Arrow-Debreu Equilibria for Rank-Dependent Utilities with Heterogeneous Probability Weighting

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Abstract

We study Arrow–Debreu equilibria for a one-period-two-date pure exchange economy with rank-dependent utility agents having heterogeneous probability weighting and outcome utility functions. In particular, we allow the economy to have a mix of expected utility agents and rank-dependent utility ones, with non-convex probability weighting functions. The standard approach for convex economy equilibria fails due to the incompatibility with second-order stochastic dominance. The representative agent approach devised in Xia and Zhou (2016) does not work either due to the heterogeneity of the weighting functions. We overcome these difficulties by considering the comonotone allocations, on which the rank-dependent utilities become concave. Accordingly, we introduce the notion of comonotone Pareto optima, and derive their characterizing conditions. With the aid of the auxiliary problem of price equilibria with transfers, we provide a sufficient condition in terms of the model primitives under which an Arrow–Debreu equilibrium exists, along with the explicit expression of the state-price density in equilibrium. This new, general sufficient condition distinguishes the paper from previous related studies with homogeneous and/or convex probability weightings.

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1 Introduction

Rank-dependent utility theory (RDUT), proposed by Quiggin (1982/1993) and developed by many economists [among others Schmeidler (1989) and Abdellaoui (2002)], is one of the most notable theories on preference and choice that depart from the classical expected utility theory (EUT). RDUT has two components: a concave outcome utility function as in EUT, and a probability weighting function that transforms objective probabilities. The latter captures individuals' strong preference for improbable large gains and dislike for improbable large losses, in the form of overweighing both tails of the distribution of the future payoff. This theory has successfully explained many phenomena and paradoxes that are puzzling within the framework of EUT, such as the Allais paradox, the simultaneous risk-averse and risk-seeking behavior, and poor diversification and low stock market participation of households.[1]

Existence of Arrow-Debreu equilibrium for expected utilities was established by Arrow and Debreu (1954); see also, among many others, Werner (1987), Dana (1993a, 1993b), and Aase (1993, 2010) for more development. Existing literature on non-EUT equilibria has almost exclusively focused on utilities that are still concave and hence compatible with the second-order stochastic dominance. This concavity assumption enables one to apply the standard argument as in the EUT case to establish equilibria. Specifically, by Lansberger and Meilijson (1994) and its extensions, any Pareto optimal allocation is comonotone and, moreover, Pareto optima can be characterized as maximizing a weighted sum of utilities. As a result, Pareto optima can be found in the class of comonotone allocations in a more tractable way. Examples include Carlier and Dana (2008) for a two-agent model with the so-called rank-linear utilities that generalize rank-dependent utilities but are still concave, Dana (2011) for a multi-agent economy allowing short-selling with concave law-invariant utilities, Tsanakas and Christofides (2006) for rank-dependent utilities with convex probability weighting functions, and Boonen (2015, 2017) for expected utilities and dual utilities of Yaari (1987) with convex probability weighting functions.

Xia and Zhou (2016) study full Arrow-Debreu equilibria and pricing for a single-period RDUT economy, without assuming any shape of the probability weighting

[1] The other important non-EUT theory is Kahneman and Tversky's cumulative prospect theory (CPT), which also includes probability weighting as a key component.

function. Thus, their utilities are generally non concave and do not necessarily preserve the second-order stochastic dominance. Under the homogeneity of the agents' weighting functions (albeit allowing for heterogeneous outcome utilities), they provide sufficient conditions under which the Arrow-Debreu equilibrium uniquely exists, and derive the state-price density explicitly.

Xia and Zhou (2016) further show that, in equilibrium, an RDUT economy is equivalent to an EUT one where the two components of the rank-dependent utility, namely the outcome utility function and the probability weighting function, are merged into a *revised* outcome utility function for a representative agent. They derive the explicit form of the Arrow-Pratt relative risk aversion index of the representative agent in terms of the original outcome utility and probability weighting. Based on this form, they further derive a consumption-based CAPM, which shows that the excess return of a security must not only compensate for the risk of mean-preserving spread, but also account for the "hope and fear" arising from the exaggeration of tiny probabilities of extremely good and bad states of nature. This, as discussed in Xia and Zhou (2016), may shed some light on resolving the equity premium and risk-free rate puzzles.

There is, however, a major assumption in Xia and Zhou (2016) that the agents' probability weighting is homogeneous. The main approach in that paper depends *crucially* on the explicit representation of an (RDUT) representative agent, which can be provided only under the homogeneity of the weighting function. Nevertheless, empirical studies (Bruhin, Fehr-Duda and Epper 2010, Conte, Hey and Moffatt 2011) show that there is a mix of EUT type and RDUT type of risk-taking in a population, and within the same type of RDUT agents the probability weighting functions are generally different.

The present paper aims to remove the homogeneity assumption of probability weighting while still allowing for any shape of the weighting function in the study of Arrow-Debreu equilibria.[2] In particular, the economy under consideration in this paper allows for the existence of both EUT and RDUT agents.

Under heterogeneity of the weighting function, we have to resort to a different approach than that of the representative agent. In a classical equilibrium framework, the monotonicity of the preference implies that equilibrium allocations are Pareto optimal. If the utilities are concave, which is the case in the classical EUT setting, the Pareto optima can be characterized as solutions of maximizing a weighted sum of utilities. The notion of "price equilibria with transfers" is then introduced to approach the general equilibria; see Mas-Colell et al (1995).

[2]Permitting an arbitrary shape of the weighting function—in particular, the so-called inverse-S shaped one—is motivated by economics reasons rather than mathematical generality.

However, this classical approach for convex economies does not apply to RDUT directly, as the rank-dependent utility is *not* concave in future consumption due to the probability weighting. In this case, unlike the case of concave utilities that are compatible with the second-order stochastic dominance, a Pareto optimal allocation is not necessarily comonotone. However, as it turns out, when the equilibrium state-price density is atomless, the equilibrium future consumptions are increasing functions[3] of the aggregate future endowment (see Lemma 3.2 below); hence (by Landsberger and Meilijson (1994), Dhaene et al (2002)) must be comonotone. Consequently, we need only consider the comonotone allocations when studying Arrow-Debreu equilibria. Restricted to comonotone allocations, the rank-dependent utility becomes concave.

Based on this idea, we introduce the new notion of "comonotone Pareto optima," and reformulate the comonotone Parato optimal problem as a convexly weighted utility optimization problem in terms of quantiles of the future consumptions. This enables us to follow the classical approach to consider price equilibria with transfers, and then to give a complete characterization of a comonotone Pareto optimum to be supported by an atomless state-price density. This characterization is equivalent to the *strict* monotonicity of the state-price density with respect to the aggregate future endowment, a condition that is nonetheless difficult to verify. In this paper, we manage to find a sufficient condition in terms of the primitives of the model under which the aforementioned characterizing condition holds true, along with the explicit expression of the state-price density in equilibrium. This condition reduces to the familiar sufficient conditions when the probability weightings are homogeneous and/or convex; yet in the general case it distinguishes this paper from previous related studies.

The remainder of this paper is organized as follows. In Section 2, we define the economy under consideration and its Arrow–Debreu equilibria. In Section 3, we introduce the notion of comonotone Pareto optima and establish the existence and characterization of the corresponding weighted optimization problem. Section 4 is devoted to the problem of price equilibria with transfers, where we derive necessary and sufficient conditions for a comonotone Pareto optimum to be supported by an atomless state-price density, which is given in explicit form. In Section 5 we discuss a key sufficient condition via several concrete examples. Finally in Section 6 we establish the existence of Arrow–Debreu equilibria and suggest a numerical algorithm for computing the state-price density and equilibrium allocation. Some technical preliminaries and proofs are placed in Appendices.

[3]Throughout the paper "increasing" means "nondecreasing" and "decreasing" means "nonincreasing."

2 The Economy

We consider a one-period-two-date pure exchange economy under uncertainty with a single perishable consumption good. Agents choose their consumption for today, say date $t = 0$, and choose contingent claims on consumption for tomorrow, say date $t = 1$. Without loss of generality, the single consumption good is used as the numeraire throughout the paper. The set of possible states of nature at date 1 is Ω and the set of events at date 1 is a σ-algebra $\mathcal{F}$ of subsets of Ω. There are a finite number of agents indexed by $i = 1, \dots, I$. Each agent i has an endowment $(e_{0i}, \tilde{e}_{1i})$, where e_{0i} is number of units of the good today and the $\mathcal{F}$-measurable random variable $\tilde{e}_{1i}$ is the number of units of the good tomorrow. The aggregate endowment is

$$(e_0, \tilde{e}_1) \triangleq \left(\sum_{i=1}^{I} e_{0i}, \sum_{i=1}^{I} \tilde{e}_{1i} \right).$$

The consumption plan of an agent i is a pair $(c_{0i}, \tilde{c}_{1i})$, where c_{0i} is the number of units of the good consumed today and the $\mathcal{F}$-measurable random variable $\tilde{c}_{1i}$ is the number of units of the good to be consumed tomorrow. The preference of each agent i over consumption plans $(c_{0i}, \tilde{c}_{0i})$ is represented by

$$V_i(c_{0i}, \tilde{c}_{1i}) \triangleq u_{0i}(c_{0i}) + \beta_i \int u_{1i}(\tilde{c}_{1i})\, d(w_i \circ \mathbf{P}),$$

where:

- $\mathbf{P}$ is the belief about the states of the nature;
- u_{0i} is the utility function for consumption today;
- β_i is the time discount factor representing the time impatience for consumption;
- $\int u_{1i}(\tilde{c}_{1i})\, d(w_i \circ \mathbf{P}) \triangleq \int u_{1i}(c)\, d\bar{w}_i(F_{\tilde{c}_{1i}}(c))$ is the rank-dependent utility with outcome utility function u_{1i} for consumption $\tilde{c}_{1i}$ tomorrow and probability weighting function w_i.[4]

In the above (and hereafter) $F_{\tilde{x}}$ denotes the cumulative distribution function (CDF) of a random variable $\tilde{x}$, and $\bar{w}_i$ denotes the dual of a probability weighting function w_i given by

$$\bar{w}_i(p) \triangleq 1 - w_i(1-p) \quad \text{for all } p \in [0, 1].$$

We make the following standing assumption on the economy

[4] If w_i is continuously differentiable, then $\int u_{1i}(\tilde{c}_{1i})\, d(w_i \circ \mathbf{P}) = \int u_{1i}(c) w_i'(1 - F_{\tilde{c}_{1i}}(c))\, dF_{\tilde{c}_{1i}}(c)$. Hence, we have here an additional term $w'(1 - F_{\tilde{c}_{1i}}(c))$ serving as the weight on every consumption level c when calculating the rank-dependent utility. The weight depends on the rank $1 - F_{\tilde{c}_{1i}}(c)$ of level c over all possible realizations of $\tilde{c}_{1i}$.

Assumption 2.1.

- *The agents have homogeneous beliefs* $\mathbf{P}$ *about the states of the nature. The probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *admits no atom.*

- *For every* i*, the endowments satisfy that* $e_{0i} \geq 0$*,* $\mathbf{P}(\tilde{e}_{1i} \geq 0) = 1$*, and* $e_{0i} + \mathbf{P}(\tilde{e}_{1i} > 0) > 0$*. The CDF* $F_{\tilde{e}_1}$ *of* $\tilde{e}_1$ *is continuous and* $\mathbf{P}(\tilde{e}_1 > 0) = 1$*. Moreover,* $e_0 > 0$*.*

- *For every* i*, the functions* $u_{0i}, u_{1i} : [0, \infty) \to \mathbb{R}$ *are strictly increasing, strictly concave, continuously differentiable on* $(0, \infty)$*, and satisfy the Inada condition:* $u'_{0i}(0+) = u'_{1i}(0+) = \infty$*,* $u'_{0i}(\infty) = u'_{1i}(\infty) = 0$*. Moreover,* $u_{1i}(0) = 0$*.*

- *For every* i*, the probability weighting function* $w_i : [0, 1] \to [0, 1]$ *is strictly increasing and continuous on* $[0, 1]$ *and satisfies* $w_i(0) = 0$*,* $w_i(1) = 1$*.*

- *For every* i*, the discount factor* $\beta_i \in (0, 1]$*.*

Under Assumption 2.1, for every i, both w_i and $\bar{w}_i$ are strictly increasing and continuous, and so are their inverse functions w_i^{-1} and $\bar{w}_i^{-1}$. Moreover,

$$\bar{w}_i^{-1}(p) = 1 - w_i^{-1}(p) \quad \text{for all } p \in [0, 1].$$

Definition 2.2 (Feasible consumption plans)**.** *For every* i*, a consumption plan* $(c_{0i}, \tilde{c}_{1i})$ *is called feasible if* $c_{0i} \geq 0$ *and* $\mathbf{P}(\tilde{c}_{1i} \geq 0) = 1$*. The set of all feasible consumption plans is denoted by* $\mathscr{C}$*.*

The above economy is denoted by

$$\mathscr{E} \triangleq \left\{(\Omega, \mathcal{F}, \mathbf{P}),\ (e_{0i}, \tilde{e}_{1i})_{i=1}^I,\ \mathscr{C},\ \big(V_i(\cdot, \cdot)\big)_{i=1}^I\right\}.$$

Definition 2.3 (State-price densities)**.** *A state-price density*[5] *is an* $\mathcal{F}$*-measurable, atomless random variable* $\tilde{\rho}$ *such that* $\mathbf{P}(\tilde{\rho} > 0) = 1$*,* $\mathbf{E}[\tilde{\rho}] < \infty$ *and* $\mathbf{E}[\tilde{\rho}\tilde{e}_1] < \infty$*.*

The above definition is standard; see e.g. p. 160 of Föllmer and Schied (2011), except that as a general definition the atomless condition is not required there.

