A mean-field stochastic maximum principle via Malliavin calculus

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Abstract

This paper considers a mean-field type stochastic control problem where the dynamics is governed by a controlled Itô-Lévy process and the information available to the controller is possibly less than the overall information. All the system coefficients and the objective performance functional are allowed to be random, possibly non-Markovian. Malliavin calculus is employed to derive a maximum principle for the optimal control of such a system where the adjoint process is explicitly expressed.

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1 Introduction

Suppose the state process $X(t) = X^{(u)}(t,\omega)$; $t \geq 0$, $\omega \in \Omega$, is a controlled Itô-Lévy process in $\mathbb{R}$ of the form

$$(1.1)\qquad \begin{cases} dX(t) = b(t,X(t),u(t),\omega)dt + \sigma(t,X(t),u(t),\omega)dB(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \theta(t,X(t^-),u(t^-),z,\omega)\tilde{N}(dt,dz); \\ X(0) = x \in \mathbb{R}. \end{cases}$$

Here $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, $B(t) = B(t,\omega)$, and $\eta(t) = \eta(t,\omega)$, given by

$$(1.2)\qquad \eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz);\ t \geq 0,\ \omega \in \Omega,$$

are a 1-dimensional Brownian motion and an independent pure jump Lévy martingale, respectively, on a given filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Thus

$$(1.3)\qquad \tilde{N}(dt,dz) := N(dt,dz) - \nu(dz)dt$$

is the *compensated jump measure* of $\eta(\cdot)$, where $N(dt,dz)$ is the *jump measure* and $\nu(dz)$ the *Lévy measure* of the Lévy process $\eta(\cdot)$. The process $u(t)$ is our control process, assumed to be $\mathcal{F}_t$-adapted and have values in a given open convex set $U \subset \mathbb{R}$. The coefficients $b : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$, $\sigma : [0,T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$ and $\theta : [0,T] \times \mathbb{R} \times U \times \mathbb{R}_0 \times \Omega$ are given $\mathcal{F}_t$-predictable processes.

Let $T > 0$ be a given constant. For simplicity, we assume that

$$(1.4)\qquad \int_{\mathbb{R}_0} z^2 \nu(dz) < \infty.$$

Suppose in addition that we are given a subfiltration

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \qquad t \in [0,T]$$

representing the information available to the controller at time t and satisfying the usual conditions. For example, we could have

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+};\quad t \in [0,T],\ \delta > 0 \text{ is a constant},$$

meaning that the controller gets a delayed information compared to $\mathcal{F}_t$.

Let $\mathscr{A} = \mathscr{A}_{\mathscr{E}}$ denote a given family of controls, contained in the set of $\mathscr{E}_t$-adapted càdlàg controls $u(\cdot)$ such that (1.1) has a unique strong solution up to time T. Suppose we are given a performance functional of the form

(1.5)
$$J(u) = E\Big[\int_0^T f(t, X(t), E[f_0(X(t)], u(t), \omega)dt + g(X(T), E[g_0(X(T)], \omega)\Big]; \qquad u \in \mathscr{A}_{\mathscr{E}},$$

where $E = E_P$ denotes expectation with respect to P, $f_0 : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are given functions such that $E[|f_0(X(t)|] < \infty$ for all t and $E[|g_0(X(T)|] < \infty$, and $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ are given $\mathcal{F}_t-$adapted processes with

$$E\Big[\int_0^T |f(t, X(t), E[f_0(X(t)], u(t))|dt + |g(X(T), E[g_0(X(T)])|\Big] < \infty \qquad \text{for all} \quad u \in \mathscr{A}_{\mathscr{E}}.$$

The control problem we consider is the following:

PROBLEM 1.1 *Find $\Phi_{\mathscr{E}} \in \mathbb{R}$ and $u^* \in \mathscr{A}_{\mathscr{E}}$ (if it exists) such that*

$$\Phi_{\mathscr{E}} = \sup_{u \in \mathscr{A}_{\mathscr{E}}} J(u) = J(u^*). \tag{1.6}$$

In contrast with the standard stochastic control problems (e.g. [YZ] and [ØS] for stochastic control of Itô diffusions and jump diffusions, respectively), the performance functional (1.5) involves the mean of functions of the state variable (hence the name *mean-field*). Problems of this type occur in many applications; for example in a continuous-time Markowitz's mean–variance portfolio selection model where the variance term involves a quadratic function of the expectation. The inclusion of this mean term introduces some major technical difficulties, which include among others the *time inconsistency* leading to the failure of dynamic programming approach. Recently there has been increasing interest in the study of this type of stochastic control problems; see for example [AD], [LL] and [JMW].

On the other hand, since we allow b, σ, θ, f and g to be stochastic processes and also because our controls must be $\mathscr{E}_t-$adapted, this problem is not of Markovian type and hence cannot be solved by dynamic programming even if the mean term were not present. We instead investigate the maximum principle, and derive an explicit form for the adjoint process. The approach we employ is Malliavin calculus which enables us to express the duality involved via the Malliavin derivative. Our paper is related to the recent paper [BØ], where a maximum principle for partial information control is obtained. However, that paper assumes the existence of a solution of the adjoint equations. This is an assumption which often fails in the partial information case. Our paper is close to [AD] where a maximum principle for mean-field SDE's is obtained in terms of adjoint processes defined by backward SDE's

(BSDE's). However, since these BSDE's are non-linear, they are hard to solve. Our method, on the other hand, gives explicit expressions for associated adjoint processes without the use of BSDE's.

We emphasize that our problem should be distinguished from the *partial observation* control problems, where it is assumed that the controls are based on noisy observation of the state process. For the latter type of problems, there is a rich literature see e.g. [BEK], [B], [KX], [L], [PQ], [T]. Note that the methods and results in the partial observation case do not apply to our situation. On the other hand, there are several existing works on stochastic maximum principle (either completely or partially observed) where adjoint processes are explicitly expressed [BEK], [EK], [L], [T]. However, these works all essentially employ stochastic flows technique, over which the Malliavin calculus has an advantage in terms of numerical computations (see, e.g., [FLLL]).

2 A brief review of Malliavin calculus for Lévy processes

In this section we recall the basic definition and properties of Malliavin calculus for Lévy processes related to this paper, for reader's convenience.

In view of the Lévy–Itô decomposition theorem, which states that any Lévy process $Y(t)$ with

$$E[Y^2(t)] < \infty \quad \text{for all} \quad t$$

can be written

$$Y(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds, dz)$$

with constants a and b, we see that it suffices to deal with Malliavin calculus for $B(\cdot)$ and for

$$\eta(\cdot) := \int_0 \int_{\mathbb{R}_0} z\tilde{N}(ds, dz)$$

separately.

A general reference for this presentation is [N], [BDLØP] and [DMØP]. See also the book [DØP].

2.1 Malliavin calculus for $B(\cdot)$

A natural starting point is the Wiener-Itô chaos expansion theorem, which states that any $F \in L^2(\mathcal{F}_T, P)$ can be written

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{2.1}$$

for a unique sequence of symmetric deterministic functions $f_n \in L^2(\lambda^n)$, where λ is Lebesgue measure on $[0, T]$ and

$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \cdots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n) \tag{2.2}$$

(the n-times iterated integral of f_n with respect to $B(\cdot)$) for $n = 1, 2, \ldots$ and $I_0(f_0) = f_0$ when f_0 is a constant.

Moreover, we have the isometry

$$E[F^2] = ||F||^2_{L^2(p)} = \sum_{n=0}^{\infty} n! ||f_n||^2_{L^2(\lambda^n)}. \tag{2.3}$$

Definition 2.1 (Malliavin derivative D_t)
Let $\mathscr{D}^{(B)}_{1,2}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos expansion (2.1) satisfies

$$||F||^2_{\mathscr{D}^{(B)}_{1,2}} := \sum_{n=1}^{\infty} n n! ||f_n||^2_{L^2(\lambda^n)} < \infty. \tag{2.4}$$

For $F \in \mathscr{D}^{(B)}_{1,2}$ and $t \in [0, T]$, we define the *Malliavin derivative* of F at t (with respect to $B(\cdot)$), $D_t F$, by

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)), \tag{2.5}$$

where the notation $I_{n-1}(f_n(\cdot, t))$ means that we apply the $(n-1)$-times iterated integral to the first $n-1$ variables $t_1, \cdots, t_{n-1}$ of $f_n(t_1, t_2, \cdots, t_n)$ and keep the last variable $t_n = t$ as a parameter.

One can easily check that

$$E\Big[\int_0^T (D_t F)^2 dt\Big] = \sum_{n=1}^{\infty} n n! ||f_n||^2_{L^2(\lambda^n)} = ||F||^2_{\mathscr{D}^{(B)}_{1,2}}, \tag{2.6}$$

so $(t, \omega) \to D_t F(\omega)$ belongs to $L^2(\lambda \times P)$.

