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Evolution of the Arrow–Pratt measure of risk-tolerance for predictable forward utility processes

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Abstract We study the evolution of the Arrow–Pratt measure of risk-tolerance in the framework of discrete-time predictable forward utility processes in a complete semimartingale financial market. An agent starts with an initial utility function, which is then sequentially updated forward at discrete times under the guidance of a martingale optimality principle. We mostly consider a one-period framework and first show that solving the associated inverse investment problem is equivalent to solving some generalised integral equations for the inverse marginal function or for the conjugate function, both associated with the forward utility. We then completely characterise the class of forward utility pairs that can have a time-invariant measure of risk-tolerance and thus a preservation of preferences in time. Next, we show that in general, preferences vary over time and that whether the agent becomes more or less tolerant to risk is related to the curvature of the measure of risk-tolerance of the forward utility pair. Finally, to illustrate the obtained general results, we present an example in a binomial market model where the initial utility function belongs to the SAHARA class, and we find that this class is analytically tractable and stable in the sense that all the subsequent utility functions belong to the same class as the initial one.

Keywords Risk-aversion · Portfolio selection · Semimartingale model · Forward utility processes · Dynamic preferences · Complete financial market · Binomial model · SAHARA utility

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1 Introduction

The Arrow–Pratt measures of risk-aversion and risk-tolerance, introduced by Arrow [3, Sect. 3] and Pratt [38], have had a lasting impact on the science of decision-making. They play an important part in the theory on the economics of risk, and understanding them is crucial in numerous applications. Kenneth Arrow himself reasoned that 'the behaviour of these measures as wealth changes is of the greatest importance for prediction of economic reactions in the presence of uncertainty' (Arrow [4, page 96]). We should complement this statement by arguing that not only is the behaviour of these measures of greatest importance as wealth changes, but also as *time* passes. In this paper, we focus on studying the behaviour of the Arrow–Pratt measure of risk-tolerance under the evolution of time in the framework of discrete-time predictable forward utility (or performance) processes (*forward processes* in short) as recently introduced in Angoshtari et al. [1].

Forward criteria were introduced in continuous time and thoroughly studied in a series of papers by Musiela and Zariphopoulou [31, 32, 33, 34, 35]. The related notion of horizon-unbiased utility was developed and studied in Henderson and Hobson [20]; see also Zariphopoulou and Žitković [47] and El Karoui and Mnif [13]. Nadtochiy and Zariphopoulou [37], Choulli and Ma [12] and Liang and Zariphopoulou [30] focus on homothetic forward performance processes in models for the financial market which go beyond the original Itô process setting. Shkolnikov et al. [43], Chong et al. [10] and Geng and Zariphopoulou [14] provide an asymptotic analysis of forward performance processes. An axiomatic justification and explicit stochastic representation of forward performance criteria were obtained in Nadtochiy and Tehranchi [36]. Recently, forward processes have been further extended to situations involving model uncertainty (Källblad et al. [23], Chong and Liang [11]) and probability distortions (He et al. [19]), and applied to problems arising from pension fund strategies (Bernard and Kwak [5]) and insurance (Chong [9]).

For the purpose of investigating how preferences change over time, forward processes offer several advantages over the classical expected utility maximisation framework. An expected utility maximiser is required to *a priori* select and commit to a time horizon and a market model for the *entire* time period. He/she is also supposed to already know at time zero what his/her preferences (represented by a utility function) will be at the end of the time horizon. This approach looks at the investment problem as a closed system and assumes that the problem terminates once the horizon is reached. Under forward processes, on the other hand, the agent starts with specifying his/her preferences for today. It seems more plausible that one is able to accurately assess one's present risk-tolerance, rather than that applying at a potentially far away future time. The agent then updates his/her preferences under the guidance of a martingale optimality principle, which essentially assures time-consistent decision-making. Under this framework, preferences, investment strategies and wealth evolve together *endogenously and forward in time*.

While forward processes have so far mostly been studied in a continuous-time setting, we focus here on *discrete-time* predictable forward utility processes. The main advantage of considering a discrete-time setting is that the predictability of the forward process is *explicit*, leading to richer economic interpretations, as opposed to the continuous-time setting in which predictability is lost at the infinitesimal level and thus not transparent. We stress that the word *discrete-time* in the phrase 'discrete-time predictable forward utility processes' refers to preference *evaluation* times (namely, preferences are evaluated and updated at discrete times) and not necessarily to the trading times or the price process. Trading times and price processes can be either discrete or continuous in time. This also reflects the fact that in practice, price changes and trading can take place almost continuously while preference updates happen much less frequently.

Angoshtari et al. [1] show that when the market is described by a binomial model, the problem of determining a discrete-time predictable forward process reduces to a finite-horizon inverse investment problem which needs to be solved in a sequential manner. They then establish an equivalence between the inverse investment problem and solving a linear functional equation in the class of inverse marginal functions. As a first contribution, we generalise this one-period result from the binomial model to complete semimartingale models for the financial market, where the inverse investment problem is equivalent to solving a generalised integral equation in the class of inverse marginal functions or another integral equation in the class of conjugate functions. Showing the existence of a discrete-time predictable forward process in the general setting and constructing such processes by sequentially solving the associated generalised integral equations and showing that their solutions are predictable all remain challenging open problems not addressed in this paper. While we consider a complete semimartingale model for the financial market for most of the paper, we frequently return to the binomial case in order to connect our results with the previous work [1] and to build economic intuition.

While a theoretical framework of discrete-time forward processes has been set up in [1], the only concrete example presented there is the one with an initial isoelastic utility function in a binomial model for the financial market. In that example, the risk-tolerance of the resulting forward process remains constant over time, even when the agent updates the model for the financial market. Although such a preservation of preferences might under some circumstances be desirable or realistic, it is arguably not the most interesting case. In this paper, in a one-period framework, we first fully characterise the class of initial utility functions leading to a constant risk-tolerance measure over time in terms of a generalised integral equation for the inverse marginal function associated with the initial utility function. Inverse marginal functions associated with isoelastic utility functions always satisfy this equation, and we thus generalise the above result to the case of complete semimartingale models. We then turn our attention to the binomial model and find that in this case, preservation of preferences over time occurs if and only if the initial utility function belongs to an enlarged class of isoelastic utility functions, where the corresponding inverse marginal function might be multiplied with a doubly log-periodic factor. This suggests in turn that the preference-preserving case of isoelastic utility functions is the exception rather than the rule, and preferences do change over time in general. Returning to the one-period complete semimartingale model for the financial market, we then investigate how this happens qualitatively. We find that whether the agent becomes more or less tolerant to risk as time passes is related to the curvature of the measure of risk-tolerance corresponding to the forward utility. The result is consistent with a similar result in the context of (classical) expected utility maximisation of Gollier and Zeckhauser [16], namely that the younger of two agents who are identical except for the length of their time horizons is more (less) risk-averse if and only if the measure of risk-tolerance associated with his/her utility function is concave (convex).

In the last part of the paper, we study in detail one example, the class of symmetric asymptotic hyperbolic absolute risk aversion (SAHARA) utility functions in the binomial market model. The name SAHARA utility was coined by Chen et al. [8] in the context of a classical (backward) utility maximisation problem. A special instance of SAHARA functions was extensively studied for continuous-time forward criteria in Zariphopoulou and Zhou [46]; see also the examples in Musiela and Zariphopoulou [31, 33]. We show that this class of utility functions is analytically tractable and stable under the framework of predictable forward utility processes, in the sense that if the initial utility function is of the SAHARA class, then so are the members of the entire forward process. Moreover, only the scale parameter, but not the risk-aversion parameter and the threshold wealth, are updated over time in a forward process. As time passes, the risk-tolerance of the corresponding predictable forward utility process converges to a linear function, conforming to an HARA utility. The class of SAHARA utility functions thus truly deserves its name: An SAHARA utility is an HARA utility not only *asymptotically in wealth*, but also *asymptotically in time*.

The remainder of this paper is organised as follows. We review the definition and extend some of the main results on discrete-time predictable forward utility pairs to complete semimartingale market models in Sect. 2. In Sect. 3, we provide a complete characterisation of initial utility functions leading to time-invariant risk-tolerance of the corresponding forward pairs. The general study in a one-period framework of time-varying preferences and the relation to the curvature of the measure of risk-tolerance takes place in Sect. 4. In Sect. 5, we study the example of predictable forward utility processes when the initial utility is an SAHARA utility function. We conclude in Sect. 6.

2 Discrete-time predictable forward utility processes

In this section, we review the definition of discrete-time predictable forward utility processes and extend some of the main results of Angoshtari et al. [1] from a binomial model to complete semimartingale models for the financial market.

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. On this probability space, there is a sequence of *performance (or preference) evaluation times* $(\tau_n)_{n\in\mathbb{N}_0}$, where $\mathbb{N}_0$ denotes the nonnegative integers. The performance evaluation times are stopping times such that $\tau_0 = 0$, τ_{n+1} is $\mathcal{F}_{\tau_n}$-measurable, and they satisfy $\tau_n < \tau_{n+1}$ for every $n \in \mathbb{N}_0$.

The discounted price processes of d risky assets is represented by a d-dimensional semimartingale $S = (S_t)_{t\geq 0}$. The market further contains a risk-free asset with a constant discounted price process 1.

Trading strategies are described by means of d-dimensional, predictable and S-integrable processes $\vartheta = (\vartheta_t)_{t\geq 0}$, where ϑ_t^i denotes the number of shares of the ith risky asset held at time $t \geq 0$. Given an initial wealth $x > 0$ and trading strategy ϑ, the wealth process $X^\vartheta = (X_t^\vartheta)_{t\geq 0}$ evolves according to $X_t^\vartheta = x + \int_0^t \vartheta_u dS_u$, $t \geq 0$. A trading strategy ϑ as well as the associated wealth process X^ϑ are called *admissible* if X^ϑ is nonnegative. We denote by $\mathcal{A}(t,x)$ and $\mathcal{X}(t,x)$ the set of trading strategies $(\vartheta_s)_{s\geq t}$ and associated wealth processes $(X_s^\vartheta)_{s\geq t}$ starting from $X_t^\vartheta = x$, $t \geq 0$, and abbreviate $\mathcal{A}(x) = \mathcal{A}(0,x)$ and $\mathcal{X}(x) = \mathcal{X}(0,x)$.

