Two explicit Skorokhod embeddings for simple symmetric random walk

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Abstract

Motivated by problems in behavioural finance, we provide two explicit constructions of a randomized stopping time which embeds a given centered distribution $\mu$ on integers into a simple symmetric random walk in a uniformly integrable manner. Our first construction has a simple Markovian structure: at each step, we stop if an independent coin with a state-dependent bias returns tails. Our second construction is a discrete analogue of the celebrated Azéma–Yor solution and requires independent coin tosses only when excursions away from maximum breach predefined levels. Further, this construction maximizes the distribution of the stopped running maximum among all uniformly integrable embeddings of $\mu$.

Keywords: Skorokhod embedding, simple symmetric random walk, randomized stopping time, Azéma-Yor stopping time.

1. Introduction

We contribute to the literature on Skorokhod embedding problem (SEP). The SEP, in general, refers to the problem of finding a stopping time $\tau$, such that a given stochastic process $X$, when stopped, has the prescribed distribution $\mu$: $X_{\tau} \sim \mu$. When such a $\tau$ exists we say that $\tau$ embeds $\mu$ into $X$. This problem was first formulated and solved by Skorokhod (1965) when $X$ is a standard Brownian motion. It has remained an active field of study ever since, see Ob\l{o}j (2004) for a survey, and has recently seen a revived interest thanks to an intimate connection with the Martingale Optimal Transport, see Beiglböck et al. (2016) and the references therein.

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In this paper, we consider the SEP for the simple symmetric random walk. Our interest arose from a casino gambling model of Barberis (2012) in which the gambler’s cumulative gain and loss process is modeled by a random walk. The gambler has to decide when to stop gambling and her preferences are given by cumulative prospect theory (Tversky and Kahneman 1992). Such preferences lead to dynamic inconsistency, so this optimal stopping problem cannot be solved by the classical Snell envelope and dynamic programming approaches. By applying the Skorokhod embedding result we obtain here, He et al. (2016) convert the optimal stopping problem into an infinite-dimensional optimization problem, find the optimal stopping time, and study the gambler’s behavior in the casino.

To discuss about our results, let us first introduce some notation. We let \( X = S = (S_t : t \geq 0) \) be a simple symmetric random walk defined on a filtered probability space \((\Omega, F, \mathbb{F}, P)\), where \( \mathbb{F} = (\mathbb{F}_t)_{t \geq 0} \). We work in discrete time so here, and elsewhere, \( \{t \geq 0\} \) denotes \( t \in \{0, 1, 2, \ldots\} \). We let \( T(\mathbb{F}) \) be the set of \( \mathbb{F} \)-stopping times and say that \( \tau \in T(\mathbb{F}) \) is uniformly integrable (UI) if the stopped process \( (S_t \wedge \tau : t \geq 0) \) is UI. Here, and more generally, when \( X \) is a martingale, one typically restricts the attention to UI stopping times to avoid trivialities and to obtain solutions which are of interest and use. We let \( \mathbb{Z} \) denote the set of integers and \( \mathcal{M}_0(\mathbb{Z}) \) the set of probability measures on \( \mathbb{Z} \) which admit finite first moment and are centered. Our prime interest is in stopping times which solve the SEP:

\[
\text{SEP}(\mathbb{F}, \mu) := \{ \tau \in T(\mathbb{F}) : S_\tau \sim \mu \text{ and } \tau \text{ is UI} \}.
\]

Clearly if \( \text{SEP}(\mathbb{F}, \mu) \neq \emptyset \) then \( \mu \in \mathcal{M}_0(\mathbb{Z}) \). For embeddings in a Brownian motion, the analogue of (1) has a solution if and only if \( \mu \in \mathcal{M}_0(\mathbb{R}) \). However, in the present setup, the reverse implication depends on the filtration \( \mathbb{F} \). If we consider the natural filtration \( \mathbb{F}^S = (\mathbb{F}^S_t : t \geq 0) \) where \( \mathbb{F}^S_t = \sigma(S_u : u \leq t) \), then Cox and Obloj (2008) showed that the set of probability measures \( \mu \) on \( \mathbb{Z} \) for which \( \text{SEP}(\mathbb{F}^S, \mu) \neq \emptyset \) is a fractal subset of \( \mathcal{M}_0(\mathbb{Z}) \). In contrast, Rost (1971) and Dinges (1974) showed how to solve the SEP using randomized stopping times, so that if \( \mathbb{F} \) is rich enough then \( \text{SEP}(\mathbb{F}, \mu) \neq \emptyset \) for all \( \mu \in \mathcal{M}_0(\mathbb{Z}) \). We note also that the introduction of external randomness is natural from the point of view of applications. In the casino model of Barberis (2012) mentioned above, He et al. (2017) and Henderson et al. (2017) showed that the gambler is strictly better off when she uses extra randomness, such as a coin toss, in her stopping strategy instead of relying on \( \tau \in T(\mathbb{F}^S) \). Similarly, randomized stopping times are useful in solving optimal stopping problems, see e.g. Belomestny and Krätschmer (2017) and He et al. (2016).

Our contribution is to give two new constructions of \( \tau \in \text{SEP}(\mathbb{F}, \mu) \) with certain desirable optimality properties. Our first construction, in Section 2 below, has minimal (Markovian) dependence property: decision to stop only depends on the current state of \( S \) and an independent coin toss. The coins are suitably biased, with state dependent probabilities, which can be readily computed using an explicit algorithm we provide. Such a strategy is easy to compute and easy to
implement, which is important for applications, e.g. to justify its use by economic agents, see [He et al. (2017)]. Our second construction, presented in Section 3, is a discrete–time analogue of the [Azéma and Yor (1979)] embedding. It is also explicit and its applied appeal lies in the fact that it requires significantly less coin tossing than the first embedding. More importantly however, it has an inherent probabilistic interest: it maximizes $\mathbb{P}(\max_{i \leq \tau} S_i \geq x)$, simultaneously for all $x \in \mathbb{R}$, among all $\tau \in T(\mathbb{F})$, attaining the classical [Blackwell et al. (1963)] bound for $x \in \mathbb{N}$. We conclude the paper with explicit examples worked out in Section 4.

2. Randomized Markovian Solution to the SEP

To formalize our first construction, consider $r = (r_x : x \in \mathbb{Z}) \in [0, 1]^{\mathbb{Z}}$ and a family of Bernoulli random variables $\xi = \{\xi_{t,x}\}_{t \geq 0, x \in \mathbb{Z}}$ with $\mathbb{P}(\xi_{t,x} = 0) = r_x = 1 - \mathbb{P}(\xi_{t,x} = 1)$, which are independent of each other and of $S$. Each $\xi_{t,x}$ stands for the outcome of a coin toss at time $t$ when $S_t = x$ with 1 standing for heads and 0 standing for tails. To such $\xi$ we associate

$$\tau(r) := \inf\{t \geq 0 | \xi_{t,S_t} = 0\},$$

which is a stopping time relative to $\mathbb{P}^{S,\xi} = (F_t^{S,\xi} : t \geq 0)$ where $F_t^{S,\xi} := \sigma(S_u, \xi_{u,S_u}, u \leq t)$. The decision to stop at time $t$ only depends on the state $S_t$ and an independent coin toss. Accordingly, we refer to $\tau(r)$ as a randomized Markovian stopping time. It is clear however that the distribution of $S_{\tau(r)}$ is a function of $r$ and does not depend on the particular choice of the random variable $\xi$. The following shows that such stopping times allow us to solve (1) for all $\mu \in M_0(\mathbb{Z})$.

**Theorem 2.1** For any $\mu \in M_0(\mathbb{Z})$, there exists $r_\mu \in [0, 1]^{\mathbb{Z}}$ such that $\tau(r_\mu)$ solves $\text{SEP}(\mathbb{P}^{S,\xi}, \mu)$.

2.1. Proof of Theorem 2.1

We present now the proof of Theorem 2.1. We first consider $\mu$ with a finite support: $\mu([A,B]) = 1$ and $\mu([A]) > 0$, $\mu([B]) > 0$ for some $A, B \in \mathbb{Z}$ such that $A < 0$ and $B > 0$. Let $H_{[A,B]} = \inf\{t \geq 0 | S_t \geq B \text{ or } S_t \leq A\}$. Consider the set of $r$ which embed less probability mass than prescribed by $\mu$ on $(A, B)$, i.e.,

$$\mathcal{R}_\mu = \{r \in [0, 1]^{\mathbb{Z}} : r_i = 1 \text{ if } i \notin (A, B) \text{ and } \mathbb{P}(S_{\tau(r)} = i) \leq \mu\{i\} \text{ if } i \in (A, B)\}.$$  

(3)

By choosing $r_i = 0$ for $i \in (A, B)$ and $r_i = 1$ for $i \notin (A, B)$, we have $\mathbb{P}(S_{\tau(r)} = i) = 0$ for $i \in (A, B)$, so $\mathcal{R}_\mu$ is non-empty. Also, by definition, $\tau(r) \leq H_{[A,B]}$ for any $r \in \mathcal{R}_\mu$. We start with two lemmas which together imply that $\mathcal{R}_\mu$ has a maximal element.

