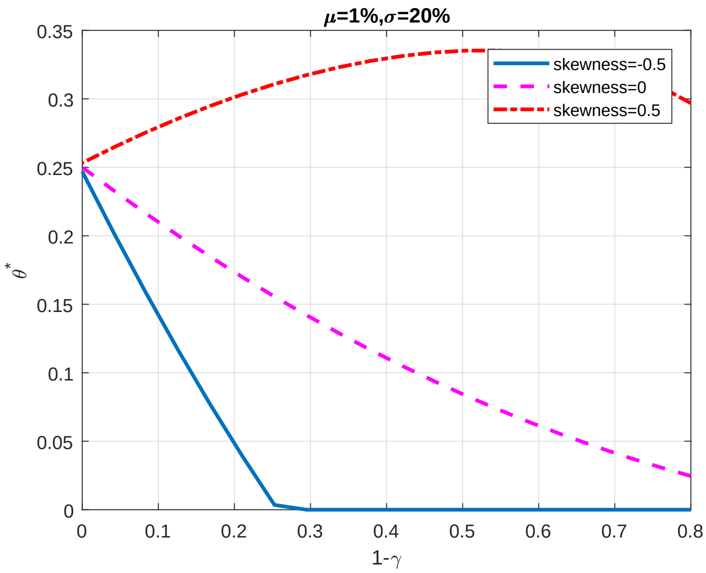

FIGURE 4. Optimal dollar amount θ^* invested in the risky asset with respect to different degrees of inverse S-shape of the probability weighting function. The utility function is the exponential one in (10) with $\beta = 1$. The probability weighting function is given by (9), so $1-\gamma$ represents the degree of inverse S-shape of the probability weighting function. We set $\delta = 1$. The excess return of the risky asset R follows a skew-normal distribution with mean $\mu = 1\%$ and standard deviation $\sigma = 20\%$. The skewness takes three values -0.5, 0, and 0.5, corresponding to the solid line, dashed line, and dash-dotted line, respectively.

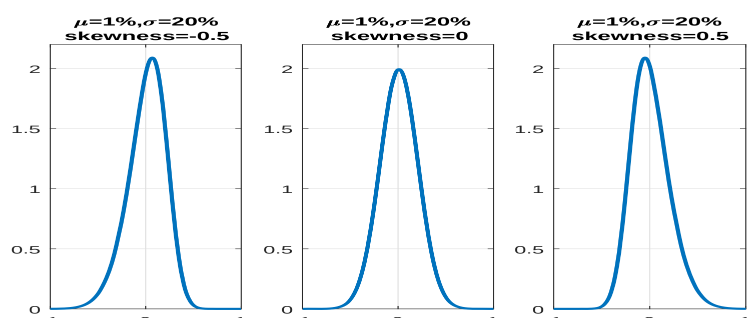

FIGURE 5. Probability density function of the excess return R of the risky asset when R follows a skew-normal distribution. The mean and standard deviation of R are set to be $\mu = 1\%$ and $\sigma = 20\%$, respectively, and the skewness of R takes three values: -0.5, 0, and 0.5, corresponding to the probability density functions in the left, middle, and right panes, respectively.

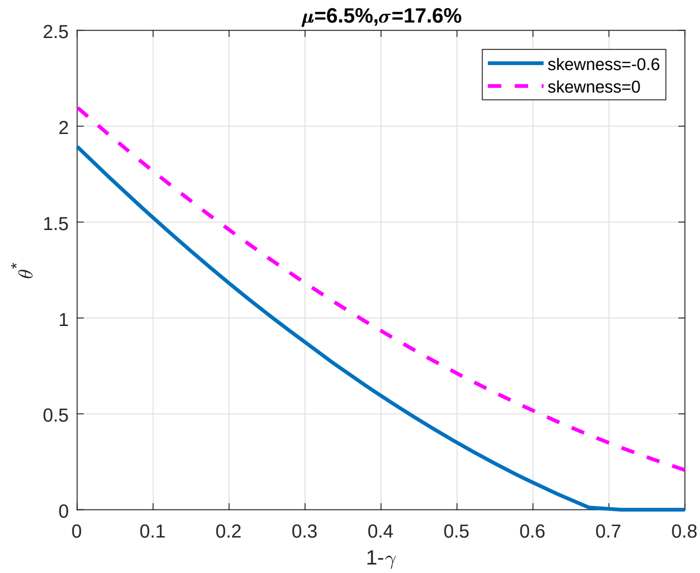

FIGURE 6. Optimal dollar amount θ^* invested in the U.S. stock market as a function of the degree of inverse S-shape of the probability weighting function. The utility function is the exponential one in (10) with $\beta = 1$. The probability weighting function is given by (9), so $1 - \gamma$ represents the degree of inverse S-shape of the weighting function. We set $\delta = 1$. The excess return of the risky asset R follows a skew-normal distribution with mean $\mu = 6.5\%$, standard deviation $\sigma = 17.6\%$ and skewness of -0.6, based on historical data of excess returns for the U.S. stock market (1962–2016). The solid line shows the optimal allocation when the skewness is -0.6 as in the historical data, while the dotted line shows the optimal allocation when skewness is 0 for comparison sake.

deviation $\sigma = 17.6\%$ and skewness of -0.6. We calibrate a skew-normal distribution with these parameter values and then calculate the optimal dollar amount invested in the risky asset for an investor with exponential utility function with $\beta = 1$ and the Goldstein-Einhorn probability weighting function with parameters $\delta = 1$ and varying γ. Figure 6 shows the optimal allocation to the U.S. stock market as a function of $(1 - \gamma)$. We observe as before that more inverse S-shaped probability weighting $(1 - \gamma)$ leads to a lower stock allocation, and even a zero weight in more extreme cases. In line with the previous results of [24], we see that high levels of inverse S-shaped weighting may explain the non-participation of general households in the stock market.

