# Remarks on optimal controls of stochastic partial differential equations 

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Abstract: For optimal controls of stochastic partial differential equations, the relationship between the maximum principle and dynamic programming is given in terms of Crandall and Lions' super- and sub-differential, without assuming the smoothness of the value function. A comparison between the super-, sub-differential and Clarke's generalized gradient is also discussed.

Keywords: Maximum principle; dynamic programming; adjoint process; value function; super- and sub-differential; generalized gradient.

## 1. Introduction

In this paper we will consider the following kind of stochastic control problems. For $s \in[0,1]$, by admissible controls $U_{\text {ad }}[s, 1]$ we mean the collection of (1) a standard probability space ( $\Omega, \mathscr{F}, P$ ) and a $d^{\prime}$-dimensional Brownian motion $\{W(t): s \leqslant t \leqslant 1\}$ with $W(s)=0 ;(2)$ a $\Gamma$-valued $\mathscr{F}_{t}^{s}$-adapted measurable process $\{U(t): s \leqslant t \leqslant 1\}$, where $\mathscr{F}_{t}^{s}:=\sigma\{W(r): s \leqslant r \leqslant t\}$ and $\Gamma$ is a given Borel set in some $R^{m}$. We denote $(\Omega, \mathscr{F}, P, W, U) \in U_{\text {ad }}[s, 1]$, but sometimes we will only write $U \in U_{\text {ad }}[s, 1]$ if no ambiguity arises.

Let $(s, \phi) \in[0,1] \times H^{0}\left(H^{0}:=L^{2}\left(R^{d}\right)\right)$ be fixed. For each $(\Omega, \mathscr{F}, P, W, U) \in U_{\mathrm{ad}}[s, 1]$, there is a cost functional

$$
J(s, \phi ; U):=E\left\{\int_{s}^{1} F(r, q(r, \cdot), U(r)) \mathrm{d} r+G(q(1))\right\},
$$

where $q$ is the solution (the precise meaning of the 'solution' will be given later on) of the following stochastic partial differential equation (SPDE in short) on the space ( $\Omega, \mathscr{F}_{\mathcal{F}}, P ; \mathscr{F}_{t}^{s}$ ):

$$
\begin{align*}
\mathrm{d} q(t, x)= & {\left[\partial_{i}\left(a^{i j}(t, x, U(t)) \partial_{j} q(t, x)\right)+b^{i}(t, x, U(t)) \partial_{i} q(t, x)\right.} \\
& +c(t, x, U(t)) q(t, x)+f(t, x, U(t))] \mathrm{d} t \\
& +\left[\sigma^{i k}(t, x, U(t)) \partial_{i} q(t, x)+h^{k}(t, x, U(t)) q(t, x)+g^{k}(t, x, U(t))\right] \mathrm{d} W_{k}(t) \\
x \in R^{d}, t \in & {[s, 1], }  \tag{1.1a}\\
q(s, x)= & \phi(x), \quad x \in R^{d} \tag{1.1b}
\end{align*}
$$

where $\left(W_{1}, W_{2}, \ldots, W_{d^{\prime}}\right):=W, \partial_{i}:=\partial / \partial x_{i}, i=1,2, \ldots, d$. Note here and in the sequel we always use the conventional repeated indices for summation.

The optimal control problem is to minimize $J(s, \phi ; U)$ over $U_{\mathrm{ad}}[s, 1]$. The problem is denoted by $C_{s, \phi}$ to recall the dependence on the initial time $s$ and initial state $\phi$. The value function is defined as

$$
\begin{equation*}
V(s, \phi):=\inf \left\{J(s, \phi ; U):(\Omega, \mathscr{F}, P, W, U) \in U_{\mathrm{ad}}[s, 1]\right\} \tag{1.2}
\end{equation*}
$$

As is well known, there are two important approaches to study the optimal control problems: Pontryagin's maximum principle [10] and Bellman's dynamic programming [1], which have been developed separately in the literature. The classical result on the relationship between these two approaches is that the partial differential of the value function in the state variable along the optimal path turns out the adjoint function/process involved in the maximum principle. Recently, Clarke and Vinter [3] obtained a nonsmooth version of this result, for the controls of ODE, by employing the notion of Clarke's generalized gradient [2]. On the other hand, Zhou also interpreted the above classical result in the language of Crandall and Lions' super- and sub-differential [5], for both ODE cases [11] and SDE cases [12]. In this paper, we shall investigate the SPDE cases, and discuss some relationships between the super-, sub-differential and the generalized gradient.

