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THE CONNECTION BETWEEN THE MAXIMUM PRINCIPLE AND DYNAMIC PROGRAMMING IN STOCHASTIC CONTROL*

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There are usually two ways to study optimal stochastic control problems: Pontryagin's maximum principle and Bellman's dynamic programming, involving an adjoint process ψ and the value function V, respectively. The classical result on the connection between the maximum principle and dynamic programming is known as $\psi(t)=V_x(t,\hat{x}(t))$, where $\hat{x}(\cdot)$ is the optimal path. In this paper we establish a *nonsmooth version* of the classical result by employing the notions of super- and sub-differential introduced by Crandall and Lions. Thus the illusory assumption that $V(\cdot,\cdot)$ is differentiable is dispensed with.

KEY WORDS: Optimal stochastic control, maximum principle, dynamic programming, super- and sub-differential, viscosity solution.

1. INTRODUCTION

For $s\in[0,1]$, the set of admissible controls $U_{ad}[s,1]$ will be the collection of (1) a standard probability space $(\Omega,\mathscr{F},P)$ and a r-dimensional Brownian motion $\{B(t):s\leqq t\leqq 1\}$ with $B(s)=0$; (2) a Γ-valued $\mathscr{F}_t^s$-adapted measurable process $u(\cdot,\cdot)$ in $[s,1]$, where $\mathscr{F}_t^s=\sigma\{B(r):s\leqq r\leqq t\}$ and Γ is a prescribed compact set in a metric space. We denote $(\Omega,\mathscr{F},P,B(t);u)\in U_{ad}[s,1]$, but sometimes we will only write $u\in U_{ad}[s,1]$ without mentioning the probability space and the Brownian motion, if no ambiguity arises.

Let $(s,y)\in[0,1]\times R^d$ be fixed. For each $(\Omega,\mathscr{F},P,B(t);u)\in U_{ad}[s,1]$, there is a cost functional $J(s,y;u)=E\{\int_s^1 L(r,x(r),u(r))\,dr+h(x(1))\}$, where $x(\cdot)$ is the solution of the following SDE on the space $(\Omega,\mathscr{F},P;\mathscr{F}_t^s)$:

$$\begin{cases} dx(t)=\sigma(t,x(t))\,dB(t)+f(t,x(t),u(t))\,dt, & s\leqq t\leqq 1,\\ x(s)=y. \end{cases} \tag{1.1}$$

The optimal control problem is to minimize $J(s,y;u)$ among all admissible controls.

We denote the above problem by $C_{s,y}$ to recall the dependence on the initial time s and initial state y. The value function is then defined as

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$$V(s, y) = \inf\{J(s, y; u) : u \in U_{ad}[s, 1]\} \tag{1.2}$$

$(\hat{x}, \hat{u})$ is called an optimal pair of the problem $C_{s,y}$ if $\hat{x}$ is the corresponding solution process of (1.1) for $\hat{u} \in U_{ad}[s, 1]$ and $J(s, y; \hat{u}) = V(s, y)$.

To study the above optimal stochastic control problem, most researches were along either of two lines: maximum principle (MP in short) and dynamic programming (DP), which go back to Pontryagin [9] and Bellman [1], respectively.

The MP says (the precise statement will be given later on): if $(\hat{x}, \hat{u})$ is an optimal pair for $C_{s,y}$, then there exists a $\mathscr{F}_t^s$-adapted process ψ, called the *adjoint process*, such that

$$EH(t, \hat{x}(t), \hat{u}(t), \psi(t)) = \max_{u \in \Gamma} EH(t, \hat{x}(t), (u, \psi(t)) \text{ a.e. } t \in [s, 1], \tag{1.3}$$

where H is the Hamiltonian defined as:

$$H(t, x, u, p) = -p \cdot f(t, x, u) - L(t, x, u) \quad \text{for} \quad (t, x, u, p) \in [0, 1] \times R^d \times \Gamma \times R^d. \tag{1.4}$$

On the other hand, the classical theory of DP asserts that if $V(\cdot, \cdot) \in C^{1,2}$, then it satisfies the following Hamilton–Jacobi–Bellman (H–J–B) equation: [5, 8]

$$-\frac{\partial V}{\partial t}(t, x) - \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}(t, x) + \sup_{u \in \Gamma} H\left(t, x, u, \frac{\partial V}{\partial x}(t, x)\right) = 0, \tag{1.5}$$

where

$$a_{ij} = 1/2 \sum_{k=1}^{r} \sigma_{ik} \sigma_{jk}, \quad i, j = 1, 2, \ldots, d.$$

But this theory is highly unsatisfactory because the value function V is *not* smooth even in the simplest case, unless the diffusion term in (1.1) is nondegenerate; see [5]. Recently, however, there has been a significant development in the study of DP due to the new conception of the viscosity solution introduced by Crandall–Lions [4]. It asserts that V is a solution of (1.5) in viscosity sense, which is not required to be in $C^{1,2}$ [7]. Hence the result in [7] may be viewed as a nonsmooth version of the classical theory.