However, investors who overweight the tails of the distribution may still be tempted to invest in an *individual* stock with positive skewness that offers them a small chance to become rich. Indeed, many individual investors try hard to find an exchange-listed company that will become ‘the next Apple’ or ‘the next Google’. As an example, we consider the historical stock return distribution of Apple, which has been listed since 1980. In the period 1980 to 2016, the average annual excess return of Apple’s stock was $\mu = 29.5\%$, with standard deviation $\sigma = 70.5\%$ and

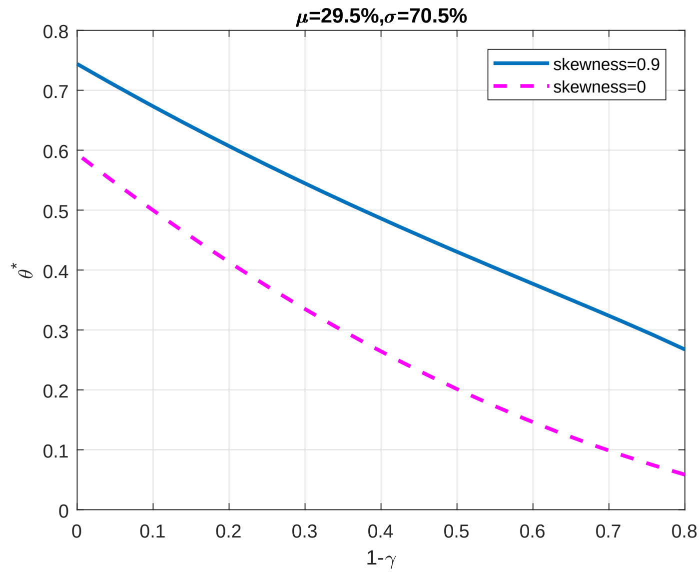

FIGURE 7. Optimal dollar amount θ^* invested in Apple as a function of the degree of inverse S-shape of the probability weighting function. The utility function is the exponential one in (10) with $\beta = 1$. The probability weighting function is given by (9), so $1-\gamma$ represents the degree of inverse S-shape of the weighting function. We set $\delta = 1$. The excess return of the risky asset R follows a skew-normal distribution with mean $\mu = 29.5\%$, standard deviation $\sigma = 70.5\%$ and skewness of 0.9, based on historical data of excess returns for the stock of the company Apple (1980–2016). The solid line shows the optimal allocation when the skewness is 0.9 as in the historical data, while the dotted line shows the optimal allocation when skewness is 0 for comparison sake.

skewness of 0.9. Clearly, Apple's stock offered both a very high average return and high positive skewness. Figure 7 shows the optimal allocation to Apple as a function of $(1-\gamma)$, when we calibrate a skew-normal distribution with these parameter values ($\mu = 29.5\%$, $\sigma = 70.5\%$, $skewness = 0.9$). Consistent with the finding in Section 5.1.1, investors with a more inverse S-shaped probability weighting functions (higher $1-\gamma$) have a *lower* demand for Apple stocks. Although investors with a more inverse S-shaped probability weighting functions appreciate a stock offering positive skewness such as Apple, they also overweight losses in the left tail and underweight the middle region of the return distribution more, and these latter two effects dominate.

How can we reconcile the previous finding that investors with a more inverse S-shaped probability weighting functions have a lower demand for Apple stocks with the often mentioned intuition that more inverse S-shaped probability weighting is associated with a higher demand for individual stocks offering a positively skewed return distribution? For example, [24] finds that RDU investors with stronger inverse S-shaped probability weighting tend to invest a *higher* portfolio weight in a single, randomly selected, individual U.S. stock, contrary to our results for Apple.

In the same spirit of the discussion related to Figure 3, Polovnichenko's results are probably driven by the fact that a randomly selected individual stock has a lower average annual return than Apple, while also having a return distribution with much stronger positive skewness.

[4] estimates the return distribution when an investor randomly samples one individual stock from the universe of all U.S. listed stocks, and holds it for one year, similar to [24]. The average excess return to such a strategy is $\mu = 11.3\%$, with $\sigma = 82.0\%$ and skewness of 19.9. The distribution has an extremely long right tail, reflecting that a few selected stocks perform extremely well. Figure 8 shows the optimal stock allocation as a function of $(1-\gamma)$, when we calibrate a skew-normal distribution with these parameter values ($\mu = 11.3\%$, $\sigma = 82.0\%$), while setting skewness at highest feasible value of 0.99 for a skew-normal distribution. Indeed, now investors with a more inverse S-shaped probability weighting functions (higher $1-\gamma$) demand more stock. A similar result is shown in [24], but here we add the insight that such a positive relation only occurs because a randomly sampled company stock offers a return distribution with extremely high positive skewness and a relatively poor expected return per unit of standard deviation (Sharpe ratio).