Example 2.1 If $F = \int_0^T f(t)dB(t)$ with $f \in L^2(\lambda)$ deterministic, then

$$D_t F = f(t) \text{ for } a.a.\, t \in [0,T].$$

More generally, if $u(s)$ is Skorohod integrable, $u(s) \in \mathscr{D}_{1,2}$ for $a.a.$ s and $D_t u(s)$ is Skorohod integrable for $a.a.$ t, then

$$D_t\Big(\int_0^T u(s)\delta B(s)\Big) = \int_0^T D_t u(s)\delta B(s) + u(t) \text{ for a.a. } (t,\omega), \tag{2.7}$$

where $\int_0^T \psi(s)\delta B(s)$ denotes the Skorohod integral of ψ with respect to $B(\cdot)$. (See [N], page 35–38 for a definition of Skorohod integrals and for more details.)

Some other basic properties of the Malliavin derivative D_t are the following:

(i) **Chain rule** ([N], page 29)
Suppose $F_1, \ldots, F_m \in \mathscr{D}_{1,2}^{(B)}$ and that $\psi : \mathbb{R}^m \to \mathbb{R}$ is C^1 with bounded partial derivatives. Then $\psi(F_1, \cdots, F_m) \in \mathscr{D}_{1,2}$ and

$$D_t\psi(F_1, \cdots, F_m) = \sum_{i=1}^m \frac{\partial \psi}{\partial x_i}(F_1, \cdots, F_m) D_t F_i. \tag{2.8}$$

(ii) **Integration by parts/duality formula** ([N], page 35)
Suppose $u(t)$ is $\mathcal{F}_t$−adapted with $E[\int_0^T u^2(t)dt] < \infty$ and let $F \in \mathscr{D}_{1,2}^{(B)}$. Then

$$E[F \int_0^T u(t)dB(t)] = E[\int_0^T u(t) D_t F dt]. \tag{2.9}$$

2.2 Malliavin calculus for $\tilde{N}(\cdot)$

The construction of a stochastic derivative/Malliavin derivative in the pure jump martingale case follows the same lines as in the Brownian motion case. In this case the corresponding Wiener-Itô chaos expansion theorem states that any $F \in L^2(\mathcal{F}_T, P)$ (where in this case $\mathcal{F}_t = \mathcal{F}_t^{(\tilde{N})}$ is the $\sigma-$algebra generated by $\eta(s) := \int_0^s \int_{\mathbb{R}_0} z\tilde{N}(dr,dz);\ 0 \le s \le t$) can be written as

$$F = \sum_{n=0}^\infty I_n(f_n);\ f_n \in \hat{L}^2((\lambda \times \nu)^n), \tag{2.10}$$

where $\hat{L}^2((\lambda \times \nu)^n)$ is the space of functions $f_n(t_1, z_1, \ldots, t_n, z_n)$; $t_i \in [0,T]$, $z_i \in \mathbb{R}_0$ such that $f_n \in L^2((\lambda \times \nu)^n)$ and f_n is symmetric with respect to the pairs of variables $(t_1, z_1), \ldots, (t_n, z_n)$.

It is important to note that in this case the $n-$times iterated integral $I_n(f_n)$ is taken with respect to $\tilde{N}(dt, dz)$ and not with respect to $d\eta(t)$. Thus, we define

$$I_n(f_n) = n! \int_0^T \int_{\mathbb{R}_0} \int_0^{t_n} \int_{\mathbb{R}_0} \cdots \int_0^{t_2} \int_{\mathbb{R}_0} f_n(t_1, z_1, \cdots, t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n) \tag{2.11}$$

for $f_n \in \hat{L}^2((\lambda \times \nu)^n)$.

The Itô isometry for stochastic integrals with respect to $\tilde{N}(dt, dz)$ then gives the following isometry for the chaos expansion:

$$||F||^2_{L^2(P)} = \sum_{n=0}^{\infty} n! ||f_n||^2_{L^2((\lambda\times\nu)^n)}. \tag{2.12}$$

As in the Brownian motion case we use the chaos expansion to define the Malliavin derivative. Note that in this case there are two parameters t, z, where t represents time and $z \neq 0$ represents a generic jump size.

Definition 2.2 **(Malliavin derivative $D_{t,z}$)** [BDLØP], [DMØP] Let $\mathscr{D}^{(\tilde{N})}_{1,2}$ be the space of all $F \in L^2(\mathcal{F}_T, P)$ such that its chaos expansion (2.10) satisfies

$$||F||^2_{\mathscr{D}^{(\tilde{N})}_{1,2}} := \sum_{n=1}^{\infty} n n! ||f_n||^2_{L^2((\lambda\times\nu)^2)} < \infty. \tag{2.13}$$

For $F \in \mathscr{D}^{(\tilde{N})}_{1,2}$, we define the Malliavin derivative of F at (t, z) (with respect to $\tilde{N}(\cdot)$), $D_{t,z}F$, by

$$D_{t,z}F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, z)), \tag{2.14}$$

where $I_{n-1}(f_n(\cdot, t, z))$ means that we perform the $(n-1)-$times iterated integral with respect to $\tilde{N}$ to the first $n-1$ variable pairs $(t_1, z_1), \cdots, (t_n, z_n)$, keeping $(t_n, z_n) = (t, z)$ as a parameter.

In this case we get the isometry.

$$E[\int_0^T \int_{\mathbb{R}_0} (D_{t,z}F)^2 \nu(dz)dt] = \sum_{n=0}^{\infty} n n! ||f_n||^2_{L^2((\lambda\times\nu)^n)} = ||F||^2_{\mathscr{D}^{(\tilde{N})}_{1,2}}. \tag{2.15}$$

(Compare with (2.6).)

Example 2.2 If $F = \int_0^T \int_{\mathbb{R}_0} f(t, z) \tilde{N}(dt, dz)$ for some deterministic $f(t, z) \in L^2(\lambda \times \nu)$, then

$$D_{t,z}F = f(t, z) \text{ for } a.a. \, (t, z).$$

More generally, if $\psi(s,\zeta)$ is Skorohod integrable with respect to $\tilde{N}(\delta s, d\zeta)$, $\psi(s,\zeta) \in \mathscr{D}_{1,2}^{(\tilde{N})}$ for $a.a.\, s, \zeta$ and $D_{t,z}\psi(s,\xi)$ is Skorohod integrable for $a.a.\,(t,z)$, then

$$(2.16)\qquad D_{t,z}\Big(\int_0^T \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(\delta s, d\zeta)\Big) = \int_0^T \int_{\mathbb{R}} D_{t,z}\psi(s,\zeta)\tilde{N}(\delta s, d\zeta) + u(t,z) \text{ for } a.a.\, t, z,$$

where $\int_0^T \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(\delta s, d\zeta)$ denotes the *Skorohod integral* of ψ with respect to $\tilde{N}(\cdot,\cdot)$. (See [DMØP] for a definition of such Skorohod integrals and for more details.)

The properties of $D_{t,z}$ corresponding to the properties (2.8) and (2.9) of D_t are the following:

(i) **Chain rule([I], [DMØP])** Suppose $F_1, \cdots, F_m \in \mathscr{D}_{1,2}^{(\tilde{N})}$ and that $\phi : \mathbb{R}^m \to \mathbb{R}$ is continuous and bounded. Then $\phi(F_1, \cdots, F_m) \in \mathscr{D}_{1,2}^{(\tilde{N})}$ and

$$(2.17)\qquad D_{t,z}\phi(F_1, \cdots, F_m) = \phi(F_1 + D_{t,z}F_1, \ldots, F_m + D_{t,z}F_m) - \phi(F_1, \ldots, F_m).$$

(ii) **Integration by parts/duality formula [DMØP]** Suppose $\Psi(t,z)$ is $\mathcal{F}_t$-adapted and $E[\int_0^T \int_{\mathbb{R}_0} \psi^2(t,z)\nu(dz)dt] < \infty$ and let $F \in \mathscr{D}_{1,2}^{(\tilde{N})}$. Then

$$(2.18)\qquad E\Big[F \int_0^T \int_{\mathbb{R}_0} \Psi(t,z)\tilde{N}(dt,dz)\Big] = E\Big[\int_0^T \int_{\mathbb{R}_0} \Psi(t,z) D_{t,z}F \nu(dz)dt\Big].$$

3 The stochastic maximum principle

We now return to Problem 1.1 given in the introduction. We make the following assumptions:

Assumption 3.1

(3.1) The functions $b(t,x,u,\omega) : [0,T]\times\mathbb{R}\times U\times\Omega \to \mathbb{R}$, $\sigma(t,x,u,\omega) : [0,T]\times\mathbb{R}\times U\times\Omega \to \mathbb{R}$, $\theta(t,x,u,z,\omega) : [0,T]\times\mathbb{R}\times U\times\mathbb{R}_0\times\Omega \to \mathbb{R}$, $f(t,x,x_0,u,\omega) : [0,T]\times\mathbb{R}\times\mathbb{R}\times U\times\Omega \to \mathbb{R}$, $g(x,x_0,\omega) : \mathbb{R}\times\mathbb{R}\times\Omega \to \mathbb{R}$, and $f_0(x_0) : \mathbb{R} \to \mathbb{R}$, $g_0(x_0) : \mathbb{R} \to \mathbb{R}$ are all continuously differentiable (C^1) with respect to the arguments (if depending on them) $x \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $u \in U$ for each $t \in [0,T]$ and a.a. $\omega \in \Omega$.

