

Existence of Solutions to a Class of Indefinite Stochastic Riccati Equations

ZHONGMIN QIAN* AND XUN YU ZHOU[†]

Abstract. An indefinite stochastic Riccati Equation is a matrix-valued, highly non-linear backward stochastic differential equation together with an algebraic, matrix positive definiteness constraint. We introduce a new approach to solve a class of such equations (including the existence of solutions) driven by one-dimensional Brownian motion. The idea is to replace the original equation by a system of BSDEs (without involving any algebraic constraint) whose existence of solutions automatically enforces the original algebraic constraint to be satisfied.

Key words. Stochastic Riccati equation, backward stochastic differential equation, stochastic differential equation

AMS Classification. 60H10, 60H30, 60J45

*Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, UK.

[†]Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong, and Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, UK. X. Y. Zhou acknowledges the support from GRF grant #CUHK419511, a start-up fund of Oxford, as well as from the Nomura Centre and OMI.

1 Introduction

Stochastic matrix Riccati equations were first introduced by Bismut [1] in his study of some stochastic control problems. A very special class of these equations is the so-called quadratic backward stochastic differential equation (BSDE). The existence and uniqueness of solutions for such BSDEs remain a largely open problem, particularly for BSDE systems; but see [6], [5], [2] and the references therein for recent progress. The *indefinite* stochastic Riccati equations (SRE) were first formulated in [3], motivated by the introduction of the indefinite stochastic linear-quadratic control problems. Such an equation is typically matrix-valued, highly nonlinear, and involves a matrix positive definite constraint in addition to the backward equation. In [4], the uniqueness of solutions to the SRE was established in the greatest generality based on a control argument, but the existence was solved only for several very special cases. The general existence remains to this date a significant open problem.

The SRE is a BSDE over a running time interval $[0, T]$:

$$\begin{aligned}
 dP = & \sum_{j=1}^k \Lambda_j dW^j - \left[PA + A'P + \sum_{j=1}^k (\Lambda_j C_j + C_j' \Lambda_j + C_j' P C_j) + Q \right] dt \\
 & + \left[PB + \sum_{j=1}^k (C_j' P + \Lambda_j) D_j \right] K^{-1} \left[B' P + \sum_{j=1}^k D_j' (P C_j + \Lambda_j) \right] dt \quad (1.1)
 \end{aligned}$$

subject to the constraint that

$$K = R + \sum_{j=1}^k D_j' P D_j > 0 \quad (1.2)$$

in the matrix sense, and subject to the terminal condition that $P(T) = H$ which is \mathcal{F}_T -measurable. In this formulation, the time parameter t is omitted for simplicity, the capital letters $A, B, C, D, \Lambda, Q, R$ and P are real matrix valued (adapted) processes and D' means the transpose of D etc. All the matrix processes are square with the same dimension n , P and Λ_j , $j = 1, 2, \dots, k$, are unknowns and all the other parameters are given, and $W = (W^1, \dots, W^k)$ is k -dimensional standard Brownian motion.

The given matrix R in defining K is called the gauge matrix, which is an adapted process. The SRE (1.1)–(1.2) is indefinite, if the gauge matrix R is allowed to be indefinite, i.e., R can have zero or negative eigenvalues.

The problem is to look for square integrable adapted processes P and (Λ_j) satisfying the corresponding stochastic integral equations as well as the constraint (1.2). Moreover, in view of the proved uniqueness of solutions, a solution matrix P must be symmetric as long as the parameters Q , R and H are symmetric.

The existence for the general SRE (1.1)–(1.2) in high dimensions will remain to be an open question for some time. The main challenge, apart from the highly nonlinear nature of the BSDE (1.1) and the fact that the equation is matrix-valued, stems from the presence of an additional algebraic constraint (1.2). In this note, we develop a new approach to solve a class of SREs driven by one-dimensional Brownian motion. The main idea is to consider a system of BSDEs that the pair (K, K^{-1}) satisfies, without any algebraic constraint. It turns out that the existence of solutions to the equation satisfied by K^{-1} can be established independently, which in turn will ensure the validity of the original constraint $K > 0$.

2 The main result

In this paper we consider the SRE driven by a one-dimensional Brownian motion, with the matrix process D being invertible.