Throughout this paper, a *utility function* is a function $U : \mathbb{R}_+ \to \mathbb{R}$ which is twice continuously differentiable, strictly increasing, strictly concave and satisfies the Inada conditions $U'(0) := \lim_{x\to 0+} U'(x) = \infty$ and $U'(\infty) := \lim_{x\to\infty} U'(x) = 0$. We also recall that an *inverse marginal function* is a function $I : \mathbb{R}_+ \to \mathbb{R}_+$, which is continuously differentiable, strictly decreasing and satisfies $\lim_{y\to 0+} I(y) = \infty$ and $\lim_{y\to\infty} I(y) = 0$. It is well known that if two functions $U, I : \mathbb{R}_+ \to \mathbb{R}$ satisfy $I = (U')^{-1}$, then U is a utility function if and only if I is an inverse marginal function.

We further recall that the *conjugate function* of a utility function U is given by

$$V(y) = \sup_{x>0} \big(U(x) - xy\big)$$

for $y > 0$. The function V is C^2, strictly decreasing, strictly convex with $V'(0) = -\infty$, $V'(\infty) = 0$, $V(0) = U(\infty)$, $V(\infty) = U(0)$, and we have the bidual relation

$$U(x) = \inf_{y>0} \big(V(y) + xy\big)$$

for $x > 0$. A utility function, its inverse marginal function and its conjugate function are related by $(U')^{-1}(x) = -V'(x) = I(x)$, $x > 0$. We denote the set of utility functions by $\mathcal{U}$, the set of inverse marginal functions by $\mathcal{I}$, and the set of conjugate functions by $\mathcal{V}$.

Definition 2.1 A sequence of random functions $U_n : \mathbb{R}_+ \times \Omega \to \mathbb{R}$, $n \in \mathbb{N}_0$, is called a *discrete-time predictable forward utility process* (a *forward process* in short) with respect to the performance evaluation times $(\tau_n)_{n\in\mathbb{N}_0}$ and filtration $\mathbb{F}$ if the following conditions hold:

(i) $U_0(x,\cdot)$ is constant and $U_n(x,\cdot)$ is $\mathcal{F}_{\tau_{n-1}}$-measurable for every $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$.

(ii) $U_n(\cdot,\omega) \in \mathcal{U}$ for almost all $\omega \in \Omega$ and all $n \in \mathbb{N}_0$.

(iii) For any initial wealth $x > 0$, $n \in \mathbb{N}$ and admissible wealth process $X \in \mathcal{X}(x)$,

$$U_{n-1}(X_{\tau_{n-1}}) \geq \mathbb{E}[U_n(X_{\tau_n})|\mathcal{F}_{\tau_{n-1}}].$$

(iv) For any initial wealth $x > 0$, there exists an admissible wealth process $X^* \in \mathcal{X}(x)$ such that for any $n \in \mathbb{N}$,

$$U_{n-1}(X^*_{\tau_{n-1}}) = \mathbb{E}[U_n(X^*_{\tau_n})|\mathcal{F}_{\tau_{n-1}}].$$

We stress again that the word *discrete-time* in the phrase 'discrete-time predictable forward utility processes' refers to performance evaluation times, and not necessarily to the trading times or the price process. The above general formulation allows both continuous-time and, by means of the standard embedding of discrete-time markets into a continuous-time framework, discrete-time financial markets.

Definition 2.1 is analogous to that in the continuous-time setting (Musiela and Zariphopoulou [31]), except for point (i) which explicitly spells out the predictability feature unique to the discrete-time setting. Points (iii) and (iv) constitute a martingale optimality principle (see e.g. Korn [26]) that guides the evolution of preferences.

According to Angoshtari et al. [1], the general scheme for constructing discrete-time forward processes is as follows. Starting with an initial utility function U_0, one iteratively discovers U_n from U_{n-1} together with an optimal strategy $(\vartheta_t)_{\tau_{n-1}\leq t\leq\tau_n}$ and wealth process $(X^*_t)_{\tau_{n-1}\leq t\leq\tau_n}$ over the investment interval $[\![\tau_{n-1}, \tau_n]\!]$ by solving

$$U_{n-1}(X^*_{\tau_{n-1}}) = \operatorname*{ess\,sup}_{\vartheta\in\mathcal{A}(\tau_{n-1}, X^*_{\tau_{n-1}})} \mathbb{E}\bigg[U_n\bigg(X^*_{\tau_{n-1}} + \int_{\tau_{n-1}}^{\tau_n} \vartheta_u dS_u\bigg)\bigg|\mathcal{F}_{\tau_{n-1}}\bigg]. \tag{2.1}$$

Recall that both U_n and τ_n are $\mathcal{F}_{\tau_{n-1}}$-measurable and thus known at time τ_{n-1}. So while it is conceptually important to regard discrete-time forward processes in a dynamic setting evolving forward in time, mathematically the construction of such a process reduces to the following inverse investment problem over a finite horizon: Given an initial utility function U_0 and a model for the financial market $S = (S_t)_{0\leq t\leq 1}$, we look for a utility function U_1 such that for any $x > 0$,

$$U_0(x) = \max_{\vartheta\in\mathcal{A}(x)} \mathbb{E}\bigg[U_1\bigg(x + \int_0^1 \vartheta_u dS_u\bigg)\bigg]. \tag{2.2}$$

Remark 2.2 Recall that for $\mathbb{P}$-almost all $\omega_0 \in \Omega$,

$$\begin{aligned}1 = \mathbb{P}\big[\big\{\omega\in\Omega : {} & U_{n-1}\big(X^*_{\tau_{n-1}}(\omega), \omega\big) = U_{n-1}\big(X^*_{\tau_{n-1}}(\omega_0), \omega_0\big),\\ & X^*_{\tau_{n-1}}(\omega) = X^*_{\tau_{n-1}}(\omega_0),\\ & \tau_{n-1}(\omega) = \tau_{n-1}(\omega_0), \tau_n(\omega) = \tau_n(\omega_0)\big\}\big|\mathcal{F}_{\tau_{n-1}}\big](\omega_0);\end{aligned}$$

cf. for example Yong and Zhou [45, Proposition 2.13]. The Inverse investment problem (2.1) therefore reduces to (2.2) under a regular conditional distribution $\mathbb{P}[\cdot|\mathcal{F}_{\tau_{n-1}}](\omega_0)$ for $\mathbb{P}$-almost all $\omega_0 \in \Omega$. However, if one wants to construct a forward process by iteratively solving (2.2) forward in time, one not only needs to determine a solution $U_n(\cdot, \omega_0)$ to (2.2) with $\mathbb{P}$ replaced by $\mathbb{P}[\cdot|\mathcal{F}_{\tau_{n-1}}](\omega_0)$ for $\mathbb{P}$-almost all fixed $\omega_0 \in \Omega$, but also must argue that the resulting $U_n(x, \cdot)$ is $\mathcal{F}_{\tau_{n-1}}$-measurable for any fixed $x > 0$. **In this paper, we neither derive results on existence and uniqueness of solutions to (2.2), nor do we provide conditions for the required measurability of the solution in case it exists.** These issues can be addressed on an ad hoc basis for special cases for the financial market or the initial utility function. For example, Angoshtari et al. [1] provide conditions for existence and uniqueness in the binomial model for the financial market, and in that setting, the required measurability is naturally satisfied. For isoelastic initial utility functions, we provide an explicit solution in Proposition 3.1 and note that this solution also satisfies the required measurability condition.

Since we only consider the *inverse investment problem* (2.2) in studying a forward process, we henceforth limit the analysis to the one-period problem (2.2) and refer to (U_0, U_1) as a forward pair. Occasionally, we call U_0 the initial utility (function) and U_1 the forward utility (function). We also drop the generic subscript n from all the variables and parameters in the model.

One of the main results in [1] is that in a binomial market, the inverse investment problem (2.2) is equivalent to a functional equation for the inverse marginal functions $I_0 = (U_0')^{-1}$ and $I_1 = (U_1')^{-1}$ of the initial and forward utility function. As a first contribution of this paper, we generalise this one-period result to any complete semimartingale market model. **For the remainder of this paper,** we in particular make the following assumption ensuring absence of arbitrage and market completeness.

Standing Assumption 2.3 There exists a unique equivalent local martingale measure $\mathbb{Q}$ for S.

We denote the Radon–Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}_1)$ by ρ.

Theorem 2.4 *Let $U_0, U_1 \in \mathcal{U}$ be utility functions with inverse marginals $I_0, I_1 \in \mathcal{I}$ and conjugate functions $V_0, V_1 \in \mathcal{V}$. The following are equivalent*:

i) *The utility functions U_0 and U_1 solve the inverse investment problem* (2.2).

ii) *I_0 and I_1 satisfy the generalised integral equation*

$$I_0(y) = \mathbb{E}[I_1(y\rho)\rho], \qquad y > 0, \tag{2.3}$$

and the utility functions are normalised to

$$U_0(1) = \mathbb{E}\Big[U_1\Big(I_1\big(U_0'(1)\rho\big)\Big)\Big]. \tag{2.4}$$

iii) *The conjugate functions satisfy the generalised integral equation*

$$V_0(y) = \mathbb{E}[V_1(y\rho)], \qquad y > 0. \tag{2.5}$$

Moreover, if one and thus all of the above statements hold true, the unique optimal wealth solving (2.2) *is given by $X_1^*(x) = I_1(\rho U_0'(x))$ and the corresponding wealth process $X^*(x)$ is a uniformly integrable martingale under $\mathbb{Q}$ on* $[0, 1]$.