(3.2) For all $t, r \in (0,T)$, $t \le r$, and all bounded $\mathscr{E}_t-$measurable random variables $\alpha = \alpha(\omega)$ the control

$$\beta_\alpha(s) = \alpha(\omega)\chi_{[t,r]}(s); \quad s \in [0,T]$$

belongs to $\mathscr{A}_{\mathscr{E}}$.

(3.3) For all $u, \beta \in \mathscr{A}_{\mathcal{E}}$ with β bounded, there exists $\delta > 0$ such that

$$u + y\beta \in \mathscr{A}_{\mathcal{E}} \quad \text{for all } y \in (-\delta, \delta)$$

Further, if we define

$$\widetilde{f}(t, X(t), E[f_0(X(t)], u(t)) := \frac{\partial f}{\partial x}(t, X(t), E[f_0(X(t)], u(t)) + E[\frac{\partial f}{\partial x_0}(t, X(t), E[f_0(X(t)], u(t))] f_0'(X(t)) \tag{3.1}$$

$$\widetilde{g}(X(T), E[g_0(X(T)]) := \frac{\partial g}{\partial x}(X(T), E[g_0(X(T)]) + E[\frac{\partial g}{\partial x_0}(X(T), E[g_0(X(T)])] g_0'(X(T)), \tag{3.2}$$

the family

$$\Big\{\widetilde{f}(t, X^{u+y\beta}(t), E[f_0(X^{u+y\beta}(t)], u(t) + y\beta(t))\frac{d}{dy}X^{u+y\beta}(t) + \frac{\partial f}{\partial u}(t, X^{u+y\beta}(t), u(t) + y\beta(t))\beta(t)\Big\}_{y\in(-\delta,\delta)}$$

is $\lambda \times P$-uniformly integrable and the family

$$\Big\{\widetilde{g}(X^{u+y\beta}(T), E[g_0(X^{u+y\beta}(T)])\frac{d}{dy}X^{u+y\beta}(T)\Big\}_{y\in(-\delta,\delta)}$$

is P-uniformly integrable.

(3.4) For all $u, \beta \in \mathscr{A}_{\mathcal{E}}$ with β bounded the process $Y(t) = Y^{(\beta)}(t) = \frac{d}{dy}X^{(u+y\beta)}(t)|_{y=0}$ exists and satisfies the equation

$$\begin{aligned} dY(t) &= Y(t^-)\Big[\frac{\partial b}{\partial x}(t, X(t), u(t))dt + \frac{\partial \sigma}{\partial x}(t, X(t), u(t))dB(t) \\ &+ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t, X(t^-), u(t^-), z)\tilde{N}(dt, dz)\Big] \\ &+ \beta(t^-)\Big[\frac{\partial b}{\partial u}(t, X(t), u(t))dt + \frac{\partial \sigma}{\partial u}(t, X(t), u(t))dB(t) \\ &+ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(t, X(t^-), u(t^-), z)\tilde{N}(dt, dz)\Big]; \\ Y(0) &= 0. \end{aligned}$$

(3.5) For all $u \in \mathscr{A}_{\mathcal{E}}$, with definitions (3.1) and (3.2), the following processes

$$K(t) := \widetilde{g}(X(T), E[g_0(X(T)]) + \int_t^T \widetilde{f}(s, X(s), E[f_0(X(s)], u(s))ds,$$

$$D_tK(t) := D_t\widetilde{g}(X(T), E[g_0(X(T))]) + \int_t^T D_t\widetilde{f}(s, X(s), E[f_0(X(s)], u(s))ds,$$

$$D_{t,z}K(t) := D_{t,z}\widetilde{g}(X(T), E[g_0(X(T))]) + \int_t^T D_{t,z}\widetilde{f}(s, X(s), E[f_0(X(s)], u(s))ds,$$

$$H_0(s, x, u) := K(s)b(s, x, u) + D_sK(s)\sigma(s, x, u) + \int_{\mathbb{R}_0} D_{s,z}K(s)\theta(s, x, u, z)\nu(dz),$$

$$G(t, s) := \exp\Big(\int_t^s \Big\{ \frac{\partial b}{\partial x}(r, X(r), u(r), \omega) - \tfrac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2(r, X(r), u(r), \omega)\Big\}dr + \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), u(r), \omega)dB(r) + \int_t^s \int_{\mathbb{R}_0} \Big\{ \ln\Big(1 + \frac{\partial \theta}{\partial x}(r, X(r), u(r), z, \omega)\Big) - \frac{\partial \theta}{\partial x}(r, X(r), u(r), z, \omega)\Big\}\nu(dz)dr + \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \frac{\partial \theta}{\partial x}(r, X(r^-), u(r^-), z, \omega)\Big)\tilde{N}(dr, dz)\Big),$$

(3.6)
$$p(t) := K(t) + \int_t^T \frac{\partial H_0}{\partial x}(s, X(s), u(s))G(t, s)ds,$$

(3.7)
$$q(t) := D_tp(t)\,, \qquad \text{and}$$

(3.8)
$$r(t, z) := D_{t,z}p(t)$$

all exist for $0 \le t \le s \le T,\ \ z \in \mathbb{R}_0$.

We now define the *Hamiltonian* for this general problem:

Definition 3.2 (The general stochastic Hamiltonian) *The general stochastic Hamiltonian is the process*

$$H(t, x, u, \omega) : [0, T] \times \mathbb{R} \times U \times \Omega \to \mathbb{R}$$

defined by

$$H(t, x, u, \omega) = f(t, x, E[f_0(X(t)], u, \omega) + p(t)b(t, x, u, \omega) + q(t)\sigma(t, x, u, \omega) + \int_{\mathbb{R}_0} r(t, z)\theta(t, x, u, z, \omega)\nu(dz). \tag{3.9}$$

Remark 3.3 In the classical Markovian case and with $f_0 \equiv 0$ and $g_0 \equiv 0$, the Hamiltonian $H_1 : [0, T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathcal{R} \to \mathbb{R}$ is defined by

$$H_1(t, x, u, p, q, r) = f(t, x, u) + pb(t, x, u) + q\sigma(t, x, u) + \int_{\mathbb{R}_0} r(z)\theta(t, x, u, z)\nu(dz), \tag{3.10}$$

where $\mathcal{R}$ is the set of functions $r : \mathbb{R}_0 \to \mathbb{R}$; see [FØS]. Thus in this case the relation between H_1 and H is that:

$$H(t, x, u, \omega) = H_1(t, x, u, p(t), q(t), r(t, \cdot)) \tag{3.11}$$

where $p(t), q(t)$ and $r(t, z)$ are given by (3.6)–(3.8).

We can now formulate our stochastic maximum principle:

THEOREM 3.4 (MAXIMUM PRINCIPLE)
(i) *Suppose $\hat{u} \in \mathscr{A}_{\mathscr{E}}$ is a critical point for $J(u)$, in the sense that*

$$\frac{d}{dy} J(\hat{u} + y\beta)|_{y=0} = 0 \quad \text{for all bounded } \beta \in \mathscr{A}_{\mathscr{E}}. \tag{3.12}$$

Then

$$E[\frac{\partial \hat{H}}{\partial u}(t, \hat{X}(t), \hat{u}(t)) | \mathscr{E}_t] = 0 \text{ for a.a. } t, \omega, \tag{3.13}$$

where

$$\begin{aligned}
&\hat{X}(t) = X^{(\hat{u})}(t) \,, \\
&\hat{H}(t, \hat{X}(t), u) = f(t, \hat{X}(t), E[f_0(\hat{X}(t)], u) + \hat{p}(t) b(t, \hat{X}(t), u) + \hat{q}(t) \sigma(t, \hat{X}(t), u) \\
&\qquad + \int_{\mathbb{R}_0} \hat{r}(t, z) \theta(t, \hat{X}(t), u, z) \nu(dz) \,,
\end{aligned}$$

with

$$\begin{aligned}
&\hat{p}(t) = \hat{K}(t) + \int_t^T \frac{\partial H_0}{\partial x}(s, \hat{X}(s), \hat{u}(s)) \hat{G}(t, s) ds \,, \\
&\hat{q}(t) := D_t \hat{p}(t) \,, \qquad \text{and} \\
&\hat{r}(t, z) := D_{t,z} \hat{p}(t)
\end{aligned}$$

and

$$\begin{aligned}
\hat{G}(t, s) = \exp\Big(&\int_t^s \Big\{ \frac{\partial b}{\partial x}(r, \hat{X}(r), u(r), \omega) - \tfrac{1}{2} \Big(\frac{\partial \sigma}{\partial x} \Big)^2 (r, \hat{X}(r), u(r), \omega) \Big\} dr \\
&+ \int_t^s \frac{\partial \sigma}{\partial x}(r, \hat{X}(r), u(r), \omega) dB(r) \\
&+ \int_t^s \int_{\mathbb{R}_0} \Big\{ \ln \Big(1 + \frac{\partial \theta}{\partial x}(r, \hat{X}(r), u(r), z, \omega) \Big) - \frac{\partial \theta}{\partial x}(r, \hat{X}(r), u(r), z, \omega) \Big\} \nu(dz) dr
\end{aligned}$$

$$+\int_t^s \int_{\mathbb{R}_0} \ln\Big(1+\frac{\partial\theta}{\partial x}(r,\hat{X}(r^-),u(r^-),z,\omega)\Big)\tilde{N}(dr,dz)\Big)\,,$$

$$\hat{K}(t) = K^{(\hat{u})}(t) = \widetilde{g}(\hat{X}(T),E[f_0(\hat{X}(T)]) + \int_t^T \widetilde{f}(s,\hat{X}(s),E[f_0(\hat{X}(t)],\hat{u}(s))ds.$$

(ii) *Conversely, suppose there exists* $\hat{u} \in \mathscr{A}_{\mathscr{E}}$ *such that (3.13) holds. Then* $\hat{u}$ *satisfies (3.12).*

PROOF.