Finally, we would like to illustrate that inverse S-shaped probability weighting can lead investors to invest in stocks that have negative expected excess returns, as long as the return distribution is positively skewed. As explained earlier, more inverse S-shaped weighting not only puts more weight on extremes in the tails, but also diminishes the weights assigned to the middle region of the return distribution. This makes the investor less sensitive to the expected return offered by the stock, and in some cases willing to accept negative expected returns in exchange for some exposure to positive skewness.

As an example we take the portfolio of the so-called 'lottery stocks' described in [20]. [20] defines a lottery stock as a stock with relatively high volatility, high skewness and low price. He constructs a well-diversified portfolio of U.S. lottery stocks using data from 1991 to 1996. This portfolio has a negative expected excess return of $\mu = -0.3\%$, a volatility of $\sigma = 27.5\%$ and skewness of 0.33. The solid line in Figure 9 shows the allocation to Kumar's lottery stock portfolio as a function of $(1-\gamma)$. It is seen that for $1-\gamma > 0.1$ investors are willing to allocate some money to this portfolio with negative excess return, because it offers a moderate amount of positive skewness that they prefer.

To further illustrate the effect of changes in the expected return, in Figure 9 we also show the demand for the lottery stock portfolio when μ is -1% (dashed line), 1% (dashed-dotted line) and 3% (dotted line). We observe that investors with more inverse S-shaped weighting functions are increasingly less sensitive to the mean return, in stark contrast to expected utility maximizers. This illustrates why lottery-type stocks can have such low or even negative average returns in the market, as the mean return is a secondary concern for investors who strongly overweight the tails of the distribution. A large number of empirical studies show evidence that securities with positive skewness or a high probability of extreme positive outcomes have low subsequent average returns: see, for example, [20], [6], [2], [10], [7] and [11]. We refer to [3] for an equilibrium model of stock pricing in the presence of investors with inverse S-shaped probability weighting functions that can explain the overpricing of stocks with positive skewness, especially when short-selling by rational agents is limited or risky.

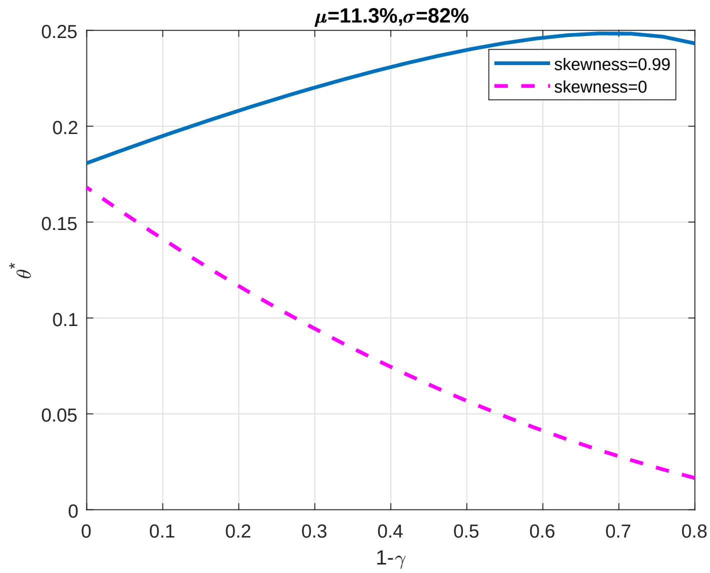

FIGURE 8. Optimal dollar amount θ^* invested in one randomly selected U.S. stock as a function of the degree of inverse S-shape of the probability weighting function. The utility function is the exponential one in (10) with $\beta = 1$. The probability weighting function is given by (9), so $1 - \gamma$ represents the degree of inverse S-shape of the weighting function. We set $\delta = 1$. The excess return of the risky asset R follows a skew-normal distribution with mean $\mu = 11.3\%$, standard deviation $\sigma = 82.0\%$ and skewness of 0.99 (the highest feasible value), based on the annual excess return distribution when one U.S. listed stock is picked randomly and held for one year, from [4]. The solid line shows the optimal allocation when the skewness is 0.99 as in the historical data, while the dotted line shows the optimal allocation when skewness is 0 for comparison sake.

6. **Conclusions.** In this paper we analyzed how changes in inverse S-shaped probability weighting influence optimal portfolio choice, an open question in the literature. For this purpose we studied a general one-period portfolio choice problem with one risky asset (e.g., a stock) and one riskless asset, when the investor's preferences are modelled by rank-dependent utility. We first derived two sufficient conditions for the existence of an optimal solution of the investment problem. We then generalized an existing result of [9] to non-continuous return distributions, showing that an investor with a more risk averse RDU preference relation will always invest less in the risky asset.