Therefore, in the remainder of this note, W is a *one-dimensional* standard Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $(\mathcal{F}_t)_{t \geq 0}$ is the Brownian filtration generated by W . Without lose of generality, we may assume that $D = I$, and we study the following BSDE

$$\begin{aligned} dP &= \Lambda dW - [PA + A'P + C'PC + \Lambda C + C'\Lambda + Q] dt, \\ &\quad + [PB + C'P + \Lambda] (R + P)^{-1} [B'P + PC + \Lambda] dt, \quad t \in [0, T], \\ P(T) &= H, \end{aligned} \tag{2.1}$$

where $T > 0$ is fixed throughout the paper and the random symmetric matrix H is bounded and \mathcal{F}_T -measurable, subject to the constraint that

$$K \equiv R + P > 0. \quad (2.2)$$

For simplicity we assume that the coefficients A, B, C, Q are bounded, square $n \times n$ matrix valued adapted processes, and in addition Q is symmetric.

We are interested in the indefinite gauge case, that is, R may have zero or negative eigenvalues, not necessarily being positive definite. The most interesting case in applications is when R is a matrix-valued Itô process; so we assume that it has the representation

$$R(t) = R(0) + \int_0^t F ds + \int_0^t G dW \quad (2.3)$$

where F, G are bounded, adapted and measurable symmetric matrix-valued processes.

Definition 2.1 *By a solution (P, Λ) to (2.1) we mean a pair of (\mathcal{F}_t) -adapted, measurable, square integrable and matrix-valued processes $P = (P(t))_{t \in [0, T]}$ and $\Lambda = (\Lambda(t))_{t \in [0, T]}$ such that $K(t) = R(t) + P(t) > 0$ in the matrix sense for all $t \in [0, T]$, a.s., and*

$$\begin{aligned} P(t) &= H - \int_t^T \Lambda dW \\ &\quad + \int_t^T [PA + A'P + C'PC + \Lambda C + C'\Lambda + Q] ds \\ &\quad - \int_t^T [PB + C'P + \Lambda] K^{-1} [B'P + PC + \Lambda] ds \end{aligned} \quad (2.4)$$

for $t \in [0, T]$, a.s., where the stochastic integral is understood in the Itô sense. A solution (P, Λ) is called bounded if P is bounded.

Clearly, if (P, Λ) is a solution, then P must be a continuous matrix-valued semimartingale.

Due to the presence of K^{-1} in (2.1), it is natural to rewrite (2.1) in terms of $K = R + P$ and $\tilde{\Lambda} = \Lambda + G$. This can be achieved by making substitutions in (2.1): P by $K - R$ and Λ by $\tilde{\Lambda} - G$, leading to the following BSDE

$$\begin{aligned} dK &= \tilde{\Lambda}dW - \left[K\tilde{A} + \tilde{A}'K + \tilde{Q} \right] dt + \left[KB + \tilde{\Lambda} - \tilde{R} \right] K^{-1} \left[B'K + \tilde{\Lambda} - \tilde{R}' \right] dt, \\ K(T) &= R(T) + H, \end{aligned} \quad (2.5)$$

where

$$\tilde{Q} = Q - F + C'RC + R(BC - A) + (C'B' - A')R \quad (2.6)$$

and

$$\tilde{A} = A - BC, \quad \tilde{R} = RB + C'R + G. \quad (2.7)$$

We are now in a position to state our main result.

Theorem 2.2 *Assume*

(i)

$$\tilde{R} = RB + C'R + G = 0 \quad (2.8)$$

and

$$\tilde{Q} = Q - F + C'RC + R(BC - A) + (C'B' - A')R \geq 0, \quad (2.9)$$

(ii) G and F are bounded adapted measurable processes such that $R(T) + H > 0$, and there is a constant $\delta > 0$ such that $(R(T) + H)^{-1} \geq \delta I$.

Then there is a unique solution (P, Λ) to the SRE (2.1)–(2.2). Moreover, $P + R$ is bounded.

The uniqueness has been established in [4], Theorem 3.2. The existence, on the other hand, is known in the so-called definite case, namely, when $R > 0$, $Q \geq 0$, $H \geq 0$; see [7] (where there is an additional assumption that $D = 0$). The existence when all these matrices are allowed to

be indefinite is investigated in [4] for several special cases. It should be noted that the existence of (2.1) is by no means unconditional; the problem is to find *sufficient conditions* under which the existence holds. One of the conditions, (2.8), of Theorem 2.2 stipulates that R satisfies the following Itô equation

$$dR = -(RB + C'R)dW + Fdt \quad (2.10)$$

where F can be arbitrary (up to the required Lebesgue integrability). The other condition, (2.9), requires an “overall” positive semidefiniteness in place of that of individual matrices.