## 2. Preliminaries

Let $\Gamma$ be a given Borel set in $R^{m}$. We define a family of second-order differential operators $\{A(t, u)$ : $t \in[0,1], u \in \Gamma\}$ and a family of first-order differential operators $\left\{M^{k}(t, u): t \in[0,1], u \in \Gamma, k=\right.$ $\left.1,2, \ldots, d^{\prime}\right\}$ by

$$
\begin{align*}
& A(t, u) \phi(x):=\partial_{i}\left(a^{i j}(t, x, u) \partial_{j} \phi(x)\right)+b^{i}(t, x, u) \partial_{i} \phi(x)+c(t, x, u) \phi(x),  \tag{2.1}\\
& M^{k}(t, u) \phi(x):=\sigma^{i k}(t, x, u) \partial_{i} \phi(x)+h^{k}(t, x, u) \phi(x), \quad x \in R^{d}, \tag{2.2}
\end{align*}
$$

where $a^{i /}, b^{i}, c, \sigma^{i k}$ and $h^{k}$ are real valued functions, for $i, j=1,2, \ldots, d ; k=1,2, \ldots, d^{\prime}$.
We will also consider the formal adjoints of (2.1), (2.2):

$$
\begin{align*}
A^{*}(t, u) \phi(x):= & \partial_{i}\left(a^{i j}(t, x, u) \partial_{j} \phi(x)\right)-b^{i}(t, x, u) \partial_{i} \phi(x) \\
& +\left[c(t, x, u)-\partial_{i} b^{i}(t, x, u)\right] \phi(x),  \tag{2.3}\\
M^{k *}(t, u) \phi(x):= & -\sigma^{i k}(t, x, u) \partial_{i} \phi(x)+\left[h^{k}(t, x, u)-\partial_{i} \sigma^{i k}(t, x, u)\right] \phi(x), \quad x \in R^{d} . \tag{2.4}
\end{align*}
$$

We denote by $H^{r}$ the Sobolev space

$$
H^{r}:=\left\{\phi: D^{\alpha} \phi \in L^{2}\left(R^{d}\right), \text { for any } \alpha,|\alpha| \leqslant r\right\}, \quad r=0,1,2, \ldots,
$$

with the Sobolev norm

$$
\|\phi\|_{r}:=\left\{\sum_{|\alpha| \leqslant r} \int_{R^{d}}\left|D^{\alpha} \phi(x)\right|^{2} \mathrm{~d} x\right\}^{1 / 2}, \quad \text { for } \phi \in H^{r} .
$$

We denote by $H^{r}$ the dual space of $H^{-r}$ for $r=-1,-2, \ldots$
In this paper we will consider the triplet $\left(H^{-1}, H^{0}, H^{1}\right)$ under $\left(H^{0}\right)^{*}=H^{0}$. We will denote by $\langle\cdot, \cdot\rangle$ the duality pairing between $H^{-1}$ and $H^{1}$, and by $(\cdot, \cdot)$ the inner product in $H^{0}$.

For any second-order differential operator $L$ which has the same form as (2.1), when we write $\langle L \phi, \psi\rangle$, then $L$ is understood to be an operator from $H^{1}$ to $H^{-1}$ by using formally Green's formula. For example, for the operator $A(t, u)$, we have, for $\phi, \psi \in H^{1}$,

$$
\begin{equation*}
\langle A(t, u) \phi, \psi\rangle:=-\left(a^{i j}(t, \cdot, u) \partial_{j} \phi, \partial_{i} \psi\right)+\left(b^{i}(t, \cdot, u) \partial_{i} \phi, \psi\right)+(c(t, \cdot, u) \phi, \psi) . \tag{2.5}
\end{equation*}
$$

Remark 2.1. We have obviously $\langle A(t, u) \phi, \psi\rangle=\left\langle\phi, A^{*}(t, u) \psi\right\rangle$ and $\left(M^{k}(t, u) \phi, \psi\right)=\left(\phi, M^{k *}(t, u) \psi\right)$ for $\phi, \psi \in H^{1}$.