Proof We first show that i) implies ii). According to Kramkov and Schachermayer [27, 28], the optimal wealth solving (2.2) exists and is given by $X_1^*(x) = I_1(y\rho)$ with $y = U_0'(x)$. Indeed, because U_0 is a utility function and V_0 its conjugate function, $\inf\{y > 0 : V_0(y) < \infty\} = 0$. This in particular implies that

$$U_0(x) = \mathbb{E}\Big[U_1\Big(I_1\big(U_0'(x)\rho\big)\Big)\Big]$$

for any $x > 0$ and thus in particular for $x = 1$ yielding (2.4). By [27, Theorem 2.0], we further have that

$$U_0'(x) = \mathbb{E}\bigg[\frac{X_1(x)U_1'(X_1(x))}{x}\bigg] = \frac{1}{x}\mathbb{E}\big[I_1\big(U_0'(x)\rho\big)U_0'(x)\rho\big].$$

Thus

$$x = \mathbb{E}\big[I_1\big(U_0'(x)\rho\big)\rho\big]$$

and the change of variable $x = I_0(y)$ yields (2.3).

Next, we show that ii) implies iii). Let $\widetilde{U}_0 : (0,\infty) \to \mathbb{R} \cup \{\infty\}$ be the value function of the expected utility maximisation problem with respect to U_1,

$$\widetilde{U}_0(x) = \sup_{X \in \mathcal{Z}(x)} \mathbb{E}[U_1(X)],$$

where $\mathcal{Z}(x) = \{x + \int_0^1 \vartheta_u dS_u : \vartheta \in \mathcal{A}(x)\}$ is the set of all admissible wealths at time 1. Let $x_1 = \mathbb{E}[\rho I_1(U_0'(1)\rho)]$ which is positive and finite by (2.3). Because $U_1(x) \le U_1(y) + U_1'(y)(x-y)$ for any $x, y \in (0,\infty)$,

$$\mathbb{E}[U_1(X)] \le \mathbb{E}\Big[U_1\Big(I_1\big(U_0'(1)\rho\big)\Big)\Big] + U_0'(1)\Big(\mathbb{E}[\rho X] - \mathbb{E}\big[\rho I_1\big(U_0'(1)\rho\big)\big]\Big) \le U_0(1)$$

for any $X \in \mathcal{Z}(x_1)$, where the second inequality follows from (2.4) and the fact that $\mathbb{E}[\rho X] \le x_1$ which holds because $\int \vartheta dS$ is bounded below and thus a $\mathbb{Q}$-local martingale and then a $\mathbb{Q}$-supermartingale by the lemma of Ansel and Stricker [2]. We therefore have $\widetilde{U}_0(x_1) < \infty$ and then, again by [27], that there exists a $y_0 > 0$ such that $\mathbb{E}[V_1(y\rho)] < \infty$ for $y \ge y_0$. For $y < y_0$, we have

$$\begin{aligned}
\mathbb{E}[V_1(y\rho)] &= \mathbb{E}\bigg[-\int_y^{y_0}\Big(\frac{d}{dz}V_1(z\rho)\Big)dz + V_1(y_0\rho)\bigg]\\
&= \mathbb{E}\bigg[\int_y^{y_0}\rho I_1(z\rho)dz\bigg] + \mathbb{E}[V_1(y_0\rho)]\\
&= \int_y^{y_0} I_0(z)dz + \mathbb{E}[V_1(y_0\rho)]\\
&\le (y_0 - y)I_0(y) + \mathbb{E}[V_1(y_0\rho)],
\end{aligned}$$

where the third equality follows from Tonelli's theorem and (2.3). We in particular have that $\mathbb{E}[V_1(y\rho)] < \infty$ for any $y > 0$. By [28] and with the same arguments as above, we thus have that

$$\widetilde{I}_0(y) = \mathbb{E}[I_1(y\rho)\rho] = I_0(y),$$

where $\widetilde{I}_0(y) = (\widetilde{U}_0')^{-1}(y)$, $y > 0$. Hence, it holds that $\widetilde{U}_0(x) = U_0(x) + c$ for some $c \in \mathbb{R}$. By (2.4), we have $c = 0$ and can in particular conclude for any $y > 0$ that $U_0(y) = \widetilde{U}_0(y) = \mathbb{E}[U_1(I_1(\rho U_0'(y)))]$. Now recall that $V_i(y) = U_i(I_i(z)) - zI_i(z)$, $z > 0$, $i \in \{0,1\}$; see e.g. Rockafellar [40, Theorem 26.4]. Equation (2.3) thus becomes

$$\frac{1}{y}\Big(U_0\big(I_0(y)\big) - V_0(y)\Big) = \mathbb{E}\Big[(y\rho)^{-1}\Big(U_1\big(I_1(y\rho) - V_1(y\rho)\big)\rho\Big)\Big]$$

which simplifies to (2.5) because $U_0(y) = \mathbb{E}[U_1(I_1(\rho U_0'(y)))]$.

Finally, we show that iii) implies i). For any $x > 0$ and $\vartheta \in \mathcal{A}(x)$,

$$\begin{aligned}\mathbb{E}\left[U_1\left(x+\int_0^1 \vartheta_u dS_u\right)\right] &= \mathbb{E}\left[\inf_{y>0} V_1(y) + \left(x+\int_0^1 \vartheta_u dS_u\right)y\right] \\ &\le \mathbb{E}\left[V_1(\rho) + \left(x+\int_0^1 \vartheta_u dS_u\right)\rho\right] \\ &\le V_0(1) + x,\end{aligned}$$

where the last inequality follows from (2.5) and the fact that wealth processes are $\mathbb{Q}$-supermartingales as argued above. Therefore, the function $\widetilde{U}_0 : (0,\infty) \to \mathbb{R}$ defined as before by $\widetilde{U}_0(x) := \sup_{X\in\mathcal{Z}(x)} \mathbb{E}[U_1(X)]$ is finite everywhere. By [27, Theorems 2.0 and 2.2],

$$\widetilde{U}_0(x) = \inf_{y\ge 0}\big(V_0(y) + xy\big) = U_0(x),$$

and we conclude that U_0 and U_1 solve (2.2) and the maximum in (2.2) is attained.

The facts that the optimal wealth solving (2.2) is given by $X_1^*(x) = I_1(\rho U_0'(x))$ and that the corresponding wealth process is a uniformly integrable martingale under $\mathbb{Q}$ on [0, 1] follow directly from [27, 28]. □

Theorem 2.4 provides a method to construct a forward pair in complete financial markets. One starts with a utility function U_0 and defines the corresponding inverse marginal function $I_0(y) = (U_0')^{-1}(y)$, $y > 0$, or, alternatively, the corresponding conjugate function $V_0(y) = \sup_{x>0}(U_0(x) - xy)$, $y > 0$. If one is able to solve the generalised integral equation (2.3) and show that the solution I_1 is in $\mathcal{I}$, one can define $U_1 : \mathbb{R}_+ \to \mathbb{R}$ by

$$U_1(x) = U_0(1) + \mathbb{E}\left[\int_{I_1(\rho U_0'(1))}^{x} I_1^{-1}(\xi)d\xi\right].$$

Then U_1 satisfies $(U_1')^{-1} = I_1$ and (2.4) by construction and we can thus conclude that (U_0, U_1) is a forward pair. We point out again that in order to generalise this construction method to the dynamic setting, one would also have to argue the predictability of the forward utility as we highlighted in Remark 2.2. Analogously, if one can find a solution V_1 to the generalised integral equation (2.5) and show that V_1 is in $\mathcal{V}$, one can define $U_1 : \mathbb{R}_+ \to \mathbb{R}$ by $U_1(x) = \inf_{y>0}(V_1(y) + xy)$ and conclude that (U_0, U_1) is a forward pair.

Remark 2.5 There is no one-to-one relationship between utility functions and their inverse marginal functions since an inverse marginal function only determines the corresponding utility function up to a constant. It is thus not surprising that we need some kind of normalisation in (2.4), since for (U_0, U_1) to be a forward pair, we need U_0 to be the value function of the expected utility maximisation problem with utility function U_1.

Remark 2.6 Theorem 2.4 provides characterisations of a forward pair (U_0, U_1). These also turn out to be useful to study the properties of forward processes in terms of implied risk attitudes, which is the main topic of this paper. However, the theorem itself does not provide conditions under which forward processes exist and are unique. While this question has been addressed for the binomial setting in [1], the general case remains a challenging open problem. Theorem 2.4 implies that the question is related to the existence and uniqueness of a solution to the generalised integral equations (2.3) or (2.5), given an initial utility function U_0. This latter problem is nonetheless challenging since inverse marginal functions and conjugate functions are typically not Lebesgue-summable, i.e., the Lebesgue integral of their absolute value is not finite, and thus standard approaches via Fourier transforms cannot be directly applied. Moreover, even if solutions to the generalised integral equations exist, one furthermore has to argue that the corresponding forward utility has the required measurability properties; cf. Remark 2.2.

The case of a *binomial model for the financial market* was extensively studied in Angoshtari et al. [1], where discrete-time predictable forward utility processes have been introduced. We point out that the term binomial model in the context of discrete-time predictable forward processes always refers to the setting where trading times coincide with performance evaluation times. To the best of our knowledge, there are so far no results on this framework going beyond the binomial setting. We thus frequently return to the binomial model, on the one hand to connect our results with the previous work of [1], and on the other to illustrate our findings in a more tractable and explicit setting.

Under the binomial model and in a one-period framework, the financial market consists of a single stock whose price follows a Bernoulli distribution. Specifically, we suppose that $S_0 = 1$ and that $S_1 = u$ with probability p and $S_1 = d$ with probability $1 - p$, where $d < 1 < u$ and $0 < p < 1$. For this market, there is a unique equivalent martingale measure $\mathbb{Q}$ given by $q = \mathbb{Q}[S_1 = u] = \frac{1-d}{u-d}$ and with pricing kernel $\rho = \frac{q}{p}\mathbf{1}_{\{S_1=u\}} + \frac{1-q}{1-p}\mathbf{1}_{\{S_1=d\}}$. The following straightforward corollary to Theorem 2.4 restates some of the main results in [1, Theorems 5.1 and 5.2]. We remark that the functional equation for the conjugate functions has not been established in [1] and is new to the best of our knowledge.