Now a natural question arises: what is the relationship between MP and DP? In *deterministic* control ($\sigma = 0$), the well-known result is that if $V(\cdot, \cdot) \in C^{1,2}$, then

$$\psi(t) = V_x(t, \hat{x}(t)), \tag{1.6}$$

see [5]. But it is also based on the assumption that V is smooth. Recently, this is improved by the author [11], by showing that (1.6) can be interpreted as

$$D_x^- V(t, \hat{x}(t)) \subset \{\psi(t)\} \subset D_x^+ V(t, \hat{x}(t)) \tag{1.7}$$

where D_x^- and D_x^+ is the sub and super differential evoked in defining the viscosity solution [4]. Some related work by Clarke and Vinter [3, 10] is discussed below (see Remark 3.4).

The present work is a continuation of the study to the connection between the MP and DP in stochastic control. After precise statements of the MP and DP in Section 2, we will give a stochastic version of (1.7) in Section 3, under the usual assumptions appearing in proofs of the MP and DP. Therefore, the MP, DP and their connection can be established within a *unified* framework of nonsmooth analysis: the framework of viscosity solution.

2. MAXIMUM PRINCIPLE AND DYNAMIC PROGRAMMING

Throughout this work, we impose the following assumptions:

(A_1) $\sigma(\cdot,\cdot)$ is a continuous mapping from $[0,1]\times R^d$ to $R^{d\times r}$; sometimes we write $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_r)$, where σ_i is R^d-valued, $i=1,2,\ldots,r$;

(A_2) $f(\cdot,\cdot,\cdot)$ and $L(\cdot,\cdot,\cdot)$ are continuous mappings from $[0,1]\times R^d\times\Gamma$ to R^d and R^1 respectively; moreover, f and L are continuous with respect to (t,x), uniformly in $u\in\Gamma$;

(A_3) $\forall(t,u)\in[0,1]\times\Gamma$, $f(t,\cdot,u)$, $L(t,\cdot,u)$, $\sigma(t,\cdot)$ and $h(\cdot)$ are continuously differentiable, where $h(\cdot)$ is R^1-valued;

(A_4) $\exists$ const $K>0$ such that

$$|f(t,x,u)-f(t,y,u)|+|L(t,x,u)-L(t,y,u)|+|\sigma(t,x)-\sigma(t,y)|$$

$$+|h(x)-h(y)|\leqq K|x-y|,\ \forall(t,x,y,u)\in[0,1]\times R^d\times R^d\times\Gamma; \tag{2.1}$$

$$|f(t,x,u)|+|L(t,x,u)|+|\sigma(t,x)|+|h(x)|\leqq K(1+|x|)$$

$$\forall(t,x,u)\in[0,1]\times R^d\times\Gamma. \tag{2.2}$$

LEMMA 2.1 *There exists a constant $C>0$ such that*

$$|J(t,x;u)-J(s,y;u)|\leqq C(|t-s|^{1/2}+|x-y|),\ \forall(t,x),(s,y)\in[0,1]\times R^d,\ \forall u\in U_{ad}; \tag{2.3}$$

$$|V(t,x)-V(s,y)|\leqq C(|t-s|^{1/2}+|x-y|),\ \forall(t,x),(s,y)\in[0,1]\times R^d. \tag{2.4}$$

Proof The proof is conventional and easy, hence omitted.

THEOREM 2.1 (MP) *Suppose $(\hat{x},\hat{u})$ is an optimal pair for the problem $C_{s,y}$, then there exists an adjoint process $\psi(t)$ on $[s,1]$ which is $\mathscr{F}_t^s$-adapted such that*

$$EH(t,\hat{x}(t),\hat{u}(t),\psi(t))=\max_{u\in\Gamma}EH(t,\hat{x}(t),u,\psi(t)),\ \text{a.e. } t\in[s,1] \tag{2.5}$$

$$H(t,\hat{x}(t),\hat{u}(t),\psi(t))=\max_{u\in\Gamma}H(t,\hat{x}(t),u,\psi(t)),\ \text{w.p.1; a.e. } t\in[s,1]. \tag{2.6}$$

Furthermore, the process ψ can be represented as

$$\psi(t)=E(\int_t^1 L_x(r,\hat{x}(r),\hat{u}(r))\Phi(r,t)\,dr+h_x(\hat{x}(1))\Phi(1,t)|\mathscr{F}_t^s),\quad t\geqq s,\ \text{w.p.1},\tag{2.7}$$

where L_x and h_x denote respectively the gradients of L and h, and the random matrix Φ satisfies the following SDE

$$\begin{aligned}\Phi(t,\tau)&=I+\int_\tau^t f_x(r,\hat{x}(r),\hat{u}(r))\Phi(r,\tau)\,dr\\&\quad+\sum_{i=1}^{r}\int_\tau^t \sigma_{ix}(r,\hat{x}(r))\Phi(r,\tau)\,dB(r),\quad\text{for}\quad s\leqq\tau\leqq t;\\ \Phi(t,\tau)&=0,\quad\text{for}\quad s\leqq t<\tau,\end{aligned}\tag{2.8}$$

here f_x and σ_{ix} denote respectively the Jacobian matrices of f and σ_i.

