FAILURE OF SMOOTH PASTING PRINCIPLE AND NONEXISTENCE OF EQUILIBRIUM STOPPING RULES UNDER TIME-INCONSISTENCY*

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Abstract. This paper considers time-inconsistent stopping problems in which the inconsistency arises from a class of non-exponential discount functions called the weighted discount functions. We show that the smooth pasting (SP) principle, the main approach that is used to construct explicit solutions for the classical time-consistent optimal stopping problems, may fail under time-inconsistency. Specifically, a *mere* change of the discount function from exponential to non-exponential (everything else being the same) will fail the SP approach. In general, we prove that the SP solves a time-inconsistent problem, within the intra-personal game theoretic framework with a general nonlinear cost functional and a geometric Brownian motion, if and only if certain inequalities on the model primitives are satisfied. In the special case of a real options problem, we show that while these inequalities hold trivially for the exponential discount function, they may not hold even for very simple non-exponential discount functions. Moreover, we show that the real options problem actually does not admit any equilibrium whenever the SP fails. The negative results in this paper caution blindly extending the classical approach for time-consistent stopping problems to their time-inconsistent counterparts.

Key words. optimal stopping, weighted discount function, time inconsistency, equilibrium stopping, intra-personal game, smooth pasting, real options.

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1. Introduction. A crucial assumption imposed on classical optimal stopping models is that an agent has a constant time preference rate and hence discounts her future payoff exponentially. When this assumption is violated, an optimal stopping problem becomes generally time-inconsistent in that any optimal stopping rule obtained today may no longer be optimal from the perspective of a future date. The problem then becomes largely descriptive rather than normative because there is generally no *dynamically* optimal solution that can be used to guide the agent's decisions. Different agents may react differently to a same time-inconsistent problem, and a goal of the study is to *describe* the different behaviors. [34] is the first to observe that non-constant time preference rates result in time-inconsistency, and to categorize three types of agents when facing such time-inconsistency. One of the types is called a "non-committed, sophisticated agent" who, at *any* given time, optimizes the underlying objective taking as constraints the stopping decisions chosen by her future selves. Such a problem has been formulated within an intra-personal game theoretic framework and the corresponding equilibria are used to describe the behaviors of this type of agents; see, for example, [31]; [26]; [30]; [24] and [29]. An extended dynamic programming equation for continuous-time deterministic equilibrium controls is derived in [11], followed by a stochastic version in [2] and application to a mean–variance portfolio model in [3].

This paper studies a time-inconsistent stopping problem in continuous time within the intra-personal game framework, in which the source of time-inconsistency is the

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so-called weighted discount function (WDF), a very general class of non-exponential discount functions.[1] We make two main contributions. First, we demonstrate that the smooth pasting (SP) principle, which is almost the *exclusive* approach in solving classical optimal stopping problems, may fail *simply* because time-consistency is lost. Second, for a stopping model whose time-consistent counterpart is the well-studied real options problem, we establish a condition under which no equilibrium stopping rule exists. These results are constructive and they caution blindly extending the SP principle to time-inconsistent stopping problems.

Let us now elaborate on the first contribution. Recall that the SP is an *Ansatz* used to derive (often explicit) solutions to conventional, time-consistent optimal stopping problems. It conjectures a candidate solution to the underlying Bellman equation (or variational inequalities), which is a free boundary PDE, based on the C^1 smooth pasting around the free boundary, and then checks that it solves the PDE under some standard regularity/convexity conditions on the model primitives. Finally it verifies that the first hitting time of the free boundary indeed solves the optimal stopping problem using the standard verification technique. Recently, [14] and [17], among others, extend the application of the SP principle to solving time-inconsistent stopping problems. While the SP *happened to* work in the specific settings of these papers, it is more a lucky exception than a rule. Indeed, in the present paper we show that, for a geometric Brownian motion with a nonlinear cost functional, while the SP always yields a candidate solution, the latter actually gives rise to an equilibrium stopping rule if and only if certain inequalities on the model primitives are satisfied. These inequalities hold trivially for the time-consistent exponential discount case, but does not in general for its time-inconsistent non-exponential counterpart, *even if all the other parameters and assumptions (state dynamics, running cost, etc.) are identical.* Indeed, the violation of such inequalities is not rare even in very simple cases. For example, we show that in the special case of a real options problem with some WDFs including the pseudo-exponential discount function ([11]; [23]; [16]), the inequalities do not hold for plausible sets of parameter values of the chosen discount functions. The bottom line is that one cannot blindly apply the SP to any stopping model when time-inconsistency is present, even if the SP does work for its time-consistent counterpart.