Corollary 2.7 *Consider the one-period binomial model for the financial market, and let $U_0, U_1 \in \mathcal{U}$ be utility functions with inverse marginals $I_0, I_1 \in \mathcal{I}$ and conjugate functions $V_0, V_1 \in \mathcal{V}$. Then the following are equivalent*:

i) *The utility functions U_0 and U_1 solve the inverse investment problem* (2.2).

ii) *I_0 and I_1 satisfy the linear functional equation*

$$I_0(y) = qI_1\left(y\frac{q}{p}\right) + (1-q)I_1\left(y\frac{1-q}{1-p}\right), \qquad y > 0, \tag{2.6}$$

and the utility functions are normalised to

$$U_0(1) = pU_1\left(I_1\left(U_0'(1)\frac{q}{p}\right) + (1-p)U_1\left(I_1\left(U_0'(1)\frac{1-q}{1-p}\right)\right)\right). \tag{2.7}$$

iii) *The conjugate functions satisfy the generalised integral equation*

$$V_0(y) = pV_1\left(y\frac{q}{p}\right) + (1-p)V_1\left(y\frac{1-q}{1-p}\right), \qquad y > 0.$$

Moreover, if one and thus all of the above statements hold true, the unique optimal wealth solving (2.2) *is given by* $X_1^*(x) = I_1(\rho U_0'(x))$.

Remark 2.8 It is straightforward to show that (2.6) is equivalent to the linear functional equation reported in [1], namely

$$I_1(ay) + bI_1(y) = (1+b)I_0(cy), \qquad y > 0, \tag{2.8}$$

where

$$a = \frac{1-p}{p}\frac{q}{1-q}, \qquad b = \frac{1-q}{q}, \qquad c = \frac{1-p}{1-q}. \tag{2.9}$$

3 Time invariant risk-tolerance of forward pairs

In this section, we characterise the set of initial utility functions under which the measure of risk-tolerance can be preserved over time along a forward pair.

Recall that the *Arrow–Pratt measure of (absolute) risk-aversion* of a utility function U is defined by

$$A(x) = -\frac{U''(x)}{U'(x)},$$

and its reciprocal, the *Arrow–Pratt measure of (absolute) risk-tolerance*, is given by

$$T(x) = \frac{1}{A(x)} = -\frac{U'(x)}{U''(x)}.$$

These equivalent measures fully characterise the preferences of the agent. Indeed, if A_0 and A_1 denote the measures of risk-aversion of two utility functions U_0 and U_1, then it follows from $A(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\log(U'(x))$ that $A_0(x) = A_1(x)$ for all $x > 0$ if and only if there are constants $C > 0$ and $D \in \mathbb{R}$ such that

$$U_1(x) = CU_0(x) + D.$$

In this case, the preferences represented by the two utility functions are identical as they are related by an affine transformation.

It is our goal to investigate, within the framework of predictable forward utility, how preferences evolve when time passes and wealth changes. To this end, we first characterise initial utility functions U_0 under which preferences can be preserved over time and are thus a function of wealth alone. If (U_0, U_1) is a predictable forward utility pair, this is the case if and only if the corresponding measures of risk-tolerance coincide, i.e., $T_0(x) = T_1(x)$ for all $x > 0$. In the binomial market, one such example is the case of an isoelastic initial utility function exhibiting constant relative risk-aversion, $U_0(x) = \frac{1}{\gamma}x^\gamma$ with $\gamma \in (-\infty, 1) \setminus \{0\}$. As shown in Angoshtari et al. [1], we then have $U_1(x) = \delta U_0(x)$ with $\delta = \frac{1+b}{c^{1/(1-\gamma)}(a^{-1/(1-\gamma)}+b)}$, where a, b, c are given by (2.9). Since the market parameters enter only into the factor δ, preferences in this example remain constant over time even if market parameters were to be updated. The following proposition extends this result to general one-period complete financial markets.

Proposition 3.1 *If the initial utility is* $U_0(x) = \frac{1}{\gamma}x^\gamma$ *for some* $\gamma \in (-\infty, 1) \setminus \{0\}$, *then a forward utility is given by* $U_1(x) = \delta U_0(x)$, *where*

$$\delta = \left(\mathbb{E}\left[\rho^{\frac{\gamma}{\gamma-1}}\right]\right)^{\gamma-1}.$$

If $U_0(x) = \log x$, *then a forward utility is given by* $U_1(x) = \log x + \mathbb{E}[\log \rho]$.

Proof Let $\gamma \in (-\infty, 1) \setminus \{0\}$. The conjugate of $U_0(x) = \frac{1}{\gamma}x^\gamma$ is clearly given by $V_0(y) = (\frac{1}{\gamma} - 1)y^{\frac{\gamma}{\gamma-1}}$. Hence we need to find $V_1 \in \mathcal{V}$ such that (2.5) holds, i.e., such that

$$\left(\frac{1}{\gamma} - 1\right)y^{\frac{\gamma}{\gamma-1}} = \mathbb{E}[V_1(y\rho)]. \tag{3.1}$$

A natural candidate is given by $V_1(\xi) = c\xi^{\frac{\gamma}{\gamma-1}}$, which solves (3.1) if and only if $c = (\frac{1}{\gamma} - 1)/\mathbb{E}[\rho^{\frac{\gamma}{\gamma-1}}]$. We obtain the corresponding forward utility as

$$\begin{aligned} U_1(x) &= \inf_{y>0}\left(\frac{\frac{1}{\gamma}-1}{\mathbb{E}[\rho^{\frac{\gamma}{\gamma-1}}]}y^{\frac{\gamma}{\gamma-1}} + xy\right) \\ &= \frac{\frac{1}{\gamma}-1}{\mathbb{E}[\rho^{\frac{\gamma}{\gamma-1}}]}\left(\left(\mathbb{E}\left[\rho^{\frac{\gamma}{\gamma-1}}\right]x\right)^{\gamma-1}\right)^{\frac{\gamma}{\gamma-1}} + x\left(\mathbb{E}\left[\rho^{\frac{\gamma}{\gamma-1}}\right]x\right)^{\gamma-1} \\ &= \frac{1}{\gamma}\left(\mathbb{E}\left[\rho^{\frac{\gamma}{\gamma-1}}\right]\right)^{\gamma-1}x^\gamma. \end{aligned}$$

The case $U_0(x) = \log x$ is proved similarly. □

Remark 3.2 Proposition 3.1 resembles a result of Choulli and Ma [12] which shows that the measure of risk-tolerance of isoelastic utility functions is independent of both the time and state in a continuous-time forward setting with a general locally bounded semimartingale market model.

Proposition 3.1 shows that preferences can be preserved when the initial utility belongs to the class of isoelastic utility functions. It is natural to ask whether this is the only class under which preferences are preserved. To this end, we consider a forward pair (U_0, U_1) with $T_0(x) = T_1(x)$ for all $x > 0$. Then $U_1'(x) = CU_0'(x)$ for some constant $C > 0$, and the inverse marginal functions corresponding to the utility functions U_0 and U_1 therefore satisfy the relationship $I_1(y) = I_0(\frac{y}{C})$. The generalised integral equation for the inverse marginal function (2.3) thus becomes a necessary and sufficient condition on the inverse marginal function associated with the initial utility function,

$$I_0(y) = \mathbb{E}\left[I_0\left(\frac{y}{C}\rho\right)\rho\right], \qquad y > 0. \tag{3.2}$$

Whether (3.2) allows solutions going beyond the class of inverse marginal functions corresponding to isoelastic utility functions depends on the financial market under consideration. We illustrate this in the case of the one-period binomial model for the financial market, **which we consider for the remainder of this section.** Studying this question for other important models for the financial market is left as an open problem for future research. For the one-period binomial setting, (3.2) becomes

$$I_0\left(\frac{a}{C}y\right) + bI_0\left(\frac{1}{C}y\right) = (1+b)I_0(cy), \qquad y > 0. \tag{3.3}$$

The following lemma determines all inverse marginal functions solving (3.3) for a given $C > 0$.

Lemma 3.3 *Let $C > 0$. We consider the following three cases*:

(i) *If $p = q$, then every inverse marginal function I_0 satisfies* (3.3) *when $C = 1$, and no inverse marginal function I_0 satisfies* (3.3) *otherwise.*

(ii) *If $p \neq q$ and either $C \le \min(\frac{q}{p}, \frac{1-q}{1-p})$ or $C \ge (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$, then there is no inverse marginal function I_0 satisfying* (3.3).

(iii) *Let $p \neq q$ and $\min(\frac{q}{p}, \frac{1-q}{1-p}) < C < (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$. Then there is a unique negative solution β_1 to*

$$1 = \frac{1}{1+b}\left(\frac{a}{Cc}\right)^{\beta} + \frac{b}{1+b}\left(\frac{1}{Cc}\right)^{\beta} \tag{3.4}$$

and we have:

(a) *If $\log\frac{a}{Cc}$ and $\log\frac{1}{Cc}$ are not commensurable, i.e., $\log\frac{a}{Cc}/\log\frac{1}{Cc}$ is not rational, then I_0 is an inverse marginal function satisfying* (3.3) *if and only if there is a $k > 0$ such that $I_0(x) = kx^{\beta_1}$.*

(b) *If $\log\frac{a}{Cc}$ and $\log\frac{1}{Cc}$ are commensurable, then I_0 is an inverse marginal function satisfying* (3.3) *if and only if there is a positive, continuously differentiable function $k(x)$ satisfying $k(x) = k(\frac{a}{Cc}x) = k(\frac{1}{Cc}x)$ and $xk'(x) < -\beta_1 k(x)$ for all $x > 0$ such that $I_0(x) = k(x)x^{\beta_1}$.*