We introduced and defined the notion of a *more inverse S-shaped* probability weighting function, using a concave-convex transformation function, to study the impact of overweighting the probability of unlikely and extreme events on portfolio choice. In the special case when the risky asset return follows a skewed Bernoulli distribution, we proved analytically that the impact of a more inverse S-shaped weighting function on the stock allocation depends on the skewness parameter of

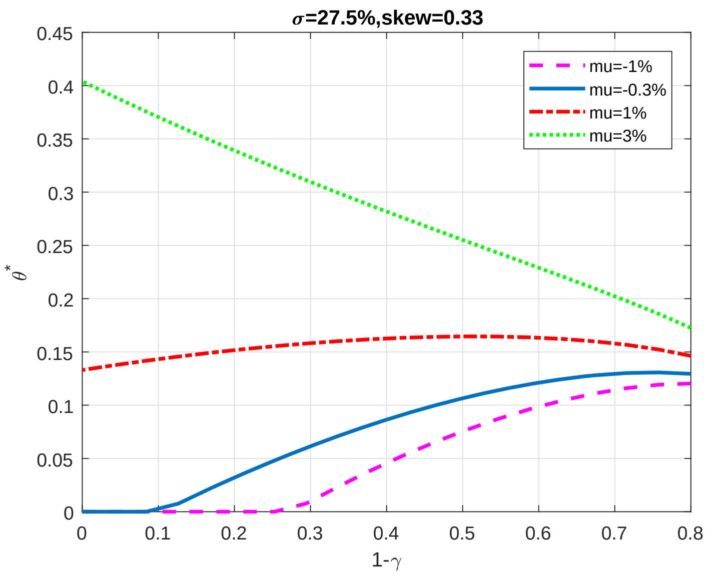

FIGURE 9. Optimal dollar amount θ^* invested in a portfolio of U.S. lottery stocks as a function of the degree of inverse S-shape of the probability weighting function. The utility function is the exponential one in (10) with $\beta = 1$. The probability weighting function is given by (9), so $1 - \gamma$ represents the degree of inverse S-shape of the weighting function. We set $\delta = 1$. The excess return of the risky asset R follows a skew-normal distribution with mean $\mu = -0.3\%$, standard deviation $\sigma = 27.5\%$ and skewness of 0.33, based on a portfolio of U.S. lottery stocks described in [20]. The solid line shows the optimal allocation when the mean excess return is $\mu = -0.3$ as estimated by [20]. The other lines show the portfolio allocation for other levels of expected return: $\mu = -1\%$ (dashed line), $\mu = 1\%$ (dashed-dotted line) and $\mu = 3\%$ (dotted line), while keeping σ and skewness constant.

the distribution. For general return distributions the effect of more inverse S-shaped weighting on the risky asset allocation is more complex, depending on the interplay between the weights given to the tails and the middle region of the return distribution, as well as the curvature of the utility function.

We illustrated these effects numerically using simulations of a skew-normal return distribution. The main finding is that an increase in inverse S-shaped weighting typically leads to a *lower* allocation to the risky asset, regardless of whether the return distribution is skewed left or right, as long as the asset offers a non-negligible positive risk premium. For the historical return distribution of the U.S. stock market, which is skewed to the left, a more inverse S-shaped weighting function leads to a lower optimal allocation. Even for an individual stock like Apple with a return distribution that is strongly skewed to the right, we found that more inverse S-shaped weighting leads to a lower optimal weight. Only for stocks offering poor expected returns (e.g., negative), or very extreme positive skewness, we found that an increase in inverse S-shaped weighting can lead to larger portfolio allocations.

The implications of our results are that investors with more inverse S-shaped probability weighting are less likely to invest in well-diversified stock market funds that offer a reasonable risk-return trade-off, and more likely to buy stocks with lottery-like features that offer poor risk-adjusted returns [20]. These results can be used to set testable hypotheses for the empirical household finance literature, where datasets with measurements of inverse S-shaped probability weighting in the general population are available; see e.g., [5] and [13].

For future research an extension of our work to multiple risky assets is of interest, to study the effect of inverse S-shaped weighting on portfolio diversification. An extension to risky assets with non-linear payoff functions, such as derivatives and structured products, is relevant as well.

Appendix A. **Concave transformation.** A function $\phi(\cdot):\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is a *concave transformation* of (or *more concave* than) another one $\psi(\cdot):\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ if there exists a concave function $h(\cdot):\mathbb{R}\cup\{-\infty\}\to\mathbb{R}\cup\{-\infty\}$ such that $h(-\infty)=h(-\infty)$ and $\phi(\cdot)=h(\psi(\cdot))$. A function $\phi(\cdot):\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ is a *convex transformation* of (or *more convex* than) another one $\psi(\cdot):\mathbb{R}\to\mathbb{R}\cup\{+\infty\}$ if $-\phi(\cdot)$ is more concave than $-\psi(\cdot)$.

The following theorem provides a full charaterization of concave transformation, which, in contrast to the literature, does not necessarily rely on the assumption of twice differentiable functions.