We will make further comments on the conditions of the preceding theorem in Section 5.

3 A linear equation

We need a result about the representation for a linear matrix-valued BSDE. Consider

$$\begin{aligned} dY &= \sum_{j=1}^k U_j dW^j - \left[Y\hat{A} + \hat{A}'Y + \sum_{i=1}^m \hat{E}'_i Y \hat{E}_i + \sum_{j=1}^k (U_j \hat{C}_j + \hat{C}'_j U_j + \hat{C}'_j Y \hat{C}_j) + \hat{Q} \right] dt, \\ Y_T &= \hat{H}, \end{aligned} \quad (3.1)$$

where \hat{A} , \hat{C}_j , \hat{E}_i ($j = 1, \dots, k$, $i = 1, \dots, m$) and \hat{Q} are $n \times n$ matrix valued, adapted and bounded, \hat{H} is bounded and \mathcal{F}_T -measurable, and W is a standard Brownian motion of dimension k . The BSDE is linear so there is a unique solution. Choose a standard Brownian motion \hat{W} of dimension m , which is independent of W . For any $p \in \mathbb{R}^n$ and $0 \leq t < T$, let ξ be the solution to the linear stochastic differential equation

$$d\xi = \hat{A}\xi ds + \sum_{j=1}^k \hat{C}_j \xi dW^j + \sum_{i=1}^m \hat{E}_i \xi d\hat{W}^i, \quad \xi_t = p. \quad (3.2)$$

Lemma 3.1 *Under the above assumptions and notations, we have*

$$p'Y(t)p \equiv \xi_t'Y(t)\xi_t = \mathbb{E} \left[\xi_T' \hat{H} \xi_T + \int_t^T \xi_s' \hat{Q} \xi_s ds \middle| \mathcal{F}_t \right]. \quad (3.3)$$

Proof. Applying Itô's formula to

$$J(s) = \xi_s'Y(s)\xi_s = (Y(s)\xi_s, \xi_s) \quad (3.4)$$

we obtain

$$dJ = -\xi' \hat{Q} \xi ds + \xi' \sum_{j=1}^k \left(U_j + \hat{C}_j' Y + Y \hat{C}_j \right) dW^j \xi + \xi' \sum_{i=1}^m \left(Y \hat{E}_i + \hat{E}_i' Y \right) d\hat{W}^i \xi. \quad (3.5)$$

Integrating from t to T , and conditional on \mathcal{F}_t we obtain (3.3). ■

Lemma 3.2 *If in addition $\hat{H} \geq \delta I$ for some constant $\delta > 0$ and $\hat{Q} \geq 0$, then the solution Y to (3.1) satisfies*

$$Y(t) \geq \delta e^{-\beta(T-t)} I \quad \forall t \in [0, T], \quad a.s. \quad (3.6)$$

where

$$\beta = \text{ess sup}_{\omega \in \Omega, s \leq T} \{ -2 \inf_{|v|=1} v' \hat{A}(s, \omega) v, 0 \}. \quad (3.7)$$

Proof. Let $p \in \mathbb{R}^n$ and ξ solve (3.2). Applying Itô's formula to $e^{\beta s} |\xi|^2$ to obtain

$$\begin{aligned} d(e^{\beta s} |\xi|^2) &= e^{\beta s} \left(2\xi' \hat{A} \xi + \sum_{j=1}^k |\hat{C}_j \xi|^2 + \sum_{i=1}^m |\hat{E}_i \xi|^2 + \beta |\xi|^2 \right) ds \\ &\quad + e^{\beta s} \left(2 \sum_{j=1}^k \xi' \hat{C}_j \xi dW^j + 2 \sum_{i=1}^m \xi' \hat{E}_i \xi d\hat{W}^i \right). \end{aligned}$$

Integrating over $[t, T]$ and taking conditional expectation on \mathcal{F}_t we obtain

$$\mathbb{E} [|\xi_T|^2 | \mathcal{F}_t] \geq |p|^2 e^{-\beta(T-t)}.$$

Finally, it follows from Lemma 3.1 that

$$p'Y(t)p \geq \mathbb{E} \left[\xi'_T \hat{H} \xi_T \mid \mathcal{F}_t \right] \geq \delta |p|^2 e^{-\beta(T-t)}$$

which implies (3.6). ■

4 Proof of Theorem 2.2

The remainder of the paper is devoted to the proof of Theorem 2.2.