The second contribution is on the nonexistence of an intra-personal equilibrium. For a time-consistent stopping problem, optimal stopping rules exist when the cost functional and the underlying process satisfy some mild regularity conditions (see, e.g., [33]). However, this is no longer the case for the time-inconsistent counterpart. To demonstrate this, we again take the real options problem with a WDF. For such a problem, we prove that there simply does not exist any equilibrium stopping rule whenever the aforementioned inequalities are violated and hence the SP principle fails. Our result therefore reveals that equilibrium stopping rules within the intra-personal game theoretic framework may not exist no matter what regularity conditions are imposed on the underlying models.

There are studies in the literature on time-inconsistent stopping including nonexistence results, albeit in considerably different settings especially in terms of the source of time-inconsistency. [5] and [6] study continuous-time stopping problems where the

[1]The WDF, proposed in [10], is a weighted average of a set of exponential discount functions. It has been shown in [10] that it can be used to model the time preference of a group of individuals as well as that of behavioral agents, and that most commonly used non-exponential discount functions are WDFs.

time-inconsistency follows from the types of payoff functions (mean-variance or endogenous habit formation). In particular, [5] shows that the candidate solution derived from the SP may not lead to an equilibrium stopping for some range of parameters for a mean–variance stopping problem. However, apart from being a different model, the reason for the failure of the SP therein does not seem to be the time-inconsistency; see Remark 3.7 for a detailed discussion. [6] and[5] also consider mixed strategies as opposed to the pure strategies studied in our paper and many other papers. Time-inconsistent problems using mixed strategies are interesting, and it is possible that no equilibrium may be found even in the class of mixed strategies. However, the main point of this paper is to show that a mere change of discounting factor from exponential to non-exponential may cause a stopping problem that has an equilibrium to one that does not, *even though both are using pure strategies*. [18] and [19] investigate continuous-time stopping problems with non-exponential discount functions and with probability distortion respectively. They define equilibria via a fixed point of a mapping, which is essentially based on a zeroth-order condition and hence is different from our definition. Under their settings, immediately stopping is always a (trivial) equilibrium (so there is no issue of nonexistence), which is not the case according to our definition.

The remainder of the paper is organized as follows. In section 2, we recall the definition and some important properties of the WDF introduced by [10], formulate a general time-inconsistent stopping problem within the intra-personal game theoretic framework, and characterize the equilibrium stopping rules by a Bellman system and provide the verification theorem. In section 3 we consider the case when the state process is a geometric Brownian motion, apply the SP principle to derive a candidate solution and establish certain equivalent conditions for the derived candidate solution to actually solve the Bellman system. Then we present a real options problem, in which the aforementioned equivalent conditions reduce to a single inequality, failing when there is simply no equilibrium at all. Finally, section 4 concludes the paper. Appendix A contains proofs of some results.

2. The Model .

2.1. Time preferences. Throughout this paper we consider weighted discount functions defined as follows.

Definition 2.1 ([10]). *Let $h : [0, \infty) \to (0, 1]$ be strictly decreasing with $h(0) = 1$. We call h a weighted discount function (WDF) if there exists a distribution function F concentrated on $[0, \infty)$ such that*

$$h(t) = \int_0^\infty e^{-rt} dF(r). \tag{2.1}$$

Moreover, we call F the weighting distribution *of h.*

Many commonly used discount functions can be represented in weighted form. For example, exponential function $h(t) = e^{-rt}, r > 0$ ([32]) and pseudo-exponential function $h(t) = \delta e^{-rt} + (1-\delta)e^{-(r+\lambda)t}, 0 < \delta < 1, r > 0, \lambda > 0$ ([11]; [23]) are WDFs with degenerate and binary distributions respectively. A more complicated example is the generalized hyperbolic discount function ([28]) with parameters $\gamma > 0, \beta > 0$, which can be represented as