Definition 2.4 (Arrow-Debreu equilibria)**.** *An Arrow-Debreu equilibrium of the economy* $\mathscr{E}$ *is a collection*

$$\{\tilde{\rho},\ (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I\}$$

consisting of a state-price density $\tilde{\rho}$ *and a collection* $(c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I$ *of feasible consumption plans that satisfies the following conditions:*

[5] Also sometimes termed "pricing kernel" or "stochastic discount factor (SDF)" in the literature.

(i) ***Individual rationality***: *For every* i, $(c^*_{0i}, \tilde{c}^*_{1i})$ *is an optimal consumption plan of agent* i *subject to her budget constraint; that is,* $(c^*_{0i}, \tilde{c}^*_{1i})$ *solves the following problem*

$$\begin{aligned} &\mathit{Maximize} \quad V_i(c_{0i}, \tilde{c}_{1i}) \\ &\mathit{subject\ to} \quad \begin{cases} (c_{0i}, \tilde{c}_{1i}) \in \mathscr{C}, \\ c_{0i} + \mathbf{E}[\tilde{\rho}\tilde{c}_{1i}] \le e_{0i} + \mathbf{E}[\tilde{\rho}\tilde{e}_{1i}]. \end{cases} \end{aligned} \tag{2.1}$$

(ii) ***Market clearing***: $\sum_{i=1}^I c^*_{0i} = e_0$ *and* $\sum_{i=1}^I \tilde{c}^*_{1i} = \tilde{e}_1$.

3 Comonotone Pareto Optima

Definition 3.1 (Feasible allocations)**.** *A feasible allocation* $(c_{0i}, \tilde{c}_{1i})_{i=1}^I$ *consists of a profile of feasible consumption plans* $(c_{0i}, \tilde{c}_{1i})$ *such that* $\sum_{i=1}^I c_{0i} = e_0$ *and* $\sum_{i=1}^I \tilde{c}_{1i} = \tilde{e}_1$.[6] *The set of all feasible allocations is denoted by* $\mathscr{A}$.

In an Arrow-Debreu equilibrium $\left\{\tilde{\rho},\ (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I\right\}$ of the economy, the equilibrium allocation $(c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I$ must be Pareto optimal over $\mathscr{A}$, if the preferences of all the agents are strictly monotone (see, e.g., Mas-Colell et al. (1995) Section 16.C).

If the functionals $V_i(c_{0i}, \tilde{c}_{1i})$ were concave in $(c_{0i}, \tilde{c}_{1i})$, then a Pareto optimal allocation would in turn solve the following optimization problem

$$\begin{aligned} &\text{Maximize} \quad \textstyle\sum_{i=1}^I \theta_i V_i(c_{0i}, \tilde{c}_{1i}) \\ &\text{subject to} \quad (c_{0i}, \tilde{c}_{1i})_{i=1}^I \in \mathscr{A}, \end{aligned} \tag{3.1}$$

for some $\theta \triangleq (\theta_1, \ldots, \theta_I) \in \Delta$, where

$$\Delta \triangleq \left\{ (\theta_1, \ldots, \theta_I) :\ \theta_i \ge 0 \text{ for all } i \text{ and } \sum_{i=1}^I \theta_i = 1 \right\}.$$

The Arrow-Debreu equilibrium thus could be identified from the set of all Pareto optimal allocations.

Unfortunately, due to the presence of probability weighting functions w_i, our functionals $V_i(c_{0i}, \tilde{c}_{1i})$ are generally *not* concave in $(c_{0i}, \tilde{c}_{1i})$. So a Pareto optimal allocation does not necessarily solve the optimization problem (3.1).

[6] It usually requires in the literature that the total consumption not exceed the total endowment, namely, $\sum_{i=1}^I c_{0i} \le e_0$ and $\sum_{i=1}^I \tilde{c}_{1i} \le \tilde{e}_1$, in defining the feasibility of allocations. By introducing a dummy agent we can require without loss of generality that the total consumption equals the total endowment in a feasible allocation.

However, we can show that when the equilibrium state-price density is atomless, the equilibrium future consumptions are necessarily increasing functions of the aggregate future endowment. To make it precise, let us define

$$\mathscr{A}_c \triangleq \left\{ (c_{0i}, \tilde{c}_{1i})_{i=1}^I \in \mathscr{A} \;\middle|\; \begin{array}{l} \tilde{c}_{1i} = f_i(\tilde{e}_1) \text{ for some increasing and} \\ \text{continuous function } f_i \text{ for all } i = 1, \ldots, I \end{array} \right\}.$$

Also, hereafter, $Q_{\tilde{x}}$ (resp. $Q^-_{\tilde{x}}$) denotes the upper (resp. lower) quantile function of a random variable $\tilde{x}$.[7]

Lemma 3.2. *Under Assumption 2.1, let* $\{\tilde{\rho},\, (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I\}$ *be an Arrow-Debreu equilibrium. Then* $(c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I \in \mathscr{A}_c$.

Proof. We use an argument similar to the one in Theorem 5.2 of Xia and Zhou (2016) to prove this lemma. It has been established in Jin and Zhou (2008) that, when the state-price density is atomless, an individual agent's optimal future consumption is always a decreasing function of the state-price density. As a result, the aggregate future endowment $\tilde{e}_1$, being the sum of $\tilde{c}^*_{1i}$, is also a decreasing function of $\tilde{\rho}$. Thus

$$F_{\tilde{e}_1}(\tilde{e}_1) = g(\tilde{\rho}) \quad \text{a.s.}$$

for a decreasing function g. Without loss of generality, we can assume that g is right-continuous because $\tilde{\rho}$ is atomless. Considering the quantile functions of both sides in the above, it follows from Assumption 2.1 and the continuity of $F_{\tilde{e}_1}$ that

$$p = g(Q_{\tilde{\rho}}(1-p)) \;\; \forall p \in [0, 1],$$

implying

$$1 - F_{\tilde{\rho}}(\tilde{\rho}) = g(Q_{\tilde{\rho}}(F_{\tilde{\rho}}(\tilde{\rho}))) = g(\tilde{\rho}) = F_{\tilde{e}_1}(\tilde{e}_1) \quad \text{a.s.}$$

Consequently,

$$\tilde{\rho} \equiv Q_{\tilde{\rho}}(F_{\tilde{\rho}}(\tilde{\rho})) = Q_{\tilde{\rho}}(1 - F_{\tilde{e}_1}(\tilde{e}_1)) \quad \text{a.s.}$$

Therefore, $\tilde{\rho}$ is a decreasing function of $\tilde{e}_1$. Recalling that $\tilde{c}^*_{1i}$ are decreasing functions of $\tilde{\rho}$, we conclude that $\tilde{c}^*_{1i}$ are increasing functions of $\tilde{e}_1$. Finally, because the sum of $\tilde{c}^*_{1i} \geq 0$ is $\tilde{e}_1$, all of $\tilde{c}^*_{1i}$ must be continuous functions of $\tilde{e}_1$. $\square$

By p. 99 of Landsberger and Meilijson (1994) and Theorem 2 of Dhaene et al (2002), $\mathscr{A}_c$ is nothing but the set of comonotone allocations. Hence, Lemma 3.2 suggests that it suffices to consider only the comonotone allocations $(c_{0i}, \tilde{c}_{1i})_{i=1}^I \in \mathscr{A}_c$ when investigating Arrow-Debreu equilibrium.

To summarize, an equilibrium allocation $(c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I$ must be comonotone Pareto optimal in the sense of the following definition.

[7]See Appendix A.3 of Föllmer and Schied (2011) for definitions and properties of upper/lower quantile functions.

Definition 3.3 (Comonotone Pareto optimality)**.** *An allocation* $(c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I \in \mathscr{A}_c$ *is called comonotone Pareto optimal if there is no* $(c_{0i}, \tilde{c}_{1i})_{i=1}^I \in \mathscr{A}_c$ *such that*

$$\begin{cases} V_i(c_{0i}, \tilde{c}_{1i}) \geq V_i(c_{0i}^*, \tilde{c}_{1i}^*) & \textit{for all } i, \\ V_i(c_{0i}, \tilde{c}_{1i}) > V_i(c_{0i}^*, \tilde{c}_{1i}^*) & \textit{for some } i. \end{cases}$$

Let $\mathbb{G}$ denote the set of upper quantile functions of nonnegative random variables, that is,

$$\mathbb{G} \triangleq \{G : [0,1) \to [0,\infty) \text{ nondecreasing and right-continuous}\}.$$

Moreover, let

$$\mathscr{H} \triangleq \left\{ (c_{0i}, G_i)_{i=1}^I : \; c_{0i} \geq 0 \text{ and } G_i \in \mathbb{G} \text{ for all } i, \sum_{i=1}^I c_{0i} = e_0, \sum_{i=1}^I G_i = Q_{\tilde{e}_1} \right\}.$$

The set $\mathscr{A}_c$ can be characterized in terms of (upper) quantile functions by the following lemma, which is straightforward to establish.

Lemma 3.4. *Under Assumption 2.1, we have*

$$\mathscr{A}_c = \left\{ (c_{0i}, G_i(F_{\tilde{e}_1}(\tilde{e}_1)))_{i=1}^I : \; (c_{0i}, G_i)_{i=1}^I \in \mathscr{H} \right\}.$$

If G_i is the upper quantile of $\tilde{c}_{1i}$, then it is not hard to check (see Xia and Zhou (2016)) that

$$\int u_{1i}(\tilde{c}_{1i}) \, d(w_i \circ \mathbf{P}) = \int_{(0,1)} u_{1i}(G_i(p)) \, d\bar{w}_i(p).$$

Moreover,

$$U_i(c_{0i}, G_i) \triangleq u_{0i}(c_{0i}) + \beta_i \int_{(0,1)} u_{1i}(G_i(p)) \, d\bar{w}_i(p) = V_i(c_{0i}, \tilde{c}_{1i})$$

is concave in (c_{0i}, G_i).

Definition 3.5. *A* $(c_{0i}^*, G_i^*)_{i=1}^I \in \mathscr{H}$ *is called Pareto optimal over* $\mathscr{H}$ *if there is no* $(c_{0i}, G_i)_{i=1}^I \in \mathscr{H}$ *such that*

$$\begin{cases} U_i(c_{0i}, G_i) \geq U_i(c_{0i}^*, G_i^*) & \textit{for all } i, \\ U_i(c_{0i}, G_i) > U_i(c_{0i}^*, G_i^*) & \textit{for some } i. \end{cases}$$

For any allocation $(c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I \in \mathscr{A}_c$, let $G_i^* = Q_{\tilde{c}_{1i}^*}$ for all i. Then it is obvious that $(c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I$ is comonotone Pareto optimal if and only if $(c_{0i}^*, G_i^*)_{i=1}^I \in \mathscr{H}$ is Pareto optimal over $\mathscr{H}$. Moreover, in this case, it follows from the concavity of functional $U_i(c_{0i}, G_i)$ that $(c_{0i}^*, G_i^*)_{i=1}^I$ must solve the following problem

$$\begin{aligned} &\text{Maximize} \quad \textstyle\sum_{i=1}^I \theta_i U_i(c_{0i}, G_i) \\ &\text{subject to} \quad (c_{0i}, G_i)_{i=1}^I \in \mathscr{H}, \end{aligned} \tag{3.2}$$

for some $\theta \triangleq (\theta_1, \ldots, \theta_I) \in \Delta$.