Proof Case (i) is trivial because $a = c = 1$ when $p = q$. For the remainder of this proof, we thus assume that $p \neq q$. Note that a strictly decreasing solution to (3.3) can only exist when $a < Cc < 1$ if $p > q$ or $a > Cc > 1$ if $p < q$. A solution to (3.3) within the class of inverse marginal functions can thus only exist when $\min(\frac{q}{p}, \frac{1-q}{1-p}) < C < \max(\frac{q}{p}, \frac{1-q}{1-p})$, which we assume from now on. This shows that there is no inverse marginal function satisfying (3.3) when $C > \min(\frac{q}{p}, \frac{1-q}{1-p})$, but because $(\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q} < \max(\frac{q}{p}, \frac{1-q}{1-p})$ not yet that $C < (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$ is also necessary for the existence of an inverse marginal function satisfying (3.3). To investigate (3.4), we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(\beta) := \frac{1}{1+b}\left(\frac{a}{Cc}\right)^{\beta} + \frac{b}{1+b}\left(\frac{1}{Cc}\right)^{\beta} - 1.$$

Clearly, f is strictly convex, $f(0) = 0$ and

$$\begin{aligned} f'(0) &= \frac{1}{1+b}\log\frac{a}{Cc} + \frac{b}{1+b}\log\frac{1}{Cc} \\ &= q\log\frac{q}{p} + (1-q)\log\frac{1-q}{1-p} - \log C. \end{aligned}$$

Hence $f'(0)$ is positive when $C < (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$, zero when $C = (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$ and negative when $C > (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$. Since $\min(\frac{q}{p}, \frac{1-q}{1-p}) < C < \max(\frac{q}{p}, \frac{1-q}{1-p})$, we either have $\frac{a}{Cc} < 1 < \frac{1}{Cc}$ or $\frac{1}{Cc} < 1 < \frac{a}{Cc}$, and f thus converges to infinity when β goes to plus or minus infinity. Hence, the solutions of (3.4) are $\beta_0 = 0$ and a unique negative β_1 when $C < (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$. When $C = (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$, f is nonnegative and $\beta_0 = 0$ is the only solution to (3.4). In the case where $\max(\frac{q}{p}, \frac{1-q}{1-p}) > C > (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$, the solutions to (3.4) are given by $\beta_0 = 0$ and a unique positive β_1. Now note that equation (3.3) is equivalent to

$$I_0(y) = \frac{1}{1+b}I_0\left(\frac{a}{Cc}y\right) + \frac{b}{1+b}I_0\left(\frac{1}{Cc}y\right).$$

As $\beta_0 = 0$ and β_1 are the only solutions to the characteristic equation (3.4) of the above functional equation, according to Laczkovich [29], if I_0 is a nonnegative and measurable solution of (3.3), then there are nonnegative measurable functions k_0 and k with $k_0(x) = k_0(\frac{a}{Cc}x) = k_0(\frac{1}{Cc}x)$, respectively $k(x) = k(\frac{a}{Cc}x) = k(\frac{1}{Cc}x)$, for all $x > 0$, and k_0, k are constant whenever $\log\frac{a}{Cc}$ and $\log\frac{1}{Cc}$ are not commensurable, such that

$$I_0(x) = k_0(x) + k(x)x^{\beta_1}.$$

When we require I_0 to be an inverse marginal function, we have $\lim_{x\to\infty} I_0(x) = 0$ which implies that $k_0(x) = 0$. Moreover, I_0 also must be strictly decreasing, and β_1 thus must be negative because of the double log-periodicity of k. We conclude (ii). The properties of an inverse marginal function further require that k is continuously differentiable and that

$$0 > I_0'(x) = \big(k'(x)x + \beta_1 k(x)\big)x^{\beta_1 - 1}$$

which results in the requirement that $xk'(x) < -\beta_1 k(x)$ for all $x > 0$. On the other hand, it is easy to verify that when $I_0(x) = k(x)x^{\beta_1}$, where k satisfies the properties as above and β_1 solves (3.4), then I_0 satisfies (3.3). □

Remark 3.4 It is straightforward to show that $\min(\frac{q}{p}, \frac{1-q}{1-p}) \leq 1 \leq (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$ for any $0 \leq p, q \leq 1$ and that both inequalities are strict if and only if $p \neq q$. When $p \neq q$ and $C = 1$, we are thus in case (iii) of Lemma 3.3, and (3.4) becomes

$$1 = q\left(\frac{q}{p}\right)^{\beta} + (1-q)\left(\frac{1-q}{1-p}\right)^{\beta}.$$

This equation is obviously solved by $\beta_0 = 0$ and $\beta_1 = -1$. The unique negative solution to (3.4) is thus given by $\beta_1 = -1$, which corresponds to the extended class of logarithmic utility functions. Because the function f defined in the proof of Lemma 3.3 is increasing in the parameter C for a fixed negative β, we conclude that $\min(\frac{q}{p}, \frac{1-q}{1-p}) < C < 1$ corresponds to $\beta_1 < -1$ and thus to utility functions with $U(+\infty) = +\infty$, whereas $1 < C < (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$ corresponds to $-1 < \beta_1 < 0$ and thus to utility functions with $U(+\infty) < +\infty$.

The following example shows the existence of functions k having the properties as described in part (iii) (b) of Lemma 3.3.

Example 3.5 We here give an example showing that there exist positive, continuously differentiable functions $k : \mathbb{R}_+ \to \mathbb{R}$ such that $k(x) = k(\frac{a}{Cc}x) = k(\frac{1}{Cc}x)$ and $xk'(x) < -\beta_1 k(x)$ for all $x > 0$ when $\log\frac{a}{Cc}/\log\frac{1}{Cc} = -n/m$ for some $n, m \in \mathbb{N}$. Indeed, it is easy to see that

$$k(x) = K + \sin\left(\frac{2\pi m}{|\log\frac{1}{Cc}|}\log x\right)$$

satisfies the above properties for large enough $K > 0$.

The following theorem provides a full characterisation of all initial utility functions leading to time-invariance of preferences under the framework of discrete-time predictable forward utility processes in a one-period binomial model for the financial market.

Theorem 3.6 *Suppose $p \neq q$ and let U_0 be a utility function with risk-tolerance T_0 and inverse marginal I_0. Preferences are time-invariant, i.e., there is a forward utility U_1 for U_0 with risk-tolerance T_1 satisfying $T_0(x) = T_1(x)$ for all $x > 0$, if and only if I_0 has one of the following two forms: either $I_0(x) = kx^{\beta_1}$ for some constants $k > 0$ and $\beta_1 < 0$, or $I_0(x) = k(x)x^{\beta_1}$ where the function $k : (0,\infty) \to (0,\infty)$ is continuously differentiable and satisfies $k(x) = k(\frac{a}{Cc}x) = k(\frac{1}{Cc}x)$ and $xk'(x) < -\beta_1 k(x)$ for all $x > 0$ for some constants $C > 0$ and $\beta_1 < 0$ satisfying (3.4).*

Proof For the ‘only if’ direction, let U_1 be a forward utility for U_0 with risk-tolerance $T_1(x) = T_0(x)$ for all $x > 0$ and inverse marginal I_1. As we have seen at the beginning of Sect. 3, $T_1(x) = T_0(x)$ for all $x > 0$ implies that there exists a $C > 0$ such that $I_1(y) = I_0(\frac{y}{C})$. Since U_1 is a forward utility for U_0, I_0 satisfies (3.3). It follows from Lemma 3.3 that $\min(\frac{q}{p}, \frac{1-q}{1-p}) < C < (\frac{q}{p})^q(\frac{1-q}{1-p})^{1-q}$ and there exist a $\beta_1 < 0$ solving (3.4) as well as a function $k : (0,\infty) \to (0,\infty)$ which is constant when $\log\frac{a}{Cc}$

and $\log \frac{1}{Cc}$ are not commensurable, and continuously differentiable as well as satisfying $k(x) = k(\frac{a}{Cc}x) = k(\frac{1}{Cc}x)$ for all $x > 0$ and $xk'(x) < -\beta_1 k(x)$ when $\log \frac{a}{Cc}$ and $\log \frac{1}{Cc}$ are commensurable such that $I_0(x) = k(x)x^{\beta_1}$.

We next show the 'if' direction. In the first case where $I_0(x) = kx^{\beta_1}$ for some $k > 0$ and $\beta_1 < 0$, let

$$C = \left(\frac{1}{1+b}\left(\frac{a}{c}\right)^{\beta_1} + \frac{b}{1+b}\left(\frac{1}{c}\right)^{\beta_1}\right)^{1/\beta_1}.$$

Then I_0 satisfies (3.3). In the second case, I_0 satisfying (3.3) is implied by the assumptions and Lemma 3.3. Set $I_1(y) := I_0(\frac{y}{C})$. Then I_1 satisfies (2.8) and according to Theorem 2.4 is thus the inverse marginal function corresponding to the forward utility for U_0. Moreover, $I_1(y) = I_0(\frac{y}{C})$ yields $U_1'(x) = CU_0'(x)$ which in turn implies $T_1(x) = T_0(x)$. □

Theorem 3.6 shows that for the case of a one-period binomial market, preferences can be preserved over time if and only if the initial utility function belongs to an extended class of isoelastic utility functions. Indeed, an inverse marginal of the form $I(x) = kx^{\beta}$ with constants $k > 0$ and $\beta < 0$ corresponds to a utility function $U(x) = k^{1/|\beta|}x^{1-1/|\beta|} + D$ for some constant $D \in \mathbb{R}$. This case thus coincides with the example studied in Angoshtari et al. [1]. The case where k is a doubly log-periodic function can only occur when $\log \frac{a}{Cc}$ and $\log \frac{1}{Cc}$ are commensurable. We emphasise here that C and β_1 depend on one another as well as on the market parameters via (3.4). Hence, C is *not* free once one fixes a preference parameter β_1 and the market parameters. Since the rational numbers are a Lebesgue nullset, the extended class of isoelastic utility functions where the inverse marginal might be multiplied with a doubly log-periodic factor actually reduces to the standard class of isoelastic utility functions for almost all values of the preference and market parameters. This in particular implies that the preference-preserving power utility functions are the exception rather than the rule: In general, preferences described by predictable forward processes *will* indeed change as time evolves, and this change potentially depends on the dynamically evolving conditions of the market.