$$h(t) = \frac{1}{(1+\gamma t)^{\frac{\beta}{\gamma}}} \equiv \int_0^\infty e^{-rt} f\left(r; \frac{\beta}{\gamma}, \gamma\right) dr \tag{2.2}$$

where $f(r;k,\theta) = \frac{r^{k-1}e^{-\frac{r}{\theta}}}{\theta^k\Gamma(k)}$ is the density function of the Gamma distribution with parameters k and θ, and $\Gamma(k) = \int_0^\infty x^{k-1}e^{-x}dx$ the Gamma function evaluated at k. See [10] for more examples and discussions about the types of discount functions that are of weighted form.

The following result is a restatement of the well-known Bernstein's theorem in terms of WDFs, which actually provides a characterization of the latter.

THEOREM 2.2 ([1]). *A discount function h is a WDF if and only if it is continuous on $[0,\infty)$, infinitely differentiable on $(0,\infty)$, and satisfies $(-1)^n h^{(n)}(t) \geq 0$, for all non-negative integers n and for all $t > 0$.*

Bernstein's theorem can be used to examine if a given function is a WDF without necessarily representing it in the form of (2.1). For example, it follows from this theorem that the constant sensitivity discount function ([9]) $h(t) = e^{-at^k}, a > 0, 0 < k < 1$, and the constant absolute decreasing impatience discount function ([4]) $h(t) = e^{e^{-ct}-1}, c > 0$, are both WDFs.

2.2. Stopping rules and equilibria. On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, P)$ there lives a one dimensional Brownian motion W, and a family of Markov diffusion processes $X = X^x$ parameterized by the initial state $X_0 = x \in \mathbb{R}$ and governed by the following stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \sigma(X_t)W_t, \quad X_0 = x, \tag{2.3}$$

where b, σ are Lipschitz continuous functions, i.e., there exists an $L > 0$ such that for any $x \neq y$

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|. \tag{2.4}$$

We assume that $\mathbb{F}$ is the P-augmentation of the natural filtration generated by X. To avoid an uninteresting case we also assume that $|\sigma(x)| \geq c > 0\ \forall x \in \mathbb{R}$ so that X is non-degenerate.[2]

For any fixed $x \in \mathbb{R}$, an agent monitors the process $X = X^x$ and aims to minimize the following cost functional

$$J(x;\tau) = \mathbb{E}\left[\int_0^\tau h(s)f(X_s)ds + h(\tau)g(X_\tau)\Big|X_0 = x\right] \tag{2.5}$$

by choosing $\tau \in \mathcal{T}$, the set of all $\mathbb{F}$-stopping times. Here h is a WDF with a weighting distribution F, g is continuous and bounded, and f is continuous with polynomial growth, i.e., there exists $m \geq 1$ and $C > 0$ such that

$$|f(x)| \leq C(|x|^m + 1). \tag{2.6}$$

Moreover, we assume that there exists $n \geq 1, C(r) > 0$ satisfying $\int_0^\infty C(r)dF(r) + \int_0^\infty rC(r)dF(r) < \infty$ such that

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-rs}|f(X_s)|ds + e^{-r\tau}|g(X_\tau)|\Big|X_0 = x\right] \leq C(r)(|x|^n + 1), \quad \forall r \in \operatorname{supp}(F). \tag{2.7}$$

[2]Here we assume that the Brownian motion is one dimensional just for notational simplicity. There is no essential difficulty with a multi-dimensional Brownian motion.

This is a (weak) assumption to ensure that the optimal value of the stopping problem is finite, and hence the problem is well-posed.

We now define stopping rules which are essentially binary feedback controls. These stopping rules induce Markovian stopping times for any given Markov process.