Theorem A.1. *(i) If $\phi(\cdot)$ is a concave transformation of $\psi(\cdot)$, then the domain of $\phi(\cdot)$ is contained in the domain of $\psi(\cdot)$.*

(ii) Suppose $\psi(\cdot)$ is strictly increasing and continuous in its domain. Then, $\phi(\cdot)$ is a concave transformation of $\psi(\cdot)$ if and only if the domain of $\phi(\cdot)$ is connected and contained in the domain of $\psi(\cdot)$ and

$$\frac{\phi(x_3)-\phi(x_2)}{\psi(x_3)-\psi(x_2)}\le\frac{\phi(x_2)-\phi(x_1)}{\psi(x_2)-\psi(x_1)}$$

for any $x_1<x_2<x_3$ in the domain of $\phi(\cdot)$.

(iii) Suppose both $\phi(\cdot)$ and $\psi(\cdot)$ are strictly increasing and continuous in their domains. Then, $\phi(\cdot)$ is a concave transformation of $\psi(\cdot)$ if and only if the domain of $\phi(\cdot)$ is contained in the domain of $\psi(\cdot)$ and

$$\frac{\phi(x_3)-\phi(x_2)}{\phi(x_2)-\phi(x_1)}\le\frac{\psi(x_3)-\psi(x_2)}{\psi(x_2)-\psi(x_1)}$$

for any $x_1<x_2<x_3$ in the domain of $\phi(\cdot)$.

(iv) Suppose that $\psi(\cdot)$ is strictly increasing in its domain and that both $\phi(\cdot)$ and $\psi(\cdot)$ are absolutely continuous in their domains. Then, $\phi(\cdot)$ is a concave transformation of $\psi(\cdot)$ if and only if the domain of $\phi(\cdot)$ is connected and contained in the domain of $\psi(\cdot)$ and

$$\frac{\phi'(x)}{\psi'(x)}$$

is decreasing in the interior of the domain of $\phi(\cdot)$.

(v) Suppose both $\phi(\cdot)$ and $\psi(\cdot)$ are strictly increasing and absolutely continuous in their domains. In addition, $\phi'(\cdot)$ and $\psi'(\cdot)$ are absolutely continuous. Then, $\phi(\cdot)$ is a concave transformation of $\psi(\cdot)$ if and only if the domain of $\phi(\cdot)$ is

contained in the domain of $\psi(\cdot)$ *and*

$$-\frac{\phi''(x)}{\phi'(x)} \geq -\frac{\psi''(x)}{\psi'(x)}$$

for almost everywhere x *in the interior of the domain of* $\phi(\cdot)$.

Proof. (i) For any $x \in \mathbb{R}$ such that $\phi(x) > -\infty$, we have $h(\psi(x)) = \phi(x) > -\infty$. Because $h(-\infty) = -\infty$, we must have $\psi(x) > -\infty$.

(ii) We first prove the necessity. Suppose $\phi(\cdot)$ is a concave transformation of $\psi(\cdot)$, i.e., there exists a concave function $h(\cdot)$ such that $h(-\infty) = -\infty$ and $\phi(x) = h(\psi(x)), x \in \mathbb{R}$. We have concluded that the domain of $\phi(\cdot)$ must be contained in the domain of $\psi(\cdot)$. In addition, for any $x < y$ in the domain of $\phi(\cdot)$ and any $\alpha \in (0, 1)$, we have, from the concavity of $h(\cdot)$, that

$$\begin{aligned}
&\phi(\alpha x + (1-\alpha)y) = h(\psi(\alpha x + (1-\alpha)y)) \\
&= h\left(\frac{\psi(y) - \psi(\alpha x + (1-\alpha)y)}{\psi(y) - \psi(x)}\psi(x) + \frac{\psi(\alpha x + (1-\alpha)y) - \psi(x)}{\psi(y) - \psi(x)}\psi(y)\right) \\
&\geq \frac{\psi(y) - \psi(\alpha x + (1-\alpha)y)}{\psi(y) - \psi(x)}h(\psi(x)) + \frac{\psi(\alpha x + (1-\alpha)y) - \psi(x)}{\psi(y) - \psi(x)}h(\psi(y)) \\
&= \frac{\psi(y) - \psi(\alpha x + (1-\alpha)y)}{\psi(y) - \psi(x)}\phi(x) + \frac{\psi(\alpha x + (1-\alpha)y) - \psi(x)}{\psi(y) - \psi(x)}\phi(y) \\
&> -\infty.
\end{aligned}$$

Therefore, the domain of $\phi(\cdot)$ is connected. On the other hand, for any $x_1 < x_2 < x_3$ in the domain of $\phi(\cdot)$, we have $\psi(x_1) < \psi(x_2) < \psi(x_3)$. In addition, $\phi(x_i) = h(\psi(x_i)), i = 1, 2, 3$ and thus $\psi(x_i), i = 1, 2, 3$ are in the domain of $h(\cdot)$. Then, by the concavity of $h(\cdot)$, we have

$$\frac{h(\psi(x_3)) - h(\psi(x_2))}{\psi(x_3) - \psi(x_2)} \leq \frac{h(\psi(x_2)) - h(\psi(x_1))}{\psi(x_2) - \psi(x_1)}.$$

Thus, we have

$$\frac{\phi(x_3) - \phi(x_2)}{\psi(x_3) - \psi(x_2)} \leq \frac{\phi(x_2) - \phi(x_1)}{\psi(x_2) - \psi(x_1)}.$$

Next, we show the sufficiency. Suppose the domain of $\phi(\cdot)$ is connected and contained in the domain of $\psi(\cdot)$ and the above condition holds. Define $h(y) := \phi(\psi^{-1}(y))$ if $\psi^{-1}(y)$ is in the domain of $\phi(\cdot)$ and define $h(y) = -\infty$ otherwise. Then, by definition, $h(-\infty) = -\infty$. Moreover, for any x in the domain of $\phi(\cdot)$, we have

$$h(\psi(x)) = \phi(\psi^{-1}(\psi(x))) = \phi(x).$$

For any x not in the domain of $\phi(\cdot)$, by definition we have $h(\psi(x)) = -\infty = \phi(x)$. Thus, we only need show that $h(\cdot)$ is concave.