Proof See [12] as one of the special cases.

Remark 2.1 The forms of the MP are somewhat different depending on authors [2, 6, 12]. In [2], the adjoint process ψ is described as a solution to some SDE. But in order to investigate the connection between the MP and DP, it is convenient to write the ψ explicitly down as (2.7). See Section 3 for details.

Before going to the statement of the DP, let us briefly recall the definition of the viscosity solution of a second-order nonlinear partial differential equation [7].

DEFINITION 2.1 Let Q be a smooth domain in R^n and $v\in C(Q)$, the second-order super- (resp. sub-) differential of v at $x\in Q$, denoted by $D^{2,+}v(x)$ (resp. $D^{2,-}$) is the *set* defined as

$$D^{2,+}v(x)=\{(A,p)\in R^{n\times n}\times R^n:\overline{\lim_{y\to x}}\,[v(y)-v(x)-p\cdot(y-x)-1/2(A(y-x),y-x)]/|y-x|^2\leqq 0\}.$$

$$(\text{resp. } D^{2,-}v(x)=\{(A,p)\in R^{n\times n}\times R^n:\underline{\lim_{y\to x}}\,\{\cdots\}\geqq 0\}).$$

DEFINITION 2.2 Let $G\in C(Q\times R\times R^n\times R^{n\times n})$ and $v\in C(Q)$, v is called a viscosity solution of the equation $G(x,v,\partial v,\partial^2 v)=0$, if

$$G(x,v(x),p,A)\leqq 0,\ \forall\,(A,p)\in D^{2,+}v(x),\ \forall\,x\in Q;\tag{2.9}$$

$$G(x,v(x),p,A)\geqq 0,\ \forall\,(A,p)\in D^{2,-}v(x),\ \forall\,x\in Q.\tag{2.10}$$

Remark 2.2 The first-order super- and sub-differential of $v\in C(Q)$ are now denoted by $D^{1,+}v(x)$ and $D^{1,-}v(x)$ respectively. See [4, 11]. It is easy to see that if $(A,p)\in D^{2,+}v(x)$, then $p\in D^{1,+}v(x)$, etc.

Remark 2.3 For a function $v\in C([0,1]\times Q)$, when we write $D_x^{2,+}v(t,x)$, it is understood to be the super-differential with respect to the x-variable, etc.

THEOREM 2.2 (*DP*) *The value function* V *is a viscosity solution of the H–J–B equation* (1.5) *with the* end *condition*

$$V(1,x)=h(x). \tag{2.11}$$

Remark 2.4 Our definition of the value function (1.2) follows that in [5], which is somewhat different from that in Lions [7] and Nisio [8]. The difference lies in that our value function satisfies a *backward* H–J–B equation while the one in [7, 8] satisfies a *forward* H–J–B equation. Thus the proof of Theorem 2.2 is also somewhat different from that in [7].

Proof of Theorem 2.2 For fixed $(t,x)\in[0,1]\times R^d$, let $(A,p)\in D^{2,-}_{t,x}V(t,x)$ where $p\in R^{d+1}$, $A\equiv(A_{ij})\in R^{(d+1)\times(d+1)}$, $i,j=0,1,2,\ldots,d$. By the property of sub-differential [7], there exists $F\in C^2_b((0,1)\times R^d)\cap C_b([0,1)\times R^d)$ such that

$$F(t,x)=V(t,x),\ (F_t(t,x),\ F_x(t,x))=(p_0,p_1)\equiv p, F_{tt}(t,x)=A_{00},$$

$$F_{xx}(t,x)=(A_{ij}),\ i,j=1,2,\ldots,d; \tag{2.12}$$

$$F(s,y)<V(s,y),\ \forall(s,y)\neq(t,x). \tag{2.13}$$

$\forall\varepsilon>0$, $\exists(\Omega,\mathscr{F},P,B(r);u_\varepsilon)\in U_{ad}[t,1]$ such that $V(t,x)\geqq J(t,x;u_\varepsilon)-\varepsilon^2$. Note by the definition of the admissible control, for P-a.s. $\omega\in\Omega$,

$$(\Omega,\mathscr{F},P(\cdot|\mathscr{F}^t_{t+\varepsilon})(\omega),B(r)-B(t+\varepsilon);u_\varepsilon|_{[t+\varepsilon,1]})\in U_{ad}[t+\varepsilon,1],$$