Definition 2.3 (Stopping rule). *A stopping rule is a measurable function* $u : \mathbb{R} \to \{0,1\}$ *where* 0 *indicates "continue" and* 1 *indicates "stop". For any given Markov process* $X = \{X_t\}_{t\geq 0}$*, a stopping rule* u *defines a Markovian stopping time*

$$\tau_u = \inf\{t \geq 0, u(X_t) = 1\}. \tag{2.8}$$

Given a stopping rule u, we can define the stopping region $\mathcal{S}_u = \{x \in (0,\infty) : u(x) = 1\}$. For any $x \in \bar{\mathcal{S}_u}$, since the underlying process X is non-degenerate, a standard result (e.g., Chapter 3 of [22]) yields that $\mathbb{P}(\tau_u = 0|X_0 = x) = 1$, and hence $J(x;\tau_u) = g(x)$. This means that the agent stops immediately once the process reaches at any point in $\bar{\mathcal{S}_u}$. As a result, in the setting of this paper, the continuation region is $\mathcal{C}_u = \bar{\mathcal{S}_u}^c$.

As discussed earlier the non-exponential discount function h in the cost functional (2.5) renders the underlying optimal stopping problem generally time-inconsistent. In this paper we consider a sophisticated and non-committed agent who is aware of the time-inconsistency but unable to control her future actions. In this case, she seeks to find the so-called equilibrium strategies within the intra-personal game theoretic framework, in which the individual is represented by different players at different dates.[3]

We now give the precise definition of an equilibrium stopping rule $\hat{u}$, which essentially entails a solution to a game in which no self at any time (or, equivalently in the current setting, at any state) is willing to deviate from $\hat{u}$.

Definition 2.4 (Equilibrium stopping rule). *The stopping rule* $\hat{u}$ *is an equilibrium stopping rule if*

$$\limsup_{\epsilon\to 0+} \frac{J(x;\tau_{\hat{u}}) - J(x;\tau^{\epsilon,a})}{\epsilon} \leq 0, \;\; \forall x \in \mathbb{R}, \; \forall a \in \{0,1\}, \tag{2.9}$$

where

$$\tau^{\epsilon,a} = \begin{cases} \inf\{t \geq \epsilon, \hat{u}(X_t) = 1\} & \text{if } a = 0, \\ 0 & \text{if } a = 1 \end{cases} \tag{2.10}$$

with $\{X_t\}_{t\geq 0}$ *being the solution to* (2.3).

This definition of an equilibrium is consistent with the majority of definition for time-inconsistent control problems in the literature (see, e.g., [2]; [12]; and [3]) when a stopping rule is interpreted as a binary control. Indeed, $\tau^{\epsilon,a}$ is a stopping time that might be different from $\tau_{\hat{u}}$ only in the very small initial time interval $[0,\epsilon)$; hence it is a "perturbation" of the latter.[4]

[3]Given the infiniteness of the time horizon, the stationarity of the process X as well the time-homogeneity of the running objective function f, each self at any given time t faces exactly the same decision problem as the others, which only depends on the current state $X_t = x$, but not on time t directly. We can thus identify self t by the current state $X_t = x$. That is why we need to consider only stationary stopping rules u, which are functions of the state variable x only. For details on this convention, see, e.g. [14]; [12]; [16] and, in particular, Section 3.2 of [10].

[4]This definition is based on the first-order condition with respect to the perturbation. [21] calls it a *weak equilibrium*, and proposes a *strong equilibrium* via a direct comparison of objective functional values for a time-homogeneous, continuous-time, finite-state Markov chain. [20] further extends the notion of strong equilibria to a general diffusion framework, and shows that a strong equilibrium is very restrictive leading to nonexistence for a number of problems; see Proposition 4.9 therein.

2.3. Equilibrium characterization. The following result, Theorem 2.5, formally establishes the Bellman system and provides the verification theorem for verifying equilibrium stoppings.