For $\alpha, \beta \in(-\infty,+\infty)$ with $\alpha<\beta$, we are given a filtered probability space ( $\Omega, \mathscr{F}, P, \mathscr{F}: \alpha \leqslant t \leqslant \beta$ ) and a Hilbert space $X$. For $p \in[1,+\infty]$, define

$$
\begin{gathered}
L_{\mathscr{F}}^{p}(\alpha, \beta ; X):=\left\{\phi: \phi \text { is an } X \text {-valued } \mathscr{F}_{i} \text {-adapted process on }[\alpha, \beta],\right. \\
\text { and } \left.\phi \in L^{p}([\alpha, \beta] \times \Omega ; X)\right\} .
\end{gathered}
$$

We identify $\phi$ and $\phi^{\prime}$ in $L_{\phi}^{p}(\alpha, \beta ; X)$ if $E \int_{\alpha}^{\beta}\left\|\phi(t)-\phi^{\prime}(t)\right\|^{p} \mathrm{~d} t=0$.
Given $(s, \phi) \in[0,1] \times H^{0}$ and $(\Omega, \mathscr{F}, P, W, U) \in U_{\text {ad }}[s, 1]$, we rewrite (1.1) in the following form, omitting to write the variable $x$ :

$$
\begin{align*}
\mathrm{d} q(t)= & {[A(t, U(t)) q(t)+f(t, U(t))] \mathrm{d} t } \\
& +\left[M^{k}(t, U(t)) q(t)+g^{k}(t, U(t))\right] \mathrm{d} W_{k}(t), \quad t \in[s, 1], \tag{2.6a}
\end{align*}
$$

$$
\begin{equation*}
q(s)=\phi . \tag{2.6b}
\end{equation*}
$$

A process $q=q^{u} \in L_{\mathscr{F}}^{2}\left(s, 1 ; H^{1}\right)$ is called a solution of (2.6) or a response for the control $U$, if for any $\eta \in C_{0}^{\infty}\left(=\operatorname{smooth}\right.$ function on $R^{d}$ with compact support) and almost all $(t, \omega) \in[s, 1] \times \Omega$,

$$
\begin{align*}
(q(t), \eta)= & (\phi, \eta)+\int_{s}^{t}\langle A(r, U(r)) q(r)+f(r, U(r)), \eta\rangle \mathrm{d} r \\
& +\int_{s}^{t}\left(M^{k}(r, U(r)) q(r)+g^{k}(r, U(r)), \eta\right) \mathrm{d} W_{k}(r) \tag{2.7}
\end{align*}
$$

Let us fix two positive constants $K, \delta$. We introduce the following conditions on the data appearing in our problem:
(A1) $a^{i j}, b^{i}, c, \sigma^{i k}, h^{k}:[0,1] \times R^{d} \times \Gamma \rightarrow R^{1}$ are measurable in $(t, x, u)$ and continuous in $u$; the functions $a^{i j}, b^{i}, c, \sigma^{i k}, h^{k}, \partial_{j} \sigma^{i k}$ and $\partial_{j} h^{k}$ and their derivatives in $x$ up to the second order do not exceed $K$ in absolute value.
(A2) $a^{i j}=a^{i j}, i, j=1,2, \ldots, d$; the matrix $\left(A^{i j}\right):=\left(a^{i j}-\frac{1}{2} \Sigma_{k=1}^{d^{\prime}} \boldsymbol{\sigma}^{i k} \sigma^{j k}\right) \geqslant 0$ is uniformly positive definite:

$$
A^{i} \xi_{i} \xi_{j} \geqslant \delta|\xi|^{2}, \quad \text { for any }(t, x, u) \text {, and any } \xi \in R^{d} .
$$

(A3) $F:[0,1] \times H^{1} \times \Gamma \rightarrow R^{1}$ is continuous; fix $(t, u)$; for each $\phi \in H^{1}$, there is an element in $H^{-1}$, denoted by $\nabla F(t, \phi, u)$, such that

$$
F(t, \psi, u)-F(t, \phi, u)=\langle\nabla F(t, \phi, u), \psi-\phi\rangle+o\left(\|\psi-\phi\|_{1}\right) ;
$$

moreover, $\nabla F(t, \cdot, u): H^{1} \rightarrow H^{-1}$ is continuous and $\|\nabla F(t, \phi, u)\|_{-1} \leqslant K$ for any $\phi \in H^{1}$.
(A4) $G: H^{0} \rightarrow R^{1}$ is continuously Fréchet differentiable; its Fréchet differential $\nabla G$ satisfies $\|\nabla G(\phi)\|_{0}$ $\leqslant K$ for any $\phi \in H^{0}$.
(A5) $f, g^{k}:[0,1] \times R^{d} \times \Gamma \rightarrow R^{1}$ are measurable in $(t, x, u)$ and continuous in $u$, for $k=1,2, \ldots, d^{\prime}$; $f(t, \cdot, u) \in H^{-1}, \mathrm{~g}^{\mathrm{k}}(t, \cdot, u) \in H^{0}$, and

$$
|f(t, x, u)|+\left|g^{k}(t, x, u)\right|+\|f(t, \cdot, u)\|_{-1}+\left\|g^{k}(t, \cdot, u)\right\|_{0} \leqslant K, \quad k=1,2, \ldots, d^{\prime} .
$$

Remark 2.2. (A3) is satisfied if $F(t, \cdot, u)$ is continuously Fréchet differentiable on $H^{0}$ and if the $H^{0}$-norm of the Fréchet differential is bounded by $K$; on the other hand, (A3) also includes the following kind of functional which appears frequently in some explicit problems:

$$
F(t, \phi, u):=\left(F^{0}(t, u), \phi\right)+\left(F^{i}(t, u), \partial_{i} \phi\right)
$$

where $F^{i}$ is $H^{0}$-valued, $i=0,1,2, \ldots, d$.