To handle the positive definiteness constraint (2.2), we couple the BSDE (2.5) together with another BSDE for $X = K^{-1}$, and consider the resulting system of BSDEs *without* the explicit constraint $K > 0$. This last constraint will be *implied* by the existence of solutions to this system of BSDEs.

Therefore we next derive the BSDE for $X = K^{-1}$ which can be obtained from the identities $XK = KX = I$. In fact, by integrating by parts,

$$dX = -X(dK)X - \langle dX, dK \rangle X. \quad (4.1)$$

In particular the martingale part of X is $-X\tilde{\Lambda}X dW$; so $\langle dX, dK \rangle = -X\tilde{\Lambda}X\tilde{\Lambda}dt$. Substituting this equation into (4.1) to obtain

$$dX = -X(dK)X + X\tilde{\Lambda}X\tilde{\Lambda}Xdt. \quad (4.2)$$

Using (2.5) we obtain a BSDE that $X = K^{-1}$ must satisfy, that is

$$\begin{aligned} dX &= -X\tilde{\Lambda}X dW + X \left[K\tilde{A} + \tilde{A}'K + \tilde{Q} \right] X dt \\ &\quad - X \left[KB - \tilde{R} + \tilde{\Lambda} \right] X \left[B'K - \tilde{R}' + \tilde{\Lambda} \right] X dt + X\tilde{\Lambda}X\tilde{\Lambda}X dt. \end{aligned} \quad (4.3)$$

Setting $Z = -X\tilde{\Lambda}X$, and using the fact that $KX = XK = I$, we obtain its equation

$$\begin{aligned} dX &= ZdW + \left[\tilde{A}X + X\tilde{A}' - BXB' + BZ + ZB' \right] dt - \left[Z\tilde{R}'X + X\tilde{R}Z \right] dt \\ &\quad + \left[X\tilde{Q}X + BX\tilde{R}'X + X\tilde{R}XB' - X\tilde{R}X\tilde{R}'X \right] dt, \\ X(T) &= (R(T) + H)^{-1}. \end{aligned} \tag{4.4}$$

Notice that all the terms involving $\tilde{\Lambda}$ have been canceled out thanks to the assumption that the driving noise W is one dimensional; so (4.4) no longer contains $\tilde{\Lambda}$. This reveals another significant feature of the SRE, that is, equation (4.4) for the inverse matrix $X = K^{-1}$ is itself closed, in the sense that it does not depend on K or $\tilde{\Lambda}$. Therefore we can solve (4.4) independently without the prior knowledge that X is the inverse of K . Let us call (4.4) *the inverse equation* associated with the SRE (2.1)–(2.2).

Therefore, if we are able to solve (4.4) to get (X, Z) with $X > 0$ on $[0, T]$, then (P, Λ) , where $P = X^{-1} - R$ and $\Lambda = -X^{-1}ZX - G$, is a solution to (2.1). In particular, $R + P \equiv K \equiv X^{-1} > 0$ is satisfied automatically.

Now we return to the BSDE (2.5) for K and we wish to rewrite it in terms of (Z, X) . There are several ways to do this because of the relations $XK = KX = I$, and we will choose one which will serve our propose in this paper. In (2.5) replace K^{-1} by X and replace $X\tilde{\Lambda}$ by $-ZK$ to obtain

$$\begin{aligned} dK &= \tilde{\Lambda}dW - \left[K\tilde{A} + \tilde{A}'K + \tilde{Q} \right] dt - \left[KB + \tilde{\Lambda} - \tilde{R} \right] X\tilde{R}'dt \\ &\quad + \left[KB + \tilde{\Lambda} - \tilde{R} \right] (XB' - Z)Kdt, \\ K(T) &= R(T) + H. \end{aligned} \tag{4.5}$$

We consider (4.4) and (4.5) together as a single system of BSDEs, and ignore the fact that X is the inverse matrix of K as well as the constraint $K > 0$. This system can be solved one by one: we can solve (4.4) first to obtain (X, Z) , and then solve (4.5) regarding (X, Z) as known parameters. This is actually the approach we will follow.