hence

$$\begin{aligned}
F(t,x)&=V(t,x)\geqq J(t,x;u_\varepsilon)-\varepsilon^2\\
&=E\int_t^{t+\varepsilon}L(s,x_\varepsilon(s),u_\varepsilon(s))\,ds+E\left\{E\int_{t+\varepsilon}^1 L(s,x_\varepsilon(s),u_\varepsilon(s))\,ds+h(x_\varepsilon(1))|\mathscr{F}^t_{t+\varepsilon}\right\}-\varepsilon^2\\
&\geqq E\int_t^{t+\varepsilon}L(s,x_\varepsilon(s),u_\varepsilon(s))\,ds+EV(t+\varepsilon,x_\varepsilon(t+\varepsilon))-\varepsilon^2\\
&\geqq E\int_t^{t+\varepsilon}L(s,x_\varepsilon(s),u_\varepsilon(s))\,ds+EF(t+\varepsilon,x_\varepsilon(t+\varepsilon))-\varepsilon^2,
\end{aligned} \tag{2.14}$$

where x_ε is the solution of (1.1) for the control u_ε with initial (t,x). Applying Itô's formula to $F(\cdot,\cdot)$, we get from (2.14) that

$$-\int_t^{t+\varepsilon}E\left\{F_s(s,x_\varepsilon(s))+\sum_{i,j=1}^d a_{ij}(s,x_\varepsilon(s))\frac{\partial^2F}{\partial x_i\,\partial x_j}\right\}ds$$

$$\geqq \int_t^{t+\varepsilon} E\{L(s, x_\varepsilon(s), u_\varepsilon(s)) + F_x(s, x_\varepsilon(s)) \cdot f(s, x_\varepsilon(s), u_\varepsilon(s))\}\, ds - \varepsilon^2$$

$$\geqq \int_t^{t+\varepsilon} E\{L(t, x, u_\varepsilon(s)) + F_x(t, x) \cdot f(t, x, u_\varepsilon(s))\}\, ds - o(\varepsilon)$$

$$\geqq -\varepsilon \sup_{u \in \Gamma} H(t, x, u, F_x(t, x)) - o(\varepsilon). \tag{2.15}$$

Dividing (2.15) by ε and letting ε tend to 0, we arrive at

$$-p_0 - \sum_{i,j=1}^{d} a_{ij}(t, x) A_{ij} + \sup_{u \in \Gamma} H(t, x, u, p_1) \geqq 0. \tag{2.16}$$

Now let $(A, p) \in D_{t,x}^{2,+} V(t, x)$, there is also a function $F \in C_b^2((0, 1) \times R^d) \cap C_b([0, 1) \times R^d)$ with the property (2.12) and

$$F(s, y) > V(s, y), \ \forall (s, y) \neq (t, x). \tag{2.17}$$

$\forall \varepsilon > 0$, by Lemma 2.1, $\exists \delta = \delta(t, \varepsilon)$ such that

$$|J(t+\varepsilon, y; u) - J(t+\varepsilon, z; u)| + |V(t+\varepsilon, y) - V(t+\varepsilon, z)| < \varepsilon^2,$$

$$\forall u \in U_{ad}[t+\varepsilon, 1], \text{ if } |y - z| < \delta. \tag{2.18}$$

Let $\{D_j\}$ be a Borel partition of R^d with diameter $|D_j| < \delta$. Choose $x_j \in D_j$. For each j, $\exists (\Omega_j, \mathscr{F}_j, P_j, B_j(r); v_j) \in U_{ad}[t+\varepsilon, 1]$ such that

$$J(t+\varepsilon, x_j; v_j) \leqq V(t+\varepsilon, x_j) + \varepsilon^2, \tag{2.19}$$

hence for any $y \in D_j$,

$$J(t+\varepsilon, y; v_j) \leqq J(t+\varepsilon, x_j; v_j) + \varepsilon^2 \leqq V(t+\varepsilon, x_j) + 2\varepsilon^2 \leqq V(t+\varepsilon, y) + 3\varepsilon^2. \tag{2.20}$$

Since v_j is $\sigma\{B_j(r), t+\varepsilon \leqq r \leqq s\}$-adapted, so there exists a measurable $\tilde{v}_j$: $[t+\varepsilon, 1] \times C([t+\varepsilon, 1] \to R^r) \to \Gamma$ such that $v_j(r, \omega) = \tilde{v}_j(r, B_j(\cdot \wedge r, \omega))$. Now for any $u \in \Gamma$, let $x_\varepsilon(\cdot)|_{[t, t+\varepsilon]}$ be the solution of (1.1) with the initial (t, x) and any admissible control $(\Omega, \mathscr{F}, P, B(r); u(t) \equiv u)$ on $[t, t+\varepsilon]$. Define

$$u_\varepsilon(s, \omega) = \begin{cases} u, & \text{if } s \in [t, t+\varepsilon) \\ \tilde{v}_j(s, B(\cdot \wedge s, \omega) - B(t+\varepsilon, \omega)), & \text{if } s \in [t+\varepsilon, 1] \text{ and } x_\varepsilon(t+\varepsilon) \in D_j. \end{cases} \tag{2.21}$$