**Lemma 2.2** If $r, r' \in \mathcal{R}_\mu$ then so does their maximum: $\tilde{r} \in \mathcal{R}_\mu$, where $\tilde{r}_i = r_i \lor r'_i$, $i \in \mathbb{Z}$.
Proof. Fix \( r, r' \in \mathcal{R}_\mu \) and define \( \tilde{r} = \{ \tilde{r}_i \}_{i \in \mathbb{Z}} \) with \( \tilde{r}_i = r_i \lor r'_i \). Let \( \{ \xi_{t,i} \}_{t \geq 0, i \in \mathbb{Z}} \) be a sequence of Bernoulli random variables used to construct \( \tau(r) \) and consider another sequence of Bernoulli random variables \( \{ \varepsilon_{t,i} \}_{t \geq 0, i \in \mathbb{Z}} \) which are independent of each other and of \( S \), and satisfy

\[
P(\varepsilon_{t,i} = 0) = 1 - P(\varepsilon_{t,i} = 1) = \frac{(r'_i - r_i)'}{1 - r_i} 1_{r_i < 1}, \quad t \geq 0, i \in \mathbb{Z}.
\]

Define \( \tilde{\varepsilon}_{t,i} := \xi_{t,i} \varepsilon_{t,i} \) and observe that

\[
P(\tilde{\varepsilon}_{t,i} = 0) = P(\xi_{t,i} = 0) + P(\xi_{t,i} = 1)P(\varepsilon_{t,i} = 0) = r_i + (1 - r_i) \frac{(r'_i - r_i)'}{1 - r_i} 1_{r_i < 1} = r_i \lor r'_i.
\]

It follows that \( \tau(\tilde{r}) \) can be constructed as

\[
\tau(\tilde{r}) = \inf \{ t \geq 0 | \tilde{\varepsilon}_{t,i} = 0 \} = \inf \{ t \geq 0 | \xi_{t,S_t} \varepsilon_{t,S_t} = 0 \}
\]

and in particular \( \tau(\tilde{r}) \leq \tau(r) \). Consider any \( i \in \mathbb{Z} \cap (A, B) \) such that \( r_i \geq r'_i \), i.e., \( \tilde{r}_i = r_i \) and thus \( \varepsilon_{t,i} = 1 \) a.s. Then, for any \( t \geq 0 \),

\[
\{ S_{\tau(\tilde{r})} = i, \tau(\tilde{r}) = t \} = \{ \xi_{u,S_u} \varepsilon_{u,S_u} = 1, u \leq t - 1, \xi_{t,i} \varepsilon_{t,i} = 0, S_t = i \} \\
\subseteq \{ \xi_{u,S_u} = 1, u \leq t - 1, \xi_{t,i} = 0, S_t = i \} = \{ S_{\tau(r)} = i, \tau(r) = t \},
\]

where the set inclusion is the case because \( \varepsilon_{t,i} = 1 \). We conclude \( \{ S_{\tau(\tilde{r})} = i \} \subseteq \{ S_{\tau(r)} = i \} \), so \( P(S_{\tau(\tilde{r})} = i) \leq P(S_{\tau(r)} = i) \leq \mu(\{i\}) \). An analogous argument starting with \( \tau(r') \) shows that \( P(S_{\tau(r')} = i) \leq \mu(\{i\}) \) for any \( i \in \mathbb{Z} \cap (A, B) \) such that \( r_i < r'_i \). Furthermore, we clearly have \( \tilde{r}_i = 1, i \notin (A, B) \) so that \( \tilde{r} \in \mathcal{R}_\mu \).

Lemma 2.3 The distribution of \( S_{\tau(r)} \) is continuous in \( r \in \mathcal{R} := \{ r \in [0, 1]^2 : r_i = 1, i \notin (-B, A) \} \).

Proof. We need to prove that \( P(S_{\tau(r)} = i) \) is continuous in \( r \) for any \( i \in \mathbb{Z} \).

For any \( t \geq 0 \),

\[
P(S_{\tau(r)} = i, \tau(r) = t) = P(\xi_{u,S_u} = 1, u \leq t - 1, \xi_{t,i} = 0, S_t = i) \\
= E E[1_{\xi_{u,S_u} = 1, u \leq t - 1, \xi_{t,i} = 0, S_t = i} | \sigma(S_u : u \leq t)]).
\]

For each realization of \( S_u, u \leq t \), the above conditional probability is continuous in \( r \). Because the number of realizations of \( S_u, u \leq t \) is finite, we conclude that \( P(S_{\tau(r)} = i, \tau(r) = t) \) is continuous in \( r \) for any \( t \geq 0 \). Finally, \( \tau(r) \leq H_{[A,B]} \) for any \( r \in \mathcal{R} \), so that

\[
\sup_{r \in \mathcal{R}} P(S_{\tau(r)} = i, \tau(r) \geq t) \leq P(H_{[A,B]} \geq t) \xrightarrow{t \to \infty} 0
\]

and we conclude that \( P(S_{\tau(r)} = i) \) is continuous in \( r \in \mathcal{R} \).
Proposition 2.4 Consider $\mu \in \mathcal{M}_0(\mathbb{Z})$ with a finite support: $\mu([A, B]) = 1$ and $\mu(\{A\}) > 0$, $\mu(\{B\}) > 0$ for some $A, B \in \mathbb{Z}$ with $A < 0$ and $B > 0$, and recall $\mathcal{R}_\mu$ in [3]. Then there exists a maximal element $r^\text{max} \in \mathcal{R}_\mu$, i.e. $r^\text{max}_i \geq r_i$ for any $i \in \mathbb{Z}$ and $r \in \mathcal{R}_\mu$, and it solves the Skorokhod embedding problem: $\tau(r^\text{max}) \in \text{SEP}(\mathbb{P}^{\mathcal{S}_k}, \mu)$.

Proof. Existence of $r^\text{max}$ follows directly from Lemmas 2.2 and 2.3. By definition, $r^\text{max}_i = 1$ for $i \notin (A, B)$ so that $\tau(r^\text{max}) \leq H_{[A, B]}$ which in particular implies that $\tau(r^\text{max})$ is UI and $S_{\tau(r^\text{max})}$ is supported on $[A, B]$ and is centred. It remains to argue that $S_{\tau(r^\text{max})} \sim \mu$. Suppose otherwise. Then there exists $i_0 \in (A, B)$ such that $\mathbb{P}(S_{\tau(r^\text{max})} = i_0) < \mu(\{i_0\})$. Consider $r^\star$ with $r^\star_i = r^\text{max}_i$ for $i \neq i_0$ and $r^\star_{i_0} = r^\text{max}_{i_0} + \epsilon$. It follows that $\mathbb{P}(S_{\tau(r^\star)} = i_0) \leq \mathbb{P}(S_{\tau(r^\text{max})} = i_0) < \mu(\{i_0\})$ for $i \neq i_0$. Further, by Lemma 2.3, $\mathbb{P}(S_{\tau(r^\star)} = i_0) \leq \mu(\{i_0\})$ for sufficiently small $\epsilon$ which contradicts the maximality of $r^\text{max}$ in $\mathcal{R}_\mu$.