Consider any $y_1 < y_2 < y_3 \in \mathbb{R}$. We only need show that

$$h(y_2) \geq \frac{y_3 - y_2}{y_3 - y_1}h(y_1) + \frac{y_2 - y_1}{y_3 - y_1}h(y_3).$$

We only need consider the case in which $h(y_1) > -\infty$ and $h(y_3) > -\infty$. In this case, we must have that y_1, y_3 are in the range of $\psi(\cdot)$ and that $\psi^{-1}(y_1)$ and $\psi^{-1}(y_3)$ are in the domain of $\phi(\cdot)$. As a result, $\psi^{-1}(y_2)$ is also in the domain of $\phi(\cdot)$ because this domain is connected and $\psi(\cdot)$ is strictly increasing. Thus, we immediately have

$$\frac{\phi(\psi^{-1}(y_3)) - \phi(\psi^{-1}(y_2))}{\psi(\psi^{-1}(y_3)) - \psi(\psi^{-1}(y_2))} \le \frac{\phi(\psi^{-1}(y_2)) - \phi(\psi^{-1}(y_1))}{\psi(\psi^{-1}(y_2)) - \psi(\psi^{-1}(y_1))},$$

i.e.,

$$\frac{h(y_3) - h(y_2)}{y_3 - y_2} \le \frac{h(y_2) - h(y_1)}{y_2 - y_1}.$$

This completes the proof.

(iii) This is a direct consequence of assertion (ii).

(iv) We first consider the necessity. Notice that $\phi(\cdot)$ and $\psi(\cdot)$ are almost everywhere differentiable in the domain of $\phi(\cdot)$. Recalling assertion (ii), we have, for any $x_1 < x_2$ in the interior of the domain of $\phi(\cdot)$ such that $\phi(\cdot)$ and $\psi(\cdot)$ are differentiable at these two points, that

$$\frac{\phi(x_2+\delta) - \phi(x_2)}{\psi(x_2+\delta) - \psi(x_2)} \le \frac{\phi(x_1+\delta) - \phi(x_1)}{\psi(x_1+\delta) - \psi(x_1)}$$

for sufficiently small $\delta > 0$. Sending δ to zero, we immediately conclude that

$$\frac{\phi'(x_2)}{\psi'(x_2)} \le \frac{\phi'(x_1)}{\psi'(x_1)}.$$

Next, we prove the sufficiency. Define $h(y) := \phi(\psi^{-1}(y))$ for any y in the range of $\psi(\cdot)$ such that $\psi^{-1}(y)$ is in the domain of $\phi(\cdot)$ and define $h(y) = -\infty$ otherwise. Similar to the argument in the proof of assertion (ii), we can show that $h(-\infty) = -\infty$ and $\phi(\cdot) = h(\psi(\cdot))$. Because $\psi(\cdot)$ is strictly increasing and absolutely continuous, according to Lemma A.2, $\psi^{-1}(\cdot)$ is absolutely continuous and its derivative is equal to $1/\psi'(\psi^{-1}(y))$. Because $\phi(\cdot)$ is also absolutely continuous and $\psi^{-1}(\cdot)$ is increasing, we conclude from Lemma A.3 that $h(\cdot)$ is absolutely continuous in its domain and

$$h'(y) = \frac{\phi'(\psi^{-1}(y))}{\psi'(\psi^{-1}(y))}.$$

Because $\frac{\phi'(x)}{\psi'(x)}$ is decreasing, so is $h'(\cdot)$. As a consequence, $h(\cdot)$ is a concave function, so $\phi(\cdot)$ is more concave than $\psi(\cdot)$.

(v) From Lemma A.3, we conclude that $\log \phi'(x)$ and $\log \psi'(\cdot)$ are absolutely continuous and their derivatives are equal to $\frac{\phi''(x)}{\phi'(x)}$ and $\frac{\psi''(x)}{\psi'(x)}$, respectively. As a result,

$$\frac{d}{dx} \log\left[\frac{\phi'(x)}{\psi'(x)}\right] = \frac{\phi''(x)}{\phi'(x)} - \frac{\psi''(x)}{\psi'(x)}$$

for almost everywhere x in the domain of $\phi(\cdot)$. Consequently, $\frac{\phi'(x)}{\psi'(x)}$ is decreasing if and only if $-\frac{\phi''(x)}{\phi'(x)} \ge -\frac{\psi''(x)}{\psi'(x)}$ for almost everywhere x in the domain of $\phi(\cdot)$.