Therefore, like in the classical Arrow-Debreu equilibrium literature with monotonicity and concavity of preferences, we can derive an equilibrium from solutions of Problem (3.2). This problem, in turn, can be divided into two separated problems. The first one is

$$\begin{aligned} &\text{Maximize} \quad \textstyle\sum_{i=1}^I \theta_i u_{0i}(c_{0i}) \\ &\text{subject to} \quad \textstyle\sum_{i=1}^I c_{0i} = e_0 \text{ and } c_{0i} \geq 0 \text{ for all } i, \end{aligned} \tag{3.3}$$

and the second one is

$$\begin{aligned} &\text{Maximize} \quad \textstyle\sum_{i=1}^I \int_{(0,1)} \theta_i \beta_i u_{1i}(G_i(p))\, d\bar{w}_i(p) \\ &\text{subject to} \quad (G_i)_{i=1}^I \in \mathscr{G}, \end{aligned} \tag{3.4}$$

where

$$\mathscr{G} \triangleq \left\{ (G_i)_{i=1}^I : \ G_i \in \mathbb{G} \text{ for all } i \text{ and } \sum_{i=1}^I G_i = Q_{\tilde{e}_1} \right\}.$$

Obviously, the solution $(c_{0i}^*)_{i=1}^I$ of problem (3.3) is given by

$$c_{0i}^* = (u_{0i}')^{-1}\left(\frac{\mu_\theta}{\theta_i}\right) \quad (i = 1, \ldots, I), \tag{3.5}$$

where $\mu_\theta \in (0, \infty)$ is uniquely determined by the following equation

$$\sum_{i=1}^I (u_{0i}')^{-1}\left(\frac{\mu_\theta}{\theta_i}\right) = e_0. \tag{3.6}$$

In the next part of this section, we investigate the solution of optimization problem (3.4). First of all, we introduce the following assumption

Assumption 3.6. *In addition to Assumption 2.1, for every $i = 1, \ldots, I$, the agent i's rank-dependent utility for tomorrow's total endowment $\tilde{e}_1$ is finite, namely,*

$$\int_{(0,1)} u_{1i}(Q_{\tilde{e}_1}(p))\, d\bar{w}_i(p) < \infty. \tag{3.7}$$

Notation. *For a function $G \in \mathbb{G}$, we need to consider the Lebesgue-Stieltjes measure dG on $[0,1)$. The measure of point 0 is set to be $G(0)$ by convention. For any Borel function $f : [0,1) \to \mathbb{R}$, the (Lebesgue-Stieltjes) integration $\int_{[0,1)} f(p)\, dG(p)$ is then given by*

$$\int_{[0,1)} f(p)\, dG(p) \triangleq f(0)G(0) + \int_{(0,1)} f(p)\, dG(p).$$

For every $(G_i)_{i=1}^I \in \mathscr{G}$, it is easy to see that the measure dG_i $(i = 1, \dots, I)$ is absolutely continuous w.r.t. the measure $dQ_{\tilde{e}_1}$. Let g_i denotes the density of dG_i w.r.t. $dQ_{\tilde{e}_1}$, that is,

$$g_i = \frac{dG_i}{dQ_{\tilde{e}_1}} \quad \text{for all } i = 1, \dots, I.$$

Then $\sum_{i=1}^I g_i(p) = 1$ and $g_i(p) \geq 0$ $(i = 1, \dots, I)$ for $dQ_{\tilde{e}_1}$-almost-every $p \in [0,1)$. Thus each g_i lies in the closed unit ball in $L^\infty([0,1), dQ_{\tilde{e}_1})$.

Lemma 3.7. *Under Assumption 3.6, the optimization problem* (3.4) *admits a unique solution for every $\theta \in \Delta$.*

Proof. The uniqueness of the solution obviously follows from the strict concavity of u_{1i} $(i = 1, \dots, I)$. Now we show the existence. Let $v(\theta)$ denote the value of optimization problem (3.4), that is,

$$v(\theta) \triangleq \sup_{(G_i)_{i=1}^I \in \mathscr{G}} \sum_{i=1}^I \theta_i \beta_i \int_{(0,1)} u_{1i}(G_i(p))\, d\bar{w}_i(p).$$

Then by (3.7),

$$v(\theta) \leq \sum_{i=1}^I \theta_i \beta_i \int_{(0,1)} u_{1i}(Q_{\tilde{e}_1}(p))\, d\bar{w}_i(p) < \infty.$$

Hence, there exists a sequence $\{(G_i^n)_{i=1}^I, n \geq 1\} \subseteq \mathscr{G}$ such that

$$\sum_{i=1}^I \theta_i \beta_i \int_{(0,1)} u_{1i}(G_i^n(p))\, d\bar{w}_i(p) \to v(\theta)$$

as $n \to \infty$. Let g_i^n denote the densities of dG_i^n w.r.t. $dQ_{\tilde{e}_1}$, that is,

$$g_i^n = \frac{dG_i^n}{dQ_{\tilde{e}_1}}.$$

Each g_i^n lies in the closed unit ball of $L^\infty([0,1), dQ_{\tilde{e}_1})$ for all $n \geq 1$ and for all $i = 1, \dots, I$. Moreover, it is well known that the closed unit ball is sequentially compact under the weak* topology $\sigma\left(L^\infty([0,1), dQ_{\tilde{e}_1}), L^1([0,1), dQ_{\tilde{e}_1})\right)$. Thus there exists a subsequence of $\{(g_i^n)_{i=1}^I, n \geq 1\}$, which is still denoted by $\{(g_i^n)_{i=1}^I, n \geq 1\}$

for notational simplicity, such that the subsequence $\{g_i^n, n \geq 1\}$ is weak* convergent to some $g_i^\infty \in L^\infty([0,1), dQ_{\tilde{e}_1})$ for every $i = 1, \dots, I$. Let G_i^∞ be given by

$$G_i^\infty(p) = \int_{[0,p]} g_i^\infty(q)\, dQ_{\tilde{e}_1}(q) \quad \text{for all } p \in [0,1).$$

Then by the weak* convergence of $\{g_i^n, n \geq 1\}$, we know

$$G_i^n(p) = \int_{[0,p]} g_i^n(q)\, dQ_{\tilde{e}_1}(q) \to \int_{[0,p]} g_i^\infty(q)\, dQ_{\tilde{e}_1}(q) = G_i^\infty(p) \quad \text{for all } p \in [0,1)$$

as $n \to \infty$. It is easy to check that $(G_i^\infty)_{i=1}^I \in \mathscr{G}$. Moreover, by (3.7) and the dominated convergence theorem, we have

$$\sum_{i=1}^I \theta_i \beta_i \int_{(0,1)} u_{1i}(G_i^n(p))\, d\bar{w}_i(p) \to \sum_{i=1}^I \theta_i \beta_i \int_{(0,1)} u_{1i}(G_i^\infty(p))\, d\bar{w}_i(p)$$

as $n \to \infty$, which implies that $v(\theta) = \sum_{i=1}^I \theta_i \beta_i \int_{(0,1)} u_{1i}(G_i^\infty(p))\, d\bar{w}_i(p)$, that is, $(G_i^\infty)_{i=1}^I$ solves problem (3.4). $\square$

Next we are going to derive characterizations of the solution to (3.4) that will eventually lead to an expression of the pricing kernel. First, the following technical result is in order.

Lemma 3.8. *Under Assumption 3.6, given $G \in \mathbb{G}$, if $G(p) \leq Q_{\tilde{e}_1}(p)$ for all $p \in [0,1)$, then*

$$\int_{(0,1)} u'_{1i}(G(p))G(p)\, d\bar{w}_i(p) < \infty \quad \textit{for all } i = 1, \cdots, I.$$

Proof. Let $G \in \mathbb{G}$ with $G(p) \leq Q_{\tilde{e}_1}(p)$ for all p. By the concavity of u_{1i} and the fact that $u_{1i}(0) = 0$, we have

$$u'_{1i}(x)x \leq u_{1i}(x) \quad \text{for all } x > 0. \tag{3.8}$$

Then by (3.7), we have

$$\begin{aligned} \int_{(0,1)} u'_{1i}(G(p))G(p)\, d\bar{w}_i(p) &\leq \int_{(0,1)} u_{1i}(G(p))\, d\bar{w}_i(p) \\ &\leq \int_{(0,1)} u_{1i}(Q_{\tilde{e}_1}(p))\, d\bar{w}_i(p) \\ &< \infty \end{aligned}$$

for all i. $\square$

If $(G_i^*)_{i=1}^I$ solves problem (3.4) and $\theta_j = 0$ for an agent j, then it is obvious that $G_j^*(p) \equiv 0$ and therefore agent j's consumption for tomorrow is zero. We will thus focus on those agents $i \in \mathcal{I}_\theta$, where

$$\mathcal{I}_\theta \triangleq \{i : \ \theta_i > 0, \ i = 1, \ldots, I\}.$$

The solution of problem (3.4) can be characterized by the following lemma, which can be proved, in view of Lemma 3.8, similarly to the proof of Proposition 3.2 in Xia and Zhou (2016).

Proposition 3.9. *Under Assumption 3.6, let $(G_i^*)_{i=1}^I \in \mathscr{G}$. Then*

$$\sum_{i \in \mathcal{I}_\theta} \int_{(0,1)} \theta_i \beta_i u'_{1i}(G_i^*(p)) G_i^*(p) \, d\bar{w}_i(p) < \infty \tag{3.9}$$

and the following statements are equivalent:

(i) $(G_i^*)_{i=1}^I$ *solves problem* (3.4)*;*

(ii) *For all $(G_i)_{i=1}^I \in \mathscr{G}$,*

$$\sum_{i \in \mathcal{I}_\theta} \int_{(0,1)} \theta_i \beta_i u'_{1i}(G_i^*(p)) G_i(p) \, d\bar{w}_i(p) \leq \sum_{i \in \mathcal{I}_\theta} \int_{(0,1)} \theta_i \beta_i u'_{1i}(G_i^*(p)) G_i^*(p) \, d\bar{w}_i(p). \tag{3.10}$$

Lemma 3.10. *Under Assumption 3.6, if $(G_i^*)_{i=1}^I \in \mathscr{G}$ solves problem* (3.4) *then $G_j^*(p) > 0$ for all $j \in \mathcal{I}_\theta$ and for all $p \in (0,1)$.*

Proof. It is easy to see that $Q_{\tilde{e}_1}$ is strictly increasing as $F_{\tilde{e}_1}$ is continuous. Thus $Q_{\tilde{e}_1}(p) > 0$ for all $p \in (0,1)$. Now assume $(G_i^*)_{i=1}^I \in \mathscr{G}$ solves problem (3.4) and $j \in \mathcal{I}_\theta$. Consider a $(G_i)_{i=1}^I \in \mathscr{G}$ which is defined as follows

$$G_i = \begin{cases} 0, & \text{if } i \notin \mathcal{I}_\theta, \\ \frac{Q_{\tilde{e}_1}}{|\mathcal{I}_\theta|}, & \text{if } i \in \mathcal{I}_\theta, \end{cases}$$

where $|\mathcal{I}_\theta|$ denotes the cardinality of the set $\mathcal{I}_\theta$. By Proposition 3.9, we have

$$\sum_{i \in \mathcal{I}_\theta} \int_{(0,1)} \theta_i \beta_i u'_{1i}(G_i^*(p)) \frac{Q_{\tilde{e}_1}(p)}{|\mathcal{I}_\theta|} \, d\bar{w}_i(p) < \infty.$$

Then $u'_{1i}(G^*_i(p)) < \infty$ for all $p \in (0,1)$ and for all $i \in \mathcal{I}_\theta$, by the fact that $\bar{w}_i$ is strictly increasing and $u'_{1i}(G^*(\cdot))$ is decreasing. Thus, $G^*_i(p) > 0$ for all $p \in (0,1)$ and for all $i \in \mathcal{I}_\theta$, in view of the Inada condition on u_{1i}. $\square$

Based on Lemma 3.10, to find the solution of problem (3.4), we need to consider only those $(G^*_i)_{i=1}^I \in \mathscr{G}_\theta$, where

$$\mathscr{G}_\theta \triangleq \left\{(G_i)_{i=1}^I \in \mathscr{G} :\ G_i \equiv 0 \text{ for } i \notin \mathcal{I}_\theta,\ \text{and } G_i(p) > 0 \text{ for } i \in \mathcal{I}_\theta \text{ and } p \in (0,1)\right\}.$$

Given a $(G^*_i)_{i=1}^I \in \mathscr{G}_\theta$, define

$$M^*_i(q) \triangleq -\int_{(q,1)} \theta_i\beta_i u'_{1i}(G^*_i(p))\, d\bar{w}_i(p),\ q \in [0,1),\ i \in \mathcal{I}_\theta. \tag{3.11}$$

The function $-M^*_i$ is the marginal utility of agent i in terms of the quantile function. To see this, consider a perturbed quantile $G^*_i + \varepsilon\Delta$ where $\Delta(p) = \mathbf{1}_{p\geq q}$ with the parameter $q \in [0,1]$. Then the marginal utility is

$$\frac{d}{d\varepsilon}\int_0^1 \theta_i\beta_i u_{1i}(G^*_i(p) + \varepsilon\mathbf{1}_{p\geq q})d\bar{w}_i(p)\Big|_{\varepsilon=0},$$

whose value is exactly $-M^*_i(q)$.

Obviously, for any $i \in \mathcal{I}_\theta$, $M^*_i(q) \in (-\infty, 0)$ for all $q \in (0,1)$ and M^*_i is continuous and strictly increasing on $(0,1)$. We now extend the definition of M^*_i by setting

$$M^*_i(1) \triangleq \lim_{q\uparrow 1} M^*_i(q) = 0. \tag{3.12}$$

With this extension, M^*_i is continuous on $[0,1]$ when $M^*_i(0) > -\infty$ or extended continuous otherwise.[8] Furthermore, M^*_i is strictly increasing on $[0,1]$. Finally, define

$$M^*(q) \triangleq \min_{i\in\mathcal{I}_\theta} M^*_i(q) \quad \text{for all } q \in [0,1]. \tag{3.13}$$

It turns out that the functions M^*_i and M^* play a critical role in deriving the equilibria and the pricing kernels. Important properties of these functions are presented in Appendix A.

4 Price Equilibria with Transfers

In this section we investigate the relationship between Pareto optima and price equilibria with transfers, which generalize the notion of Arrow-Debreu equilibria.[9] The

[8]Function M^*_i is called extended continuous at point $p = 0$ if $M^*_i(0) = -\infty = \lim_{p\downarrow 0} M^*_i(p)$.