Proof. [Proof of Theorem 2.1] Proposition 2.4 establishes Theorem 2.1 for $\mu$ with bounded support. The final step is to extend the result to general $\mu \in \mathcal{M}_0$ through a limiting and coupling procedure. We fix such $\mu \in \mathcal{M}_0$ and, without loss of generality, assume that its support is unbounded on the right, i.e., $\mu((-\infty, n]) < 1$ for all $n \geq 0$. For each integer $n \geq 1$, we construct $\mu_n$ such that $\mu_n$ is supported on $[-n, n]$, agrees with $\mu$ on $(-n, n)$, and has zero mean, where integer $n$ is to be determined. The constraints imply that

$$
\begin{align*}
\mu_n(\{-n\}) + \mu((n, -n)) + \mu_n(\{n\}) &= 1, \\
-n\mu_n(\{-n\}) + n\mu_n(\{n\}) + \sum_{i=-n+1}^{n-1} i\mu(\{i\}) &= 0,
\end{align*}
$$

from which we can solve

$$
\begin{align*}
\mu_n(\{n\}) &= \frac{1}{n + n} \left[ n(1 - \mu((-n, n))) - \sum_{i=-n+1}^{n-1} i\mu(\{i\}) \right], \\
\mu_n(\{-n\}) &= \frac{1}{n + n} \left[ n(1 - \mu((-n, n))) + \sum_{i=-n+1}^{n-1} i\mu(\{i\}) \right].
\end{align*}
$$

To ensure that $\mu_n$ is a probability measure we define $n$ as

$$
n := \min \{ k \in \mathbb{N} | f(n, k) > 0 \}, \quad f(n, k) := k(1 - \mu((-k, n))) - \sum_{i=-k+1}^{n-1} i\mu(\{i\}).
$$

For a fixed $n$, $f(n, k)$ is increasing in $k$ and $\lim_{k \to \infty} f(n, k) = +\infty$ because $\mu((-\infty, n]) < 1$. It follows that $n$ is well-defined and $\mu_n(\{n\})$, given in (4), is positive. Furthermore, $n > 0$ because $f(n, 0) \leq 0$. At the same time

$$
n(1 - \mu((-n, n))) + \sum_{i=-n+1}^{n-1} i\mu(\{i\}) = -f(n, n - 1) + (n + n - 1)(1 - \mu((-n, n))) > 0,
$$
where the inequality follows because \( f(n, n - 1) \leq 0 \) by the definition of \( n \), \( \mu((-\infty, n)) < 1 \), \( n \geq 1 \), and \( n > 0 \). We conclude that \( \mu_n((-\infty)) \), given in \([4]\), is positive and hence \( \mu_n \) is a probability measure with all the desired properties.

Finally, because \( f(n, k) \) is decreasing in \( n \) for each \( k \), we observe that \( \bar{n} \) is increasing in \( n \). We claim that \( -\bar{n} \) converges to the left end of the support of \( \mu \) as \( n \) goes to infinity, i.e., \( \lim_{n \to \infty} -\bar{n} = \sup\{i \in \mathbb{Z} : \mu((-\infty, -i]) > 0\} \). Otherwise, there exists a positive integer \( m < \sup\{i \in \mathbb{Z} : \mu((-\infty, -i]) > 0\} \) such that \( \bar{n} \leq m \) for any \( n \). Because \( f(n, k) \) is increasing in \( k \), we have \( f(n, m) > 0 \) for any \( n \). However,

\[
\lim_{n \to \infty} f(n, m) = m(1 - \mu((\!-\!m, +\infty))) - \sum_{i=-m+1}^{\infty} i\mu(\{i\}) < 0,
\]

where the last inequality follows since \( \sum_{i=-m+1}^{\infty} i\mu(\{i\}) = 0 \) and \( m < \sup\{i \in \mathbb{Z} : \mu((-\infty, -i]) > 0\} \), which gives the desired contradiction.

Let \( r^n = \{r^n_i\}_{i \in \mathbb{Z}} \) be the maximal element of \( R_{\mu_n} \), as given by Proposition 2.3. We claim that \( r^n_i \) is decreasing in \( n \) for each fixed \( i \in \mathbb{Z} \). To this end, we only need to show that \( r^n_i \geq r^{n+1}_i \) for any \( i \in (-\bar{n}, n) \). Suppose this is not the case and there exist \( i_0 \in (-\bar{n}, n) \) such that \( r^{n_i}_0 < r^{n+1}_i \). We define \( \tilde{r} = \{\tilde{r}_i\}_{i \in \mathbb{Z}} \) with \( \tilde{r}_i := 1 \) for \( i \not\in (-\bar{n}, n) \) and \( \tilde{r}_i := r^{n+1}_i \) for \( i \in (-\bar{n}, n) \). It is obvious that for any \( i \in (-\bar{n}, n) \),

\[
\mathbb{P}(S_{\tau(\tilde{r})} = i) \leq \mathbb{P}(S_{\tau(r^{n+1})} = i) = \mu(\{i\}),
\]

so that \( \tilde{r} \in R_{\mu_n} \), which contradicts the maximality of \( r^n \) in \( R_{\mu_n} \).

Because \( r^n_i \) is decreasing in \( n \) for each \( i \), we can define \( r_i := \lim_{n \to \infty} r^n_i \) and denote \( r = \{r_i\}_{i \in \mathbb{Z}} \). It remains to show that the statement of Theorem 2.1 holds with \( r_n \). To this end, we construct suitable versions of \( \tau(r^n) \)'s. Construct Bernoulli random variables \( \{\xi^n_{t,i}\}_{t \geq 0, n \geq 1, i \in \mathbb{Z}} \) which are independent of each other and of \( S \) and satisfy

\[
\mathbb{P}(\xi^n_{t,i} = 1) = r^n_i, \quad \mathbb{P}(\xi^n_{t,i} = 1) = \frac{r^n_i}{r^n_i - 1} \mathbf{1}_{r^n_i > 0}, \quad n \geq 2.
\]

Define \( \xi^n_{t,i} = 1 - \prod_{k=1}^{n} \xi^n_{t,i} \), \( n \geq 1 \). Then, it is straightforward to verify that \( \mathbb{P}(\xi^n_{t,i} = 0) = r^n_i \) and we can use them to define \( \tau(r^n) = \inf\{t \geq 0 : \xi^n_{t,i} = 0\} \). Then, by definition, \( \tau(r^n) \) is increasing in \( n \) and \( \lim_{n \to \infty} \tau(r^n) = \inf\{t \geq 0 : \xi_{t,i} = 0\} \), where \( \xi_{t,i} := \lim_{n \to \infty} \xi^n_{t,i} \). Note that \( \xi_{t,i}, t \geq 0, i \in \mathbb{Z} \) are well-defined Bernoulli random variables, independent of each other and of \( S \) with

\[
\mathbb{P}(\xi_{t,i} = 0) = \lim_{n \to \infty} \mathbb{P}(\xi^n_{t,i} = 0) = \lim_{n \to \infty} r^n_i = r_i,
\]

so \( \inf\{t \geq 0 : \xi_{t,i} = 0\} \) is a version of \( \tau(r) \). With this version, we have \( \tau(r) = \lim_{n \to \infty} \tau(r^n) \).

We now show that \( \tau(r) < \infty \) a.s. and \( \tau(r) \in \text{SEP} \left( \mathbb{P}^S, \xi, \mu \right) \). To this end, let \( F_t := |S_t| - \sum_{j=0}^{t} 1_{S_j = 0}, \ t \in \mathbb{N} \). One easily checks that \( \{F_t\}_{t \geq 0} \) is a martingale. By the optional sampling theorem, using \( \tau(r^n) \leq H_{[-\bar{n}, n]} \), we conclude
Theorem 2.1, it is straightforward to argue that for any $2.2$. Algorithmic computation of the stopping probabilities

\[ \mathbf{E}[F_{r^n}] = 0. \]

This yields the first equality in the following computation:

\[
\mathbf{E} \left[ \sum_{j=0}^{\tau^n-1} 1_{S_j=0} \right] = \mathbf{E}[|S_{r^n}|] = \sum_{i \in \mathbb{Z}} |i| \mu_n \{ \{ i \} \} \leq \sum_{i \in \mathbb{Z}} |i| \mu(\{ i \}) < \infty. \quad (5)
\]

Letting $n \to \infty$, by the monotone convergence theorem, $\mathbf{E}[\sum_{j=0}^{\tau-1} 1_{S_j=0}] < \infty$ and it follows that $\tau(\tau) < \infty$ a.s. Observe also that for each $i \neq \inf\{ j \in \mathbb{Z} | \mu(\{ j \}) > 0 \}$, we have $\mathbb{P}(S_{r^n} = i) = \mu_n(\{ i \}) = \mu(\{ i \})$ for sufficiently large $n$. It now follows that $\mathbb{P}(S_{r^n} = i) = \lim_{n \to \infty} \mathbb{P}(S_{r^n} = i) = \mu(\{ i \})$ and hence $S_{r^n} \sim \mu$. Finally,

\[
\sum_{i \in \mathbb{Z}} |i| \mu(\{ i \}) = \mathbb{E}[|S_{r^n}|] \leq \liminf_{n \to \infty} \mathbb{E}[|S_{r^n}|] \leq \sum_{i \in \mathbb{Z}} |i| \mu(\{ i \}) < \infty,
\]

where the first inequality is due to Fatou’s lemma and the second inequality is because of (6). Therefore, we have $\lim_{n \to \infty} \mathbb{E}[|S_{r^n}|] = \mathbb{E}[|S_{r^n}|] < \infty$. It follows by Scheffe’s lemma that $S_{r^n} \to S_{r^n}$ in $L^1$, and hence $\mathbb{E}[S_{r^n} | F_{r^n}] = S_{r^n}$ almost surely (a.s.). By the martingale convergence theorem and the tower property of conditional expectation,

\[
\mathbb{E}[S_{r^n} | F_{r^n} \wedge t] = \lim_{n \to \infty} \mathbb{E}[S_{r^n} | F_{r^n} \wedge t] = \lim_{n \to \infty} \mathbb{E}[\mathbb{E}[S_{r^n} | F_{r^n}] | F_{r^n} \wedge t] = \lim_{n \to \infty} \mathbb{E}[S_{r^n} | F_{r^n} \wedge t] = S_{r^n} \wedge t,
\]

and in particular $\{S_{r^n} \wedge t\}_{t \geq 0}$ is uniformly integrable as required.