Under our assumption that $\tilde{R} = 0$, our basic BSDEs (4.4) and (4.5) are significantly simplified. In fact

$$dX = ZdW + \left[\tilde{A}X + X\tilde{A}' - BXB' + BZ + ZB' \right] dt + X\tilde{Q}Xdt, \quad X(T) = (R(T) + H)^{-1}, \quad (4.6)$$

and

$$dK = \tilde{\Lambda}dW - \left[K\tilde{A} + \tilde{A}'K + \tilde{Q} \right] dt + (KB + \tilde{\Lambda})(XB' - Z)Kdt, \quad K(T) = R(T) + H. \quad (4.7)$$

The following lemma will be used in the proof of Lemma 4.4.

Lemma 4.1 *Suppose (X, Z) is a solution to (4.6). Then $(K, \tilde{\Lambda})$ is a solution to (4.7) if and only if $XK = KX = I$.*

Proof. If $K = X^{-1}$, then Itô's formula yields

$$dK = -K(dX)K + KZKZKdt.$$

A straightforward computation, noting that $XK = KX = I$, deduces that $(K, \tilde{\Lambda}) \equiv (K, -KZK)$ solves (4.7).

Conversely, suppose $(K, \tilde{\Lambda})$ is a solution to (4.7). Let $Y = KX - I$. Then applying Itô's formula to (4.6) and (4.7) we obtain

$$\begin{aligned} dY &= K(dX) + (dK)X + \langle dK, dX \rangle \\ &= UdW + UB'dt + Y\tilde{A}'dt - \tilde{A}'Ydt + Y\tilde{Q}Xdt \\ &\quad + (\tilde{\Lambda}XB' - KBZ - \tilde{\Lambda}Z)Ydt \\ &= UdW + UB'dt + Y \left(\tilde{A}' + \tilde{Q}X \right) dt \\ &\quad + \left[\tilde{\Lambda}XB' - \tilde{A}' - (KB + \tilde{\Lambda})Z \right] Ydt, \end{aligned}$$

where $U = KZ + \tilde{\Lambda}X$. This is a linear BSDE with the terminal value $Y(T) = K(T)X(T) - I = 0$. The uniqueness of solution to the linear BSDE then yields $Y = 0$. ■

BSDE (4.6) is matrix valued with a quadratic term in the drift. If $\tilde{Q} > 0$, then it is a special case of a *definite* SRE whose solvability has been established by Bismut [1] and Peng [7]. In our case where $\tilde{Q} \geq 0$, we use an approximation scheme, adapted from [7], to prove the existence of (4.6).

Lemma 4.2 *Let η be a bounded, $n \times n$ symmetric matrix-valued \mathcal{F}_T -measurable random variable. Then there is a unique adapted bounded solution (X, Z) to the BSDE*

$$\begin{aligned} dX &= ZdW + \left(\tilde{A}X + X\tilde{A}' - BXB' + BZ + ZB' \right) dt + X\tilde{Q}Xdt, \\ X(T) &= \eta. \end{aligned} \tag{4.8}$$

If in addition $\eta > 0$, then $X(t) > 0$ for all $t \in [0, T]$.

Proof. To show the existence of the BSDE (4.8), we consider the following iteration: $X^{(0)} \equiv \eta$, and

$$\begin{aligned} dX^{(n+1)} &= Z^{(n+1)}dW + \left[\tilde{A}X^{(n+1)} + X^{(n+1)}\tilde{A}' - BX^{(n+1)}B' + BZ^{(n+1)} + Z^{(n+1)}B' \right] dt \\ &\quad + X^{(n+1)}\tilde{Q}X^{(n)}dt + X^{(n)}\tilde{Q}X^{(n+1)}dt - X^{(n)}\tilde{Q}X^{(n)}dt, \\ X^{(n+1)}(T) &= \eta, \quad n = 0, 1, 2, \dots \end{aligned}$$

For each n , the above equation is a linear BSDE, whose unique solution defines $(X^{(n+1)}, Z^{(n+1)})$. Since $\eta > 0$, according to Lemma 3.1, each $X^{(n)} \geq 0$.