Finally let us recall the notion of first-order super- and sub-differential of a functional on Hilbert space (Lions [8]).

Definitions 2.1. Let $v \in C\left(H^{0} ; R^{1}\right)$. The first-order super- (resp. sub-) differential of $v$ at $\phi \in H^{0}$, denoted by $\mathrm{D}^{+} v(\phi)$ (resp. $\left.\mathrm{D}^{-1} v(\phi)\right)$ is the set defined as

$$
\mathrm{D}^{+} v(\phi):=\left\{p \in H^{0}: \lim _{\phi \rightarrow \phi}[v(\psi)-v(\phi)-(p, \psi-\phi)] /\|\psi-\phi\|_{0} \leqslant 0\right\}
$$

$\left(\right.$ resp. $\mathrm{D}^{-} v(\phi):=\left\{p \in H^{0}:{\left.\left.\lim \inf _{\psi \rightarrow \phi}[\cdots] \geqslant 0\right\}\right) .}\right.$

## 3. Value function and adjoint process

In this section, we suppose $\phi \in H^{0}$ is fixed and $(\Omega, \mathscr{F}, P, W, \hat{U}) \in U_{\text {ad }}[0,1]$ is an optimal control of the problem $C_{0, \phi}$ and $\hat{q}$ is the corresponding state. We will denote $\hat{A}(t):=A(t, \hat{U}(t)), \hat{F}(t):=$ $F(t, \hat{q}(t), \hat{U}(t)), \nabla \hat{F}(t):=\nabla F(t, \hat{q}(t), \hat{U}(t))$, etc. to simplify the notations. We also denote $\mathscr{F}_{t}:=\mathscr{F}_{t}^{0}$ for $t \in[0,1]$.

According to Zhou [13, Theorem 4.1], there exists uniquely a pair $(\lambda, r) \in L_{\mathscr{F}}^{2}\left(0,1 ; H^{1}\right) \times$ $\left[L_{. \mathscr{F}}^{2}\left(0,1 ; H^{0}\right)\right]^{d^{\prime}}$ such that

$$
\begin{align*}
& \mathrm{d} \lambda(t)=-\left[\hat{A}^{*}(t) \lambda(t)+\sum_{k=1}^{d^{\prime}} \hat{M}^{k *}(t) r^{k}(t)+\nabla \hat{F}(t)\right] \mathrm{d} t+\sum_{k=1}^{d^{\prime}} r^{k}(t) \mathrm{d} W_{k}(t), \quad t \in[0,1],  \tag{3.1a}\\
& \lambda(1)=\nabla G(\hat{q}(1)) . \tag{3.1b}
\end{align*}
$$

Remark 3.1. The above result was proved in [13] when the given $\lambda(1)$ is deterministic. But the proof adapts easily to the present case. Equation (3.1) is called the adjoint equation of the controlled system (2.6), and the $\mathscr{F}_{i}$-adapted process $\lambda$ is called the adjoint process.

On the other hand, it is routine to prove that

$$
\begin{equation*}
|V(t, \psi)-V(t, \phi)| \leqslant N_{1}\|\psi-\phi\|_{0}, \quad \text { for any } \psi, \phi \in H^{0}, \tag{3.2}
\end{equation*}
$$

where $N_{1}$ depends only on $K, \delta$. Hence the super- and sub-differential of $V(t, \cdot)$, which are denoted by $\mathrm{D}^{ \pm} V(t, \cdot)$, are well defined according to Definition 2.1.