2.2. Algorithmic computation of the stopping probabilities $r_\mu$

In this section, we work under the assumptions of Theorem 2.1 and provide an algorithmic method for computing $\{r_i\}_{i \in \mathbb{Z}}$ obtained therein. We let $\tau = \tau(r_\mu)$ and $g_i$ denote the expected number of visits of $S$ to state $i$ strictly before $\tau$, i.e.:

\[
g_i := \mathbb{E} \left[ \sum_{t=0}^{\tau^n-1} 1_{S_t=i} \right] = \sum_{t=0}^{\infty} \mathbb{P}(\tau > t, S_t = i). \quad (6)
\]

Denote $a^+ := \max\{a, 0\}$. Similarly to the process $F$ defined in the proof of Theorem 2.1, it is straightforward to argue that for any $i \leq 0 \leq j$ the processes

\[
(i - S_t)^+ - \frac{1}{2} \sum_{u=0}^{t-1} 1_{S_u = i}, \quad (S_t - j)^+ - \frac{1}{2} \sum_{u=0}^{t-1} 1_{S_u = j}, \quad t \geq 0,
\]

are martingales. To compute $g_i$, we then apply the optional sampling theorem at $\tau \wedge t$ and let $t \to \infty$. Using the fact that $\{S_{\tau \wedge t}\}$ is a UI family of random variables together with monotone convergence theorem, we deduce that

\[
g_i = 2 \mathbb{E}[(S_{\tau} - i)^+] = 2 \sum_{k=i}^{\infty} \mathbb{P}(S_{\tau} \geq k + 1), \quad i = 0, 1, 2, \ldots, \quad (7)
\]

\[
g_i = 2 \mathbb{E}[(i - S_{\tau})^+] = 2 \sum_{k=-\infty}^{i} \mathbb{P}(S_{\tau} \leq k - 1), \quad i = 0, -1, -2, \ldots \quad (8)
\]
Writing \( p_i := \mu(\{i\}) = \mathbb{P}(S_\tau = i) \), we now compute

\[
p_i = \mathbb{P}(S_\tau = i) = \sum_{t=0}^{\infty} \mathbb{P}(\tau = t, S_t = i)
\]

\[
= \sum_{t=0}^{\infty} \mathbb{P}(\xi_u, S_u = 1, u = 0, 1, \ldots, t - 1, \xi_t, S_t = 0, S_t = i), \quad \text{and by conditioning}
\]

\[
= \sum_{t=0}^{\infty} \mathbb{P}(\xi_u, S_u = 1, u = 0, 1, \ldots, t - 1, S_t = i) r_i
\]

\[
= r_i \sum_{t=0}^{\infty} \mathbb{P}(\tau \geq t, S_t = i) = r_i (g_i + p_i).
\]

Therefore, if \( p_i + g_i > 0 \), we must have \( r_i = \frac{p_i}{p_i + g_i} \). If \( p_i + g_i = 0 \), then \( r_i \) can take any value in \([0, 1]\). Because we consider \( r_\mu \) to be the maximal in \( \mathcal{R}_\mu \), we set \( r_i = 1 \). Thus \( r_\mu = (r_i) \) in Theorem 2.1 is given by

\[
r_i = \frac{p_i}{p_i + g_i} 1_{\{p_i + g_i > 0\}} + 1_{\{p_i + g_i = 0\}}, \quad i \in \mathbb{Z}.
\]

(9)

### 3. Randomized Azéma-Yor solution to the SEP

Let us recall the celebrated Azéma-Yor solution to the SEP for a standard Brownian motion \((B_u : u \geq 0)\). We reserve \( t \) for the discrete time parameter and use \( u \in [0, \infty) \) for the continuous time parameter. To a centered probability measure \( \mu \) on \( \mathbb{R} \) we associate its barycenter function

\[
\psi_\mu(x) := \frac{1}{\bar{\mu}(x)} \int_{[x, +\infty)} yd\mu(y), \quad x \in \mathbb{R},
\]

(10)

where \( \bar{\mu}(x) := \mu([x, +\infty)) \) and \( \psi_\mu(x) := x \) for \( x \) such that \( \mu([x, +\infty)) = 0 \). We let \( b_\mu \) denote the right-continuous inverse of \( \psi_\mu \); i.e., \( b_\mu(y) := \sup\{x : \psi_\mu(x) \leq y\}, \quad y \geq 0 \). Then

\[
T_{\mu}^{\text{AY}} := \inf\{u \geq 0 : B_u \leq b_\mu(B^*_u)\}, \quad \text{where } B^*_u := \sup_{s \leq u} B_s,
\]

(11)

satisfies \( B_{T_{\mu}^{\text{AY}}} \sim \mu \) and \((B_{u\wedge T_{\mu}^{\text{AY}}} : u \geq 0)\) is UI. Furthermore, for any other such solution \( \bar{T} \) to the SEP, and any \( x \geq 0 \), we have \( \mathbb{P}(B^*_\bar{T} \geq x) \leq \mathbb{P}(B^*_T \geq x) = \bar{\mu}^{HL}(x) \); hence \( T_{\mu}^{\text{AY}} \) maximizes the distribution of the maximum in the stochastic order. Here \( \bar{\mu}^{HL} \) is the Hardy-Littlewood transform of \( \mu \) and the bound \( \bar{\mu}^{HL}(x) \) is due to Blackwell et al. (1963), and has been extensively studied since; see e.g. Carraro et al. (2012) for details.

A direct transcription of the Azéma-Yor embedding to the context of a simple symmetric random walk only works for measures \( \mu \in \mathcal{M}_0(\mathbb{Z}) \) for which \( \psi_\mu(x) \in \mathbb{N} \) for all \( x \in \mathbb{R} \), which is a restrictive condition; see Cox and Obloj...
for details. For a general $\mu \in \mathcal{M}_0(\mathbb{Z})$ we should seek instead to emulate the structure of the stopping time: the process stops when its drawdown hits a certain level, i.e., when $B^*_{n} - B_{n} \geq B^*_{n} - b_{\mu}(B^*_{n})$, or when it reaches a new maximum at which time a new maximum drawdown level is set, whichever comes first. Only in general, we expect to use an independent randomization when deciding to stop or to continue an excursion away from the maximum. Surprisingly, this can be done explicitly and the resulting stopping time maximizes stochastically the distribution of the running maximum of the stopped random walk among all solutions in \([1]\).

Before we state the theorem we need to introduce some notation. Let $\mu \in \mathcal{M}_0(\mathbb{Z})$ and denote $\bar{x} := \sup \{ n \in \mathbb{Z} | \mu(\{n\}) > 0 \}$ and $\underline{x} := \inf \{ n \in \mathbb{Z} | \mu(\{n\}) > 0 \}$ the bounds of the support of $\mu$. The barycentre function $\psi_{\mu}$ is piece-wise constant on $(-\infty, \bar{x}]$ with jumps in atoms of $\mu$, is non-decreasing, left-continuous with $\psi_{\mu}(x) > x$ for $x < \bar{x}$, $\psi_{\mu}(x) = x$ for $x \in [\bar{x}, +\infty)$, and $\psi_{\mu}(-\infty) = 0$. The inverse function $b_{\mu}$ is right-continuous, non-decreasing with $b_{\mu}(0) = \underline{x}$, and is integer valued on $(0, \bar{x})$. Further, $b_{\mu}(y) < y$ for $y < \bar{x}$ and $b_{\mu}(y) = y$ for $y \in [\bar{x}, +\infty)$; in particular $b_{\mu}(n) \leq n - 1$ for $n \in \mathbb{N} \cap [0, \underline{x})$. Moreover, for any $n \in \mathbb{Z} \cap [\underline{x}, \bar{x}]$, $\mu(\{n\}) > 0$ if and only if $n$ is in the range of $b_{\mu}$; consequently, $\mu(\{y\}) = 0$ for any $y$ that is not in the range of $b_{\mu}$.