(i): Suppose $\hat{u} \in \mathscr{A}_{\mathscr{E}}$ is a critical point for $J(u)$. Choose an arbitrary $\beta \in \mathscr{A}_{\mathscr{E}}$ bounded and let $\delta > 0$ be as in (3.3) of Assumption 3.1.

For simplicity of notation we write $\hat{u} = u$, $\hat{X} = X$ and $\hat{Y} = Y$ in the following. By (3.3) and Fubini we have

(3.14)

$$\begin{aligned}
0 &= \tfrac{d}{dy}J(u+y\beta)|_{y=0} \\
&= E[\textstyle\int_0^T \Big\{\tfrac{\partial f}{\partial x}(t,X(t),E[f_0(X(t)],u(t))Y(t) + \tfrac{\partial f}{\partial x_0}(t,X(t),E[f_0(X(t)],u(t))E[f_0'(X(t))Y(t)] \\
&\qquad + \tfrac{\partial f}{\partial u}(t,X(t),E[f_0(X(t)],u(t))\beta(t)\Big\}\,dt \\
&\qquad + \tfrac{\partial g}{\partial x}(X(T),E[g_0(X(T)])Y(T) + \tfrac{\partial g}{\partial x_0}(X(T),E[g_0(X(T)])E[g_0'(X(T))Y(T)] \\
&= E[\textstyle\int_0^T \Big\{\tfrac{\partial f}{\partial x}(t,X(t),E[f_0(X(t)],u(t))Y(t) + E[\tfrac{\partial f}{\partial x_0}(t,X(t),E[f_0(X(t)],u(t))]f_0'(X(t))Y(t) \\
&\qquad + \tfrac{\partial f}{\partial u}(t,X(t),E[f_0(X(t)],u(t))\beta(t)\Big\}\,dt \\
&\qquad + \tfrac{\partial g}{\partial x}(X(T),E[g_0(X(T)])Y(T) + E[\tfrac{\partial g}{\partial x_0}(X(T),E[g_0(X(T)])]g_0'(X(T))Y(T)] \\
&= E[\textstyle\int_0^T \Big\{\widetilde{f}(t,X(t),E[f_0(X(t)],u(t))Y(t) + \tfrac{\partial f}{\partial u}(t,X(t),E[f_0(X(t)],u(t))\beta(t)\Big\}\,dt \\
&\qquad + \widetilde{g}(X(T),E[g_0(X(T)])Y(T)]\,,
\end{aligned}$$

where we recall the definition (3.1) and (3.2)

$$\begin{aligned}
\widetilde{f}(t,X(t),E[f_0(X(t)],u(t)) &:= \frac{\partial f}{\partial x}(t,X(t),E[f_0(X(t)],u(t)) \\
&\qquad + E[\frac{\partial f}{\partial x_0}(t,X(t),E[f_0(X(t)],u(t))]f_0'(X(t)) \\
\widetilde{g}(X(T),E[g_0(X(T)]) &:= \frac{\partial g}{\partial x}(X(T),E[g_0(X(T)]) \\
&\qquad + E[\frac{\partial g}{\partial x_0}(X(T),E[g_0(X(T)])]g_0'(X(T))
\end{aligned}$$

and

$$(3.15)\quad \begin{aligned} Y(t) &= Y^{(\beta)}(t) = \tfrac{d}{dy}X^{(u+y\beta)}(t)|_{y=0} \\ &= \int_0^t \{\tfrac{\partial b}{\partial x}(s,X(s),u(s))Y(s) + \tfrac{\partial b}{\partial u}(s,X(s),u(s))\beta(s)\}ds \\ &+ \int_0^t \{\tfrac{\partial \sigma}{\partial x}(s,X(s),u(s))Y(s) + \tfrac{\partial \sigma}{\partial u}(s,X(s),u(s))\beta(s)\}dB(s) \\ &+ \int_0^t \int_{\mathbb{R}_0} \{\tfrac{\partial \theta}{\partial x}(s,X(s),u(s),z)Y(s) + \tfrac{\partial \theta}{\partial u}(s,X(s),u(s),z)\beta(s)\}\tilde{N}(ds,dz). \end{aligned}$$

If we use the short hand notation

$$\widetilde{f}(t,X(t),E[f_0(X(t)],u(t)) = \widetilde{f}(t), \quad \frac{\partial f}{\partial u}f(t,X(t),E[f_0(X(t)],u(t)) = \frac{\partial f}{\partial u}(t)$$

and similarly for $\frac{\partial b}{\partial x}, \frac{\partial b}{\partial u}, \frac{\partial \sigma}{\partial x}, \frac{\partial \sigma}{\partial u}, \frac{\partial \theta}{\partial x}$, and $\frac{\partial \theta}{\partial u}$, we can write

$$(3.16)\quad \begin{aligned} dY(t) &= \{\tfrac{\partial b}{\partial x}(t)Y(t) + \tfrac{\partial b}{\partial u}(t)\beta(t)\}dt + \{\tfrac{\partial \sigma}{\partial x}(t)Y(t) + \tfrac{\partial \sigma}{\partial u}(t)\beta(t)\}dB(t) \\ &+ \int_{\mathbb{R}_0} \{\tfrac{\partial \theta}{\partial x}(t)Y(t) + \tfrac{\partial \theta}{\partial u}(t)\beta(t)\}\tilde{N}(dt,dz); \\ Y(0) &= 0. \end{aligned}$$

By the duality formulas (2.9) and (2.18), we get

(3.17)

$$\begin{aligned} &E[\widetilde{g}(X(T),E[g_0(X(T)])Y(T)] \\ &= E\Big[\widetilde{g}(X(T),E[g_0(X(T)])\Big(\int_0^T \{\tfrac{\partial b}{\partial x}(t)Y(t) + \tfrac{\partial b}{\partial u}(t)\beta(t)\}dt \\ &\quad + \int_0^T \{\tfrac{\partial \sigma}{\partial x}(t)Y(t) + \tfrac{\partial \sigma}{\partial u}(t)\beta(t)\}dB(t) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \{\tfrac{\partial \theta}{\partial x}(t)Y(t) + \tfrac{\partial \theta}{\partial u}(t)\beta(t)\}\tilde{N}(dt,dz)\Big)\Big] \\ &= E\Big[\int_0^T \{\widetilde{g}(X(T),E[g_0(X(T)])\tfrac{\partial b}{\partial x}(t)Y(t) + \widetilde{g}(X(T),E[g_0(X(T)])\tfrac{\partial b}{\partial u}(t)\beta(t) \\ &\quad + D_t(\widetilde{g}(X(T),E[g_0(X(T)]))\tfrac{\partial \sigma}{\partial x}(t)Y(t) + D_t(\widetilde{g}(X(T),E[g_0(X(T)]))\tfrac{\partial \sigma}{\partial u}(t)\beta(t) \\ &\quad + \int_{\mathbb{R}_0} [D_{t,z}(\widetilde{g}(X(T),E[g_0(X(T)]))\tfrac{\partial \theta}{\partial x}(t)Y(t) + D_{t,z}(\widetilde{g}(X(T),E[g_0(X(T)]))\tfrac{\partial \theta}{\partial u}(t)\beta(t)]\nu(dz)\}dt\Big]. \end{aligned}$$