4 Time varying risk-tolerance of forward pairs

We have seen in the previous section that initial utility functions belonging to the class of isoelastic utility functions lead to a preservation of preferences in a one-period framework, and that at least in the binomial market model, any initial utility function under which preferences can be preserved as time evolves must belong to an extended class of isoelastic utility functions. It is the objective of this section to study how risk-tolerance is updated under the framework of forward pairs once one goes beyond this extended class of isoelastic utility functions, and thus to understand how preferences evolve over time in general. We again denote the risk-tolerance corresponding to U_0 (resp. U_1) by T_0 (resp. T_1). The following theorem shows that for the general one-period complete semimartingale market model, how risk-tolerance changes as time passes depends on the curvature of the risk-tolerance measure.

Theorem 4.1 *Let* (U_0, U_1) *be a forward pair such that* $\mathbb{E}[\rho T_1(X_1^*(x))] < \infty$ *for any* $x > 0$. *The following hold*:

(i) *If* T_1 *is* (*strictly*) *concave*, *then* $T_0(x) \le T_1(x)$ $(T_0(x) < T_1(x))$ *for all* $x \in \mathbb{R}$.

(ii) *If* T_1 *is* (*strictly*) *convex*, *then* $T_0(x) \ge T_1(x)$ $(T_0(x) > T_1(x))$ *for all* $x \in \mathbb{R}$.

Proof Let $x > 0$ be fixed. According to Kramkov and Schachermayer [27, Theorem 2.0], we have

$$U_0'(x) = \mathbb{E}\left[\frac{I_1(U_0'(x)\rho)U_0'(x)\rho}{x}\right].$$

The above equality implies that

$$x = \mathbb{E}\big[I_1\big(U_0'(x)\rho\big)\rho\big]. \tag{4.1}$$

Differentiating with respect to x yields

$$\begin{aligned}
1 &= \lim_{h\to 0}\frac{1}{h}\Big(\mathbb{E}\big[I_1\big(U_0'(x+h)\rho\big)\rho\big] - \mathbb{E}\big[I_1\big(U_0'(x)\rho\big)\rho\big]\Big)\\
&= \lim_{h\to 0}\mathbb{E}\left[\rho\frac{I_1(U_0'(x+h)\rho) - I_1(U_0'(x)\rho)}{h}\right]\\
&= \lim_{h\to 0}\mathbb{E}\left[\rho^2\frac{U_0''(\xi(h))}{U_1''(I_1(U_0'(\xi(h))\rho))}\right]\\
&= \lim_{h\to 0}\mathbb{E}\left[\rho^2\frac{U_0''(\xi(h))}{U_1''(I_1(U_0'(\xi(h))\rho))}\frac{U_1'(I_1(U_0'(\xi(h))\rho))}{U_0'(\xi(h))\rho}\right]\\
&= \lim_{h\to 0}\mathbb{E}\left[\rho\frac{T_1(I_1(U_0'(\xi(h))\rho))}{T_0(\xi(h))}\right]\\
&= \mathbb{E}\left[\rho\frac{T_1(I_1(U_0'(x)\rho))}{T_0(x)}\right],
\end{aligned}$$

where $\xi(h) \in (x, x+h)$ exists by the mean-value theorem, the fourth equality holds because $U_1' \circ I_1 = \mathrm{id}$, and the last step follows from the dominated convergence theorem because $\mathbb{E}[\rho T_1(X_1^*(x))] < \infty$ (recall that $X_1^*(x) = I_1(\rho U_0'(x))$ by Theorem 2.4), along with the continuity of T_1, I_1, U_0' and T_0.

We obtain between the risk-tolerance of the original utility function and the risk-tolerance of the forward utility the relationship

$$T_0(x) = \mathbb{E}\Big[\rho T_1\Big(I_1\big(U_0'(x)\rho\big)\Big)\Big]. \tag{4.2}$$

In Case (i), (4.1), (4.2) and Jensen's inequality imply that $T_0(x) \le T_1(x)$ for any $x > 0$ and the inequality is strict when T_1 is strictly concave. Case (ii) is analogous with the opposite inequality. □

Remark 4.2 The measure of risk-tolerance of isoelastic utility functions is linear and thus in particular both convex and concave. Theorem 4.1 then implies that its risk-tolerance will remain constant, consistent with our earlier results in Proposition 3.1.

It is natural to ask whether the conclusion of Theorem 4.1 still holds when one starts from the assumption that T_0 is concave or convex. When T_0 is concave, using first (4.2) and then Jensen's inequality together with (4.1) yields for any $x > 0$ that

$$\mathbb{E}\Big[\rho T_1\Big(I_1\big(U_0'(x)\rho\big)\Big)\Big] = T_0(x) \geq \mathbb{E}\Big[\rho T_0\Big(I_1\big(U_0'(x)\rho\big)\Big)\Big]. \tag{4.3}$$

Since $U_0' : (0, \infty) \to (0, \infty)$ is a bijection by the Inada conditions, (4.3) gives a general direction for the future behaviour of the agent. Nonetheless, (4.3) does not exclude the possibility that $T_1(\xi) < T_0(\xi)$ at some particular $\xi > 0$, and so the general answer is no. The reason is that the local behaviour of T_0 at a given point $\xi > 0$ is not related to T_1 at the same point, but instead to the risk-neutral average over the risk-tolerance evaluated on the optimal wealth corresponding to the initial wealth ξ; cf. also Example 4.4 below. In the case where T_0 is convex, one obtains the opposite inequality in (4.3).

On the other hand, if we have convexity/concavity in risk-tolerance of each utility function in a forward pair, Theorem 4.1 dictates whether the agent becomes more risk tolerant or otherwise as he/she grows older. Turning this observation around, given an agent exhibiting monotonicity in time for his/her risk-tolerance, Theorem 4.1 may help choose a suitable parametrised family of utility functions to model his/her risk preferences. In the next section, we present a forward utility process with consistently convex risk-tolerance measures.

Remark 4.3 Gollier and Zeckhauser [16] study the implication of the curvature of the measure of risk-tolerance on the relation between horizon length and willingness to take risk in a model of a fixed time horizon. They compute the risk-tolerance of the intermediate value function in a two-step setting, and show that convexity of the absolute risk-tolerance is necessary and sufficient for a younger agent to take more risk than an older one. Clearly, our result in Theorem 4.1 (ii) is consistent with their finding. A similar result relating the risk-tolerance to the curvature of the utility function applied at the end of a fixed time horizon was obtained by Källblad and Zariphopoulou [24] for classical expected utility maximisation in a continuous-time lognormal market.

There is no consensus in the literature as to whether absolute risk-tolerance should be concave or convex. Hennessy and Lapan [21] show that a concave risk-tolerance implies the notion of standard risk-aversion of Kimball [25], which in turn implies proper risk-aversion (Pratt and Zeckhauser [39]) and risk vulnerability (Gollier and Pratt [15]). However, none of these concepts conversely implies concavity of the risk-tolerance. Gollier and Zeckhauser [16] point out that there are equally appealing arguments for both a concave or a convex risk-tolerance. Guiso and Paiella [18] suggest a concave measure of risk-tolerance through an empirical study and find that quite contrary to conventional wisdom, the portfolio share of risky assets is increasing in age (see also Guiso et al. [17] for an earlier result in this direction). This is consistent with our finding in Theorem 4.1, which implies an increasing risk-tolerance of the forward pair when the risk-tolerance is concave.

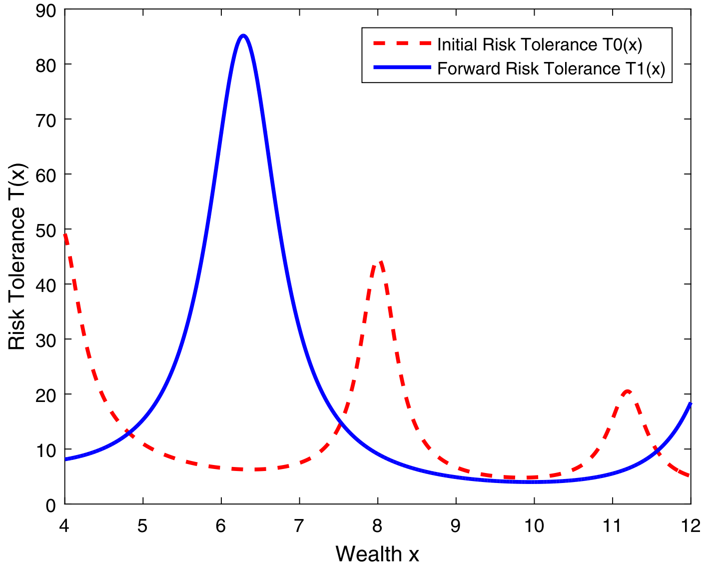

Fig. 1 *Counterexample against time-monotonicity of the measure of risk-tolerance.* This figure shows the measure of risk-tolerance of the initial and forward utility when the risk-tolerance of the former is oscillating in a market with $p = 0.6$, $u = 1.1$ and $d = 0.9$. It illustrates that the risk-tolerance is not necessarily monotonic in time.

Example 4.4 This example serves to explicate that in general, there need not be a uniform order of elevation between the risk-tolerance measures of the two neighbouring utility functions (U_0, U_1) in a forward pair. We again work in the one-period binomial model for the financial market. Intuitively, U_0 and U_1 are related by $U_0(x) = pU_1(x + \pi^*(u-1)) + (1-p)U_1(x + \pi^*(d-1))$. So the local behaviour of U_0 at a wealth level x is not related to that of U_1 at the same wealth level x, but instead to U_1 evaluated at two different wealth levels, one above and the other below x. If the risk-tolerance of U_1 is fluctuating in some region of the state, then that of U_0 might be fluctuating in a completely different region. We illustrate this phenomenon numerically. We take

$$U_1(x) = -\frac{(1+\delta)}{2}x^2 + Kx - \cos x,$$

for $x \in (a, b)$, parametrised by $\delta, K > 0$, where $0 < a < b < +\infty$ is such that U_1 is increasing in that interval. The function U_1 can be extended to be a globally increasing and strictly concave utility on the whole real line satisfying the Inada conditions, but it is not important for our purpose to specify exactly how this is done as the local behaviour of the utility function in a sufficiently large region fully specifies an expected utility maximisation problem in a binomial market within some smaller region. For the numerical example, we consider $p = 0.6$, $u = 1.1$, $d = 0.9$, $\delta = 0.15$ and $K = 20$. Then U_1 is strictly increasing between $a = 1$ and $b = 17$. The absolute risk-tolerance for U_1 is given by

$$T_1(x) = \frac{-(1+\delta)x + \sin x + K}{1 + \delta - \cos x}.$$

We then solve the classical expected utility maximisation problem to obtain the value function $U_0(x)$. Clearly, U_0 and U_1 are two neighbouring utility functions in a forward pair. Since it is difficult to compute the inverse marginal utility function I_1 explicitly and thus to obtain analytical results for the measure of risk-tolerance of the initial utility function U_0, we solve the expected utility maximisation problem numerically using MATLAB and then compute a central difference of second order accuracy to estimate the risk-tolerance of the initial utility. We do so for initial wealth levels between 4 and 12 so that the resulting terminal wealth under the optimal strategy remains between a and b. Figure 1 compares the risk-tolerance of the two utility functions. As expected, the risk-tolerance of U_0 is lower than that of U_1 at some wealth levels, but the opposite is the case at other wealth levels.