Theorem 3.1. Assume (A1)-(A5); then for a.e. $t \in[0,1]$,

$$
\begin{equation*}
\mathrm{D}^{-} V(t, \hat{q}(t)) \subset\{\lambda(t)\} \subset \mathrm{D}^{+} V(t, \hat{q}(t)), \quad P \text {-a.s. } \tag{3.3}
\end{equation*}
$$

Proof. To avoid notational complexity, we will prove the Theorem for $d^{\prime}=1$ (there is no essential difficulty when $d^{\prime}>1$ ). Thus the index $k$ will be dropped. Appealing to Krylov and Rozovskii [7], $\hat{q} \in L_{; \vec{F}}^{2}\left(0,1 ; H^{1}\right) \cap L^{2}\left(\Omega ; C\left(0,1 ; H^{0}\right)\right)$. Hence for a.e. $t \in[0,1]$,

$$
\begin{equation*}
\|\hat{q}(t)\|_{1}^{2}+\|\lambda(t)\|_{1}^{2}<+\infty, \quad P \text {-a.s. } \tag{3.4}
\end{equation*}
$$

Fix a $t \in[0,1)$ such that (3.4) holds. For any $\psi \in H^{0}$, let $q(\cdot ; \psi)$ satisfy the following SPDE on $[t, 1]$ :

$$
\begin{align*}
q(s ; \psi)= & \psi+\int_{t}^{s}[\hat{A}(p) q(p ; \psi)+\hat{f}(p)] \mathrm{d} p \\
& +\int_{t}^{s}[\hat{M}(p) q(p ; \psi)+\hat{g}(p)] \mathrm{d} W(p), \quad s \in[t, 1] . \tag{3.5}
\end{align*}
$$

Define $\xi(s ; \psi):=q(s ; \psi)-\hat{q}(s)$ for $s \in[t, 1]$. Then $\xi$ satisfies the following SPDE:

$$
\begin{align*}
& \mathrm{d} \xi(s ; \psi)=\hat{A}(s) \xi(s ; \psi) \mathrm{d} s+\hat{M}(s) \xi(s ; \psi) \mathrm{d} W(s), \quad s \in[t, 1],  \tag{3.6a}\\
& \xi(t ; \psi)=\psi-\hat{q}(t) . \tag{3.6b}
\end{align*}
$$

Since (3.6) may be regarded as an SPDE on the probability space ( $\Omega, \mathscr{F}, P\left(\cdot \mid \mathscr{F}_{t}\right)(\omega) ; \mathscr{F}_{s}: t \leqslant s \leqslant 1$ ) for $P$-a.s. $\omega$, so [7, Corollary 2.2 y yields

$$
\begin{equation*}
E\left(\sup _{t \leqslant s \leqslant 1}\|\xi(s ; \psi)\|_{0}^{2}+\delta \int_{t}^{1}\|\xi(s ; \psi)\|_{1}^{2} \mathrm{~d} s \mid \mathscr{F _ { t }}\right) \leqslant N_{2}\|\psi-\hat{q}(t)\|_{0}^{2}, \quad P \text {-a.s. } \tag{3.7}
\end{equation*}
$$

where $N_{2}$ depends only on $K, \delta$.
Applying Ito's formula to (3.1) and (3.6), we get

$$
\begin{aligned}
& \mathrm{d}(\lambda(s), \xi(s ; \psi)) \\
&=-\left\langle\hat{A}^{*}(s) \lambda(s)+\hat{M}^{*}(s) r(s)+\nabla \hat{F}(s), \xi(s ; \psi)\right\rangle \mathrm{d} s+(r(s), \xi(s ; \psi)) \mathrm{d} W(s) \\
&+\langle\lambda(s), \hat{A}(s) \xi(s ; \psi)\rangle \mathrm{d} s+(\lambda(s), \hat{M}(s) \xi(s ; \psi)) \mathrm{d} W(s)+(r(s), \hat{M}(s) \xi(s ; \psi)) \mathrm{d} s, \\
&=-\langle\nabla \hat{F}(s), \xi(s ; \psi)\rangle \mathrm{d} s+\left(r(s)+\hat{M}^{*}(s) \lambda(s), \xi(s ; \psi)\right) \mathrm{d} W(s)
\end{aligned}
$$

Consequently,

$$
\begin{equation*}
E\left(\int_{t}^{1}\langle\nabla \hat{F}(s), \xi(s ; \psi)\rangle \mathrm{d} s+(\nabla G(\hat{q}(1)), \xi(1 ; \psi)) \mid \mathscr{F}_{t}\right)=(\lambda(t), \psi-\hat{q}(t)), \quad P \text {-a.s. } \tag{3.8}
\end{equation*}
$$