[9]The notion of price equilibria with transfers is well known in microeconomic theory; see, e.g., Mas-Colell et al. (1995).

concept of Arrow-Debreu equilibrium applies to the case of a private ownership economy, in which an agent's wealth is derived from the ownership of endowments. The more general notion of a price equilibrium with transfers allows for an arbitrary distribution of wealth among agents. The results on equilibria with transfers, while interesting in their own rights, will help us establish the existence of Arrow-Debreu equilibria.

Definition 4.1 (Price equilibria with transfers)**.** A price equilibrium with transfers of the economy $\mathscr{E}$ is a collection

$$\left\{\tilde{\rho},\ (c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I\right\}$$

consisting of a state-price density $\tilde{\rho}$ and a collection $(c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I$ of feasible consumption plans that satisfy the following conditions:

(i) For every i, the feasible consumption plan $(c_{0i}^*, \tilde{c}_{1i}^*)$ maximizes the preference of agent i in the budget set

$$\{(c_{0i}, \tilde{c}_{1i}) \in \mathscr{C} :\ c_{0i} + \mathbf{E}[\tilde{\rho}\tilde{c}_{1i}] \le c_{0i}^* + \mathbf{E}[\tilde{\rho}\tilde{c}_{1i}^*]\}\,;$$

that is

$$\begin{aligned} V_i(c_{0i}^*, \tilde{c}_{1i}^*) = &\max_{(c_{0i},\tilde{c}_{1i})\in\mathscr{C}} V_i(c_{0i}, \tilde{c}_{1i}) \\ &\text{subject to } c_{0i} + \mathbf{E}[\tilde{\rho}\tilde{c}_{1i}] \le c_{0i}^* + \mathbf{E}[\tilde{\rho}\tilde{c}_{1i}^*]. \end{aligned} \tag{4.1}$$

(ii) $\sum_{i=1}^I c_{0i}^* = e_0$ and $\sum_{i=1}^I \tilde{c}_{1i}^* = \tilde{e}_1$.

Clearly, an Arrow-Debreu equilibrium is also an equilibrium with transfers, but not vice versa. On the other hand, it is well known that if $\left\{\tilde{\rho},\ (c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I\right\}$ is a price equilibrium with transfers, then $(c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I$ is Pareto optimal over $\mathscr{A}$; see, e.g., Mas-Colell at al. (1995) Section 16.C. Now we consider a somewhat inverse question: what is the condition under which a Pareto optimal $(c_{0i}^*, G_i^*)_{i=1}^I \in \mathscr{H}$ can be supported by an atomless state-price density in the sense of the following definition?

Definition 4.2 (Supporting state-price densities)**.** A $(c_{0i}^*, G_i^*)_{i=1}^I \in \mathscr{H}$ is supported by a state-price density $\tilde{\rho}$ if there exist $\tilde{c}_{1i}^*$ ($i = 1, \ldots, I$) that satisfy the following conditions:

(i) For every $i = 1, \ldots, I$, G_i^* is the upper quantile function of $\tilde{c}_{1i}^*$;

(ii) $\left\{\tilde{\rho},\ (c_{0i}^*, \tilde{c}_{1i}^*)_{i=1}^I\right\}$ is a price equilibrium with transfers.

The following gives a complete characterization for a comonotone Pareto optimum to be supported by an atomless state-price density. Moreover, it identifies the state-price density explicitly in terms of the right derivative of the function M^*.

Theorem 4.3. *Suppose $Q_{\tilde{e}_1}(0) > 0$, w_i ($i = 1, \ldots, I$) is continuously differentiable on $(0,1)$, and Assumption 3.6 holds. Assume $(c^*_{0i}, G^*_i)_{i=1}^I \in \mathscr{H}$ solves problem (3.2). Let M^*_i ($i \in \mathcal{I}_\theta$) be given by (3.11)–(3.12) and M^* be given by (3.13). Then $(c^*_{0i}, G^*_i)_{i=1}^I$ is supported by an atomless state-price density $\tilde{\rho}$ if and only if M^* is strictly concave. Moreover, in this case a supporting state-price density $\tilde{\rho}$ is given by*

$$\tilde{\rho} = \frac{1}{\mu_\theta}(M^*)'_+(F_{\tilde{e}_1}(\tilde{e}_1)), \tag{4.2}$$

where $(M^*)'_+$ denotes the right derivative of M^* and μ_θ is determined by equation (3.6).

A proof is placed in Appendix B.

So the strict concavity of M^* is a characterizing condition for a comonotone Pareto optimum to be supported by an atomless state-price density. This condition does not hold automatically, and it requires a certain "alignment" between the agent preferences and the aggregate random endowment. To see this, let us examine the special case when there is only one agent with the utility functions u_0, u_1 and the probability weighting function w. In this case $M^*(q) = -\int_q^1 \beta_i u'_1(Q_{\tilde{e}_1}(p))d\bar{w}(p)$, and M^* is strictly concave if and only if $u'_1(Q_{\tilde{e}_1}(p))\bar{w}'(p)$ is strictly decreasing.

In the rest of this section, we focus on the search of some sufficient condition on the model primitives to guarantee the strict concavity of M^*. To this end, for any $p \in (0,1)$ and any subset $\mathcal{I}$ of $\{1, \cdots, I\}$, set

$$\Gamma(p, \mathcal{I}) \triangleq \sup\left\{\sum_{i\in\mathcal{I}} -\frac{u'_{1i}(x_i)}{u''_{1i}(x_i)}\frac{\bar{w}''_i(p)}{\bar{w}'_i(p)} : \sum_{i\in\mathcal{I}} x_i = Q_{\tilde{e}_1}(p) \text{ and } x_i > 0 \text{ for all } i \in \mathcal{I}\right\},$$

and

$$\Gamma(p) \triangleq \max_{\mathcal{I}\subseteq\{1,\cdots,I\}} \Gamma(p, \mathcal{I}). \tag{4.3}$$

Notation. *Consider a function $G \in \mathbb{G}$, a Lebesgue integrable function $f : [0,1) \to (-\infty, \infty)$, and a nonnegative Borel function $g : [0,1) \to [0,\infty)$. If*

$$\int_B dG(p) > \int_B f(p)\,dp$$

for all non-null Borel set $B \subseteq (0,1)$,[10] *then we write $dG(p) \succ f(p)\,dp$. Similarly, if*

$$\int_B g(p)\,dG(p) \le \int_B f(p)\,dp$$

for all Borel set $B \subseteq (0,1)$, then we write $g(p)\,dG(p) \preceq f(p)\,dp$.

Theorem 4.4. *Under Assumption 3.6, assume further that $Q_{\tilde{e}_1}(0) > 0$, w_i ($i = 1, \ldots, I$) is twice continuously differentiable on $(0,1]$, Γ is Lebesgue integrable and*

$$dQ_{\tilde{e}_1}(p) \succ \Gamma(p)\,dp, \quad p \in (0,1). \tag{4.4}$$

For any given $\theta \in \Delta$, let $(c^*_{0i}, G^*_i)_{i=1}^I \in \mathscr{H}$ solve problem (3.2). Let M^*_i ($i \in \mathcal{I}_\theta$) be given by (3.11)–(3.12) and M^* be given by (3.13). Then M^* is strictly concave.

A proof is given in Appendix C.

Assumption (4.4) is an abstract condition to be discussed next.

5 Condition (4.4)

In this section, we discuss the implications of the sufficient condition (4.4) in concrete examples.

To start, we note that although (4.4) seems to be abstract, it is nothing more than the strict monotonicity condition of $u_1'(Q_{\tilde{e}_1}(p))\bar{w}'(p)$ in the single-agent case generalized to the multi-agent case. To see this, consider a single agent with the utility functions u_0, u_1 and the probability weighting function w. Assuming that $Q_{\tilde{e}_1}(\cdot)$ is differentiable and both u_1 and w are twice differentiable, M^* is strictly concave if and only if $u_1'(Q_{\tilde{e}_1}(p))\bar{w}'(p)$ is strictly decreasing, or

$$\begin{aligned} & 0 > \frac{d}{dp}\{u_1'(Q_{\tilde{e}_1}(p))\bar{w}'(p)\} = u_1''(Q_{\tilde{e}_1}(p))Q_{\tilde{e}_1}'\bar{w}'(p) + u_1'(Q_{\tilde{e}_1}(p))\bar{w}''(p) \\ \Leftrightarrow \quad & -\frac{u_1'(Q_{\tilde{e}_1}(p))}{u_1''(Q_{\tilde{e}_1}(p))}\frac{\bar{w}''(p)}{\bar{w}'(p)} < Q_{\tilde{e}_1}'(p), \end{aligned}$$

where the latter inequality is exactly the condition (4.4). In this case, this condition reduces to Assumption 5.6 in Xia and Zhou (2016), the key condition there to ensure the existence of equilibria.

Similarly, (4.4) specializes to Assumption 5.6 in Xia and Zhou (2016) when there are multiple agents with homogeneous probability weighting functions. Indeed, if $w_1 = w_2 = \cdots = w$, then, as discussed in Section 4, Xia and Zhou (2016), the

[10] A Borel set B is called non-null if its Lebesgue measure is strictly positive.

equilibrium problem reduces to the one in a representative agent economy in which the representative agent's preference is still represented by RDU

$$u_{0\lambda}(c_0) + \int u_{1\lambda}(\tilde{c}_1) d(w \circ \mathbf{P}).$$

Then, condition (4.4) in this economy is equivalent to the condition that $u'_{1\lambda}(Q_{\tilde{e}_1}(p))\bar{w}'(p)$ be strictly decreasing, i.e., that the function Ψ_λ defined in Section 5, Xia and Zhou (2016), be strictly increasing.

Next, let us examine a market where there are two types of agents, agent 1 with an RDUT preference and agent 2 with a classical EUT preference. Then $w_2(q) = q, \bar{w}'_2(q) = 1, \bar{w}''_2(q) = 0$. In this case, we can easily see that

$$\begin{aligned}
\Gamma(p, \{1\}) &= -\frac{u'_{11}(Q_{\tilde{e}_1}(p))}{u''_{11}(Q_{\tilde{e}_1}(p))} \frac{\bar{w}''_1(p)}{\bar{w}'_1(p)}, \\
\Gamma(p, \{2\}) &= 0, \\
\Gamma(p, \{1,2\}) &= \max\left\{0, \sup_{0<x<Q_{\tilde{e}_1}(p)} \left\{-\frac{u'_{11}(x)}{u''_{11}(x)} \frac{\bar{w}''_1(p)}{\bar{w}'_1(p)}\right\}\right\}, \\
\Gamma(p) &= \sup_{0<x\le Q_{\tilde{e}_1}(p)} \left\{-\frac{u'_{11}(x)}{u''_{11}(x)} \frac{(\bar{w}''_1(p))^+}{\bar{w}'_1(p)}\right\}.
\end{aligned}$$

Hence, given that $dQ_{\tilde{e}_1}(p) \succ 0$, condition (4.4) reduces to

$$dQ_{\tilde{e}_1}(p) \succ -\frac{u'_{11}(x)}{u''_{11}(x)} \frac{\bar{w}''_1(p)}{\bar{w}'_1(p)} dp, \quad \forall 0 < x \le Q_{\tilde{e}_1}(p).$$

Moreover, if the Arrow–Pratt measure of absolute risk aversion (ARA) of agent 1, $-\frac{u''_{11}(x)}{u'_{11}(x)}$, is decreasing in x,[11] then the above condition is equivalent to

$$dQ_{\tilde{e}_1}(p) \succ \Gamma(p, \{1\}) dp,$$

which is the same condition (4.4) for a market with *only* agent 1. In other words, the existence of the expected utility agent in this case has no bearing on condition (4.4).

Now consider an economy with two RDUT agents with utility functions u_i and weighting functions w_i, $i = 1, 2$. Denote $\mathcal{A}_i(x) = -u''_i(x)/u'_i(x)$ as the ARA, and $\mathcal{B}_i(p) = \frac{\bar{w}''_i(p)}{\bar{w}'_i(p)}$, $i = 1, 2$. Then,

$$\begin{aligned}
\Gamma(p, \{i\}) &= \frac{\mathcal{B}_i(p)}{\mathcal{A}_i(Q_{\tilde{e}_1}(p))}, \quad i = 1, 2, \\
\Gamma(p, \{1,2\}) &= \sup_{x_1+x_2<Q_{\tilde{e}_1}(p), x_1>0, x_2>0} \left\{\frac{\mathcal{B}_1(p)}{\mathcal{A}_1(x_1)} + \frac{\mathcal{B}_2(p)}{\mathcal{A}_2(x_2)}\right\} = \Gamma(p).
\end{aligned}$$

[11]Decreasing ARA, as with a power utility function, is consistent with the common investment behavior that an individual experiencing an increase in wealth will choose to increase the risky exposure in his/her portfolios.