Similarly we have, using the Fubini theorem,

$$\begin{aligned} &E[\int_0^T \widetilde{f}(t)Y(t)dt] \\ &\quad = E[\int_0^T \widetilde{f}(t)(\int_0^t \{\tfrac{\partial b}{\partial x}(s)Y(s) + \tfrac{\partial b}{\partial u}(s)\beta(s)\}ds \\ &\quad\quad + \int_0^t \{\tfrac{\partial \sigma}{\partial x}(s)Y(s) + \tfrac{\partial \sigma}{\partial u}(s)\beta(s)\}dB(s) \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}_0}\{\tfrac{\partial\theta}{\partial x}(s)Y(s)+\tfrac{\partial\theta}{\partial u}(s)\beta(s)\}\tilde{N}(ds,dz))dt]\\
&=E[\int_0^T(\int_0^t\{\widetilde{f}(t)[\tfrac{\partial b}{\partial x}(s)Y(s)+\tfrac{\partial b}{\partial u}(s)\beta(s)]\\
&+D_s(\widetilde{f}(t))[\tfrac{\partial\sigma}{\partial x}(s)Y(s)+\tfrac{\partial\sigma}{\partial u}(s)\beta(s)]\\
&+\int_{\mathbb{R}_0}D_{s,z}(\widetilde{f}(t))[\tfrac{\partial\theta}{\partial x}(s)Y(s)+\tfrac{\partial\theta}{\partial u}(s)\beta(s)]\nu(dz)\}ds)dt\\
&=E[\int_0^T\{(\int_s^T\widetilde{f}(t)dt)[\tfrac{\partial b}{\partial x}(s)Y(s)+\tfrac{\partial b}{\partial u}(s)\beta(s)]\\
&+(\int_s^T D_s\widetilde{f}(t)dt)[\tfrac{\partial\sigma}{\partial x}(s)Y(s)+\tfrac{\partial\sigma}{\partial u}(s)\beta(s)]\\
&+\int_{\mathbb{R}_0}(\int_s^T D_{s,z}\widetilde{f}(t)dt)[\tfrac{\partial\theta}{\partial x}(s)Y(s)+\tfrac{\partial\theta}{\partial u}(s)\beta(s)]\nu(dz)\}ds].
\end{aligned}
$$

Changing the notation $s\to t$, this becomes

$$
\begin{aligned}
E[\int_0^T\widetilde{f}(t)Y(t)dt]=\ &E[\int_0^T\{(\int_t^T\widetilde{f}(s)ds)[\tfrac{\partial b}{\partial x}(t)Y(t)+\tfrac{\partial b}{\partial u}(t)\beta(t)]\\
&+(\int_t^T D_t\widetilde{f}(s)ds)[\tfrac{\partial\sigma}{\partial x}(t)Y(t)+\tfrac{\partial\sigma}{\partial u}(t)\beta(t)]\\
&+\int_{\mathbb{R}_0}(\int_t^T D_{t,z}\widetilde{f}(s)ds)[\tfrac{\partial\theta}{\partial x}(t)Y(t)+\tfrac{\partial\theta}{\partial u}(t)\beta(t)]\nu(dz)\}dt].
\end{aligned}
\tag{3.18}
$$

Recall

$$
K(t):=\widetilde{g}(X(T),E[g_0(X(T))])+\int_t^T\widetilde{f}(s)ds. \tag{3.19}
$$

By combining (3.17)–(3.19), we get

$$
\begin{aligned}
&E[\int_0^T\{K(t)(\tfrac{\partial b}{\partial x}(t)Y(t)+\tfrac{\partial b}{\partial u}(t)\beta(t))\\
&+D_tK(t)(\tfrac{\partial\sigma}{\partial x}(t)Y(t)+\tfrac{\partial\sigma}{\partial u}(t)\beta(t))\\
&+\int_{\mathbb{R}_0}D_{t,z}K(t)(\tfrac{\partial\theta}{\partial x}(t)Y(t)+\tfrac{\partial\theta}{\partial u}(t)\beta(t))\nu(dz)+\tfrac{\partial f}{\partial u}(t)\beta(t)\}dt]=0.
\end{aligned}
\tag{3.20}
$$

Now apply the above to $\beta=\beta_\alpha\in\mathscr{A}_{\mathscr{E}}$ of the form $\beta_\alpha(s)=\alpha\chi_{[t,t+h]}(s)$, for some $t,h\in(0,T)$, $t+h\le T$, where $\alpha=\alpha(\omega)$ is bounded and $\mathscr{E}_t$-measurable. Then $Y^{(\beta_\alpha)}(s)=0$ for $0\le s\le t$ and hence (3.20) becomes

$$
A_1+A_2=0, \tag{3.21}
$$

where

$$
\begin{aligned}
A_1&=E\left[\int_t^T\{K(s)\frac{\partial b}{\partial x}(s)+D_sK(s)\frac{\partial\sigma}{\partial x}(s)+\int_{\mathbb{R}_0}D_{s,z}K(s)\frac{\partial\theta}{\partial x}(s)\nu(dz)\}Y^{(\beta_\alpha)}(s)ds\right],\\
A_2&=E\left[(\int_t^{t+h}\{K(s)\frac{\partial b}{\partial u}(s)+D_sK(s)\frac{\partial\sigma}{\partial u}(s)+\int_{\mathbb{R}_0}D_{s,z}K(s)\frac{\partial\theta}{\partial u}(s)\nu(dz)+\frac{\partial f}{\partial u}(s)\}ds)\alpha\right].
\end{aligned}
$$

Note that, by (3.16), with $Y(s) = Y^{(\beta_\alpha)}(s)$ and $s \geq t+h$, the process $Y(s)$ follows the following dynamics

$$dY(s) = Y(s^-)\Big\{\frac{\partial b}{\partial x}(s)ds + \frac{\partial \sigma}{\partial x}(s)dB(s) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(s)\tilde{N}(ds,dz)\Big\}, \tag{3.22}$$

for $s \geq t+h$ with initial condition $Y(t+h)$ in time $t+h$. This equation can be solved explicitly and we get

$$Y(s) = Y(t+h)G(t+h,s); \qquad s \geq t+h, \tag{3.23}$$

where, in general, for $s \geq t$,

$$\begin{aligned} G(t,s) = \exp\Big(& \int_t^s \Big\{\frac{\partial b}{\partial x}(r) - \tfrac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2(r)\Big\}dr + \int_t^s \frac{\partial \sigma}{\partial x}(r)dB(r) \\ & + \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \frac{\partial \theta}{\partial x}(r)\Big)\tilde{N}(dr,dz) \\ & + \int_t^s \int_{\mathbb{R}_0} \Big\{\ln\Big(1 + \frac{\partial \theta}{\partial x}(r)\Big) - \frac{\partial \theta}{\partial x}(r)\Big\}\nu(dz)dr\Big). \end{aligned}$$

That $Y(s)$ indeed is the solution of (3.22) can be verified by applying the Itô formula to $Y(s)$ given in (3.23). Note that $G(t,s)$ does not depend on h, but $Y(s)$ does.

Put

$$H_0(s,x,u) = K(s)b(s,x,u) + D_sK(s)\sigma(s,x,u) + \int_{\mathbb{R}_0} D_{s,z}K(s)\theta(s,x,u,z)\nu(dz), \tag{3.24}$$

and $H_0(s) = H_0^{(u)}(s) = H_0(s,X(s),u(s))$. Then

$$A_1 = E\Big[\int_t^T \frac{\partial H_0}{\partial x}(s)Y(s)ds\Big].$$

Differentiating with respect to h at $h=0$ we get

$$\frac{d}{dh}A_1\Big|_{h=0} = \frac{d}{dh}E\Big[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)Y(s)ds\Big]_{h=0} + \frac{d}{dh}E\Big[\int_{t+h}^T \frac{\partial H_0}{\partial x}(s)Y(s)ds\Big]_{h=0}. \tag{3.25}$$

Since $Y(t) = 0$ and since $\partial H_0/\partial x(s)$ is càdlàg we see that

$$\frac{d}{dh}E\Big[\int_t^{t+h} \frac{\partial H_0}{\partial x}(s)Y(s)ds\Big]_{h=0} = 0. \tag{3.26}$$

Therefore, using (3.23) and that $Y(t) = 0$,

$$\frac{d}{dh}A_1\Big|_{h=0} = \frac{d}{dh}E\Big[\int_{t+h}^T \frac{\partial H_0}{\partial x}(s)Y(t+h)G(t+h,s)ds\Big]_{h=0}$$

$$= \int_t^T \frac{d}{dh} E\Big[\frac{\partial H_0}{\partial x}(s) Y(t+h) G(t+h,s)\Big]_{h=0} ds$$

$$= \int_t^T \frac{d}{dh} E\Big[\frac{\partial H_0}{\partial x}(s) G(t,s) Y(t+h)\Big]_{h=0} ds. \tag{3.27}$$

By (3.16)

$$Y(t+h) = \alpha \int_t^{t+h} \Big\{\frac{\partial b}{\partial u}(r)dr + \frac{\partial \sigma}{\partial u}(r)dB(r) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(r)\tilde{N}(dr,dz)\Big\}$$
$$+ \int_t^{t+h} Y(r^-)\Big\{\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)dB(r) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(r)\tilde{N}(dr,dz)\Big\}. \tag{3.28}$$

Therefore, by (3.27) and (3.28),

$$\frac{d}{dh} A_1\big|_{h=0} = \Lambda_1 + \Lambda_2, \tag{3.29}$$

where

$$\Lambda_1 = \int_t^T \frac{d}{dh} E\Big[\frac{\partial H_0}{\partial x}(s) G(t,s) \alpha \int_t^{t+h} \Big\{\frac{\partial b}{\partial u}(r)dr + \frac{\partial \sigma}{\partial u}(r)dB(r)$$
$$+ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(r)\tilde{N}(dr,dz)\Big\}\Big]_{h=0} ds \tag{3.30}$$

and

$$\Lambda_2 = \int_t^T \frac{d}{dh} E\Big[\frac{\partial H_0}{\partial x}(s) G(t,s) \int_t^{t+h} Y(r^-)\Big\{\frac{\partial b}{\partial x}(r)dr + \frac{\partial \sigma}{\partial x}(r)dB(r)$$
$$+ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(r)\tilde{N}(dr,dz)\Big\}\Big]_{h=0} ds. \tag{3.31}$$