Since $H^{0}$ is separable, we may choose $\psi_{1}, \psi_{2}, \ldots, \psi_{n}, \ldots$ which is dense in $H^{0}$. Choose an $\Omega_{0} \subset \Omega$ with $P\left(\Omega_{0}\right)=1$ such that for any $\omega_{0} \in \Omega_{0},(3.4),(3.7)$ and (3.8) are satisfied for any $\psi_{n},\left(\Omega, \mathscr{F}, P\left(\cdot \mid \mathscr{F}_{i}\right)\left(\omega_{0}\right)\right.$, $\left.W(s)-W(t),\left.\hat{U}\right|_{[t, 1]}\right) \in U_{\text {ad }}[t, 1]$, and

$$
\begin{equation*}
V\left(t, \hat{q}\left(t, \omega_{0}\right)\right)=E\left(\int_{t}^{1} \hat{F}(s) \mathrm{d} s+G(\hat{q}(1)) \mid \mathscr{F}_{t}\right)\left(\omega_{0}\right) \tag{3.9}
\end{equation*}
$$

(The above (3.9) is called Bellman's principle of optimality. It was proved in Zhou [12] for the systems governed by SDE, and is extended trivally to the SPDE cases.)

Let $\omega_{0} \in \Omega_{0}$ be fixed; then for any $\psi_{n}$, we have

$$
\begin{align*}
& V\left(t, \psi_{n}\right)-V\left(t, \hat{q}\left(t, \omega_{0}\right)\right) \\
& \quad \leqslant E\left(\int_{t}^{1}\left[F\left(s, q\left(s ; \psi_{n}\right), \hat{U}(s)\right)-\hat{F}(s)\right] \mathrm{d} s+G\left(q\left(1 ; \psi_{n}\right)\right)-G(\hat{q}(1)) \mid \mathscr{F}_{t}\right)\left(\omega_{0}\right) \\
& \quad=E\left(\int_{t}^{1}\left\langle\nabla \hat{F}(s), \xi\left(s ; \psi_{n}\right)\right\rangle \mathrm{d} s+\left(\nabla G(\hat{q}(1)), \xi\left(1 ; \psi_{n}\right)\right) \mid \mathscr{F}_{t}\right)\left(\omega_{0}\right)+\varepsilon_{1 n}+\varepsilon_{2 n} \tag{3.10}
\end{align*}
$$

where

$$
\begin{aligned}
& \varepsilon_{1 n}:=E\left(\int_{t}^{1} \int_{0}^{1}\left\langle\nabla F\left(s, \hat{q}(s)+p \xi\left(s ; \psi_{n}\right), \hat{U}(s)\right)-\nabla \hat{F}(s), \xi\left(s ; \psi_{n}\right)\right\rangle \mathrm{d} p \mathrm{~d} s \mid \mathscr{F}_{t}\right)\left(\omega_{0}\right), \\
& \varepsilon_{2 n}:=E\left(\int_{0}^{1}\left(\nabla G\left(\hat{q}(1)+p \xi\left(1 ; \psi_{n}\right)\right)-\nabla G(\hat{q}(1)), \xi\left(1 ; \psi_{n}\right)\right) \mathrm{d} p \mid \mathscr{F}_{t}\right)\left(\omega_{0}\right)
\end{aligned}
$$

In view of (3.7) and the assumption (A3), (A4), it is easy to derive that

$$
\left[\left|\varepsilon_{1 n}\right|+\left|\varepsilon_{2 n}\right|\right] /\left\|\psi_{n}-\hat{q}\left(t, \omega_{0}\right)\right\|_{0} \rightarrow 0 \quad \text { as } \psi_{n} \rightarrow \hat{q}\left(t, \omega_{0}\right) \text { in } H^{0}
$$

Thus, noting (3.8), we may rewrite (3.10) as

$$
\begin{equation*}
V\left(t, \psi_{n}\right)-V\left(t, \hat{q}\left(t, \omega_{0}\right)\right) \leqslant\left(\lambda\left(t, \omega_{0}\right), \psi_{n}-\hat{q}\left(t, \omega_{0}\right)\right)+o\left(\left\|\psi_{n}-\hat{q}\left(t, \omega_{0}\right)\right\|_{0}\right) . \tag{3.11}
\end{equation*}
$$