If $u_i(x) = \frac{x^{1-\gamma_i}}{1-\gamma_i}$ for some $\gamma_i > 0$, then $\mathcal{A}_i(x) = \frac{\gamma_i}{x}$. Hence

$$\begin{aligned}
\Gamma(p, \{i\}) &= \frac{Q_{\tilde{e}_1}(p)}{\gamma_i}\mathcal{B}_i(p), \quad i = 1, 2, \\
\Gamma(p, \{1, 2\}) &= \sup_{x_1+x_2<Q_{\tilde{e}_1}(p), x_1>0, x_2>0} \left\{ x_1 \frac{\mathcal{B}_1(p)}{\gamma_1} + x_2 \frac{\mathcal{B}_2(p)}{\gamma_2} \right\} \\
&= Q_{\tilde{e}_1}(p) \max_{i=1,2} \frac{\mathcal{B}_i(p)}{\gamma_i} = \Gamma(p).
\end{aligned}$$

In this case, the condition (4.4) reduces to (assuming $Q_{\tilde{e}_1}$ is differentiable)

$$\max_{i=1,2} \frac{\mathcal{B}_i(p)}{\gamma_i} < \frac{Q'_{\tilde{e}_1}(p)}{Q_{\tilde{e}_1}(p)}$$

or

$$\max_{i=1,2} \frac{1}{\gamma_i} \frac{d}{dp} \ln \bar{w}'_i(p) < \frac{d}{dp} \ln Q_{\tilde{e}_1}(p).$$

It is easy to verify that the above condition is equivalent to the condition that $u'_i(Q_{\tilde{e}_1}(p))\bar{w}'_i(p)$ be strictly decreasing for both agent i. Refer to Example 5.8 of Xia and Zhou (2016) for a concrete example in which this condition holds with Prelec's weighting functions and Pareto distributed $\tilde{e}_1$.

Finally, if the probability weighting functions w_i are convex, $i = 1, \ldots, I$, then $\frac{\bar{w}''_i(p)}{\bar{w}'_i(p)} \leq 0$ for all p and, therefore, $\Gamma(p) \leq 0$ for all p. In this case, condition (4.4) holds automatically.

6 Arrow-Debreu Equilibria

We are now ready to present the existence result for the original Arrow–Debreu equilibria. The main idea is that a price equilibrium with transfers, $\left\{\tilde{\rho},\ (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I\right\}$, is an Arrow-Debreu equilibrium if

$$e_{0i} + \mathbf{E}[\tilde{\rho}\tilde{e}_{1i}] = c^*_{0i} + \mathbf{E}[\tilde{\rho}\tilde{c}^*_{1i}],\ \forall i.$$

As $\tilde{\rho}$, c^*_{0i} and $\tilde{c}^*_{1i}$ correspond to some $\theta \in \Delta$, the problem becomes to find $\theta \in \Delta$ such that the previous equalities hold. This can be achieved by using the standard continuity arguments and a fixed-point theorem, see, e.g., Chapter 6 of Dana and Jeanblac (2007) and Section 3.6 of Föllmer and Schied (2011). We present the main existence theorem as follows without a proof, as the arguments are standard based on Theorems 4.3 and 4.4.

Theorem 6.1. *Under Assumption 3.6, assume further that* $\mathbf{E}[\tilde{e}_1] < \infty$, $Q_{\tilde{e}_1}(0) > 0$, w_i *(*$i = 1, \ldots, I$*) is twice continuously differentiable on* $(0, 1]$, Γ *is Lebesgue integrable and* $dQ_{\tilde{e}_1}(p) \succ \Gamma(p)\, dp$. *Then there exists an Arrow-Debreu equilibrium.*

Under the assumptions of Theorem 6.1, we can devise a numerical algorithm to compute the state-price density along with the equilibrium allocation, with the following steps:

1. Choose starting weights $\theta_0 \in \Delta$ and a small positive error tolerance ε, and set $\theta = \theta_0$;
2. Solve problem (3.4) to get $G_i^*(\cdot)$ (this can be done numerically following the algorithm constructed by Carlier and Lachapelle (2011));
3. Determine $M^*(\cdot)$ via (3.12) and (3.11), μ_θ via (3.6), c_{0i} via (3.5), and $\tilde{\rho}$ via (4.2);
4. Evaluate $\epsilon_i = e_{0i} + \mathbf{E}[\tilde{\rho}\tilde{e}_{1i}] - c_{0i} - \mathbf{E}[\tilde{\rho}G_i^*(F_{\tilde{e}_1}(\tilde{e}_1))]$ for $i = 1, \cdots, I$;
5. If $\sum |\epsilon_i| < \varepsilon$, go to Step 6; otherwise, follow the standard fixed-point argument (as in Dana and Jeanblac (2007) and Föllmer and Schied (2011)) to update θ and go to Step 2;
6. $(\tilde{\rho}, (G_i^*(F_{\tilde{e}_1}(\tilde{e}_1)))_{i=1,\cdots,I})$ is the pair of state-price density and equilibrium allocation.

In terms of the numerical computation, the key step is Step 2 above to compute the comonotone Pareto optima, which was proposed by Carlier and Lachapelle (2011). As the utilities are concave in that paper, the comonotone Pareto optima are automatically supported by state-price densities; see the discussion in the last paragraph of Section 5. When utilities are not concave, as in our paper, we need the additional assumption (4.4) to ensure that the comonotone Pareto optimal allocations can be supported by atomless state-price densities, whose computation is standard.

We now use an example to illustrate the algorithm above. Consider a market with two agents. Agent 1 has RDUT preference and agent 2 follows EUT. Both agents have the same utility function at $t = 0$ and $t = 1$, which is

$$u(x) = \begin{cases} 8\sqrt{x} - 3x & \text{if } 0 \le x < 1 \\ -\frac{1}{x} + 6 & \text{if } x > 1, \end{cases}$$

which is clearly C^2 with $u(0) = 0$. Moreover, agent 1's weighting function is Prelec's

$$w(p) = e^{-\sqrt{-\ln p}}, \quad p \in (0,1).$$

We assume the two agents have exactly the same endowments

$$e_{01} = e_{02} = 1, \;\; \tilde{e}_{11} = \tilde{e}_{12} = \tilde{e}_1/2,$$

where the distribution function of $\tilde{e}_1$ is

$$F_{\tilde{e}_1}(y) = \begin{cases} 0, & \text{if } y < 1, \\ 1 - y^{-2}, & \text{if } y \geq 1. \end{cases}$$

It is not hard to check that Assumptions 2.1 and 3.6 hold. The condition $Q_{\tilde{e}_1}(0) = 1 > 0$ required in Theorem 4.4 is also obvious. We now only need to check condition (4.4).

With $w(p) = e^{-\sqrt{-\ln p}}$, it is straightforward to compute

$$\frac{\bar{w}''(p)}{\bar{w}'(p)} = \frac{1}{2}\frac{1}{1-p}\left(2 - \frac{1}{-\ln(1-p)} - \frac{1}{\sqrt{-\ln(1-p)}}\right), \; p \in (0,1).$$

Hence (4.4) is equivalent to

$$dQ_{\tilde{e}_1}(p) > \frac{Q_{\tilde{e}_1}(p)}{2}\frac{1}{2}\frac{1}{1-p}\left(2 - \frac{1}{-\ln(1-p)} - \frac{1}{\sqrt{-\ln(1-p)}}\right) dp, \; p \in (0,1). \quad (6.1)$$

Noting $F'_{\tilde{e}_1}(y) = (1 - F_{\tilde{e}_1}(y))\frac{2}{y}$ for $y \geq 1$, we have, for any $p = F_{\tilde{e}_1}(y) \in (0,1)$, $F'_{\tilde{e}_1}(Q_{\tilde{e}_1}(p)) = \frac{2(1-p)}{Q_{\tilde{e}_1}(p)}$. Hence

$$dQ_{\tilde{e}_1}(p) = \frac{dp}{F'_{\tilde{e}_1}(Q_{\tilde{e}_1}(p))} = \frac{Q_{\tilde{e}_1}(p)}{2(1-p)}dp,$$

leading immediately to (6.1). Theorem 6.1 then dictates that there exists an Arrow-Debreu equilibrium.

We have implemented the general algorithm in Matlab for this example with the initial $\theta_0 = (\frac{1}{2}, \frac{1}{2})$ and the error bound $\varepsilon = \frac{1}{100}\frac{e_0 + \mathbf{E}[\tilde{e}_1]}{2} = 0.02$. The algorithm stopped at the end of the 11th iteration when $\sum_i |\epsilon_i|$ fell below ε for the first time.

The equilibrium consumptions at $t = 0$ are $(c^*_{01}, c^*_{02}) = (1.0591, 0.9409)$. The other part of the Arrow-Debreu equilibrium, namely $(\tilde{\rho}, \tilde{c}^*_{11}, \tilde{c}^*_{12})$, consists of random variables. We plot them as functions of $\tilde{z} = F_{\tilde{e}_1}(\tilde{e}_1)$ in Figure 1.

We have also calculated the optimal weights $(\theta^*_1, \theta^*_2) = (0.5569, 0.4431)$, and on average $\mathbf{E}[\tilde{c}^*_{11}] = 0.9698 < \mathbf{E}[\tilde{c}^*_{12}] = 0.9845$.

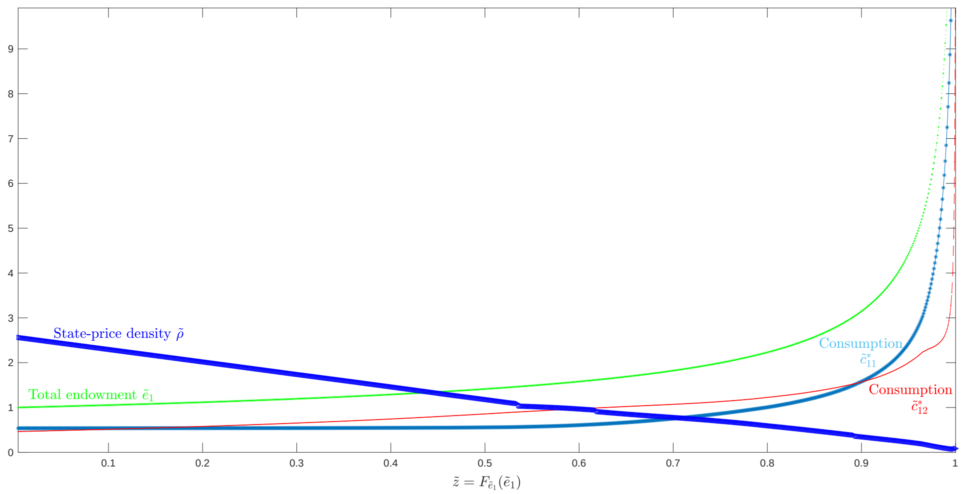

Figure 1: Plots of state-price density, total random endowment, and future consumptions of the two agents as functions of $\tilde{z} = F_{\tilde{e}_1}(\tilde{e}_1)$

7 Concluding Remarks

With the homogeneity of the weighting functions, Xia and Zhou (2016) derive an explicit formula for the state-price density $\tilde{\rho}$, expressed as a weighted marginal rate of substitution between initial and end-of-period consumption of the representative agent. This expression in turn enables them to study the impact of probability weighting on the equity premium. While (4.2) derived in the present paper is an analytically explicit formula for $\tilde{\rho}$, it involves the function M^* that is difficult to analyze. If it is too hard to have a similar result for all states of nature, we may instead attempt to derive a formula only in the so-called "tail states", namely those states in which the total endowment has extremely large or small values. After all, in typical inverse-S shaped models, probability weighting becomes impactful only in the tail states. A study on this aspect and its implications for the equity premium will be reported in a forthcoming paper.

Appendices

A The Functions M_i^* and M^*

The following result characterizes the solution to (3.4) in terms of M_i^*, $i \in \mathcal{I}_\theta$.

Lemma A.1. *Suppose Assumption 3.6 holds. Let* $(G_i^*)_{i=1}^I \in \mathscr{G}_\theta$ *and* M_i^* $(i \in \mathcal{I}_\theta)$ *be given by* (3.11)–(3.12)*. Then*

$$\sum_{i\in\mathcal{I}_\theta} \int_{[0,1)} M_i^*(q)\, dG_i^*(q) > -\infty \tag{A.1}$$

and the following statements are equivalent:

(i) $(G_i^*)_{i=1}^I$ *solves problem* (3.4)*;*

(ii) *For all* $(G_i)_{i=1}^I \in \mathscr{G}$*, we have*

$$\sum_{i\in\mathcal{I}_\theta} \int_{[0,1)} M_i^*(q)\, dG_i(q) \geq \sum_{i\in\mathcal{I}_\theta} \int_{[0,1)} M_i^*(q)\, dG_i^*(q). \tag{A.2}$$

Moreover, in the case when $Q_{\tilde{e}_1}(0) > 0$*,* (A.1)–(A.2) *imply that* $M_i^*(0) > -\infty$ *for all* $i \in \mathcal{I}_\theta$*.*

Proof. For all $(G_i)_{i=1}^I \in \mathscr{G}$ and $i \in \mathcal{I}_\theta$,[12]

$$\begin{aligned}
&\int_{(0,1)} \theta_i\beta_i u_{1i}'(G_i^*(p))G_i(p)\, d\bar{w}_i(p) \\
=\ & \int_{(0,1)} \theta_i\beta_i u_{1i}'(G_i^*(p)) \left(G_i(0) + \int_{(0,p]} dG_i(q)\right) d\bar{w}_i(p) \\
=\ & -M_i^*(0)G_i(0) + \int_{(0,1)} \left(\int_{[q,1)} \theta_i\beta_i u_{1i}'(G_i^*(p))\, d\bar{w}_i(p)\right) dG_i(q) \\
=\ & -M_i^*(0)G_i(0) - \int_{(0,1)} M_i^*(q)\, dG_i(q) \\
=\ & -\int_{[0,1)} M_i^*(q)\, dG_i(q),
\end{aligned} \tag{A.3}$$

where the second equality follows from Fubini's theorem. Thus Proposition 3.9 implies the first part of the lemma.