For each $1 \leq n < \bar{x}$, $\{b_{\mu}(y) | y \in [n, n + 1]\} \cap (-\infty, n]$ is a nonempty set of finitely many integers which we rank and denote as $x^0_1 > x^0_2 > \cdots > x^0_{m_n + 1}$. Similarly, we rank $\{b_{\mu}(y) | y \in [0, 1]\}$, which is a nonempty set of finitely many integers $x^0_1 > x^0_2 > \cdots > x^0_{m_n + 1}$ if $b_{\mu}(0) > -\infty$ and a set of countably many integers $x^0_1 > x^0_2 > \cdots$ otherwise. Then, for each $1 \leq n < \bar{x}$, $x^n_{m_n + 1} = x^0_{m_n + 1} = b_{\mu}(n) \leq n - 1$. Note that we may have $m_n = 0$, in which case $x^n_{m_n + 1} = x^0_{m_n + 1} = x^n_1$. For each $1 \leq n < \bar{x}$ and for $n = 0$ when $\underline{x} > -\infty$, define

$$
\Gamma^n := \tilde{\mu}(x^n_{m_n + 1} + 1) - x^n_{m_n + 1},
$$

$$
g^n_k := \frac{n + 1 - x^n_k}{\Gamma^n} \mu(\{x^n_k\}), \quad k = 2, 3, \ldots, m_n,
$$

$$
g^n_{m_n + 1} := \frac{n + 1 - x^n_{m_n + 1}}{\Gamma^n} \left[ \mu(\{x^n_{m_n + 1}\}) - \tilde{\mu}(x^n_{m_n + 1}) \frac{n - \psi_{\mu}(x^n_{m_n + 1})}{n - x^n_{m_n + 1}} \right],
$$

$$
f^n_{m_n + 1} := 0, \quad f^n_k = f^n_{k+1} + g^n_{k+1}, \quad k = 1, 2, \ldots, m_n, \quad f^n_0 := 1.
$$

When $\underline{x} = -\infty$, define

$$
g^n_k = (1 - x^n_k) \mu(\{x^n_k\}), \quad k \geq 2, \quad f^n_k = \sum_{i=k+1}^{\infty} g^n_i, \quad k \geq 1, \quad f^n_0 := 1.
$$

Then, for each $1 \leq n < \bar{x}$ and for $n = 0$ when $\underline{x} > -\infty$, $\rho^n_k := 1 - (f^n_k / f^n_{k-1})$ is in $[0, 1)$ for $k = 1, \ldots, m_n$ and we let $\rho^n_{m_n + 1} := 1$; when $\underline{x} = -\infty$, $\rho^n_k := 1 - (f^n_k / f^n_0)$ is in $(0, 1)$ for each $k \geq 1$ and we set $m_0 + 1 := +\infty$. Let $\eta = (\eta^n_k : 0 \leq n < \bar{x}, k \in \mathbb{Z} \cap [1, m_n + 1])$ be a family of mutually independent Bernoulli random variables, independent of $S$, with $\mathbb{P}(\eta^n_k = 0) = \rho^n_k = 1 -$
\( \mathbb{P}(\eta_n^k = 1) \). We let \( S_t^k := \sup_{0 \leq s \leq t} S_t \) and define the enlarged filtration \( \mathbb{F}^{S,n} \) via 
\[ F_t^{\mathbb{F}^{S,n}} := \sigma(S_u, \eta_k^S, : k \in \mathbb{Z} \cap [1, m + 1], u \leq t). \]

We are now ready to define our Azéma–Yor embedding for \( S \). It is an \( \mathbb{F}^{S,n} \)-stopping time which, in analogy to (11), stops when an excursion away from the maximum breaches a given level. However, since the maximum only takes integer values, we emulate the behaviour of \( B_{\mu_T^{AY}} \) between hitting times of two consecutive integers in an averaged manner, using independent randomization. Specifically, if we let \( \mathcal{H}_n := \inf\{t \geq 0 : S_t = n\} \) then after \( \mathcal{H}_n \) but before \( \mathcal{H}_{n+1} \) we may stop at each of \( x_1^n > x_2^n \ldots > x_m^n \) depending on the independent coin tosses \( \eta_k^n \), while we stop a.s. if we hit \( x_{m+1}^n \). If we first hit \( n+1 \) then a new set of stopping levels is set. Finally, we stop upon hitting \( \bar{x} \).

**Theorem 3.1** Let \( \mu \in \mathcal{M}_0(\mathbb{Z}) \) and \( \tau_{\mu}^{AY} \) be given by
\[ \tau_{\mu}^{AY} := \inf \left\{ t \geq 0 : S_t \leq x_k^S \text{ and } \eta_k^S = 0 \text{ for some } k \in \mathbb{Z} \cap [1, m_n + 1] \right\} \wedge \mathcal{H}_x. \]

Then \( \tau_{\mu}^{AY} \in \text{SEP}(\mathbb{F}^{S,n}) \) and for any \( \sigma \in \text{SEP}(\mathbb{F}, S) \)
\[ \mathbb{P}(S_{\sigma}^* \geq n) \leq \mathbb{P}(S_{\tau_{\mu}^{AY}}^{\mu} \geq n) = \hat{\mu}(b_{\mu}(n)) \frac{\psi_{\mu}(b_{\mu}(n)) - b_{\mu}(n)}{n - b_{\mu}(n)}, \quad n \in \mathbb{N}, \quad (14) \]
with the convention \( \frac{0}{0} = 1 \).

The optimality property in (14) is analogous to the optimality of \( T_{\mu}^{AY} \) in a Brownian setup, as described above and the bound in (14) coincides with \( \hat{\mu}^{HL}(n) \).

Finally, we note that by considering our solution for \( (-S_t)_{t \geq 0} \) we obtain a reversed Azéma–Yor solution which stops when the maximum drawup since the time of the historical minimum hits certain levels. It follows from (14) that such embedding maximizes the distribution of the running minimum in the stochastic order.

**Proof.** [Proof of Theorem 3.1] It is straightforward to verify that all the conclusions of the theorem hold for the case in which \( \mu(\{0\}) = 1 \), so we assume in the following that \( \mu(\{0\}) < 1 \) and thus \( \underline{x} = 0 < \bar{x} \). Throughout the proof we let \( \tau = \tau_{\mu}^{AY} \) and recall that \( \mathcal{H}_j = \inf\{t \geq 0 : S_t = j\} \). We first prove constructively that \( \mu^{\bar{x}}'s \) are well defined and the constructed stopping time \( \tau \) embeds \( \mu \) into the random walk. Specifically, we argue by induction that for any \( 0 \leq j < \bar{x} \) the stopping time \( \tau \) satisfies
\[
\begin{cases}
\mathbb{P}(S_\tau = y \text{ and } \tau < \mathcal{H}_{j+1}) = \mu(\{y\}), & \text{for } y < x_1^j \\
\mathbb{P}(S_\tau = y \text{ and } \tau < \mathcal{H}_{j+1}) = \hat{\mu}(x_1^j) + 1 - \psi_{\mu}(x_1^j), & \text{for } y = x_1^j \\
\mathbb{P}(S_\tau = y \text{ and } \tau < \mathcal{H}_{j+1}) = 0, & \text{for } y > x_1^j.
\end{cases}
\quad (15)
\]

First, we show the inductive step: we prove that (15) holds for \( j = n < \bar{x} \) given that it holds for \( j = 0, \ldots, n - 1 \). Because (15) is true for \( j = n - 1 \), we
obtain

\[ P(\tau \geq \mathcal{H}_n) = 1 - P(\tau < \mathcal{H}_n) = 1 - \left( \sum_{y < x_1^n} \mu(\{y\}) \right) - \bar{\mu}(x_1^{n-1}) \frac{n - \psi(\mu(x_1^{n-1}))}{n - x_1^{n-1}} \]

\[ = \mu(x_1^{n-1}) - \bar{\mu}(x_1^{n-1}) \frac{n - \psi(\mu(x_1^{n-1}))}{n - x_1^{n-1}} = \bar{\mu}(x_1^{n-1}) \frac{\psi(\mu(x_1^{n-1})) - x_1^{n-1}}{n - x_1^{n-1}} \]

\[ = \bar{\mu}(x_{m+n}) \frac{\psi(\mu(x_{m+n+1})) - x_{m+n+1}}{n - x_{m+n+1}} = \Gamma_n, \] \hspace{1cm} (16)

where \( \Gamma_n \) is defined as in (12). Recalling that \( x_{m+n+1} = b(\mu) \leq n - 1 < \bar{x} \) and \( \psi(\mu(x) < x \) for \( x < \bar{x} \), we conclude that \( P(\tau \geq \mathcal{H}_n) > 0 \).