□

Lemma A.2. *Suppose $f(\cdot)$ is a strictly increasing and absolutely continuous function in its domain. Then, $f'(\cdot)$ is strictly positive almost everywhere and $f^{-1}(\cdot)$ is* *absolutely continuous. In addition,*

$$\frac{d}{dy}f^{-1}(y) = \frac{1}{f'(f^{-1}(y))}.$$

Proof. It is straightforward to see that $f'(\cdot)$ is strictly positive almost everywhere. Then, by the classical result due to M.A. Zarecki (see p. 271 of [23]), $f^{-1}(\cdot)$ is absolutely continuous. Now, consider any $y_1 < y_2$ in the domain of f^{-1}. We have

$$\int_{y_1}^{y_2} \frac{1}{f'(f^{-1}(y))}dy = \int_{f^{-1}(y_1)}^{f^{-1}(y_2)} \frac{1}{f'(z)}f'(z)dz = f^{-1}(y_2) - f^{-1}(y_1).$$

This completes the proof. ☐

Lemma A.3. *Let $g(\cdot)$ be an absolutely continuous function and $f(\cdot)$ be a function whose domain contains the range of $g(\cdot)$. Then, $f(g(\cdot))$ is absolutely continuous and its derivative is $f'(g(x))g'(x)$ if one of the following two conditions is satisfied:*

(i) $f(\cdot)$ is locally Lipschitz continuous and differentiable.
(ii) $f(\cdot)$ is absolutely continuous and $g(\cdot)$ is increasing.

Proof. (i) Because $f(\cdot)$ is locally Lipschitz and $g(\cdot)$ is absolutely continuous, $f(g(\cdot))$ has bounded variation. According to a result by G.M. Fichtenholz (see Theorem IX.5 on p. 252 of [23]), $f(g(\cdot))$ is absolutely continuous. Because $g(\cdot)$ is absolutely continuous, it is differentiable almost everywhere. Fix any x at which $g(\cdot)$ is differentiable. If $g(\cdot)$ is constant in a neighbourhood of x, then we have

$$\frac{d}{dx}f(g(x)) = \lim_{\delta\to 0}\frac{f(g(x+\delta)) - f(g(x))}{\delta} = 0 = f'(g(x))g'(x).$$

Otherwise, there exists a sequence of x_n approaching x such that $g(x_n) \neq g(x)$. In this case, we have

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{n\to+\infty}\frac{f(g(x_n)) - f(g(x))}{x_n - x}\\ &= \lim_{n\to+\infty}\frac{f(g(x_n)) - f(g(x))}{g(x_n) - g(x)}\frac{g(x_n) - g(x)}{x_n - x} = f'(g(x))g'(x).\end{aligned}$$

This completes the proof.

(ii) Fix any $x_1 < x_2$ in the domain of $g(\cdot)$. We want to prove that

$$\int_{x_1}^{x_2} f'(g(x))g'(x)dx = f(g(x_2)) - f(g(x_1)).$$

If $g(x_1) = g(x_2)$, then $g'(x) = 0, x \in (x_1, x_2)$, so the above equality holds. If $g(x_1) < g(x_2)$, define $\tilde{x}_1 := \sup\{x \geq x_1 : g(x) = g(x_1)\}$ and $\tilde{x}_2 := \inf\{x \leq x_2 : g(x) = g(x_2)\}$. Obviously, we have $g'(x) = 0, x \in (x_1, \tilde{x}_1) \cup (\tilde{x}_2, x_2)$ and $x \in (\tilde{x}_1, \tilde{x}_2)$ if and only if $g(x) \in (g(\tilde{x}_1), g(\tilde{x}_2))$. Denote by $f'_+(x) := \max(f'(x), 0)$ and $f'_-(x) := \max(-f'(x), 0)$. We have

$$\begin{aligned}&\int_{x_1}^{x_2} f'_+(g(x))g'(x)dx = \int_{\tilde{x}_1}^{\tilde{x}_2} f'_+(g(x))g'(x)dx\\ &= \int f'_+(g(x))\mathbf{1}_{(\tilde{x}_1,\tilde{x}_2)}(x)g'(x)dx\end{aligned}$$

$$
\begin{aligned}
&= \int f'_+(g(x))\mathbf{1}_{(g(\tilde{x}_1),g(\tilde{x}_2))}(g(x))g'(x)dx = \int f'_+(y)\mathbf{1}_{(g(\tilde{x}_1),g(\tilde{x}_2))}(y)dy \\
&= \int_{g(\tilde{x}_1)}^{g(\tilde{x}_2)} f'_+(y)dy = \int_{g(x_1)}^{g(x_2)} f'_+(y)dy,
\end{aligned}
$$

where the fourth equality is the result of change-of-variable and the last inequality comes from the definition of $\tilde{x}_1$ and $\tilde{x}_2$. Similarly, we have

$$
\int_{x_1}^{x_2} f'_-(g(x))g'(x)dx = \int_{g(x_1)}^{g(x_2)} f'_-(y)dy.
$$

Because $f(\cdot)$ is absolutely continuous, $f'(\cdot)$ is integrable and, consequently, we have

$$
\begin{aligned}
\int_{x_1}^{x_2} f'(g(x))g'(x)dx &= \int_{x_1}^{x_2} f'_+(g(x))g'(x)dx - \int_{x_1}^{x_2} f'_-(g(x))g'(x)dx \\
&= \int_{g(x_1)}^{g(x_2)} f'_+(y)dy - \int_{g(x_1)}^{g(x_2)} f'_-(y)dy \\
&= \int_{g(x_1)}^{g(x_2)} f'(y)dy \\
&= f(g(x_2)) - f(g(x_1)).
\end{aligned}
$$

This completes the proof.