Let $Y^{(n)} = X^{(n)} - X^{(n+1)}$ and $U^{(n)} = Z^{(n)} - Z^{(n+1)}$. Then the pair $(Y^{(n)}, U^{(n)})$ satisfies the

following stochastic equation:

$$\begin{aligned}
dY^{(n)} &= U^{(n)}dW + \left[\tilde{A}Y^{(n)} + Y^{(n)}\tilde{A}' - BY^{(n)}B' + BU^{(n)} + U^{(n)}B' \right] dt \\
&\quad + Y^{(n)}\tilde{Q}X^{(n)}dt + X^{(n)}\tilde{Q}Y^{(n)}dt \\
&\quad + \left[X^{(n)}\tilde{Q}X^{(n-1)} + X^{(n-1)}\tilde{Q}X^{(n)} - X^{(n)}\tilde{Q}X^{(n)} - X^{(n-1)}\tilde{Q}X^{(n-1)} \right] dt \\
Y^{(n)}(T) &= 0.
\end{aligned}$$

Note that, since $\tilde{Q} \geq 0$, the symmetric matrix

$$X^{(n)}\tilde{Q}X^{(n-1)} + X^{(n-1)}\tilde{Q}X^{(n)} - X^{(n)}\tilde{Q}X^{(n)} - X^{(n-1)}\tilde{Q}X^{(n-1)} \equiv -(X^{(n)} - X^{(n-1)})\tilde{Q}(X^{(n)} - X^{(n-1)}) \leq 0$$

for each n , hence $Y^{(n)} \geq 0$ (Lemma 3.1). It follows that $X^{(n)}$ is decreasing in n in matrix sense, and therefore has a unique limit denoted by X . It is then routine to show that $\{Z^{(n)} : n \geq 1\}$ converges as well (in $L^2([0, T] \times \Omega, dt \otimes d\mathbb{P})$) to a limit process Z . Then (X, Z) solves (4.8).

Since $\tilde{Q} \geq 0$ and $\eta \geq 0$ in matrix sense, $X(t) \geq 0$ for all $t \in [0, T]$. Furthermore, if $\tilde{Q} \geq 0$ and $\eta > 0$, then $X(t) > 0$ for any $0 \leq t \leq T$. To see this, we apply the representation (3.3) to $k = 1$, $m = 0$, $\hat{A} = -\tilde{A} - \frac{1}{2}X\tilde{Q}$ and $\hat{C}_1 = -B$. Fix $t \leq T$ and define ξ_s by solving the corresponding SDE (3.2). If $p \neq 0$, then, by the uniqueness of linear BSDE for (3.2) with terminal condition ξ_T , we can conclude that $\xi_T \neq 0$. As a result, $\xi_T^H H \xi_T > 0$, and therefore $p'X(t)p > 0$ a.s.. ■

Lemma 4.3 *Under the same assumption as in Lemma 4.2, and if in addition $\eta \geq \delta I$ for some $\delta > 0$, then X^{-1} is bounded.*

Proof. Since X is bounded, we conclude that

$$\beta_0 = \text{ess sup}_{\omega \in \Omega, s \leq T} \left\{ - \inf_{|v|=1} \langle 2\tilde{A}v + X\tilde{Q}v, v \rangle, 0 \right\}$$

is finite. It then follows from Lemma 3.2 that $X(t) \geq \delta e^{-\beta_0 T} I$ so that X^{-1} is bounded. ■

Lemma 4.4 *Let $\eta = (R(T) + H)^{-1} \geq \delta I$ in Lemma 4.2 and (X, Z) be the corresponding solution. Then $P = X^{-1} - R$ solves the SRE (2.1)–(2.2).*

Proof. Let $K = X^{-1}$ which have been proved to be a bounded matrix valued semimartingale. Let $\tilde{\Lambda} = -KZK$. Since $KX = XK = I$, it follows from Lemma 4.1 that $(K, \tilde{\Lambda})$ satisfies (4.5). Therefore (P, Λ) , where $P = X^{-1} - R$ and $\Lambda = \tilde{\Lambda} - G$, in turn solves (2.1). Moreover, $K = X^{-1} > 0$, namely, the constraint (2.2) is satisfied. ■

The uniqueness for SRE is well known. The proof of Theorem 2.2 is complete.

5 Discussions and Examples

In this section we discuss about the assumptions of Theorem 2.2 and give examples for illustration.