5 An example with SAHARA utility functions in a binomial market

In this section, we study and solve a case where the initial utility function U_0 belongs to the class of symmetric asymptotic hyperbolic absolute risk aversion (SAHARA) utility functions in the binomial model for the financial market. The term was coined by Chen et al. [8] for the classical, backward framework. A variant of this class first appeared in Musiela and Zariphopoulou [31, Example 1] and has been extensively studied in Zariphopoulou and Zhou [46] for continuous-time forward criteria; see also Musiela and Zariphopoulou [33, Example 12].

We work in the binomial model for the financial market because there, it is straightforward to extend the results of Sect. 2, and in particular Corollary 2.7, to utility functions defined on the whole real line, i.e., utility functions $U : \mathbb{R} \to \mathbb{R}$ which satisfy the Inada condition $\lim_{x\to-\infty} U'(x) = +\infty$ at minus infinity instead of zero and otherwise have the same properties as the utility functions we studied before. For the example studied in this section, we do not require the terminal wealth to be nonnegative, nor do we impose any restriction on the set of admissible strategies. Studying discrete-time predictable forward utility processes for utility functions defined on the whole real line in general market models would be much more intricate. When wealth is allowed to become negative, it is often not clear how to define a suitable class of admissible strategies. For (backward) utility maximisation with utility functions defined on the whole real line, it is typically the case that even when the solution exists and is representable as a stochastic integral, the integrand is not within the class of admissible strategies, or that the class of admissible strategies depends on the utility function (Schachermayer [41, 42], Biagini and Frittelli [7], Biagini and Černý [6]). However, for the forward framework, the existence of an admissible optimal strategy is required by definition.

Definition 5.1 A utility function $U : \mathbb{R} \to \mathbb{R}$ is of the *SAHARA class* with *risk-aversion parameter* $\alpha > 0$, *scale parameter* $\beta > 0$ and *threshold wealth* $\xi \in \mathbb{R}$ if its absolute risk-tolerance is given by

$$T(x) = \frac{1}{\alpha}\sqrt{\beta^2 + (x-\xi)^2} > 0.$$

When wealth is far above or below the threshold wealth, the risk-tolerance behaves like a linear function of wealth, giving rise to the name of its class. When the threshold level is approached from above, the risk-tolerance decreases. Consequently, the agent would invest a smaller proportion of his/her wealth in the risky asset, reflecting his/her reluctance to fall below the threshold. However, unlike in the case where wealth is restricted to remain nonnegative, this reluctance is not absolute and the agent thus allows the possibility of falling below the threshold. Once below the threshold, the agent becomes more tolerant to risk the further away he/she gets from the threshold. Preferences reflected by an SAHARA utility function are thus resembling those by the so-called value function in the prospect theory of Kahneman and Tversky [22] and Tversky and Kahneman [44] in behavioural finance, where there is a reference point dividing absolute wealths into a gain region in which the agent is risk-averse and a loss region in which the agent is risk-seeking.

It is straightforward to compute that SAHARA utility functions are up to affine transformations of the form

$$U(x) = -\frac{1}{\alpha^2-1}\frac{(x-\xi)+\alpha\sqrt{\beta^2+(x-\xi)^2}}{((x-\xi)+\sqrt{\beta^2+(x-\xi)^2})^{\alpha}} \tag{5.1}$$

if $\alpha \neq 1$ and

$$U(x) = \frac{\log((x-\xi)+\sqrt{\beta^2+(x-\xi)^2})}{2} + \frac{x-\xi}{2\beta^2}\left(\sqrt{\beta^2+(x-\xi)^2}-(x-\xi)\right) \tag{5.2}$$

if $\alpha = 1$. The corresponding inverse marginal function is given by

$$I(y) = \xi + \frac{1}{2}(y^{-1/\alpha} - \beta^2 y^{1/\alpha}), \qquad y > 0.$$

Note that Chen et al. [8] assume for simplicity that the threshold wealth is zero. Here we allow an arbitrary threshold wealth ξ because we are interested in understanding how this threshold evolves when we go forward in time.

The following theorem provides a complete solution for discrete-time predictable forward utility processes in a binomial market when the initial utility function belongs to the SAHARA class. It constitutes in particular the discrete-time analogue to the results established in Zariphopoulou and Zhou [46].

Theorem 5.2 *Let the initial utility function U_0 be of the SAHARA class with risk-aversion parameter $\alpha > 0$, scale parameter $\beta > 0$ and threshold wealth $\xi \in \mathbb{R}$. Then the forward utility is of the form $U_1(x) = \epsilon + \lambda^{\alpha}\widehat{U}(x)$, where λ is given by*

$$\lambda = \frac{(1+b)c^{-1/\alpha}}{a^{-1/\alpha}+b}, \tag{5.3}$$

$\widehat{U}$ is of the form (5.1) respectively (5.2) with risk-aversion parameter α_1, scale parameter β_1 and threshold wealth ξ_1 updated according to

$$\alpha_1 = \alpha, \qquad \beta_1 = \frac{\beta(1+b)}{\sqrt{1+b(a^{-1/\alpha}+a^{1/\alpha})+b^2}}, \qquad \xi_1 = \xi, \tag{5.4}$$

and ϵ is given by

$$\epsilon = U_0(1) - p\lambda^\alpha \widehat{U}\Big(I_1\Big(\frac{p}{q}U_0'(1)\Big)\Big) - (1-p)\lambda^\alpha \widehat{U}\Big(I_1\Big(\frac{1-p}{1-q}U_0'(1)\Big)\Big). \tag{5.5}$$

Moreover, the optimal terminal wealth is given by

$$X_1^*(x) = X_1^{*,u}(x)\mathbf{1}_{\{S_1=u\}} + X_1^{*,d}(x)\mathbf{1}_{\{S_1=d\}}$$

with

$$X_1^{*,u}(x) = \xi + \frac{\lambda}{2}\bigg(\Big(\frac{q}{p}\Big)^{-1/\alpha}\Big(x-\xi+\sqrt{\beta^2+(x-\xi)^2}\Big) - \frac{(\frac{\beta_1}{\lambda})^2(\frac{q}{p})^{1/\alpha}}{x-\xi+\sqrt{\beta^2+(x-\xi)^2}}\bigg)$$

and

$$X_1^{*,d}(x) = \xi + \frac{\lambda}{2}\bigg(\Big(\frac{1-q}{1-p}\Big)^{-1/\alpha}\Big(x-\xi+\sqrt{\beta^2+(x-\xi)^2}\Big) - \frac{(\frac{\beta_1}{\lambda})^2(\frac{1-q}{1-p})^{1/\alpha}}{x-\xi+\sqrt{\beta^2+(x-\xi)^2}}\bigg).$$

Finally, the corresponding optimal strategy is given by

$$\pi^*(x) = \frac{1}{u-d}\big(X^{*,u}(x) - X^{*,d}(x)\big).$$

Proof A careful reading of the proofs of Theorems 5.1 and 5.2 in Angoshtari et al. [1] shows that Corollary 2.7 extends to utility functions defined on the whole real line. We thus need to find an inverse marginal function I_1 solving the functional equation

$$I_1(ay) + bI_1(y) = (1+b)\xi + \frac{1}{2}(1+b)\big(c^{-1/\alpha}y^{-1/\alpha} - \beta^2 c^{1/\alpha}y^{1/\alpha}\big) \tag{5.6}$$

for all $y > 0$. Making the ansatz $I_1(y) = \xi_1 + \frac{\lambda}{2}(y^{-1/\alpha_1} - \beta_1^2 y^{1/\alpha_1})$, one easily finds that I_1 satisfies (5.6) when the parameters are given by $\lambda = (1+b)c^{-1/\alpha}/(a^{-1/\alpha}+b)$, $\alpha_1 = \alpha$, $\tilde{\beta}_1 = \beta c^{1/\alpha}\sqrt{\frac{a^{-1/\alpha}+b}{a^{1/\alpha}+b}}$ and $\xi_1 = \xi$. We therefore have

$$U_1'(x) = \left(\frac{x-\xi_1}{\lambda} + \sqrt{\tilde{\beta}_1^2 + \left(\frac{x-\xi_1}{\lambda}\right)^2}\right)^{-\alpha_1}$$
$$= \lambda^{\alpha_1}\left(x - \xi_1 + \sqrt{(\lambda\tilde{\beta}_1)^2 + (x-\xi_1)^2}\right)^{-\alpha_1}.$$

Since $\lambda\tilde{\beta}_1 = \beta_1$ and ϵ in (5.5) is chosen such that (2.7) is satisfied, the form of the forward utility follows. The optimal terminal wealth and optimal strategy can be deduced using Corollary 2.7 and noting that

$$U_0'(x) = \left(x - \xi + \sqrt{\beta^2 + (x-\xi)^2}\right)^{-\alpha}. \qquad \square$$

It follows from Theorem 5.2 that the forward utility remains within the class of SAHARA utility functions and, more surprisingly, only the scale parameter β changes over time. In particular, the risk parameter α and the threshold wealth ξ remain constant independently of any changes in the prevailing market conditions.