It is clear that $u_\varepsilon \in U_{ad}[t, 1]$, let x_ε be the corresponding solution of (1.1) for initial (t, x) and the control u_ε. We can write

$$F(t, x) = V(t, x) \leqq J(t, x; u_\varepsilon)$$

$$= E \int_t^{t+\varepsilon} L(s, x_\varepsilon(s), u)\, ds + EJ(t+\varepsilon, x_\varepsilon(t+\varepsilon); u_\varepsilon|_{[t+\varepsilon, 1]})$$

$$\leqq E \int_t^{t+\varepsilon} L(s, x_\varepsilon(s), u)\, ds + EV(t+\varepsilon, x_\varepsilon(t+\varepsilon)) + 3\varepsilon^2 \text{ (see (2.20))}$$

$$\leqq E \int_t^{t+\varepsilon} L(s, x_\varepsilon(s), u)\, ds + EF(t+\varepsilon, x_\varepsilon(t+\varepsilon)) + 3\varepsilon^2. \tag{2.22}$$

Using a similar calculation as (2.15), we get

$$-p_0 - \sum_{i,j=1}^{d} a_{ij}(t,x)A_{ij} + \sup_{u \in \Gamma} H(t, x, u, p_1) \leqq 0. \tag{2.23}$$

By combining (2.16) and (2.23), it follows that V is a viscosity solution of (1.5). Moreover, (2.11) is clear. The proof is completed.

3. CONNECTION BETWEEN MAXIMUM PRINCIPLE AND DYNAMIC PROGRAMMING

In deterministic control, the following principle plays an important role in the study of DP: if $(\hat{x}, \hat{u})$ is optimal for the problem $C_{s,y}$, then it is also optimal for $C_{t,\hat{x}(t)}$, for each $t \in [s, 1]$. See [1, 11]. Sometimes it is called Bellman's principle of optimality. The following Lemma 3.1 is essentially a stochastic version of the Bellman's principle.

LEMMA 3.1 *Suppose $(\hat{x}, \hat{u})$, defined on a probability space $(\Omega, \mathscr{F}, P)$ with a Brownian motion $\{B(t): s \leqq t \leqq 1\}$, is an optimal pair for $C_{s,y}$, then for each $t \in [s, 1]$*

$$V(t, \hat{x}(t)) = E\left(\int_t^1 L(r, \hat{x}(r), \hat{u}(r))\, dr + h(\hat{x}(1)) \Big| \mathscr{F}_t^s\right), \text{ w.p.1.} \tag{3.1}$$

Proof Note for a.s. $\omega \in \Omega$, $(\Omega, \mathscr{F}, P(\cdot|\mathscr{F}_t^s)(\omega), B(r) - B(t); \hat{u}|_{[t,1]}) \in U_{ad}[t, 1]$, hence

$$V(t, \hat{x}(t)) \leqq E\left(\int_t^1 L(r, \hat{x}(r), \hat{u}(r))\, dr + h(\hat{x}(1)) \Big| \mathscr{F}_t^s\right), \text{ w.p.1.} \tag{3.2}$$

If

$$P\left\{\omega: V(t, \hat{x}(t,\omega)) < E\left(\int_t^1 L(r, \hat{x}(r), \hat{u}(r))\, dr + h(\hat{x}(1)) \Big| \mathscr{F}_t^s\right)(\omega)\right\} > 0 \tag{3.3}$$

then $\exists\, \varepsilon > 0$ such that

$$EV(t, \hat{x}(t)) + \varepsilon < E\left[\int_t^1 L(r, \hat{x}(r), \hat{u}(r))\, dr + h(\hat{x}(1))\right]. \tag{3.4}$$

Using a similar argument as in (2.18) $\sim$ (2.22), we can choose an admissible control v in $[t,1]$ such that

$$EJ(t,\hat{x}(t);v) < EV(t,\hat{x}(t)) + \varepsilon. \tag{3.5}$$

Define

$$u^*(r) = \begin{cases} \hat{u}(r), & r \in [s,t) \\ v(r), & r \in [t,1], \end{cases}$$

then we have

$$\begin{aligned} J(s,y;u^*) &= E\int_s^t L(r,\hat{x}(r),\hat{u}(r))\,dr + EJ(t,\hat{x}(t);v) \\ &< E\int_s^t L(r,\hat{x}(r),\hat{u}(r))\,dr + E\left[\int_t^1 L(r,\hat{x}(r),\hat{u}(r))\,dr + h(\hat{x}(1))\right] \\ &= J(s,y;\hat{u}). \end{aligned} \tag{3.6}$$

This is a contradiction to the optimality of $\hat{u}$. Hence (3.3) is false. The proof is completed.

Remark 3.1 It is easy to check that (3.1) is equivalent to that $M(t) \equiv V(t,\hat{x}(t)) + \int_s^t L(r,\hat{x}(r),\hat{u}(r))\,dr$ is an $\mathscr{F}_t^s$-martingale in $[s,1]$.

In the following we will write σ_x to stand for $\sum_{i=1}^r \sigma_{ix}$.