Hence for any $\varepsilon>0$, there exists an $\eta>0$ such that if $\left\|\psi_{n}-\hat{q}\left(t, \omega_{0}\right)\right\|_{0} \leqslant \eta$, then

$$
V\left(t, \psi_{n}\right)-V\left(t, \hat{q}\left(t, \omega_{0}\right)\right) \leqslant\left(\lambda\left(t, \omega_{0}\right), \psi_{n}-\hat{q}\left(t, \omega_{0}\right)\right)+\varepsilon_{n},
$$

where $\left|\varepsilon_{n}\right| /\left\|\psi_{n}-\hat{q}\left(t, \omega_{0}\right)\right\|_{0} \leqslant \frac{1}{3} \varepsilon$. Now choose $\eta^{\prime}:=\min \left\{1, \eta, \varepsilon /\left[3\left(N_{1}+\left\|\lambda\left(t, \omega_{0}\right)\right\|_{0}\right)\right]\right\}$ where $N_{1}$ is the constant in (3.2); then for any $\psi \in H^{0}$ with $0<\left\|\psi-\hat{q}\left(t, \omega_{0}\right)\right\|_{0}<\eta^{\prime}$ find a fixed $\psi_{N}$ such that $\left\|\psi_{N}-\hat{q}\left(t, \omega_{0}\right)\right\|_{0} \leqslant\left\|\psi-\hat{q}\left(t, \omega_{0}\right)\right\|_{0}$ and $\left\|\psi_{N}-\psi\right\|_{0} \leqslant\left\|\psi-\hat{q}\left(t, \omega_{0}\right)\right\|_{0}^{2}$. Thus we get

$$
V(t, \psi)-V\left(t, \hat{q}\left(t, \omega_{0}\right)\right) \leqslant\left(\lambda\left(t, \omega_{0}\right), \psi-\hat{q}\left(t, \omega_{0}\right)\right)+N_{3}\left\|\psi-\psi_{N}\right\|_{0}+\left(\lambda\left(t, \omega_{0}\right), \psi_{N}-\psi\right)+\varepsilon_{N} .
$$

Moreover, it is easy to compute that

$$
\left|N_{3}\right| \psi-\psi_{N}\left|+\left(\lambda\left(t, \omega_{0}\right), \psi_{N}-\psi\right)+\varepsilon_{N}\right| /\left\|\psi-\hat{q}\left(t, \omega_{0}\right)\right\|_{0} \leqslant \varepsilon .
$$

This means (3.11) holds with $\psi_{n}$ replaced by any $\psi \in H^{0}$, which immediately leads to the fact that $\lambda\left(t, \omega_{0}\right) \in \mathrm{D}^{+} V\left(t, \hat{q}\left(t, \omega_{0}\right)\right)$. On the other hand, suppose $p \in \mathrm{D}^{-} V\left(t, \hat{q}\left(t, \omega_{0}\right)\right)$. Then by the definition,

$$
\begin{aligned}
0 & \leqslant \liminf _{\psi \rightarrow \hat{q}\left(t, \omega_{0}\right)}\left[V(t, \psi)-V\left(t, \hat{q}\left(t, \omega_{0}\right)\right)-\left(p, \psi-\hat{q}\left(t, \omega_{0}\right)\right)\right] /\left\|\psi-\hat{q}\left(t, \omega_{0}\right)\right\|_{0} \\
& \leqslant \liminf _{\psi \rightarrow \hat{q}\left(t, \omega_{0}\right)}\left(\lambda\left(t, \omega_{0}\right)-p, \psi-\hat{q}\left(t, \omega_{0}\right)\right) /\left\|\psi-\hat{q}\left(t, \omega_{0}\right)\right\|_{0} .
\end{aligned}
$$

Hence $p=\lambda\left(t, \omega_{0}\right)$, which is the left part of (3.3). This concludes the proof.
Remark 3.2. Using a dynamic programing approach, it can be proved that the value function $V$ is a viscosity solution of the HJB equation on $H^{0}$ (cf. Nisio [9]). On the other hand, under some additional assumptions, the necessary conditions of optimal controls of the system (2.6) is derived in Zhou [13] in the form of a maximum principle which involves the adjoint process $\lambda$.

Remark 3.3. Now let us explain what (3.3) means. It is known in the classical PDE theory that the method of characteristics is effective in obtaining solutions of first-order nonlinear PDE. Its main idea is that if there is a $C^{2}$ solution of the PDE, then the solution can be constructed from a family of solutions of some ordinary differential equations for curves called characteristic strips (cf. Courant and Hilbert [4]). The interesting point is that, for optimal controls of ODE, the differential equations of the characteristic strips for the HJB equation are just the adjoint equations involved in the maximum principle (cf. Fleming and Rishel [6]). But this theory is very unsatisfactory, because it assumes that the value function is $C^{2}$, which is not true even in the simplest cases (cf. [11] for an example). Now (3.3) gives a relationship between the value function and the adjoint process, without assuming any smoothness of the value function. It suggests that the viscosity solution (not necessarily smooth!) of a second-order HJB equation might be constructed through a family of SDEs like (3.1).

Finally let us briefly compare the super-, sub-differential with Clarke's generalized gradient. First we recall the following definition.