Let us now turn our attention to the updating of the scale parameter β. It follows from the general fact $x + 1/x \geq 2$ for $x > 0$, with equality if and only if $x = 1$, that $\beta_1 \leq \beta$, with equality if and only if $a = 1$, i.e., if the expected return of the risky asset equals the risk-free return. Since b depends only on q, the reduction of the scale parameter is increasing in the difference between the physical and the risk-neutral probability measure for any fixed values u and d of the risky asset. Note that the risk-tolerance

$$T(x) = \frac{1}{\alpha}\sqrt{\beta^2 + (x-\xi)^2} > 0$$

of an SAHARA utility function is convex as a function of wealth and increasing in the parameter β. That the scale parameter β_1 of the forward utility is smaller than the scale parameter β_0 of the initial utility function is thus consistent with Theorem 4.1. As long as the expected excess return offered by the market remains larger than a positive, time-independent constant, the scale parameter β will converge to zero as time passes. This implies that the risk-tolerance of the forward process approximates a linear function corresponding to an HARA utility. An SAHARA utility is thus an HARA utility not only asymptotically in wealth, but also asymptotically in time.

To conclude the study of this case, we compare the optimal terminal wealth for the SAHARA preferences with the optimal terminal wealth for an isoelastic utility function. For better comparability, we suppose that the threshold wealth ξ is zero and consider an isoelastic utility function with the same risk-aversion parameter α, i.e., $U^{\text{iso}}(x) = \frac{1}{1-\alpha}x^{1-\alpha}$. For this initial utility function, the optimal terminal wealth and optimal strategy are given by $X_1^{\text{iso}}(x) = X_1^{\text{iso},u}(x)\mathbf{1}_{\{S_1=u\}} + X_1^{\text{iso},d}(x)\mathbf{1}_{\{S_1=d\}}$ and $\pi^{\text{iso}}(x) = (X^{\text{iso},u}(x) - X^{\text{iso},d}(x))/(u-d)$, respectively, where

$$X_1^{\text{iso},u} = \lambda\left(\frac{q}{p}\right)^{-1/\alpha} x, \qquad X_1^{\text{iso},d} = \lambda\left(\frac{1-q}{1-p}\right)^{-1/\alpha} x,$$

with λ as in (5.3).

Proposition 5.3 *Suppose that the expected excess return of the market is positive, i.e., $p > q$. For $x > 0$, let $\pi^*(x)$ denote the optimal strategy for a forward utility when the initial utility is an SAHARA utility function with risk-aversion parameter $\alpha > 0$, scale parameter $\beta > 0$ and threshold wealth $\xi = 0$, and let $\pi^{\text{iso}}(x)$ denote the strategy for a forward utility when the initial utility is an isoelastic utility function with risk-aversion parameter α. Then the difference between the two strategies, $D^\pi(x) = \pi^*(x) - \pi^{\text{iso}}(x)$, is positive, strictly decreasing in x and converges to zero as x goes to infinity.*

Proof Denote by

$$\begin{aligned}D^u(x) &= X_1^{*,u}(x) - X_1^{\text{iso},u} \\ &= \frac{\lambda}{2}\Big(\Big(\frac{q}{p}\Big)^{-1/\alpha}\big(\sqrt{\beta^2 + x^2} - x\big) - \Big(\frac{\beta_1}{\lambda}\Big)^2\Big(\frac{q}{p}\Big)^{1/\alpha}\frac{1}{x + \sqrt{\beta^2 + x^2}}\Big)\end{aligned}$$

the difference between the terminal wealth levels in the state $\{S_1 = u\}$. Differentiating with respect to x yields

$$\frac{d}{dx}D^u(x) = \frac{\lambda}{2}\Big(\Big(\frac{q}{p}\Big)^{-1/\alpha}\Big(\frac{x}{\sqrt{\beta^2 + x^2}} - 1\Big) + \Big(\frac{\beta_1}{\lambda}\Big)^2\Big(\frac{q}{p}\Big)^{1/\alpha}\frac{1 + \frac{x}{\sqrt{\beta^2+x^2}}}{(x + \sqrt{\beta^2 + x^2})^2}\Big).$$

Hence $\frac{d}{dx}D^u(x)$ is negative if and only if

$$x + \Big(\frac{\beta_1}{\lambda}\Big)^2\Big(\frac{q}{p}\Big)^{2/\alpha}\frac{1}{x + \sqrt{\beta^2 + x^2}} < \sqrt{\beta^2 + x^2},$$

which can be further simplified to

$$\Big(\frac{\beta_1}{\lambda}\Big)^2\Big(\frac{q}{p}\Big)^{2/\alpha} < \beta^2. \tag{5.7}$$

Using the explicit form of λ and β_1 given in (5.3) and (5.4), we have (5.7) if and only if

$$1 > \Big(\frac{q}{p}\Big)^{2/\alpha} c^{2/\alpha}\frac{a^{-1/\alpha} + b}{a^{1/\alpha} + b} = \frac{a^{1/\alpha} + ba^{2/\alpha}}{a^{1/\alpha} + b}.$$

Because $a < 1$ when $p > q$, we can conclude that D^u is strictly decreasing. Moreover, it is easy to see that D^u converges to zero when x goes to infinity and D^u is thus positive. One can similarly show that $D^d(x) = X_1^{*,d}(x) - X_1^{\text{iso},d}$ is negative and strictly increasing to zero. The conclusion follows because we have $D^\pi(x) = (D^u(x) - D^d(x))/(u - d)$. □

Proposition 5.3 is further illustrated in Fig. 2 which shows the optimal terminal wealth in both states for the SAHARA and isoelastic forward utility function. The two agents behave similarly for large wealth levels, but the agent with an isoelastic initial

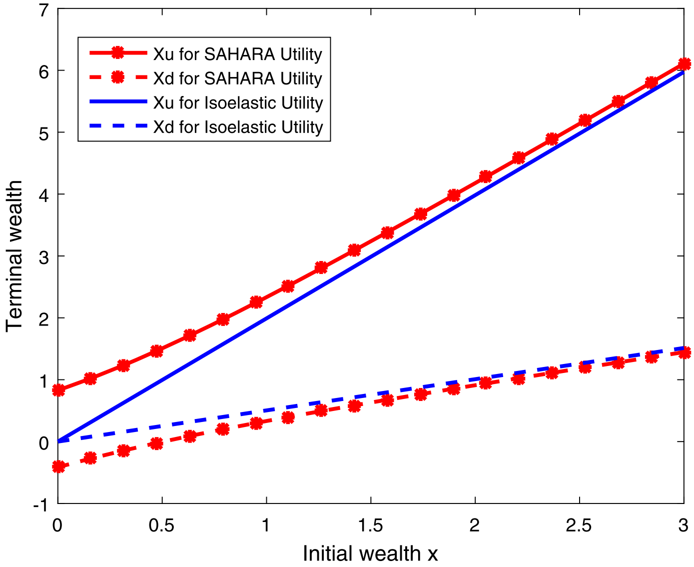

Fig. 2 This figure compares the optimal terminal wealth in both states for an SAHARA and an isoelastic utility function. The market parameters are $u = 1.1$, $d = 0.95$ and $p = 0.6$, and the preference parameters $\alpha = 0.8$, $\beta = 1$ and $\xi = 0$.

utility reduces his/her exposure to the risky asset to zero, whereas the agent with an SAHARA initial utility function keeps the exposure above a non-zero constant as the initial wealth goes to zero.

6 Conclusion

This paper is an immediate follow-up of Angoshtari et al. [1] in which the framework of discrete-time predictable forward utility processes was first introduced. One of the major results there is the reduction of a single-period inverse investment problem to a linear functional equation for the inverse marginal function when the market follows a binomial model. As a first contribution, we extend this one-period result from the binomial case to complete semimartingale models for the financial market and establish the equivalence between solutions to the inverse investment problem over a finite investment interval and generalised integral equations for either the inverse marginal function or the conjugate function. We then turn our attention to the main theme of this paper and investigate how the Arrow–Pratt measure of risk-tolerance evolves within the framework of discrete-time predictable forward utility pairs. We first investigate in a one-period framework under what conditions risk preferences can be preserved over time. We find that such a preservation holds for isoelastic utility functions in the general complete market, and we establish a complete characterisation of all initial utility functions leading to a time-constant measure of risk-tolerance for the binomial case. Moreover, for the binomial case, a preservation of preferences happens if and only if the initial utility function belongs to an enlarged class of isoelastic utility functions. Our next focus, again in a one-period framework, is on how preferences are updated qualitatively. We find that whether an agent becomes more or less tolerant to risk as time passes is related to whether the measure of risk-tolerance is concave or convex. Finally, we study an example with SAHARA utility functions in the binomial market model. This class of utility functions is found to be analytically tractable under the framework of discrete-time predictable forward utility processes. The results derived in this paper appear to be intuitively and economically interpretable, and consistent with some of the conclusions from the existing works in classical (backward) utility theory. This in turn suggests that discrete-time predictable forward utility processes may constitute a viable framework to study the dynamics of risk preferences and the associated decision-making problems.

There are several interesting directions for future research. Firstly, the question of existence and uniqueness of a forward process given an initial utility and general complete semimartingale model for the financial market remains open. This question is linked to the study of the related generalised integral equations for the inverse marginal or the conjugate function. Since it cannot be addressed by standard approaches, solving this problem will contribute to the field of integral equations in addition to the theory of forward performance processes. Secondly, the evolution of the Arrow–Pratt risk measures can be investigated in greater detail for financial markets other than the binomial model. Last but not least, studying discrete-time predictable forward processes in incomplete models remains an open and challenging research problem. In the complete market case, the convex duality approach reduces the inverse investment problem involved in the construction of forward processes to a generalised integral equation. However, this approach would lead to another inverse problem which needs to be solved over the space of equivalent supermartingale deflators if the market is incomplete. It is thus not clear whether the same approach would lead to a tractable problem, and very different techniques might be required to solve this case.

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