First of all, if $R \equiv 0$, then condition (2.8) holds automatically, and (2.9) is equivalent to $Q \geq 0$. In addition, assumption (ii) in Theorem 2.2 boils down to $H > 0$. (The condition $H^{-1} \geq \delta I$ is implied by $H > 0$ and the fact that H is bounded.) In this case, our result improves Theorem 5.2 in [4], since here we do not need to assume $C = 0$ and H^{-1} is bounded.

If R is a constant yet indefinite matrix, and $B = C = 0$, then again (2.8) is satisfied, and (2.9) reduces to $Q - RA - A'R \geq 0$. In this case we recover Theorem 5.3 of [4]. However, from Theorem 2.2 we immediately realize that the assumption $B = C = 0$ is far from being essential. Indeed, in the case when R, B and C are non-random matrices, the essential condition is $RB + C'R = 0$, which can be satisfied easily by infinitely many non-zero matrices B and C and indefinite matrices R . In this case, the condition $Q - RA - A'R \geq 0$ should be replaced by $Q - \dot{R} + C'RC + R(BC - A) + (C'B' - A')R \geq 0$, where \dot{R} denotes the derivative of R , which is zero if R is a constant matrix.

As a matter of fact, we can “generate” many generally indefinite, adapted processes R satisfying condition (2.8). To see this, let S be the solution of the following matrix-valued,

sample-wise ODE

$$dS = \left[e^{C'W} F e^{BW} - C' S B - (C'^2) S - S B^2 \right] dt$$

with any given initial state, where F is any given adapted process so that $e^{C'W} F e^{BW}$ is integrable over $t \in [0, T]$ a.s.. Define $R = e^{-C'W} S e^{-BW}$. Then Itô's formula yields that R satisfies (2.10), namely, (2.8) holds.

If condition (2.8) does not hold (i.e. $\tilde{R} \neq 0$), then, we will need to study the general inverse equation, (4.4). This is a very interesting BSDE, since it is matrix-valued involving a cubic term of X . In general, its global existence is not guaranteed. For example, suppose that $\tilde{R} = I$, $\tilde{A} = 0$, $B = 0$, and that the terminal and Q are non-random. Then (4.4) becomes

$$dX = \left[X \tilde{Q} X - X X X \right] dt$$

whose solution may explode in a finite time (and thus the corresponding SRE can not have a global solution). An interesting and challenging open problem is to identify the “weakest” condition on \tilde{R} so that the cubic BSDE (4.4) admits a solution.

On the other hand, the condition $(R(T) + H)^{-1} \geq \delta I$ for some $\delta > 0$ is more technical than essential. One can weaken this condition at the cost of involving more technicalities in our analysis. However, the main goal of this note is to introduce and highlight the main approach, that is to use a system of BSDEs to substitute the original mix of a BSDE and an algebraic constraint, to solving the indefinite SRE. Therefore, we have preferred not to let undue technicalities distract the main idea.

Finally, let us remark that the proof of Lemma 4.2, along with Lemma 4.4, has indeed suggested a numerical scheme to solve the indefinite SRE.

References

- [1] J. M. Bismut, *Linear quadratic optimal stochastic control with random coefficients*, SIAM J. Control Optimization **14** (1976), no. 3, 419–444.
- [2] P. Briand and Y. Hu, *Quadratic BSDEs with convex generators and unbounded terminal conditions*, Probab. Theory Related Fields **141** (2008), no. 3-4, 543–567.
- [3] S. Chen, X. Li, and X. Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim. **36** (1998), no. 5, 1685–1702 (electronic).
- [4] Y. Hu and X. Y. Zhou, *Indefinite stochastic Riccati equations*, SIAM J. Control Optim. **42** (2003), no. 1, 123–137 (electronic).
- [5] M. Kobylanski, J. P. Lepeltier, M. C. Quenez, and S. Torres, *Reflected BSDE with super-linear quadratic coefficient*, Probab. Math. Statist. **22** (2002), no. 1, Acta Univ. Wratislav. No. 2409, 51–83.
- [6] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth*, Ann. Probab. **28** (2000), no. 2, 558–602.
- [7] S. G. Peng, *Stochastic Hamilton-Jacobi-Bellman equations*, SIAM J. Control Optim. **30** (1992), no. 2, 284–304.