Definition 3.1 (Clarke [2]). Let $v: H^{0} \rightarrow R^{1}$ be Lipschitz continuous. The generalized gradient of $v$ at $\phi \in H^{0}$, denoted by $\partial v(\phi)$, is the set defined as

$$
\partial v(\phi):=\left\{p \in H^{0}:(p, \psi) \leqslant v^{0}(\phi ; \psi), \text { for any } \psi \in H^{0}\right\},
$$

where $v^{0}(\phi ; \psi):==\lim \sup _{\xi \rightarrow \phi, \varepsilon \rightarrow 0}[v(\xi+\varepsilon \psi)-v(\xi)] / \varepsilon$.

Remark 3.4. $\partial v(\phi)$ is a nonempty convex set and $\partial(-v)(\phi)=-\partial v(\phi)$. See [2] for a thorough treatment of the generalized gradient.

Lemma 3.1. Let $v: H^{0} \rightarrow R^{1}$ be Lipschitz continuous. Then

$$
\begin{equation*}
\mathrm{D}^{+} v(\phi) \cup \mathrm{D}^{-} v(\phi) \subset \partial v(\phi), \quad \text { for any } \phi \in H^{0} . \tag{3.12}
\end{equation*}
$$

Proof. Let $p \in \mathrm{D}^{-} v(\phi)$. Then $(p, \psi) \leqslant \lim _{\inf _{\varepsilon \rightarrow 0}}[v(\phi+\varepsilon \psi)-v(\phi)] / \varepsilon \leqslant v^{0}(\phi ; \psi)$, and hence $p \in \partial v(\phi)$. On the other hand, we have $\mathrm{D}^{+} v=-\mathrm{D}^{-}(-v) \subset-\partial(-v)=\partial v$. This yields the result.

Remark 3.5. The strict inclusion in (3.12) may occur. To see this, take

$$
v(x):= \begin{cases}x^{2} \sin (1 / x), & x \in R^{1}, x \neq 0, \\ 0, & x=0 .\end{cases}
$$

$v$ is differentiable at 0 , hence $\mathrm{D}^{+} v(0)=\mathrm{D}^{-} v(0)=\{0\}$. But $\partial v(0)$ equals the convex hull of the set of limits of the form $\lim v^{\prime}\left(h_{i}\right)$, where $h_{i} \rightarrow 0$ (cf. [2]). So $\partial v(0)=[-1,1]$. Through this example, we may catch some sense about the difference between the super-, sub-differential and the generalized gradient: if the former is a nonsmooth notion of 'differentiability', then the latter may be regarded as a nonsmooth notion of 'continuous differentiability'.

Lemma 3.2. Let $v: H^{0} \rightarrow R^{1}$ be Lipschitz continuous and semiconcave, i.e., there exists constant $\tilde{K}$ such that

$$
v(\phi+\psi)+v(\phi-\psi)-2 v(\phi) \leqslant \tilde{K}\|\psi\|_{0}^{2}, \quad \text { for any } \phi, \psi \in H^{0} .
$$

Then $\mathrm{D}^{+} v(\phi)=\partial v(\phi)$, for any $\phi \in H^{0}$.
Proof. First let us note that if $\tilde{v}$ is concave, then $p \in \partial \tilde{v}(\phi)$ iff $\tilde{v}(\psi)-\tilde{v}(\phi) \leqslant(p, \psi-\phi)$, for all $\psi \in H^{0}$ (see [2]). Hence $\mathrm{D}^{+} \tilde{v}(\phi)=\partial \tilde{v}(\phi)$ by noting Lemma 3.1.

Now $v$ is semiconcave, and it can be decomposed as $v(\phi)=\tilde{v}(\phi)+\frac{1}{2} \tilde{K}\|\phi\|_{0}^{2}$, where $\tilde{v}$ is concave. If $p \in \partial v(\phi)$, then $p-\tilde{K} \phi \in \partial \tilde{v}(\phi)=\mathrm{D}^{+} \tilde{v}(\phi)$, which yields $p \in \mathrm{D}^{+} v(\phi)$. The proof is completed.

Remark 3.6. If we assume in addition that $F(t, \cdot, u)$ and $G$ are twice Fréchet differentiable on $H^{0}$, and the norms of their Fréchet differentials up to second order are bounded uniformly, then $V(t, \cdot)$ is semiconcave on $H^{0}$ (cf. Nisio [9]). Thus in view of Lemma 3.2, Theorem 3.1 may be expressed equivalently in the language of generalized gradient as $\lambda(t) \in \partial V(t, q(t))$. This is an analogous result to that of ODE cases by Clarke and Vinter [3].

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