By the duality formulae (2.9), (2.18) we have

$$\Lambda_1 = \int_t^T \frac{d}{dh} E\Big[\alpha \int_t^{t+h} \Big\{\frac{\partial b}{\partial u}(r) F(t,s) + \frac{\partial \sigma}{\partial u}(r) D_r F(t,s)$$
$$+ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(r) D_{r,z} F(t,s)\nu(dz)\Big\}dr\Big]_{h=0} ds$$
$$= \int_t^T E\Big[\alpha\Big\{\frac{\partial b}{\partial u}(t) F(t,s) + \frac{\partial \sigma}{\partial u}(t) D_t F(t,s) + \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(t) D_{t,z} F(t,s)\nu(dz)\Big\}\Big] ds, \tag{3.32}$$

where we have put

$$F(t,s) = \frac{\partial H_0}{\partial x}(s) G(t,s). \tag{3.33}$$

Since $Y(t) = 0$ we see that

$$\Lambda_2 = 0. \tag{3.34}$$

We conclude that

$$\begin{aligned}\frac{d}{dh}A_1\big|_{h=0} &= \Lambda_1 \\ &= \int_t^T E\Big[\alpha\Big\{F(t,s)\frac{\partial b}{\partial u}(t) + D_tF(t,s)\frac{\partial \sigma}{\partial u}(t) + \int_{\mathbb{R}_0} D_{t,z}F(t,s)\frac{\partial \theta}{\partial u}(t)\nu(dz)\Big\}\Big]ds. \end{aligned} \tag{3.35}$$

Moreover, we see directly that

$$\frac{d}{dh}A_2\big|_{h=0} = E\Big[\alpha\Big\{K(t)\frac{\partial b}{\partial u}(t) + D_tK(t)\frac{\partial \sigma}{\partial u}(t) + \int_{\mathbb{R}_0} D_{t,z}K(t)\frac{\partial \theta}{\partial u}(t)\nu(dz) + \frac{\partial f}{\partial u}(t)\Big\}\Big].$$

Therefore, differentiating (3.21) with respect to h at $h=0$ gives the equation

$$\begin{aligned} E\Big[\alpha\Big\{\Big(K(t) + \int_t^T F(t,s)ds\Big)\frac{\partial b}{\partial u}(t) + D_t\Big(K(t) + \int_t^T F(t,s)ds\Big)\frac{\partial \sigma}{\partial u}(t) \\ + \int_{\mathbb{R}_0} D_{t,z}\Big(K(t) + \int_t^T F(t,s)ds\Big)\frac{\partial \theta}{\partial u}(t)\nu(dz + \frac{\partial f}{\partial u}(t))\Big\}\Big] = 0. \end{aligned} \tag{3.36}$$

We can reformulate this as follows: If we define, as in (3.6),

$$p(t) = K(t) + \int_t^T F(t,s)ds = K(t) + \int_t^T \frac{\partial H_0}{\partial x}(s)G(t,s)ds, \tag{3.37}$$

then (3.36) can be written

$$\begin{aligned} E\Big[\frac{\partial}{\partial u}\Big\{f(t,X(t),E[f_0(X(t)],u) + p(t)b(t,X(t),u) + D_tp(t)\sigma(t,X(t),u) \\ + \int_{\mathbb{R}_0} D_{t,z}p(t)\theta(t,X(t),u,z)\nu(dz)\Big\}_{u=u(t)}\alpha\Big] = 0. \end{aligned}$$

Since this holds for all bounded $\mathscr{E}_t$-measurable random variable α, we conclude that

$$E\Big[\frac{\partial}{\partial u}H(t,X(t),u)_{u=u(t)} \mid \mathscr{E}_t\Big] = 0,$$

which is (3.13). This completes the proof of (i).

(ii): Conversely, suppose (3.13) holds for some $\hat{u} \in \mathscr{A}_{\mathscr{E}}$. Then by reversing the above argument we get that (3.21) holds for all $\beta_\alpha \in \mathscr{A}_{\mathscr{E}}$ of the form

$$\beta_\alpha(s,\omega) = \alpha(\omega)\chi_{(t,t+h]}(s)$$

for some $t, h \in [0,T]$ with $t + h \leq T$ and some bounded $\mathscr{E}_t$-measurable α. Hence (3.21) holds for all linear combinations of such β_α. Since all bounded $\beta \in \mathscr{A}_{\mathscr{E}}$ can be approximated pointwise boundedly in (t,ω) by such linear combinations, it follows that (3.21) holds for all bounded $\beta \in \mathscr{A}_{\mathscr{E}}$. Hence, by reversing the remaining part of the argument above, we conclude that (3.12) holds.

□

Remark 3.5 It is natural to ask whether the explicitly constructed process $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ satisfies a BSDE, e.g., one that is similar to the adjoint equation in [AD]. In the classical case when there is no mean-field term, the adjoint equation is a linear BSDE and the answer to the above question is positive (up to some conditional expectation version of $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$). In the present case, however, the adjoint BSDE is generally nonlinear (see [AD]); hence it remains an open theoretical problem to connect the adjoint process constructed here to a BSDE. We stress that the purpose of this paper is to derive a maximum principle that is computable.

Remark 3.6 In this paper the mean-field term appears only in the performance functional. However, our results can be extended to the case where mean field terms also appear in the system dynamics without essential difficulty (other than notational complexity). Since the current paper is already notationally very heavy, we have chosen not to include this generalization.

4 Applications

In this section we illustrate the maximum principle by looking at some examples.

Example 4.1 (Optimal dividend/harvesting rate)
Suppose the cash flow $X(t) = X^{(c)}(t)$ at time t is given by

$$
\begin{aligned}
dX(t) &= (b_0(t,\omega) + b_1(t,\omega)X(t) - c(t))dt + (\sigma_0(t,\omega) + \sigma_1(t,\omega)X(t))dB(t) \\
&\quad + \textstyle\int_{\mathbf{R}_0} (\theta_0(t,z,\omega) + \theta_1(t,z,\omega)X(t))\tilde{N}(dt,dz)\,; \qquad (4.1)\\
X(0) &= x \in \mathbf{R},
\end{aligned}
$$

where

$$
\begin{aligned}
&b_0(t) = b_0(t,\omega),\ b_1(t) = b_1(t,\omega) : && [0,T] \times \Omega \mapsto \mathbf{R} \\
&\sigma_0(t) = \sigma_0(t,\omega),\ \sigma_1(t) = \sigma_1(t,\omega) : && [0,T] \times \Omega \mapsto \mathbf{R} \ \text{ and} \\
&\theta_0(t,z) = \theta_0(t,z,\omega),\ \theta_1(t,z) = \theta_1(t,z,\omega) : && [0,T] \times \mathbf{R} \times \Omega \mapsto \mathbf{R}
\end{aligned}
$$

are given $\mathcal{F}_t$-predictable processes.

Here $c(t) \geq 0$ is our control (the dividend/harvesting rate), assumed to belong to a family $\mathcal{A}_{\mathscr{E}}$ of admissible controls, contained in the set of $\mathscr{E}_t$-predictable controls.

Suppose the performance functional has the form

$$J(c) = E\left[\int_0^T \xi(s)U(c(s))ds + \zeta X^{(c)}(T)\right] \tag{4.2}$$

where $U : [0, +\infty] \mapsto \mathbf{R}$ is a C^1 utility function, $\xi(s) = \xi(s, \omega)$ is an $\mathcal{F}_t$-predictable process and $\zeta = \zeta(\omega)$ is an $\mathcal{F}_T$-measurable random variable.