LEMMA 3.2 *Let $(\hat{x},\hat{u})$ is optimal for $C_{s,y}$ and let $t\in[s,1]$. Let ξ be a $\mathscr{F}_t^s$-measurable R^d-valued random variable. Then the solution of the following SDE*

$$z(r) = \xi + \int_t^r f_x(\theta,\hat{x}(\theta),\hat{u}(\theta))z(\theta)\,d\theta + \int_t^r \sigma_x(\theta,\hat{x}(\theta))z(\theta)\,dB(\theta),\ r \geqq t \tag{3.7}$$

can be represented by

$$z(r) = \Phi(r,t)\xi,\ r\in[t,1], \tag{3.8}$$

where Φ is the solution of (2.8).

Proof It is easy to verify that the right side of (3.8) satisfies (3.7). Hence the result follows from the uniqueness of solutions of the SDE (3.7).

THEOREM 3.1 *Suppose $(\hat{x},\hat{u})$ is an optimal pair for the problem $C_{s,y}$, then for any $t\in[s,1]$*

$$D_x^{1,-}V(t,\hat{x}(t)) \subset \{\psi(t)\} \subset D_x^{1,+}V(t,\hat{x}(t)), \text{ w.p.1}, \tag{3.9}$$

where ψ is the adjoint process appearing in Theorem 2.1.

Proof $\forall\, y' \in R^d$, let $x(\cdot\,;y')$ satisfy the following SDE in $[t,1]$:

$$x(r; y') = y' + \int_t^r f(\theta, x(\theta; y'), \hat{u}(\theta))\, d\theta + \int_t^r \sigma(\theta, x(\theta; y'))\, dB(\theta), \; r \geqq t. \tag{3.10}$$

Note $m(r) \equiv \int_t^r \{\sigma(\theta, x(\theta; y')) - \sigma(\theta, \hat{x}(\theta))\}\, dB(\theta)$ is still an $\mathscr{F}_r^s$-martingale under $P(\cdot|\mathscr{F}_t^s)(\omega)$ for a.s. ω. Hence by a standard argument, we know there exists $C = C(t) > 0$ such that

$$E\left(\sup_{t \leqq \theta \leqq 1} |x(\theta; y') - \hat{x}(\theta)|^4 |\mathscr{F}_t^s\right) \leqq C|y' - \hat{x}(t)|^4, \text{ w.p.1.} \tag{3.11}$$

Now we can write

$$\begin{aligned} x(r; y') - \hat{x}(r) &= y' - \hat{x}(t) \\ &\quad + \int_t^r \{f(\theta, x(\theta; y'), \hat{u}(\theta)) - f(\theta, \hat{x}(\theta), \hat{u}(\theta))\}\, d\theta \\ &\quad + \int_t^r \{\sigma(\theta, x(\theta; y')) - \sigma(\theta, \hat{x}(\theta))\}\, dB(\theta) \\ &= y' - \hat{x}(t) + \int_t^r f_x(\theta, \hat{x}(\theta), \hat{u}(\theta))\,(x(\theta; y') - \hat{x}(\theta))\, d\theta \\ &\quad + \int_t^r \sigma_x(\theta, \hat{x}(\theta))\,(x(\theta; y') - \hat{x}(\theta))\, dB(\theta) \\ &\quad + \int_t^r \int_0^1 \{f_x(\theta, \hat{x}(\theta) + \alpha(x(\theta; y') - \hat{x}(\theta)), \hat{u}(\theta)) \\ &\quad - f_x(\theta, \hat{x}(\theta), \hat{u}(\theta))\}\, d\alpha \cdot (x(\theta; y') - \hat{x}(\theta))\, d\theta \\ &\quad + \int_t^r \int_0^1 \{\sigma_x(\theta, \hat{x}(\theta) + \alpha(x(\theta; y') - \hat{x}(\theta))) \\ &\quad - \sigma_x(\theta, \hat{x}(\theta))\}\, d\alpha \cdot (x(\theta; y') - \hat{x}(\theta))\, dB(\theta). \end{aligned} \tag{3.12}$$

Denote

$$r_1(\theta; y') = \int_0^1 \{f_x(\theta, \hat{x}(\theta) + \alpha(x(\theta; y') - \hat{x}(\theta)), \hat{u}(\theta)) - f_x(\theta, \hat{x}(\theta), \hat{u}(\theta))\}\, d\alpha,$$

$$r_2(\theta; y') = \int_0^1 \{\sigma_x(\theta, \hat{x}(\theta) + \alpha(x(\theta; y') - \hat{x}(\theta))) - \sigma_x(\theta, \hat{x}(\theta))\}\, d\alpha, \text{ for } \theta \in [t, 1].$$

Let $z(\cdot; y')$ satisfy the following SDE in $[t, 1]$:

$$z(r; y') = y' - \hat{x}(t) + \int_t^r f_x(\theta, \hat{x}(\theta), \hat{u}(\theta)) z(\theta; y')\, d\theta + \int_t^r \sigma_x(\theta, \hat{x}(\theta)) z(\theta; y')\, dB(\theta). \quad (3.13)$$