□

Appendix B. **Power utility.** As alternative for the exponential utility function, in this Appendix we assume a power utility function, i.e.,

$$
u(x) = \left(x^{1-\beta'} - 1\right)/(1-\beta'), \quad x > 0 \tag{12}
$$

for some $\beta' > 0$, where $u(x) := \log(x)$ when $\beta' = 1$. We still use the probability weighting function (9). On the other hand, we assume that $\ln(1+R)$ follows a skew-normal distribution with location parameter ξ', scale parameter ω', and shape parameter α'. As power utility is only defined for strictly positive wealth ($x > 0$), the log-transformation $\ln(1+R)$ ensures that the risky asset price stays positive. Moreover, we impose the additional constraint that the investor cannot invest more than 100% of her initial wealth in the risky asset, to avoid negative wealth when more than 100% is allocated to the risky asset and the return is close to -1.

As before, we fix the mean $\mu = 6\%$ and $\sigma = 20\%$ for the distribution of R and set three values, -0.5, 0, and 0.5, for the skewness; see Figure 11 for the probability density function of R. We set the initial wealth $x_0 = 1$ so that θ^* represents the optimal *percentage* allocation to the risky asset. We set $\beta' = 1$ and $\delta = 1$ and plot θ^* with respect to $1-\gamma$ that represents the degree of inverse S-shape of the probability weighting function. Figure 10 shows that the optimal stock weight is decreasing as the investor's weighting becomes more inverse S-shaped, similar to in the case when the utility function is exponential and the excess return of the risky asset follows a skew-normal distribution.

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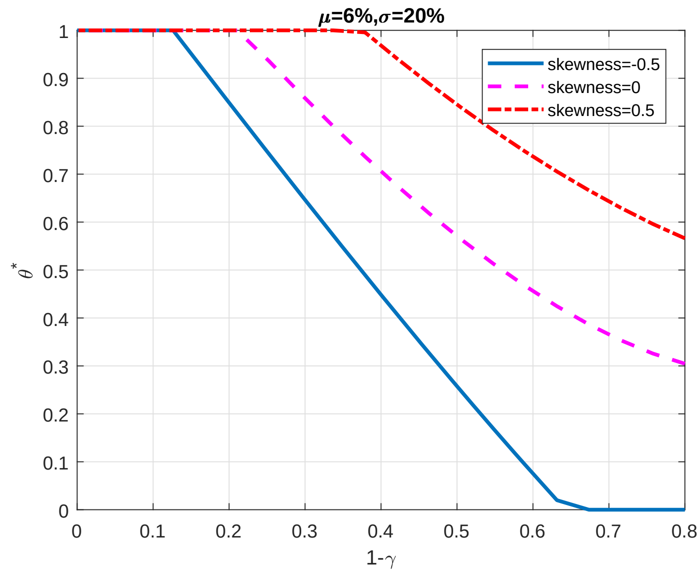

FIGURE 10. Optimal percentage allocation θ^* to the risky asset with respect to different degrees of inverse S-shape of the probability weighting function. The utility function is the power one in (12) with $\beta' = 1$. The probability weighting function is given by (9), so $1-\gamma$ represents the degree of inverse S-shape of the probability weighting function. We set $\delta = 1$. The gross return of the risky asset $R+1$ follows a log skew-normal distribution, and the mean μ and standard deviation σ of R are set to be 6% and 20%, respectively. The skewness of R takes three values -0.5, 0, and 0.5, corresponding to the solid line, dashed line, and dash-dotted line, respectively.

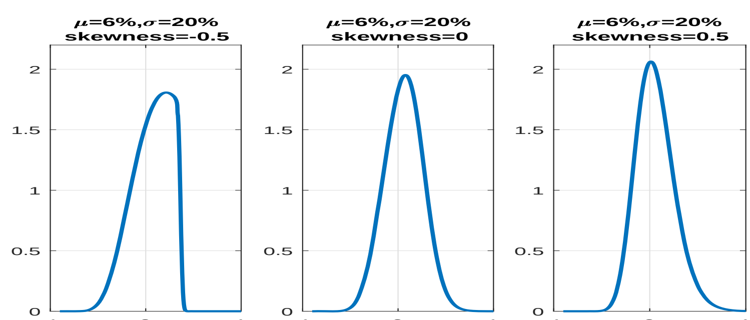

FIGURE 11. Probability density function of the excess return R of the risky asset when the gross return $R+1$ follows a log skew-normal distribution. The mean and standard deviation of R are set to be $\mu = 6\%$ and $\sigma = 20\%$, respectively, and the skewness takes three values: -0.5, 0, and 0.5, corresponding to the probability density functions in the left, middle, and right panes, respectively.

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