Consider first the case when \( m_n = 0 \), so that \( x_1^n = x_{m+n+1} = x_1^{n-1} \). and \( \tau \) stops if \( x_1^n \) is hit in between \( \mathcal{H}_n \) and \( \mathcal{H}_{n+1} \). Consequently,

\[ P(S_T = x_1^n, \tau < \mathcal{H}_{n+1}) = P(S_T = x_1^n, \tau < \mathcal{H}_n) + \mathbb{P}(S_T = x_1^n, \tau < \mathcal{H}_{n+1}|\tau \geq \mathcal{H}_n) \cdot \mathbb{P}(\tau \geq \mathcal{H}_n) \]

\[ = \mathbb{P}(S_T = x_1^n, \tau < \mathcal{H}_n) + \frac{1}{n + 1 - x_1^n} \cdot \mathbb{P}(x_{m+1}^n) \psi(\mu(x_{m+1}^n)) - x_{m+1}^n) \]

\[ = \mu(x_1^n) \frac{n - \psi(\mu(x_1^n))}{n - x_1^n} + \frac{1}{n + 1 - x_1^n} \cdot \bar{\mu}(x_1^n) \psi(\mu(x_1^n)) - x_1^n \]

\[ = \bar{\mu}(x_1^n) \frac{n + 1 - \psi(\mu(x_1^n))}{n + 1 - x_1^n}, \] and we conclude that (15) holds for \( j = n \).

Next, we consider the case in which \( m_n \geq 1 \). A direct calculation yields

\[ \mu(\{x_{m+n+1}\}) - \bar{\mu}(x_{m+n+1}) \frac{n - \psi(\mu(x_{m+n+1}))}{n - x_{m+n+1}} \]

\[ = \frac{1}{n - x_{m+n+1}} \left[ \sum_{y > x_{m+n+1}} y \mu(\{y\}) - n \mu((x_{m+n+1}, +\infty)) \right] \]

\[ = \frac{\mu((x_{m+n+1}, +\infty))}{n - x_{m+n+1}} [\psi(\mu(x_{m+n+1})) - n] = \frac{\mu((x_{m+n+1}, +\infty))}{n - x_{m+n+1}} [\psi(\mu(b(\mu(n)) + n] > 0, \]

where the inequality follows from \( x_{m+n+1} = b(\mu(n)) < n < \bar{x} \) and \( \psi(\mu(b(\mu(y)) + y \) for any \( y < \bar{x} \). Consequently, \( y_{m+n+1} > 0 \). On the other hand, because \( \mu(\{y\}) = 0 \) for any \( y \) that is not in the range of \( b(\mu) \), we conclude that

\[ \sum_{k=2}^{m_n+1} (n + 1 - x_k^n) \mu(\{x_k^n\}) = \sum_{x_{m+n+1} \leq y < x_1^n} (n + 1 - y) \mu(\{y\}) \]

\[ = (n + 1) [\bar{\mu}(x_{m+n+1}) - \bar{\mu}(x_1^n)] - \left[ \psi(\mu(x_{m+n+1})) \psi(\mu(x_{m+n+1})) - \psi(\mu(x_1^n)) \bar{\mu}(x_1^n) \right] \]

\[ = (n + 1 - \psi(\mu(x_{m+n+1}))) \bar{\mu}(x_{m+n+1}) - (n + 1 - \psi(\mu(x_1^n))) \bar{\mu}(x_1^n). \]
Consequently, we have

\[ f_1^n = \sum_{k=2}^{m+n+1} g_k \]

\[ = \frac{1}{\Gamma_n} \left[ \sum_{k=2}^{m+n+1} (n + 1 - x_k^n) \mu(\{x_k^n\}) - (n + 1 - x_{m+n+1}^n) \mu(x_{m+n+1}^n) \right] \]

\[ = \frac{1}{\Gamma_n} \left[ \mu(x_{m+n+1}^n) \psi_n(x_{m+n+1}^n) - n - x_{m+n+1}^n \right] \]

\[ = 1 - \frac{(n + 1 - \psi_n(x_1^n)) \mu(x_1^n)}{\mathbb{P}(\tau \geq \mathcal{H}_n)} \leq 1, \]

where the last inequality holds because \( \psi_n(x_1^n) = \psi_n(\min(b_n(n + 1), n)) \leq \psi_n(b_n(n + 1)) \leq n + 1. \) It follows, since \( g_k^n \geq 0, k = 2, \ldots, m+n+1, \) that \( f_k \) is strictly increasing in \( k = 1, 2, \ldots, m+n \) with \( f_{m+n}^n > 0 \) and \( f_1^n \leq 1. \) Consequently, \( \rho_k^n \) is well defined and \( \rho_k^n \in [0, 1], k = 1, \ldots, m+n. \) Recall \( \rho_{m+n+1}^n = 1 = 1 - (f_{m+n+1}^n/f_{m+n}^n). \) Set \( x_0^1 := n. \) Then, for each \( k = 1, \ldots, m+n+1, \)

\[ \mathbb{P}(S_\tau = x_k^n, \mathcal{H}_n \leq \tau < \mathcal{H}_{n+1}) = \mathbb{P}(S_\tau = x_k^n, \tau < \mathcal{H}_{n+1} | \tau \geq \mathcal{H}_n) \cdot \mathbb{P}(\tau \geq \mathcal{H}_n) \]

\[ = \left[ \prod_{j=1}^{k-1} \left( \frac{n + 1 - x_{j-1}^n}{n + 1 - x_j^n} \right) \right] \rho_k^n \mathbb{P}(\tau \geq \mathcal{H}_n) = \frac{1}{n + 1 - x_k^n} (f_{k-1}^n - f_k^n) \mathbb{P}(\tau \geq \mathcal{H}_n). \]

Therefore, recalling the definition of \( f_k^n \) and \( g_k^n, \) for \( k = 2, \ldots, m+n, \) we have \( \mathbb{P}(S_\tau = x_k^n, \mathcal{H}_n \leq \tau < \mathcal{H}_{n+1}) = \mu(\{x_k^n\}). \) Further, \( \mathbb{P}(S_\tau = x_{m+n+1}^n, \mathcal{H}_n \leq \tau < \mathcal{H}_{n+1}) = \mu(\{x_{m+n+1}^n\}) - \mu(x_{m+n+1}^n) \mathbb{P}(\tau \geq \mathcal{H}_n) \) and \( \mathbb{P}(S_\tau = x_1^n, \mathcal{H}_n \leq \tau < \mathcal{H}_{n+1}) = \frac{n+1-\psi_n(x_1^n)}{n+1-x_1^n} \mu(x_1^n) \) and we verify that (15) holds for \( j = n. \)

We move on to showing the inductive base step: we prove (15) holds for \( j = 0. \) When \( x > -\infty, \) \( x_0^n \)'s are finitely many and the proof is exactly as above. When \( x = -\infty, \) by definition, \( g_k^0 > 0 \) because \( x_0^0 \leq x_0^1 \leq 0 \) and \( \mu(\{x_0^1\}) > 0. \) Recalling that \( \mu(\{y\}) = 0 \) for any \( y \) that is not in the range of \( b_\mu, \) we have

\[
\sum_{k=2}^{\infty} g_k^0 = \sum_{k=2}^{\infty} (1 - x_k^0) \mu(\{x_k^0\}) = \sum_{y < x_1^0} (1 - y) \mu(\{y\}) = 1 - \bar{\mu}(x_1^0) - \sum_{y < x_1^0} y \mu(\{y\}) \]

\[ = 1 - \bar{\mu}(x_1^0) + \sum_{y \geq x_1^0} y \mu(\{y\}) = 1 - \bar{\mu}(x_1^0) (1 - \psi_\mu(x_1^0)). \]

Because \( x_0^0 = \min(b_\mu(1), 0) \) and \( \psi_\mu(b_\mu(y)) \leq y \) for any \( y, \) we conclude that \( \psi_\mu(x_1^0) \leq \psi_\mu(b_\mu(1)) \leq 1. \) In addition, \( \bar{\mu}(x_1^0) < 1, \) showing that \( \sum_{k=2}^{\infty} g_k^0 < 1. \) Therefore, \( f_k^0 \)'s are well defined, positive, and strictly decreasing in \( k, \) and \( f_1^0 <
1. Therefore, \( \rho_k^0 \in (0, 1) \) for each \( k \geq 1 \). Following the same arguments as previously, one concludes that (15) holds for \( j = 0 \).