We want to find $\hat{c} \in \mathcal{A}_{\mathscr{E}}$ such that

$$\sup_{c \in \mathcal{A}_{\mathscr{E}}} J(c) = J(\hat{c}). \tag{4.3}$$

Using the notation from the previous section, we note that in this case we have, with $c = u$,

$$f(t, x, x_0, c) = f(t, x, c) = \xi(t)U(c) \quad \text{and} \quad g(x) = \zeta x.$$

Hence

$$\begin{aligned}
K(t) &= \int_t^T \frac{\partial f}{\partial x}(s, X(s), c(s))ds + g'(X(T)) = \zeta\,, \\
H_0(t, x, c) &= \zeta(b_0(t) + b_1(t)x - c) + D_t\zeta(\sigma_0(t) + \sigma_1(t)x) + \int_{\mathbb{R}_0} D_{t,z}\zeta(\theta_0(t, z) + \theta_1(t, z)x)\nu(dz)\,, \\
G(t, s) &= \exp\Big(\int_t^s \Big\{b_1(r) - \tfrac{1}{2}\sigma_1^2(r)\Big\}\,dr + \int_t^s \sigma_1(r)\,dB(r) \\
&\quad + \int_t^s \int_{\mathbb{R}_0} \Big\{\ln\Big(1 + \theta_1(r, z)\Big) - \theta_1(r, z)\Big\}\nu(dz)dr + \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \theta_1(r, z)\Big)\tilde{N}(dr, dz)\Big)\,.
\end{aligned}$$

Then

$$p(t) = \zeta + \int_t^T \Big(\zeta b_1(r) + D_r\zeta\,\sigma_1(r) + \int_{\mathbb{R}_0} D_{r,z}\zeta\,\theta_1(r, z)\nu(dz)\Big)\,G(t, r)\,dr\,, \tag{4.4}$$

and the Hamiltonian becomes

$$\begin{aligned}
H(t, x, c) &= \xi(t)U(c) + p(t)(b_0(t) + b_1(t)x - c) + D_tp(t)\,(\sigma_0(t) + \sigma_1(t)x) \\
&\quad + \int_{\mathbb{R}_0} D_{t,z}p(t)\,(\theta_0(t, z) + \theta_1(t, z)x)\nu(dz).
\end{aligned} \tag{4.5}$$

Hence, if $\hat{c} \in \mathcal{A}_{\mathscr{E}}$ is optimal for the problem (4.3), we have

$$\begin{aligned}
0 &= E\left[\frac{\partial}{\partial c}H(t, \hat{X}(t), \hat{c}(t))|\mathscr{E}_t\right] \\
&= E\left[\{\xi(t)U'(\hat{c}(t)) - p(t)\}|\mathscr{E}_t\right] \\
&= U'(\hat{c}(t))E\left[\xi(t)|\mathscr{E}_t\right] - E[p(t)|\mathscr{E}_t].
\end{aligned}$$

We have proved:

THEOREM 4.1 *If there exists an optimal dividend/harvesting rate $\hat{c}(t) > 0$ for problem (4.3), then it satisfies the equation*

$$U'(\hat{c}(t))E[\xi(t)|\mathscr{E}_t] = E[p(t)|\mathscr{E}_t]\,, \tag{4.6}$$

where $p(t)$ is given by (4.4).

Example 4.2 (Optimal portfolio)

Suppose we have a financial market with the following two investment possibilities:

(i) A risk free asset, where the unit price $S_0(t)$ at time t is given by

$$dS_0(t) = \rho_t S_0(t)dt; \qquad S_0(0) = 1; \quad t \in [0,T]\,. \tag{4.7}$$

(ii) A risky asset, where the unit price $S_1(t)$ at time t is given by

$$\begin{aligned} &dS_1(t) = S_1(t^-)\Big[\alpha_t dt + \beta_t dB(t) + \textstyle\int_{\mathbb{R}_0} \zeta(t,z)\tilde{N}(dt,dz)\Big]; \qquad t \in [0,T] \\ &S_1(0) > 0. \end{aligned} \tag{4.8}$$

Here $\rho_t, \alpha_t, \beta_t$ and $\zeta(t,z)$ are bounded $\mathcal{F}_t$-predictable processes; $t \in [0,T]$, $z \in \mathbb{R}_0$ and $T > 0$ is a given constant. We also assume that

$$\zeta(t,z) \geq -1 \qquad \text{a.s. for a.a. } t, z$$

and

$$E\Big[\int_0^T \int_{\mathbb{R}_0} |\log(1+\zeta(t,z))|^2 \nu(dz)dt\Big] < \infty\,.$$

A *portfolio* in this market is an $\mathcal{E}_t$-predictable process $u(t)$ representing the amount invested in the risky asset at time t. When the portfolio $u(\cdot)$ is chosen, the corresponding wealth process $X(t) = X^{(u)}(t)$ satisfies the equation

$$\begin{aligned} dX(t) = {} & [\rho_t X(t) + (\alpha_t - \rho_t)u(t)]dt + \beta_t u(t)dB(t) \\ & + \int_{\mathbb{R}_0} \zeta(t,z)u(t^-)\tilde{N}(dt,dz)\,, \end{aligned} \tag{4.9}$$

with given initial wealth denoted by $X(0) = X_0 > 0$. *The general partial information optimal portfolio problem* is to find the portfolio $u \in \mathcal{A}_{\mathcal{E}}$ which maximizes

$$J(u) = E[g(X^{(u)}(T), E[g_0(X^{(u)}(T))], \omega)] \tag{4.10}$$

where $g(x, x_0) = g(x, x_0, \omega) : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $g_0(x) : \mathbb{R} \to \mathbb{R}$ fulfill the conditions specified in the previous section. The set $\mathcal{A}_{\mathcal{E}}$ of admissible portfolios is contained in the set of $\mathcal{E}_t$-adapted portfolios $u(t)$ such that (4.10) has a unique strong solution.

With the notation of the previous section we see that in this case we have

$$\begin{aligned} &f(t, x, u) = f(t, x, u, \omega) = 0, \\ &b(t, x, u) = \rho_t x + (\alpha_t - \rho_t)u, \\ &\sigma(t, x, u) = \beta_t u, \\ &\theta(t, x, u, z) = \zeta(t, z)u. \end{aligned}$$

Thus

$$K(t) = \widetilde{g}(X(T), E[g_0(X(T))]) = K\,, \tag{4.11}$$

and

$$\begin{aligned} H_0(t, x, u) = K(\rho_t x + (\alpha_t - \rho_t)u) + D_t K \beta_t u \\ + \int_{\mathbb{R}_0} D_{t,z} K \zeta(t, z) u \tilde{N}(dt, dz)\,, \end{aligned}$$

and

$$G(t, s) = \exp\left(\int_t^s \rho_r dr \right). \tag{4.12}$$

Thus

$$p(t) = K + \int_t^T K \rho_s \exp\left(\int_t^s \rho_r dr \right) ds\,, \tag{4.13}$$

and the Hamiltonian becomes

$$\begin{aligned} H(t, x, u) = p(t)[\rho_t x + (\alpha_t - \rho_t)u] + D_t p(t) \beta_t u \\ + \int_{\mathbb{R}_0} D_{t,z} p(t) \zeta(t, z) u \, \nu(dz). \end{aligned} \tag{4.14}$$

By the maximum principle, we now get the following condition for an optimal control.

THEOREM 4.2 *If $\hat{u}(t)$ is an optimal control with corresponding $\hat{X}(t), \hat{p}(t)$ then*

$$E\Big[\frac{d}{du} H(t, \hat{X}(t), u)_{u=\hat{u}(t)} \mid \mathcal{E}_t\Big] = 0$$

i.e.

$$E\Big[\hat{p}(t)(\alpha_t - \rho_t) + \beta_t D_t \hat{p}(t) + \int_{\mathbb{R}_0} D_{t,z} \hat{p}(t) \zeta(t, z) \, \nu(dz) \mid \mathcal{E}_t\Big] = 0. \tag{4.15}$$

Equation (4.15) is an interesting new type of equation. We could call it a *Malliavin differential type equation* in the unknown process $\hat{p}(t)$. If ρ_t is deterministic, then we see that (4.15) becomes an equation in the unknown random variable

$$\hat{K} = \widetilde{g}(\hat{X}(T), E[g_0(\hat{X}(T))])\,. \tag{4.16}$$

In the case when $\mathcal{E}_t = \mathcal{F}_t$ for all $t \in [0,T]$ this type of equation has been solved in ([ØS2], Appendix A). In this paper we will complete the example by finding a solution in the special case when

$$\nu = \rho_t = 0, \quad |\beta_t| \geq \delta > 0 \quad \text{and} \quad \mathcal{E}_t = \mathcal{F}_t;\ \ 0 \leq t \leq T, \tag{4.17}$$

where $\delta > 0$ is a given constant. In this case an optimal portfolio fulfills the SDE

$$\begin{cases} d\hat{X}(t) = \alpha_t \hat{u}(t)dt + \beta_t \hat{u}(t)dB(t); \quad t < T \\ \hat{X}(0) = X_0 \end{cases} \tag{4.18}$$

This equation can be written

$$\begin{cases} d\hat{X}(t) = \beta_t \hat{u}(t)d\tilde{B}(t); \quad t < T \\ \hat{X}(0) = X_0, \end{cases} \tag{4.19}$$

where

$$d\tilde{B}(t) = \frac{\alpha_t}{\beta_t}dt + dB(t), \tag{4.20}$$

which is a Brownian motion with respect to the probability measure Q defined by

$$dQ = N_T dP \quad \text{on } \mathcal{F}_T\,, \tag{4.21}$$

where

$$N_t = \exp\Big(-\int_0^t \frac{\alpha_s}{\beta_s}dB(s) - \tfrac{1}{2}\int_0^t \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big).$$

By the Clark-Ocone theorem under change of measure [KO] we have

$$\hat{X}(T) = E_Q[\hat{X}(T)] + \int_0^T E_Q\Big[\Big(D_t\hat{X}(T) - \hat{X}(T)\int_t^T D_t\Big(\frac{\alpha_s}{\beta_s}\Big)d\tilde{B}(s)\Big) \mid \mathcal{F}_t\Big]d\tilde{B}(t). \tag{4.22}$$