[12] As Remark A.2 below shows, it is possible that $M_i^*(0) = -\infty$ when $Q_{\tilde{e}_1}(0) = 0$. In this case $G_i^*(0) = 0$; thus the value of $M_i^*(0)G_i^*(0)$ should be clarified. Indeed,

$$\begin{aligned}
0 \geq M_i^*(0)G_i^*(0) = \lim_{q\downarrow 0} (M_i^*(q)G_i^*(q)) \quad &\geq \quad -\lim_{q\downarrow 0}\left(\int_{(q,1)} \theta_i\beta_i u_{1i}'(G_i^*(q)G_i^*(q))\, d\bar{w}_i(p)\right) \\
&\geq \quad -\lim_{q\downarrow 0}\left(\int_{(q,1)} \theta_i\beta_i u_{1i}(G_i^*(q))\, d\bar{w}_i(p)\right) = 0
\end{aligned}$$

by the concavity of u_{1i}, $u_{1i}(0) = 0$, and (3.8). Hence $M_i^*(0)G_i^*(0) = 0$.

Now assume (A.1)–(A.2) hold. For any fixed $i \in \mathcal{I}_\theta$, let $G_i = Q_{\tilde{e}_1}$ and $G_j \equiv 0$ for all $j \neq i$. Then $(G_j)_{j=1}^I \in \mathscr{G}$. By (A.1)–(A.2),

$$M_i^*(0)Q_{\tilde{e}_1}(0) + \int_{(0,1)} M_i^*(q)\, dQ_{\tilde{e}_1}(q) > -\infty,$$

which implies $M_i^*(0) > -\infty$ given that $Q_{\tilde{e}_1}(0) > 0$. □

Remark A.2. Consider the case when $Q_{\tilde{e}_1}(0) = 0$. On one hand, $M_i^*(0) = -\infty$ is possible. A simple example is when

$$\int_{(0,1)} u'_{1i}(Q_{\tilde{e}_1}(p))d\bar{w}_i(p) = +\infty,$$

which is obviously possible. On the other hand, $M_i^*(0) > -\infty$ is also possible. For example, if all the agents' preferences are identical, i.e., $u_i(\cdot) = u(\cdot), w_i(\cdot) = w(\cdot)$, and

$$\int_{(0,1)} u'\left(Q_{\frac{1}{|I|}\tilde{e}_1}(p)\right) d\bar{w}(p) < +\infty.$$

Then for $\theta_i = \frac{1}{I}$, $G_i^*(\cdot) = \frac{1}{I}Q_{\tilde{e}_1}(\cdot)$, and $M_i^*(0) = -\frac{1}{I}\int_{(0,1)} u'(Q_{\frac{1}{|I|}\tilde{e}_1}(p)d\bar{w}(p) > -\infty$. The case $M_i^*(0) = -\infty$ entails a subtle technical difficulty. As seen in Section 4, $M_i^*(0) = -\infty$ leads to the supporting pricing kernel ρ, if it exists, being not integrable, and hence not qualified as a pricing kernel.[13] This is the reason why we need to impose the condition $Q_{\tilde{e}_1}(0) > 0$ to rule out this case. Note that the last condition is equivalent to the requirement that tomorrow's aggregate endowment, $\tilde{e}_1$, has a positive lower bound almost surely, a plausible assumption.

The following characterizes problem (3.4) in terms of both M_i^* and M^*.

Proposition A.3. *Suppose $Q_{\tilde{e}_1}(0) > 0$ and Assumption 3.6 holds. Let $(G_i^*)_{i=1}^I \in \mathscr{G}_\theta$, M_i^* ($i \in \mathcal{I}_\theta$) be given by* (3.11)–(3.12)*, and M^* be given by* (3.13)*. Then the following statements are equivalent:*

(i) *$(G_i^*)_{i=1}^I$ solves problem* (3.4)*;*

(ii) *$M^*(\cdot)$ is bounded, continuous and strictly increasing on* $[0, 1]$*; for all $i \in \mathcal{I}_\theta$, $M_i^*(0) = M^*(0) > -\infty$ and*

$$\int_{(0,1)} M^*(p)\, dG_i^*(p) = \int_{(0,1)} M_i^*(p)\, dG_i^*(p). \tag{A.4}$$

Proof.

[13] Note that $-M_i^*(0)$ can be interpreted as agent i's marginal utility of future consumption (see Section 3). So if $M_i^*(0) = -\infty$, then the marginal utility is infinite, and the agent is infinitely eager to transfer consumption from time 0 to time 1. In this case, the average state-price for consumption in the future is infinity and hence non-integrable.

(i)⇒(ii) Assume (i) holds true. Then $M_i^*(0) > -\infty$ for all $i \in \mathcal{I}_\theta$, according to Lemma A.1. Thus $M^*(\cdot)$ is bounded, continuous and strictly increasing on $[0,1]$.

We now prove (A.4). Suppose, to the contrary of (A.4), that

$$\int_{(0,1)} M^*(p)\, dG_j^*(p) < \int_{(0,1)} M_j^*(p)\, dG_j^*(p). \tag{A.5}$$

for some $j \in \mathcal{I}_\theta$. Let

$$A \triangleq \{p \in (0,1) : M^*(p) < M_j^*(p)\}$$

and

$$i(p) \triangleq \min\{i \in \mathcal{I}_\theta : M_i^*(p) = M^*(p)\}$$

for all p. Let G_j° be given by

$$G_j^\circ(0) = G_j^*(0),\ dG_j^\circ(p) = \mathbf{1}_{p \notin A} dG_j^*(p)$$

and, for all $i \neq j$, let G_i° be given by

$$G_i^\circ(0) = G_i^*(0),\ dG_i^\circ(p) = dG_i^*(p) + \mathbf{1}_{p \in A}\mathbf{1}_{i=i(p)} dG_j^*(p).$$

It is easy to see that $(G_i^\circ)_{i=1}^I \in \mathscr{G}$ and

$$\begin{aligned}
& \sum_{i\in\mathcal{I}_\theta} \int_{[0,1)} M_i^*(p)\, dG_i^\circ(p) - \sum_{i\in\mathcal{I}_\theta} \int_{[0,1)} M_i^*(p)\, dG_i^*(p) \\
= & \sum_{i\in\mathcal{I}_\theta, i\neq j} \int_{(0,1)} M_i^*(p)\mathbf{1}_{p\in A}\mathbf{1}_{i=i(p)}\, dG_j^*(p) - \int_{(0,1)} M_j^*(p)\mathbf{1}_{p\in A}\, dG_j^*(p) \\
= & \sum_{i\in\mathcal{I}_\theta, i\neq j} \int_{(0,1)} M^*(p)\mathbf{1}_{p\in A}\mathbf{1}_{i=i(p)}\, dG_j^*(p) - \int_{(0,1)} M_j^*(p)\mathbf{1}_{p\in A}\, dG_j^*(p) \\
= & \int_{(0,1)} M^*(p)\mathbf{1}_{p\in A}\, dG_j^*(p) - \int_{(0,1)} M_j^*(p)\mathbf{1}_{p\in A}\, dG_j^*(p) \\
= & \int_{(0,1)} M^*(p)\, dG_j^*(p) - \int_{(0,1)} M_j^*(p)\, dG_j^*(p) \\
< & \ 0.
\end{aligned}$$

This is impossible, by Lemma A.1. Thus we have proved (A.4).

Now we prove that $M_i^*(0) = M^*(0)$ for any $i \in \mathcal{I}_\theta$. Otherwise, there exists an $i \in \mathcal{I}_\theta$ such that $M_i^*(0) > M^*(0)$. By the continuity of M_i^* and M^*, there exists a

$\delta \in (0,1)$ such that $M_i^*(p) - M^*(p) > \frac{M_i^*(0)-M^*(0)}{2} > 0$ for all $p \in [0,\delta]$. Therefore,

$$\begin{aligned} 0 &= \int_{(0,1)} (M_i^*(p) - M^*(p))\, dG_i^*(p) \\ &\geq \int_{(0,\delta]} (M_i^*(p) - M^*(p))\, dG_i^*(p) \\ &> \int_{(0,\delta]} \frac{M_i^*(0) - M^*(0)}{2}\, dG_i^*(p) \\ &= \frac{M_i^*(0) - M^*(0)}{2} (G_i^*(\delta) - G_i^*(0)) \\ &\geq 0, \end{aligned}$$

which implies that $G_i^*(0) = G_i^*(\delta) > 0$. Now take any $j \in \mathcal{I}_\theta$ such that $M_j^*(0) = M^*(0) < M_i^*(0)$, and consider a $(G_i^\varepsilon)_{i=1}^I \in \mathscr{G}$ given by

$$G_i^\varepsilon(\cdot) = G_i^*(\cdot) - \varepsilon,\ G_j^\varepsilon(\cdot) = G_j^*(\cdot) + \varepsilon,\ G_k^\varepsilon(\cdot) = G_k^*(\cdot) \text{ for all } k \notin \{i,j\},$$

where $\varepsilon \in (0, G_i^*(0))$. Then

$$\begin{aligned} &\sum_{k\in\mathcal{I}_\theta} \left\{ \int_{[0,1)} M_k^*(p)\, dG_k^\varepsilon(p) - \int_{[0,1)} M_k^*(p)\, dG_k^*(p) \right\} \\ =\ & M_j^*(0)\varepsilon - M_i^*(0)\varepsilon \\ <\ & 0, \end{aligned}$$

which contradicts (A.2) in Lemma A.1. Thus we have $M_i^*(0) = M^*(0)$ for all $i \in \mathcal{I}_\theta$.

(ii)⇒(i) Assume (ii) holds true. For any $(G_i)_{i=1}^I \in \mathscr{G}$, we have

$$\begin{aligned} &\sum_{i\in\mathcal{I}_\theta} \int_{[0,1)} M_i^*(p)\, dG_i(p) \\ =\ & \sum_{i\in\mathcal{I}_\theta} M^*(0)G_i(0) + \sum_{i\in\mathcal{I}_\theta} \int_{(0,1)} M_i^*(p)\, dG_i(p) \\ \geq\ & \sum_{i\in\mathcal{I}_\theta} M^*(0)G_i(0) + \sum_{i\in\mathcal{I}_\theta} \int_{(0,1)} M^*(p)\, dG_i(p) \\ =\ & \sum_{i\in\mathcal{I}_\theta} M^*(0)G_i^*(0) + \sum_{i\in\mathcal{I}_\theta} \int_{(0,1)} M^*(p)\, dG_i^*(p) \\ =\ & \sum_{i\in\mathcal{I}_\theta} \left\{ M_i^*(0)G_i^*(0) + \int_{[0,1)} M_i^*(p)\, dG_i^*(p) \right\} \\ =\ & \sum_{i\in\mathcal{I}_\theta} \int_{(0,1)} M_i^*(p)\, dG_i^*(p), \end{aligned}$$

which implies (i), by Lemma A.1. □

Lemma A.4. *Suppose $Q_{\tilde{e}_1}(0) > 0$ and Assumption 3.6 holds. Let $(G_i^*)_{i=1}^I \in \mathscr{G}$ solve problem (3.4). For any $i \in \mathcal{I}_\theta$, let M_i^* be given by (3.11)–(3.12) and M^* be given by (3.13), and let*

$$\begin{cases} M_i^\circ(q) = M_i^*(\bar{w}_i^{-1}(q)) \\ M_i^\bullet(q) = M^*(\bar{w}_i^{-1}(q)) \end{cases}$$

for all $q \in [0,1]$. Then M_i° is the concave envelope of $M_i^\bullet$ for all $i \in \mathcal{I}_\theta$.

Proof. Firstly, for any fixed $i \in \mathcal{I}_\theta$, both of M_i° and $M_i^\bullet$ are bounded and continuous. Moreover,

$$\begin{aligned} &M_i^\circ(1) = M_i^*(1) = 0 = M^*(1) = M_i^\bullet(1), \\ &M_i^\circ(0) = M_i^*(0) = M^*(0) = M_i^\bullet(0) > -\infty, \quad \text{(by Proposition A.3)} \end{aligned}$$

and $M_i^\circ(q) \ge M_i^\bullet(q)$ for all $q \in [0,1]$.

Secondly,

$$M_i^\circ(q) = -\int_{\left(\bar{w}_i^{-1}(q),1\right)} \theta_i\beta_i u_{1i}'(G_i^*(p))\, d\bar{w}_i(p) = -\int_{(q,1)} \theta_i\beta_i u_{1i}'(G_i^*(\bar{w}_i^{-1}(p)))\, dp. \quad \text{(A.6)}$$

Thus M_i° is concave on $[0,1]$ because $u_{1i}'(G_i^*(\bar{w}_i^{-1}(\cdot)))$ is decreasing.