Next, we show that \( S_\tau \sim \mu \). If \( \bar{x} = +\infty \), by the construction of \( \tau \), we have

\[
\mathbb{P}(\tau = +\infty) \leq \lim_{n \to +\infty} \mathbb{P}(\tau \geq \mathcal{H}_n) = \lim_{n \to +\infty} \frac{\mu(x_{m_n+1}^n) \psi_{\mu}(x_{m_n+1}^n)}{n - x_{m_n+1}^n} = 0,
\]

since \( \lim_{n \to +\infty} x_{m_n+1}^n = \lim_{n \to +\infty} b_\mu(n) = +\infty \), and \( \lim_{x \to +\infty} \mu(x) = 0 \), and \( \psi_{\mu}(x_{m_n+1}^n) \in [0, \infty) \). Consequently, \( \tau < +\infty \) a.s. and for any \( y \in \mathbb{Z} \), \( \mathbb{P}(S_\tau = y) = \lim_{n \to +\infty} \mathbb{P}(S_\tau = y, \tau < \mathcal{H}_n) = \mu(\{y\}) \). If \( \bar{x} < +\infty \), by definition, \( \tau \leq \mathcal{H}_\bar{x} < +\infty \) a.s. Moreover, because (15) is true for \( j = \bar{x} - 1 \), we have \( \mathbb{P}(S_\tau = y) = \mathbb{P}(S_\tau = y, \tau < \mathcal{H}_\bar{x}) = \mu(\{y\}) \) for any \( y < x_1^{-1} \). From the definition of \( x_1^{-1} \), we conclude that \( b_\mu \) does not take any values in \((x_1^{-1}, \bar{x})\), so \( \mu(\{n\}) = 0 \) for any integer \( n \) in this interval. Since \( S_\tau \leq \bar{x} \), it remains to argue \( \mathbb{P}(S_\tau = x_1^{-1}) = \mu(\{x_1^{-1}\}) \). This follows from (15) with \( j = \bar{x} - 1 \):

\[
\mathbb{P}(S_\tau = x_1^{-1}) = \mathbb{P}(S_\tau = x_1^{-1}, \tau < \mathcal{H}_\bar{x}) = \mu(\{x_1^{-1}\}) = \mu(\{\bar{x}\}) = \mu(\{x_1^{-1}\}),
\]

where the fourth equality follows because \( \mu(\{n\}) = 0 \) for any \( n > x_1^{-1} \) and \( n \neq \bar{x} \).

To conclude that \( \tau \in \text{SEP}(\mathbb{R}, \mu) \), it remains to argue that \( \tau \) is UI, which is equivalent to \( \lim_{K \to +\infty} K \mathbb{P}(\sup_{t \geq 0} |S_{\tau \wedge t}| \geq K) = 0 \), see e.g. [Azéma et al. 1980]. We first show that

\[
\lim_{K \to +\infty} K \mathbb{P}(\sup_{t \geq 0} |S_{\tau \wedge t}| \geq K) = \lim_{K \to +\infty} \mathbb{P}(S_\tau^* \geq K) = 0.
\]

Because \( \{S_{\tau \wedge t}\} \) never visits states outside any interval that contains the support of \( \mu \), we only need to prove this when \( \bar{x} = +\infty \), and hence \( \lim_{y \to +\infty} b_\mu(y) = +\infty \), and taking \( K \in \mathbb{N} \).

By (16) and the construction of \( \tau \), we see that for \( n \in \mathbb{N} \), \( n < \bar{x} \), we have

\[
\mathbb{P}(S_\tau^* \geq n) = \mathbb{P}(\tau \geq \mathcal{H}_n) = \mu(x_{m_n+1}^n) \psi_{\mu}(x_{m_n+1}^n) \frac{\psi_{\mu}(x_{m_n+1}^n)}{n - x_{m_n+1}^n}
\]

\[
= \mu(b_n^\tau(n)) \psi_{\mu}(b_n^\tau(n) - b_\mu(n)) \frac{\psi_{\mu}(b_n^\tau(n) - b_\mu(n))}{n - b_\mu(n)} = \mu(b_n^\tau(n) +) \psi_{\mu}(b_n^\tau(n) +) \frac{\psi_{\mu}(b_n^\tau(n) +) - b_\mu(n)}{n - b_\mu(n)}
\]

and thus

\[
n \mathbb{P}(S_\tau^* \geq n) = \mu(b_n^\tau(n) +) \psi_{\mu}(b_n^\tau(n) +) \frac{\psi_{\mu}(b_n^\tau(n) +) - b_\mu(n)}{n - b_\mu(n)}
\]

\[
= \mu(b_n^\tau(n) +) \psi_{\mu}(b_n^\tau(n) +) \frac{\psi_{\mu}(b_n^\tau(n) +) - n}{n - b_\mu(n)} + \mu(b_n^\tau(n) +) n.
\]
For $a < c < d$ the function $y \to y(d - y)/(y - a)$ attains maximum on $[c, d]$ in $y = c$. Taking $a = b_i(n) = c = \psi_\mu(b_i(n)) \leq y = n < d = \psi_\mu(b_i(n)+)$, we can bound the first term by

$$\bar{\mu}(b_i(n)+) \mu_\mu(b_i(n)+) - n \leq \bar{\mu}(b_i(n)+) \frac{\psi_\mu(b_i(n)+) - \psi_\mu(b_i(n))}{\psi_\mu(b_i(n)) - \mu_\mu(n)} \mu_\mu(b_i(n))$$

$$= \mu(\{b_i(n)\}) \mu_\mu(b_i(n)) \psi_\mu(b_i(n)) = \sum_{y \geq b_i(n)} y \mu(\{y\}),$$

which goes to zero with $n \to \infty$ since $\mu \in M_0(\mathbb{Z})$. Similarly, $\psi_\mu(b_i(n)+) > n$ gives

$$\bar{\mu}(b_i(n)+) n \leq \mu(\{b_i(n)\}) \psi_\mu(b_i(n)+) = \sum_{y \geq b_i(n)} y \mu(\{y\}) \xrightarrow{n \to \infty} 0$$

and we conclude that $nP(S^*_\tau \geq n) \to 0$ as $n \to \infty$. It remains to argue that

$$\lim_{n \to +\infty} nP\left(\inf_{t \geq 0} S_{\tau \wedge t} \leq -n\right) = 0.$$

This is trivial if $x > -\infty$. Otherwise $b_i(0) = x = -\infty$ and $x^0_i$’s are infinitely many. For $n \in \mathbb{N}$, by the construction of $\tau$, $\inf_{t \geq 0} S_{\tau \wedge t} \leq -n$ implies that $S$ visits $-n$ before hitting 1 and $S$ is not stopped at any $x^0_i > -n$. Denote the $i_n := \sup\{i \geq 1 | x^0_i > -n\}$ and note that $i_n \to \infty$ as $n \to \infty$. By construction, the probability that $S$ does not stop at $x^0_i$ given that $S$ reaches $x^0_i$ is $f_i^n/f_i^{n-1}$, $i \geq 1$. On the other hand, the probability that $S$ visits $-n$ before hitting 1 is $1/(n+1)$. Therefore, the probability that $S$ visits $-n$ before hitting 1 and $S$ is not stopped at any $x^0_i > -n$ is

$$\frac{1}{n+1} \prod_{i=1}^{i_n} f_i^n / f_i^{n-1} = \frac{1}{n+1} f_i^{i_n}.$$

From (13), $\lim_{k \to +\infty} f_k^0 = 0$ since $\mu$ has a finite first moment. Therefore,

$$\lim_{n \to +\infty} nP\left(\inf_{t \geq 0} S_{\tau \wedge t} \leq -n\right) \leq \lim_{n \to +\infty} n \frac{1}{n+1} f_i^{i_n} = 0.$$

The above concludes the proof of $\tau \in SEP(\mathbb{F}S^*, S)$.

While (14) may be deduced from known bounds, as explained before, we provide a quick self-contained proof. Fix any $n \geq 1$ and $\sigma \in SEP(\mathbb{F}, S)$. When $\bar{x} < +\infty$, by UI, we have $P(S^*_\tau \geq \bar{x}) = P(S_{\tau} = \bar{x}) = \mu(\{\bar{x}\}) = P(S_{\sigma} = \bar{x}) = P(S^*_n \geq n \geq n_0) = P(S^*_n \geq 0) = 0$ for any $n > \bar{x}$.