Comparing (4.19) and (4.22) we can express the optimal portfolio in terms of the optimal final wealth $\hat{X}(T)$:

$$\hat{u}(t) = \frac{1}{\beta_t}E_Q\Big[\Big(D_t\hat{X}(T) - \hat{X}(T)\int_t^T D_t\Big(\frac{\alpha_s}{\beta_s}\Big)d\tilde{B}(s)\Big) \mid \mathcal{F}_t\Big]. \tag{4.23}$$

In order to determine $\hat{X}(T)$ we solve the Malliavin differential equation (4.15) which in our situation simplifies to

$$\alpha_t E[K|\mathcal{F}_t] + \beta_t E[D_t K|\mathcal{F}_t] = 0\,. \tag{4.24}$$

By the Clark-Ocone theorem we have

$$K = E[K] + \int_0^T E[D_t K|\mathcal{F}_t] dB(t),$$

which implies that

$$E[K|\mathcal{F}_t] = E[K] + \int_0^t E[D_s K|\mathcal{F}_s] dB(s). \tag{4.25}$$

Define

$$M_t := E[K|\mathcal{F}_t] = E[\widetilde{g}(X(T), E[g_0(X(T))])|\mathcal{F}_t]. \tag{4.26}$$

Then by substituting (4.24) in (4.25) we get

$$M_t = E[K] - \int_0^t \frac{\alpha_s}{\beta_s} M_s dB(s)$$

or

$$dM_t = -\frac{\alpha_t}{\beta_t} M_t dB_t$$

which has the solution

$$M_t = E[K] \exp\Big(- \int_0^t \frac{\alpha_s}{\beta_s} dB(s) - \tfrac{1}{2} \int_0^t \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big). \tag{4.27}$$

This determines $\widetilde{g}(\hat{X}(T), E[g_0(\hat{X}(T))]) = M_T = K$ modulo the constant $E[\widetilde{g}(\hat{X}(T), E[g_0(\hat{X}(T))])] = M_0 = E[K]$.

We conclude by considering two choices for the function $g(x, x_0, \omega)$: First a classical utility maximazation problem without mean-field term but with possibly random utility function, and then a mean-variance optimization problem where a mean-field term enters.

Utility maximization problem: In the case of maximizing expected utility from terminal wealth we have

$$g(x, x_0, \omega) = U(x, \omega),$$

where $\omega \to U(x, \omega)$is a given $\mathcal{F}_t$-measurable random variable for each x and $x \to U(x, \omega)$ is a utility function for each ω. We assume that $x \to U(x)$ is C^1 and $U'(x)$ is strictly decreasing.

Since in this situation $\widetilde{g}(\hat{X}(T), E[g_0(\hat{X}(T))]) = U'(\hat{X}(T))$, we need to determine $M_0 = E[U'(\hat{X}(T))]$ in (4.27) in order to identify the optimal wealth candidate

$$\hat{X}(T) = (U')^{-1}(M_T) = (U')^{-1}\Big(M_0 \exp\Big(-\int_0^T \frac{\alpha_s}{\beta_s} dB(s) - \tfrac{1}{2}\int_0^T \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big)\Big) \,.$$

However, this is obtained by solving the budget constraint

$$E_Q[\hat{X}(T)] = X_0 \,.$$

i.e.

$$E_Q\Big[(U')^{-1}\Big(M_0 \exp\Big(-\int_0^T \frac{\alpha_s}{\beta_s} dB(s) - \tfrac{1}{2}\int_0^T \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big)\Big)\Big] = X_0 \,. \tag{4.28}$$

Using Bayes' rule we conclude with

THEOREM 4.3 *Suppose $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is an optimal portfolio for the problem*

$$\sup_{u \in \mathcal{A}_{\mathcal{F}}} E[U(X^{(u)}(T), \omega)] \tag{4.29}$$

with

$$dX^{(u)}(t) = \alpha_t u(t)dt + \beta_t u(t) dB(t).$$

Then

$$\hat{u}(t) = \frac{1}{\beta_t N_t} E\Big[N_T\Big(D_t\hat{X}(T) - \hat{X}(T)\int_t^T D_t\Big(\frac{\alpha_s}{\beta_s}\Big) d\tilde{B}(s)\Big) \mid \mathcal{F}_t\Big] \tag{4.30}$$

and

$$\hat{X}(T) = (U')^{-1}(M_T)\,, \tag{4.31}$$

where M_t is given by (4.27) *with M_0 implied by* (4.28).

COROLLARY 4.4 *Suppose*

$$U(x) = \frac{1}{\gamma} x^\gamma F(\omega)$$

for some $\mathcal{F}_T$-measurable bounded F. Then

$$\hat{u}(t) = \hat{X}(t)\frac{1}{\beta_t}\frac{E[N_T(D_tY - Y\int_t^T D_t(\frac{\alpha_s}{\beta_s})d\tilde{B}(s)) \mid \mathcal{F}_t]}{E[N_T Y \mid \mathcal{F}_t]} \tag{4.32}$$

where

$$Y = \Big[\frac{1}{F}\exp\Big(-\int_0^T \frac{\alpha_s}{\beta_s} dB(s) - \tfrac{1}{2}\int_0^T \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big)\Big]^{\frac{1}{\gamma-1}}. \tag{4.33}$$

PROOF. In this case we get

$$\hat{X}(T) = \Big(\frac{M_T}{F}\Big)^{\frac{1}{\gamma-1}} = M_0^{\frac{1}{\gamma-1}} Y$$

and

$$\hat{X}(t) = E_Q[\hat{X}(T) \mid \mathcal{F}_t] = \frac{M_0^{\frac{1}{\gamma-1}} E[N_T Y \mid \mathcal{F}_t]}{N_t}.$$

Therefore the result follows from (4.30).

□

Mean-variance portfolio selection: We finally consider the mean-variance portfolio selection problem as presented in [AD] (and the references therein). In this situation $X(t)$ is most often interpreted as return and not wealth process. In contrast to the original mean-variance problem, which is solved under the condition of a given expected return, we consider here the problem of minimizing the variance while maximizing the expected return. This introduces a mean-field term and the Bellman optimality principle does not hold any more due to time inconsistency. However, in our framework the solution follows immediately from the considerations above. For alternative solution methods of this problem see [AD] and the references therein.

As above let

$$dX(t) = dX^{(u)}(t) = \alpha_t u(t)dt + \beta_t u(t)dB(t).$$

The problem is thus to find a portfolio $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ such that

$$J(u) = E[X(T)] - \frac{\gamma}{2} Var[X(T)] \tag{4.34}$$

is maximized. This can be rewritten as

$$J(u) = E\Big[X(T) - \frac{\gamma}{2} X^2(T) + \frac{\gamma}{2}(E[X(T)])^2\Big]\,, \tag{4.35}$$

and thus, in this situation, $g(x, x_0) = x - \frac{\gamma}{2}x^2 + \frac{\gamma}{2}(x_0)^2$ and $g_0(x) = x$ in (4.10). From this we imply, by (3.1) and (4.26),

$$\widetilde{g}(\hat{X}(T), E[g_0(\hat{X}(T))]) = M_T = 1 - \gamma\hat{X}(T) + \gamma E[\hat{X}(T)]\,,$$

and because $M_0 = E[M_T] = 1$ we get from (4.27)

$$\hat{X}(T) = \frac{1}{\gamma} + E[\hat{X}(T)] - \exp\Big(- \int_0^T \frac{\alpha_s}{\beta_s} dB(s) - \tfrac{1}{2}\int_0^T \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big)\,. \tag{4.36}$$

Then, by the budget constraint $E_Q[\hat{X}(T)] = X_0$, we obtain

$$(4.37) \qquad E[\hat{X}(T)] = E_Q\Big[\exp\Big(-\int_0^T \frac{\alpha_s}{\beta_s} dB(s) - \tfrac{1}{2}\int_0^T \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big)\Big] - \frac{1}{\gamma}$$
$$= E\Big[\exp\Big(-2\int_0^T \frac{\alpha_s}{\beta_s} dB(s) - \int_0^T \Big(\frac{\alpha_s}{\beta_s}\Big)^2 ds\Big)\Big] - \frac{1}{\gamma}.$$

In sum, we get

Theorem 4.5 *Suppose $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is an optimal portfolio for the problem*

$$(4.38) \qquad \sup_{u \in \mathcal{A}_{\mathcal{F}}} \Big\{ E[X^{(u)}(T)] - \frac{\gamma}{2} Var[X^{(u)}(T)] \Big\}$$

Then, by (4.30),

$$(4.39) \qquad \hat{u}(t) = \frac{1}{\beta_t N_t} E\Big[N_T\Big(D_t\hat{X}(T) - \hat{X}(T)\int_t^T D_t\Big(\frac{\alpha_s}{\beta_s}\Big) d\tilde{B}(s)\Big) \mid \mathcal{F}_t\Big],$$

and $\hat{X}(T)$ is given by (4.36) *and* (4.37).

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