By assumptions $(A_1) \sim (A_4)$, (3.11) and dominated convergence theorem, we have

$$P\{\omega : E(|r_i(\theta; y')|^4 | \mathscr{F}_t^s)(\omega) \to 0 \quad \text{as} \quad y' \to \hat{x}(t, \omega), \forall \theta \in [t, 1]\} = 1, \quad i = 1, 2, \quad (3.14)$$

$$\sup_{\omega, \theta, y'} E(|r_i(\theta; y')|^4 | \mathscr{F}_t^s)(\omega) \leqq \text{const.} \quad i = 1, 2. \quad (3.15)$$

Denote $w(r; y') \equiv x(r; y') - \hat{x}(r) - z(r; y')$. With the aid of (3.11), it is not difficult to derive that

$$E\left(\sup_{t \leqq r \leqq h} |w(r; y')|^2 | \mathscr{F}_t^s \right) \leqq \text{const.} \left\{ \int_t^h E\left(\sup_{t \leqq r \leqq \theta} |w(r; y')|^2 | \mathscr{F}_t^s \right) d\theta \right\}$$

$$+ \text{const.} \left\{ \int_t^1 E(|r_1(\theta; y')|^2 | \mathscr{F}_t^s)\, d\theta \right.$$

$$\left. + \left[\int_t^1 E(|r_2(\theta; y')|^4 | \mathscr{F}_t^s)\, d\theta \right]^{1/2} \right\} |y' - \hat{x}(t)|^2, \text{ w.p.1.} \quad (3.16)$$

By virtue of (3.14) $\sim$ (3.15), we get by the Gronwall's inequality

$$P\left\{ \omega : E\left(\sup_{t \leqq r \leqq 1} |w(r; y')|^2 | \mathscr{F}_t^s \right)(\omega) = o(|y' - \hat{x}(t, \omega)|^2) \right\} = 1. \quad (3.17)$$

Now we choose an $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 1$ such that for any $\omega_0 \in \Omega_0$, (2.7), (3.1) and (3.11) are satisfied for any rational y' (i.e., all the coordinates of y' are rational numbers), and $(\Omega, \mathscr{F}, P(\cdot | \mathscr{F}_t^s)(\omega_0), B(r) - B(t); \hat{u}|_{[t,1]}) \in U_{ad}[t, 1]$. Let $\omega_0 \in \Omega_0$ be fixed, then for any rational y', we have

$$V(t, y') - V(t, \hat{x}(t, \omega_0)) \leqq E^t \left\{ \int_t^1 [L(r, x(r; y'), \hat{u}(r)) - L(r, \hat{x}(r), \hat{u}(r))]\, dr \right.$$

$$\left. + h(x(1; y')) - h(\hat{x}(1)) \right\}, \quad (3.18)$$

where $E^t \equiv E(\cdot | \mathscr{F}_t^s)(\omega_0)$. Using a similar calculation as in (3.12), we can rewrite (3.18) as

$$V(t,y')-V(t,\hat{x}(t,\omega_0))\leqq E^t\left\{\int_t^1 L_x(r,\hat{x}(r),\hat{u}(r))\,(x(r;y')-\hat{x}(r))\,dr\right.$$

$$\left.+h_x(\hat{x}(1))\,(x(1;y')-\hat{x}(1))\right\}+o(|y'-\hat{x}(t,\omega_0)|)$$

$$=E^t\left\{\int_t^1 L_x(r,\hat{x}(r),\hat{u}(r))\,[w(r;y')+\Phi(r,t)\,(y'-\hat{x}(t))]\,dr\right.$$

$$\left.+h_x(\hat{x}(1))\,[w(1;y')+\Phi(1,t)\,(y'-\hat{x}(t))]\right\}$$

$$+o(|y'-\hat{x}(t,\omega_o)|)\quad\text{(by Lemma 3.2.)}$$

$$=E^t\left\{\int_t^1 L_x(r,\hat{x}(r),\hat{u}(r))\Phi(r,t)\,dr\cdot(y'-\hat{x}(t))\right.$$

$$\left.+h_x(\hat{x}(1))\Phi(1,t)\cdot(y'-\hat{x}(t))\right\}+o(|y'-\hat{x}(t,\omega_0)|)$$

(by (3.17) which is a consequence of (3.11))

$$=E^t\left\{\int_t^1 L_x(r,\hat{x}(r),\hat{u}(r))\Phi(r,t)\,dr\right.$$

$$\left.+h_x(\hat{x}(1))\Phi(1,t)\right\}\cdot(y'-\hat{x}(t,\omega_0))+o(|y'-\hat{x}(t,\omega_0)|)$$

$$=\psi(t,\omega_0)\,(y'-\hat{x}(t,\omega_0))+o(|y'-\hat{x}(t,\omega_0)|)\quad\text{(by (2.7).)},\qquad(3.19)$$