Finally, by Proposition A.3, we have

$$\begin{aligned} &\int_{(0,1)} (M_i^\circ(q) - M_i^\bullet(q))\, dG_i^*(\bar{w}_i^{-1}(q)) \\ =&\int_{(0,1)} (M_i^*(\bar{w}_i^{-1}(q)) - M^*(\bar{w}_i^{-1}(q)))\, dG_i^*(\bar{w}_i^{-1}(q)) \\ =&\int_{(0,1)} (M_i^*(p) - M^*(p))\, dG_i^*(p) \\ =&0, \end{aligned}$$

which implies that $G_i^*(\bar{w}_i^{-1}(\cdot))$ is flat on $\{q \in (0,1) : M_i^\circ(q) > M_i^\bullet(q)\}$. Then by (A.6), M_i° is affine on $\{q \in (0,1) : M_i^\circ(q) > M_i^\bullet(q)\}$. By definition, M_i° is the concave envelope of $M_i^\bullet$. $\square$

Lemma A.5. *Suppose Assumption 3.6 holds and w_i is continuously differentiable on $(0,1)$ for all $i = 1, \ldots, I$. Let $(G_i^*)_{i=1}^I \in \mathscr{G}_\theta$ and M_i^* ($i \in \mathcal{I}_\theta$) be given by (3.11)–(3.12). Then for every $i \in \mathcal{I}_\theta$ and $p \in (0,1)$, both of the right derivative $(M_i^*)_+'(p)$ and the left derivative $(M_i^*)_-'(p)$ exist, and they are given by*

$$\begin{aligned} (M_i^*)_+'(p) &= \theta_i\beta_i u_{1i}'(G_i^*(p))\bar{w}_i'(p), \\ (M_i^*)_-'(p) &= \theta_i\beta_i u_{1i}'(G_i^*(p-))\bar{w}_i'(p). \end{aligned}$$

Proof. Obvious and omitted. $\square$

The following proposition gives the most important property of M^* essential for deriving the Arrow–Debreu equilibria, namely the right and left derivatives of M^*.

Proposition A.6. *Assume w_i is continuously differentiable on $(0,1)$ for all $i = 1,\dots,I$. Then under the conditions of Proposition A.3, we have the following statements:*

(i) *For all $p \in (0,1)$, the right derivative $(M^*)'_+(p)$ exists and*

$$(M^*)'_+(p) = \min_{i\in\mathcal{I}_\theta(p)} (M_i^*)'_+(p), \tag{A.7}$$

where $\mathcal{I}_\theta(p) \triangleq \{i \in \mathcal{I}_\theta : \ M_i^*(p) = M^*(p)\}$;

(ii) *For all $p \in (0,1)$, the left derivative $(M^*)'_-(p)$ exists and*

$$(M^*)'_-(p) = \max_{i\in\mathcal{I}_\theta(p)} (M_i^*)'_-(p); \tag{A.8}$$

(iii) *$(M^*)'_+$ is right continuous and has left limits on $(0,1)$ and $(M^*)'_+(p-) = (M^*)'_-(p)$ for all $p \in (0,1)$.*

Proof.

(i) For any fixed $p \in (0,1)$, by the continuity of M_i^* ($i \in \mathcal{I}_\theta$) and M^*, there exists a $\delta_1 \in (0, 1-p)$ such that $M_i^*(q) > M^*(q)$ for all $q \in (p, p+\delta_1)$ and for all $i \in \mathcal{I}_\theta \setminus \mathcal{I}_\theta(p)$. This combined with Lemma A.5 yields, for any $\varepsilon \in (0,\delta_1)$, that

$$\begin{aligned}
\frac{M^*(p+\varepsilon) - M^*(p)}{\varepsilon} &= \min_{i\in\mathcal{I}_\theta(p)} \frac{M_i^*(p+\varepsilon) - M^*(p)}{\varepsilon} \\
&= \min_{i\in\mathcal{I}_\theta(p)} \frac{M_i^*(p+\varepsilon) - M_i^*(p)}{\varepsilon} \\
&\xrightarrow{\varepsilon\downarrow 0} \min_{i\in\mathcal{I}_\theta(p)} (M_i^*)'_+(p).
\end{aligned}$$

(ii) For any fixed $p \in (0,1)$, by the continuity of M_i^* ($i \in \mathcal{I}_\theta$) and M^*, there exists a $\delta_1 \in (0, p)$ such that $M_i^*(q) > M^*(q)$ for all $q \in (p-\delta_1, p)$ and for all $i \in \mathcal{I}_\theta \setminus \mathcal{I}_\theta(p)$. This combined with Lemma A.5 yields, for any $\varepsilon \in (0,\delta_1)$, that

$$\begin{aligned}
\frac{M^*(p-\varepsilon) - M^*(p)}{-\varepsilon} &= \max_{i\in\mathcal{I}_\theta(p)} \frac{M_i^*(p-\varepsilon) - M^*(p)}{-\varepsilon} \\
&= \max_{i\in\mathcal{I}_\theta(p)} \frac{M_i^*(p-\varepsilon) - M_i^*(p)}{-\varepsilon} \\
&\xrightarrow{\varepsilon\downarrow 0} \max_{i\in\mathcal{I}_\theta(p)} (M_i^*)'_-(p).
\end{aligned}$$

(iii)

(a) We first prove that $(M^*)'_+$ is right continuous at any fixed $p \in (0,1)$.

Let $\delta_1 > 0$ be given as in the proof of (i). Then $\mathcal{I}_\theta(q) \subseteq \mathcal{I}_\theta(p)$ for all $q \in (p, p+\delta_1)$.

For any $\varepsilon > 0$, there exists a $\delta_2 \in (0, 1-p)$ such that

$$|(M_i^*)'_+(q) - (M_i^*)'_+(p)| < \varepsilon \text{ for all } q \in (p, p+\delta_2) \text{ and } i \in \mathcal{I}_\theta.$$

For any fixed $\{i,j\} \subseteq \mathcal{I}_\theta(p)$, if $(M_j^*)'_+(p) > (M_i^*)'_+(p)$, then by the right continuity of $(M_j^*)'_+$ and $(M_i^*)'_+$, there exists a $\delta_3(i,j) \in (0, 1-p)$ such that $(M_j^*)'_+(q) > (M_i^*)'_+(q)$ for all $q \in (p, p+\delta_3(i,j))$. Then by integrating both $(M_j^*)'_+$ and $(M_i^*)'_+$ on (p,q), we have $M_j^*(q) > M_i^*(q) \geq M^*(q)$, which implies $j \notin \mathcal{I}_\theta(q)$, for all $q \in (p, p+\delta_3(i,j))$. Let

$$\delta_3 = \min\{\delta_3(i,j) : \{i,j\} \subseteq \mathcal{I}_\theta(p) \text{ and } (M_j^*)'_+(p) > (M_i^*)'_+(p)\}.$$

If $j \in \mathcal{I}_\theta(p) \bigcap \mathcal{I}_\theta(q)$ for some $q \in (p, p+\delta_3)$, then $(M_j^*)'_+(p) = (M^*)'_+(p)$.

Now let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. For any $q \in (p, p+\delta)$, by statement (i), there exists a $j \in \mathcal{I}_\theta(q) \subseteq \mathcal{I}_\theta(p)$ such that $(M_j^*)'_+(q) = (M^*)'_+(q)$. By the above analysis, we know $(M_j^*)'_+(p) = (M^*)'_+(p)$. Therefore,

$$|(M^*)'_+(q) - (M^*)'_+(p)| = |(M_j^*)'_+(q) - (M_j^*)'_+(p)| < \varepsilon,$$

implying the right continuity of $(M^*)'_+$ at p.

(b) Now we prove that $(M^*)'_+(p-)$ exists and $(M^*)'_+(p-) = (M^*)'_-(p)$ for any fixed $p \in (0,1)$.

Let $\delta_1 > 0$ be given as in the proof of (ii). Then $\mathcal{I}_\theta(q) \subseteq \mathcal{I}_\theta(p)$ for all $q \in (p-\delta_1, p)$.

Obviously, we have $(M_i^*)'_+(p-) = (M_i^*)'_-(p)$ for all $i \in \mathcal{I}_\theta$, by Lemma A.5. Thus, for any $\varepsilon > 0$, there exists a $\delta_4 \in (0,p)$ such that

$$|(M_i^*)'_+(q) - (M_i^*)'_-(p)| < \varepsilon \text{ for all } q \in (p-\delta_4, p) \text{ and } i \in \mathcal{I}_\theta.$$

For any fixed $\{i,j\} \subseteq \mathcal{I}_\theta(p)$, if $(M_j^*)'_-(p) < (M_i^*)'_-(p)$, then by the left continuity of $(M_j^*)'_-$ and $(M_i^*)'_-$, there exists a $\delta_5(i,j) \in (0,p)$ such that $(M_j^*)'_-(q) < (M_i^*)'_-(q)$ for all $q \in (p-\delta_5(i,j), p)$. Then by integrating both $(M_j^*)'_-$ and $(M_i^*)'_-$ on (q,p), we have $M_j^*(q) > M_i^*(q) \geq M^*(q)$, which implies $j \notin \mathcal{I}_\theta(q)$, for all $q \in (p-\delta_5(i,j), p)$. Let

$$\delta_5 = \min\{\delta_5(i,j) : \{i,j\} \subseteq \mathcal{I}_\theta(p) \text{ and } (M_j^*)'_-(p) < (M_i^*)'_-(p)\}.$$

If $j \in \mathcal{I}_\theta(p) \bigcap \mathcal{I}_\theta(q)$ for some $q \in (p - \delta_5, p)$, then $(M_j^*)'_-(p) = (M^*)'_-(p)$.

Now let $\delta = \min\{\delta_1, \delta_4, \delta_5\}$. For any $q \in (p-\delta, p)$, by statement (i), there exists a $j \in \mathcal{I}_\theta(q) \subseteq \mathcal{I}_\theta(p)$ such that $(M_j^*)'_+(q) = (M^*)'_+(q)$. By the above analysis, we know $(M_j^*)'_-(p) = (M^*)'_-(p)$. Therefore,

$$|(M^*)'_+(q) - (M^*)'_-(p)| = |(M_j^*)'_+(q) - (M_j^*)'_-(p)| < \varepsilon$$

for all $q \in (p - \delta, p)$, implying that $(M^*)'_+(p-)$ exists and $(M^*)'_+(p-) = (M^*)'_-(p)$. $\square$

B A Proof of Theorem 4.3

The "if" part. Assume M^* is strictly concave, then $(M^*)'_+$ is right-continuous and strictly decreasing on $(0,1)$. Moreover, M^* is increasing and therefore $(M^*)'_+$ is strictly positive on $(0,1)$. So we know that $\tilde{\rho}$ given by (4.2) is atomless and $\mathbf{P}(\tilde{\rho} > 0) = 1$. In particular, the lower quantile function $Q^-_{\tilde{\rho}}$ of $\tilde{\rho}$ is given by

$$Q^-_{\tilde{\rho}}(p) = \frac{1}{\mu_\theta}(M^*)'_+(1-p) \quad \text{for all } p \in (0,1).$$

It is easy to see that

$$\mathbf{E}[\tilde{\rho}] = \int_{(0,1)} Q^-_{\tilde{\rho}}(p)\,dp = -\frac{M^*(0)}{\mu_\theta} < \infty.$$

Furthermore,

$$\begin{aligned}
\mathbf{E}[\tilde{\rho}\tilde{e}_1] &= \frac{1}{\mu_\theta}\int_{(0,1)} (M^*)'_+(p) Q_{\tilde{e}_1}(p)\,dp \\
&= \frac{1}{\mu_\theta}\sum_{i\in\mathcal{I}_\theta}\int_{(0,1)} G_i^*(p)\,dM^*(p) \\
&= \frac{1}{\mu_\theta}\sum_{i\in\mathcal{I}_\theta}\int_{(0,1)} \left(G_i^*(0) + \int_{(0,p]} dG_i^*(q)\right) dM^*(p) \\
&= \frac{1}{\mu_\theta}\sum_{i\in\mathcal{I}_\theta}\left\{-M^*(0)G_i^*(0) + \int_{(0,1)}\left(\int_{[q,1)} dM^*(p)\right) dG_i^*(q)\right\} \\
&= \frac{1}{\mu_\theta}\sum_{i\in\mathcal{I}_\theta}\left\{-M^*(0)G_i^*(0) - \int_{(0,1)} M^*(q)\,dG_i^*(q)\right\} \\
&= -\frac{1}{\mu_\theta}\sum_{i\in\mathcal{I}_\theta}\int_{[0,1)} M_i^*(q)\,dG_i^*(q) \\
&< \infty. \qquad\qquad (\text{B.1})
\end{aligned}$$

Consequently, we know $\tilde{\rho}$ is a state-price density.

For any $i \in \mathcal{I}_\theta$, let

$$N_i(q) = -\int_{\left(\bar{w}_i^{-1}(q),1\right)} Q_{\tilde{\rho}}^-(1-p)\,dp \quad \text{for all } q \in [0,1).$$

Then

$$N_i(q) = -\int_{\left(\bar{w}_i^{-1}(q),1\right)} \frac{1}{\mu_\theta}(M^*)'_+(p)\,dp = \frac{1}{\mu_\theta}M^*(\bar{w}_i^{-1}(q)) \quad \text{for all } q \in [0,1).$$

It is easy to see that N_i is continuous on $[0,1)$ with $N_i(1-) = 0$. Let

$$\hat{N}_i(q) = \frac{1}{\mu_\theta}M_i^*(\bar{w}_i^{-1}(q)) \quad \text{for all } q \in [0,1). \tag{B.2}$$

Then by Lemma A.4, $\hat{N}_i$ is the concave envelope of N_i.