Next, by Doob’s maximal equality and the UI of $\sigma$, $E[(S_{\sigma} - n)1_{S_{\sigma} \geq n}] = 0$ and hence, for $k \leq n \in \mathbb{N}$,

$$0 = E[(S_{\sigma} - n)1_{S_{\sigma} \geq n}] = E[(S_{\sigma} - n)1_{S_{\sigma} \geq k}] + E[(S_{\sigma} - n)(1_{S_{\sigma} \geq n} - 1_{S_{\sigma} \geq k})]$$

$$\leq E[(S_{\sigma} - n)1_{S_{\sigma} \geq k}] + (k - n)E[1_{S_{\sigma} \geq n}1_{S_{\sigma} \geq k}] - (k - n)E[1_{S_{\sigma} \geq k}1_{S_{\sigma} \geq k}]$$

$$= E[(S_{\sigma} - n)1_{S_{\sigma} \geq k}] + (k - n)E[1_{S_{\sigma} \geq n}1_{S_{\sigma} \geq k}]$$

$$= E[(S_{\sigma} - k)1_{S_{\sigma} \geq k}] - (n - k)P(S^*_\sigma \geq n). \tag{18}$$
Considering \( \mathbb{N} \ni n < \pi \) and \( k = b_\mu(n) < n \), and recalling \((17)\), we obtain

\[
\mathbb{P}(S_\sigma^* \geq n) \leq \frac{\mathbb{E}[(S_\sigma - b_\mu(n)) 1_{S_\sigma \geq b_\mu(n)}]}{n - b_\mu(n)} = \frac{\bar{\mu}(b_\mu(n)) (\psi_\mu(b_\mu(n)) - b_\mu(n))}{n - b_\mu(n)} = \mathbb{P}(S^*_r \geq n).
\]

4. Examples

We end this paper with an explicit computation of our two embeddings for two examples.

4.1. Optimal Gambling Strategy

The first example is a measure \( \mu \) arising naturally from the casino gambling model studied in He et al. (2016). Therein, a gambler whose preferences are represented by cumulative prospect theory (Tversky and Kahneman 1992) is repeatedly offered a fair bet in a casino and decides when to stop gambling and exit the casino. The optimal distribution of the gambler’s gain and loss at the exit time is a certain \( \mu \in M_0(\mathbb{Z}) \), which may be characterised explicitly, see He et al. (2016, Theorem 2). With typical model parameters\(^1\) we obtain

\[
\mu(\{n\}) = \begin{cases} 
0.4465 \times ((n^{0.6} - (n - 1)^{0.6}) \frac{10}{\pi} - ((n + 1)^{0.6} - n^{0.6}) \frac{10}{\pi}) , & n \geq 2, \\
0.3297 , & n = 1, \\
0.6216 , & n = -1, \\
0 , & \text{otherwise}. 
\end{cases}
\]

We first exhibit the randomized Markovian stopping time \( \tau(r) \) of Theorem 2.1 Using the algorithm given in Section 2.2 we compute \( r_i \), the probabilities of a coin tossed at \( S_i = i \) turning up tails, for all \( i \in \mathbb{Z} \):

\[
r_1 = 0.4040, \ r_2 = 0.0600, \ r_3 = 0.0253, \ r_4 = 0.0140, \ r_5 = 0.0089, \\
r_6 = 0.0061, \ r_7 = 0.0045, \ r_8 = 0.0034, \ r_9 = 0.0027, \ r_{10} = 0.0021, \ldots
\]

Note that \( \mu(\{0\}) = 0 \) and \( \mu(\{n\}) = 0, n \leq -2 \); so one does not stop at 0 and must stop upon reaching \(-1\). The stopping time \( \tau(r) \) is illustrated in the left pane of Figure 1. \( S \) is represented by a recombining binomial tree. Black nodes stand for “stop”, white nodes stand for “continue”, and grey nodes stand for the cases in which a random coin is tossed and one stops if and only if the coin turns up tails. The probability that the random coin turns tails is shown on the top of each grey node.

\(^1\)Specifically: \( \alpha_+ = 0.6, \delta_+ = 0.7, \alpha_- = 0.8, \delta_- = 0.7, \) and \( \lambda = 1.05 \).
Next we follow Theorem 3.1 to construct a randomized Azémia-Yor stopping time $\tau^{\text{AY}}_\mu$ embedding $\mu$. To this end, we compute $x_k^n$’s and $\rho_k^n$’s, which stand for the drawdown levels that are set after reaching maximum $n$ and the probabilities that the coins tossed at these levels turn up tails, respectively:

$m_0 = 0, x_1^0 = -1, m_1 = 1, x_1^1 = 1, \rho_1^1 = 0.2704, x_2^1 = -1,$
$m_2 = 0, x_1^2 = 1, m_3 = 0, x_3^1 = 1, m_4 = 0, x_4^1 = 1,$
$m_5 = 1, x_1^5 = 2, \rho_1^5 = 0.0049, x_5^2 = 1, m_6 = 0, x_6^1 = 1, \ldots$

The stopping time is then illustrated in the right pane of Figure 1: $S$ is represented by a non-recombining binomial tree. Again, black nodes stand for “stop”, white nodes stand for “continue”, and grey nodes stand for the cases in which a random coin is tossed and one stops if and only if the coin turns up tails. The probability that the random coin turns up tails is shown on the top of each grey node.

By definition, $\tau(r)$ is Markovian: at each time time $t$, decision to stop depends only on the current state and an independent coin toss. However, to implement the strategy, one needs to toss a coin most of the times. In contrast $\tau^{\text{AY}}_\mu$ requires less independent coin tossing: e.g. in the first five periods at most one such a coin toss, but it is path-dependent. For instance, consider $t = 3$ and $S_t = 1$. If one reaches this node along the path from $(0,0)$, through $(1,1)$ and $(2,2)$, and to $(3,1)$, then she stops. If one reaches this node along the path from $(0,0)$, through $(1,1)$ and $(2,0)$, and to $(3,1)$, then she continues. Therefore, compared to the randomized Markovian strategy, the randomized Azémia-Yor strategy involves less randomization at the cost of being path-dependent\(^2\).

4.2. Mixed Geometric Measure

The second example is a mixed geometric measure $\mu$ on $\mathbb{Z}$ with

$$
\mu(\{n\}) = \begin{cases} 
\gamma_+ [q_+(1-q_+)^{n-1}], & n \geq 1, \\
1 - \gamma_+ - \gamma_-, & n = 0, \\
\gamma_- [q_-(1-q_-)^{-n-1}], & n \leq -1,
\end{cases}
$$

where $\gamma_\pm \geq 0$, $q_\pm \in (0,1)$, $\gamma_+ + \gamma_- \leq 1$, and $\gamma_+/q_+ = \gamma_-/q_-$ so that $\mu \in \mathcal{M}_0(\mathbb{Z})$.

The randomized Markovian stopping time that embeds $\mu$ given by (20) can be derived analytically. Indeed, according to the algorithm given in Section 2.2 the probability of a coin tossed at $S_t = i$ turning up tails is

$$
r_i = \begin{cases} 
q_+^2/[(1-q_+)^2 + 1], & i \geq 1, \\
(1-\gamma_+ - \gamma_-)/[1 - \gamma_+ - \gamma_- + 2(\gamma_+/q_+)], & i = 0, \\
q_-^2/[(1-q_-)^2 + 1], & i \leq -1.
\end{cases}
$$

\(^2\)Indeed, [He et al. (2017)] showed that, theoretically, any path-dependent strategy is equivalent to a randomization of Markovian strategies.
The randomized Azéma-Yor stopping time that embeds \( \mu \) given by (20) can also be derived analytically. Because the formulae for \( x_n^+ \)'s and \( \rho_n^+ \)'s are tedious, we chose not to present them here. Instead, we illustrate the two embeddings in Figure 2 by setting \( q^+ = \gamma^+ = \frac{5}{12} \) and \( q^- = \gamma^- = \frac{13}{24} \). As in the previous example, the randomized Azéma-Yor stopping time involves less randomization than the randomized Markovian stopping time at the cost of being path-dependent.

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References

Figure 2: Randomized Markovian stopping time (left pane) and randomized Azéma-Yor stopping (right pane) embedding probability measure with $q_+ = \gamma_+ = 5/12$ and $q_- = \gamma_- = 13/24$ into the random walk $S$. Black nodes stand for “stop”, white nodes stand for “continue”, and grey nodes stand for the cases in which a random coin is tossed and one stops if and only if the coin turns tails. Each node is marked on the right by a pair $(t, x)$ representing time $t$ and $S_t = x$. Each grey node is marked on the top by a number showing the probability that the random coin tossed at that node turns up tails.