hence for any $\varepsilon>0$, there exists a $\delta>0$ such that if y'_n is rational and $|y'_n-\hat{x}(t,\omega_0)|<\delta$, then $V(t,y'_n)-V(t,\hat{x}(t,\omega_0))\leqq\psi(t,\omega_0)\,(y'_n-\hat{x}(t,\omega_0))+\varepsilon_n$ where $|\varepsilon_n|/|y'_n-\hat{x}(t,\omega_0)|<\varepsilon/3$. Now choose $\delta'=\min(1,\delta,\varepsilon/3(C+|\psi(t,\omega_0)|))$ where C is the constant in (2.4), then for any y' with $0<|y'-\hat{x}(t,\omega_0)|<\delta'$, find a fixed rational y'_N such that $|y'_N-\hat{x}(t,\omega_0)|<|y'-\hat{x}(t,\omega_0)|$ and $|y'_N-y'|<|y'-\hat{x}(t,\omega_0)|^2$. Thus we get

$$V(t,y')-V(t,\hat{x}(t,\omega_0))\leqq\psi(t,\omega_0)\,(y'-\hat{x}(t,\omega_0))+C|y'-y'_N|+\psi(t,\omega_0)\,(y'_N-y')+\varepsilon_N,$$

moreover, it is easy to compute that

$$\frac{|C|y'-y'_N|+\psi(t,\omega_0)\,(y'_N-y')+\varepsilon_N|}{|y'-\hat{x}(t,\omega_0)|}<\varepsilon.$$

This means (3.19) holds for any $y' \in R^d$, which immediately leads to the fact that $\psi(t,\omega_0) \in D_x^{1,+}V(t,\hat{x}(t,\omega_0))$. On the other hand, suppose $p \in D_x^{1,-}V(t,\hat{x}(t,\omega_0))$, then by the definition of first-order sub-differential

$$0 \leqq \lim_{y' \to \hat{x}(t,\omega_0)} \{V(t,y') - V(t,\hat{x}(t,\omega_0)) - p\cdot(y' - \hat{x}(t,\omega_0))\}/|y' - \hat{x}(t,\omega_0)|$$

$$\leqq \lim_{y' \to \hat{x}(t,\omega_0)} \{(\psi(t,\omega_0) - p)(y' - \hat{x}(t,\omega_0))\}/|y' - \hat{x}(t,\omega_0)| \quad \text{(by (3.19))}, \tag{3.20}$$

hence $p = \psi(t,\omega_0)$, which is the left half of (3.9). The proof is now completed.

Let us conclude the paper with some remarks.

Remark 3.2 By Theorem 3.1, for any $t \in [s,1]$, for a.s. $\omega \in \Omega$, if $V(t,\cdot)$ is differentiable at $\hat{x}(t,\omega)$, then $\psi(t,\omega) = V_x(t,\hat{x}(t,\omega))$. This is just a stochastic version of the classical result (1.6). Moreover, we do have examples, as we have raised in the deterministic case [11], to show that the inclusions in (3.9) may be strict. In fact, if $V(t,\cdot)$ is *not* differentiable at $\hat{x}(t,\omega)$, then (3.9) implies $D_x^{1,-}V(t,\hat{x}(t,\omega)) = \phi$ and $\psi(t,\omega) \in D_x^{1,+}V(t,\hat{x}(t,\omega))$. (Since in general if v is not differentiable at some point x, then either $D^{1,-}v(x)$ or $D^{1,+}v(x)$ will be empty.)

Remark 3.3 In the deterministic case, it can further be proved [11] that

$$D_{t,x}^{1,-}V(t,\hat{x}(t)) \subset \{(H(t,\hat{x}(t),\hat{u}(t),\psi(t)),\psi(t))\} \subset D_{t,x}^{1,+}V(t,\hat{x}(t)). \tag{3.21}$$

By virtue of (3.21) we can derive the MP *directly* from the DP. However, (3.21) fails to hold in the stochastic cases. In fact, the Hamiltonian in (3.21) should be replaced by a "generalized Hamiltonian" which is in *quadratic* form. Moreover, we are also required to study the *second-order* super- and sub-differential of $V(\cdot,\cdot)$ due to the fact that V satisfies a second-order H–J–B equation. A study of these subjects will appear in a forthcoming paper.

Remark 3.4 Here we mention another framework of nonsmooth analysis, called "generalized gradient", to investigate the connection between the MP and DP, though only of the *deterministic* cases. Clarke–Vinter [3] and Vinter [10] proves the following relationship, without assuming the differentiability of f, L and h in x-variable:

$$\psi(t) \in \partial_x V(t,\hat{x}(t)),\ (H(t,\hat{x}(t),\hat{u}(t),\psi(t)),\psi(t)) \in \partial_{t,x}V(t,\hat{x}(t)), \tag{3.22}$$

where ∂_x and $\partial_{t,x}$ denotes respectively the generalized gradient in x-variable and (t,x)-variable. As mentioned in the foregoing, the analogous result to (3.22), in the framework of "viscosity solution" and the related "super- and sub-differential", is proved in [11] (see (1.7) and (3.21)). On the other hand, the connection between these two frameworks is not clear, but at least we have $D^{1,+}v(x) \cup D^{1,-}v(x) \subset \partial v(x)$.

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