Now we are ready to show that $(c_{0i}^*, G_i^*)_{i=1}^I$ is supported by $\tilde{\rho}$. Obviously, $\hat{N}_i = \frac{1}{\mu_\theta}M_i^\circ$. Then by (A.6), the right derivative $(\hat{N}_i)'_+$ of $\hat{N}_i$ is given by

$$(\hat{N}_i)'_+(q) = \frac{\theta_i\beta_i}{\mu_\theta}u'_{1i}(G_i^*(\bar{w}_i^{-1}(q))),$$

or equivalently,

$$\frac{\mu_\theta(\hat{N}_i)'_+(\bar{w}_i(q))}{\theta_i\beta_i} = u'_{1i}(G_i^*(q)),$$

for all $q \in [0,1)$. Let

$$\tilde{c}_{1i}^* = (u'_{1i})^{-1}\left(\frac{\mu_\theta}{\theta_i\beta_i}(\hat{N}_i)'_+(\bar{w}_i(F_{\tilde{e}_1}(\tilde{e}_1)))\right),$$

then G_i^* is the upper quantile function of $\tilde{c}_{1i}^*$. Recall that $c_{0i}^* = (u'_{0i})^{-1}\left(\frac{\mu_\theta}{\theta_i}\right)$. It is not difficult to see that $(c_{0i}^*, \tilde{c}_{1i}^*)$ solves problem (4.1). Thus $(c_{0i}^*, G_i^*)_{i=1}^I$ is supported by $\tilde{\rho}$.

The "only if" part. Suppose $(c_{0i}^*, G_i^*)_{i=1}^I$ is supported by an atomless state-price density $\tilde{\rho}$, then for all $i \in \mathcal{I}_\theta$,

$$u'_{1i}(G_i^*(q)) = \lambda_i(\hat{N}_i)'_+(\bar{w}_i(q))$$

for some $\lambda_i > 0$, where $\hat{N}_i$ is the concave envelope of the function N_i given by

$$N_i(p) = -\int_{((\bar{w}_i)^{-1}(p),1)} Q_{\tilde{\rho}}^-(1-q)\,dq,$$

or equivalently,

$$N_i(\bar{w}_i(q)) = -\int_{(q,1)} Q_{\tilde{\rho}}^-(1-p)\,dp.$$

By the definitions of M_i^* and M^*, we have

$$\begin{aligned} M_i^*(q) &= -\int_{(q,1)} \theta_i\beta_i u_{1i}'(G_i^*(p))\,d\bar{w}_i(p) \\ &= -\int_{(q,1)} \theta_i\beta_i\lambda_i(\hat{N}_i)_+'(\bar{w}_i(p))\,d\bar{w}_i(p) \\ &= -\theta_i\beta_i\lambda_i\int_{(\bar{w}_i(q),1)} (\hat{N}_i)_+'(p)\,dp \\ &= \theta_i\beta_i\lambda_i\hat{N}_i(\bar{w}_i(q)). \end{aligned}$$

By Proposition A.3, we know $M_i^*(0) = M_j^*(0) > -\infty$ for all $i, j \in \mathcal{I}_\theta$. Together with the fact that $\hat{N}_i(0) = N_i(0) = -\mathbf{E}[\tilde{\rho}]$, we know $\theta_i\beta_i\lambda_i$ are same for all $i \in \mathcal{I}_\theta$. This implies $M_i^*(q) = \mu\hat{N}_i(\bar{w}_i(q))$ for some constant $\mu > 0$ and for all $i \in \mathcal{I}_\theta$.

By Proposition A.3, we know $M^*(q)$ satisfies

$$\sum_{i\in\mathcal{I}_\theta}\int_{(0,1)} [M^*(q) - M_i^*(q)]\,dG_i^*(q) = 0.$$

On the other hand, let $K(q) \triangleq -\mu\int_{(q,1)} Q_{\tilde{\rho}}^-(1-p)\,dp$, then for any $i \in \mathcal{I}_\theta$,

$$K(q) = \mu N_i(\bar{w}_i(q)) \le \mu\hat{N}_i(w_i(q)) = M_i^*(q) \tag{B.3}$$

and

$$\begin{aligned} &\int_{(0,1)} [K(q) - M_i^*(q)]\,dG_i^*(q) \\ =\;& \mu\int_{(0,1)} [N_i(\bar{w}_i(q)) - \hat{N}_i(\bar{w}_i(q))]\,dG_i^*(q) \\ =\;& \mu\int_{(0,1)} [N_i(\bar{w}_i(q)) - \hat{N}_i(\bar{w}_i(q))]\,d(u_{1i}')^{-1}(\lambda_i(\hat{N}_i)_+'(\bar{w}_i(q))) \\ =\;& 0. \end{aligned}$$

Hence

$$0 = \sum_{i\in\mathcal{I}_\theta}\int_{(0,1)} [M^*(q) - K(q)]\,dG_i^*(q) = \int_{(0,1)} [M^*(q) - K(q)]\,dQ_{\tilde{e}_1}(q).$$

From (B.3), we can see that $K(\cdot) \le M^*(\cdot)$. Because $Q_{\tilde{e}_1}$ is strictly increasing, we can conclude that

$$M^*(q) = K(q) = -\mu\int_{(q,1)} Q_{\tilde{\rho}}^-(1-p)\,dp.$$

So M^* is strictly concave as $Q_{\tilde{\rho}}^-$ is strictly increasing. $\square$

C A Proof of Theorem 4.4

It suffices to show that $(M^*)'_+$ is strictly decreasing on $(0,1)$. It is easy to see, for any fixed $p \in (0,1)$, that $Q_{\tilde{e}_1}$ is continuous at p if and only if G_i^* is continuous at p for all $i \in \mathcal{I}_\theta$. Let $Q_{\tilde{e}_1}^c$ and $G_i^{*,c}$ denote the continuous parts of $Q_{\tilde{e}_1}$ and G_i^* respectively. Then

$$dQ_{\tilde{e}_1}^c(p) = \sum_{i \in \mathcal{I}_\theta} dG_i^{*,c}(p).$$

On the other hand, by (A.6), we have $(M_i^\circ)'_+(q) = \theta_i \beta_i u'_{1i}(G_i^*(\bar{w}_i^{-1}(q)))$ for all $i \in \mathcal{I}_\theta$ and all $q \in (0,1)$, or equivalently,

$$(M_i^\circ)'_+(\bar{w}_i(p)) = \theta_i \beta_i u'_{1i}(G_i^*(p))$$

for all $p \in (0,1)$. Thus G_i^* is continuous at p if and only if $(M_i^\circ)'_+$ is continuous at $\bar{w}_i(p)$. For all $i \in \mathcal{I}_\theta$, let

$$\mathcal{C}_i \triangleq \{p \in (0,1) : \ (M_i^\circ)'_+(\bar{w}_i(p)) = (M_i^\circ)'_+(\bar{w}_i(p)-)\}$$

and

$$\mathcal{D}_i \triangleq \{p \in (0,1) : \ (M_i^\circ)(\bar{w}_i(p)) = (M_i^\bullet)(\bar{w}_i(p)\}.$$

Then for all $i \in \mathcal{I}_\theta$,

$$\begin{aligned} dG_i^{*,c}(p) &= \mathbf{1}_{p \in \mathcal{C}_i}\, d(u'_{1i})^{-1}\left(\frac{(M_i^\circ)'_+(\bar{w}_i(p))}{\theta_i \beta_i}\right) \\ &= \mathbf{1}_{p \in \mathcal{C}_i \cap \mathcal{D}_i}\, d(u'_{1i})^{-1}\left(\frac{(M_i^\circ)'_+(\bar{w}_i(p))}{\theta_i \beta_i}\right), \end{aligned}$$

where the last equality follows from the fact that M_i° is the concave envelope of $M_i^\bullet$, which implies that $(M_i^\circ)'_+(\bar{w}_i(\cdot))$ is flat outside of $\mathcal{D}_i$. Moreover, for all $i \in \mathcal{I}_\theta$, let

$$\mathcal{E}_i \triangleq \{p \in (0,1) : \ (M_i^\circ)'_+(\bar{w}_i(p)) = (M_i^\bullet)'_+(\bar{w}_i(p))\}.$$

It is easy to see, for all $i \in \mathcal{I}_\theta$, that

$$(M_i^\circ)'_+(\bar{w}_i(p)) > (M_i^\bullet)'_+(\bar{w}_i(p)) \quad \text{if } p \in \mathcal{D}_i \setminus \mathcal{E}_i,$$

as $M_i^\circ(q) \geq M_i^\bullet(q)$ for all $q \in (0,1)$. Thus for every $p \in \mathcal{D}_i \setminus \mathcal{E}_i$, there exists an interval $(p, p+\varepsilon)$ such that $(p, p+\varepsilon)$ is outside of $\mathcal{D}_i$. In this case, $(M_i^\circ)'_+(\bar{w}_i(\cdot))$ is flat on $(p, p+\varepsilon)$ because M_i° is the concave envelope of $M_i^\bullet$, resulting in

$$dG_i^{*,c}(p) = \mathbf{1}_{p \in \mathcal{C}_i \cap \mathcal{D}_i \cap \mathcal{E}_i}\, d(u'_{1i})^{-1}\left(\frac{(M_i^\circ)'_+(\bar{w}_i(p))}{\theta_i \beta_i}\right).$$

From the above analysis, we have

$$
\begin{aligned}
dQ^c_{\tilde{e}_1}(p) &= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\, d(u'_{1i})^{-1}\left(\frac{(M^\circ_i)'_+(\bar{w}_i(p))}{\theta_i\beta_i}\right)\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\big((u'_{1i})^{-1}\big)'\left(\frac{(M^\circ_i)'_+(\bar{w}_i(p))}{\theta_i\beta_i}\right)\frac{d(M^\circ_i)'_+(\bar{w}_i(p))}{\theta_i\beta_i}\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\frac{1}{u''_{1i}(G^*_i(p))}\frac{d(M^\circ_i)'_+(\bar{w}_i(p))}{\theta_i\beta_i}\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\frac{1}{u''_{1i}(G^*_i(p))}\frac{d(M^\bullet_i)'_+(\bar{w}_i(p))}{\theta_i\beta_i}\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\frac{1}{\theta_i\beta_i u''_{1i}(G^*_i(p))}\, d\left(\frac{(M^*)'_+(p)}{\bar{w}'_i(p)}\right)\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\frac{1}{\theta_i\beta_i u''_{1i}(G^*_i(p))}\,\frac{\bar{w}'_i(p)\, d(M^*)'_+(p) - (M^*)'_+(p)\bar{w}''_i(p)dp}{(\bar{w}'_i(p))^2}.
\end{aligned}
$$

Suppose on the contrary that $(M^*)'_+$ is not strictly decreasing on $(0,1)$, then there exists a non null Borel set $\mathcal{A}\subseteq(0,1)$ such that $\mathbf{1}_{p\in\mathcal{A}}\, d(M^*)'_+(p)\succeq 0$. Thus we have

$$
\begin{aligned}
\mathbf{1}_{p\in\mathcal{A}}\, dQ^c_{\tilde{e}_1}(p) &\preceq \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{A}\cap\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\frac{-1}{\theta_i\beta_i u''_{1i}(G^*_i(p))}\,\frac{(M^*)'_+(p)\bar{w}''_i(p)}{(\bar{w}'_i(p))^2}\,dp\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{A}\cap\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\frac{-1}{\theta_i\beta_i u''_{1i}(G^*_i(p))}\,\frac{(M^\bullet_i)'_+(\bar{w}_i(p))\bar{w}''_i(p)}{\bar{w}'_i(p)}\,dp\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{A}\cap\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i}\frac{-1}{\theta_i\beta_i u''_{1i}(G^*_i(p))}\,\frac{(M^\circ_i)'_+(\bar{w}_i(p))\bar{w}''_i(p)}{\bar{w}'_i(p)}\,dp\\
&= \sum_{i\in\mathcal{I}_\theta} \mathbf{1}_{p\in\mathcal{A}\cap\mathcal{C}_i\cap\mathcal{D}_i\cap\mathcal{E}_i} - \frac{u'_{1i}(G^*_i(p))}{u''_{1i}(G^*_i(p))}\,\frac{\bar{w}''_i(p)}{\bar{w}'_i(p)}\,dp\\
&\preceq \mathbf{1}_{p\in\mathcal{A}}\,\Gamma(p)\,dp. \qquad\qquad (C.1)
\end{aligned}
$$

Let $\mathcal{C}$ denotes the set of all continuous points of $Q_{\tilde{e}_1}$. Then the Lebesgue measure of $\mathcal{C}$ is one and $dQ^c_{\tilde{e}_1}(p)=\mathbf{1}_{p\in\mathcal{C}}\, dQ_{\tilde{e}_1}$. By (C.1), we can have

$$
\mathbf{1}_{p\in\mathcal{A}\cap\mathcal{C}}\, dQ_{\tilde{e}_1}(p)\preceq \mathbf{1}_{p\in\mathcal{A}\cap\mathcal{C}}\,\Gamma(p)\,dp,
$$

which contradicts (4.4). $\square$

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