THEOREM 2.5 (Equilibrium characterization). *Consider cost functional* (2.5) *with WDF* $h(t) = \int_0^\infty e^{-rt} dF(r)$, *a stopping rule* $\hat{u}$, *an underlying process* X *defined by* (2.3), *functions* $w(x;r) = \mathbb{E}\left[\int_0^{\tau_{\hat{u}}} e^{-rt} f(X_t)dt + e^{-r\tau_{\hat{u}}} g(X_{\tau_{\hat{u}}}) \Big| X_0 = x\right]$ *and* $V(x) = \int_0^\infty w(x;r)dF(r)$. *Suppose that* w *is continuous in* x *and* V *is continuously differentiable with its first-order derivative being absolutely continuous. If* $(V, w, \hat{u})$ *solves the following differential equation problem defined on* $\mathbb{R}$,

$$\min\left\{\frac{1}{2}\sigma^2(x)V_{xx}(x) + b(x)V_x(x) + f(x) - \int_0^\infty rw(x;r)dF(r), g(x) - V(x)\right\} = 0, \tag{2.11}$$

$$\hat{u}(x) = \begin{cases} 1 & \text{if } V(x) = g(x), \\ 0 & \text{otherwise}, \end{cases} \tag{2.12}$$

then $\hat{u}$ *is an equilibrium stopping rule and* V *is the value function of the problem, i.e.,* $V(x) = J(x;\tau_{\hat{u}})\ \forall x \in \mathbb{R}$.

A proof to the above proposition is relegated to the appendix.[5]

3. Failure of SP and Nonexistence of Equilibrium.

In the classical literature on (time-consistent) stopping, optimal solutions are often obtained by the SP, because the candidate solution obtained from the SP must solve the Bellman system (and hence the optimal stopping problem) under some mild conditions, such as the smoothness and convexity/concavity of the cost functions. In economics terms, the SP principle amounts to the matching of the marginal cost at the stopped state (see, e.g., [8] and [7]); hence some economists apply the SP principle without even explicitly introducing the Bellman system. However, as we will show in this section, the SP approach in the presence of time-inconsistency may *not* yield a solution to the Bellman system (and therefore *not* to the stopping problem within the game theoretic framework), no matter how smooth and convex/concave the cost functions might be.

3.1. A time-consistent benchmark. Let us start with a time-consistent optimal stopping problem which we use as a benchmark for comparison purpose and outline the way to use the SP principle in constructing explicit solutions. Consider the following classical optimal stopping problem

$$\inf_{\tau \in \mathcal{T}} \mathbb{E}\left[\int_0^\tau e^{-rs} f(X_s)ds + e^{-r\tau}K \Big| X_0 = x\right], \tag{3.1}$$

where the underlying process X is a geometric Brownian motion

$$dX_t = bX_t dt + \sigma X_t dW_t,\ x > 0, \tag{3.2}$$

and $\mathcal{T}$ is the set of all stopping times with respect to $\mathbb{F}$.

[5] A proof of this result in a different setting was provided in [10]. Here we supply a proof for reader's convenience.

In what follows we assume that the running cost f is continuously differentiable, increasing and concave. Moreover, to rule out the "trivial cases" where either immediately stopping or never stopping is optimal for this time consistent benchmark, we assume that $f(0) < rK, b < r$, and $\lim_{x\to\infty} f_x(x)x = \infty$.[6]

Define $L(x; r) = \mathbb{E}[\int_0^\infty e^{-rs} f(X_s)ds|X_0 = x]$. Noting that X is a geometric Brownian motion, we have after straightforward manipulations

$$L(x; r) = \int_0^\infty \int_0^\infty f(yx)e^{-rs}G(y, s)dyds, \tag{3.3}$$

where $G(y, s) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma y \sqrt{s}} e^{-\frac{(\ln y-(b-\frac{1}{2}\sigma^2)s)^2}{2\sigma^2 s}}$. To ensure L and L_x are well defined, we further assume that f has linear growth and $f_x(0+) < \infty$.

We now characterize the optimal stopping rule as follows.

Proposition 3.1. *There exists $x_B > 0$ such that the stopping rule $u_B(x) = \mathbf{1}_{x \geq x_B}(x)$ solves optimal stopping problem (3.1). Moreover, x_B is the unique solution of the following algebraic equation in y:*

$$\alpha(r)[K - L(y; r)] + L_x(y; r)y = 0 \tag{3.4}$$

where

$$\alpha(r) = \frac{-(b - \frac{1}{2}\sigma^2) + \sqrt{(b - \frac{1}{2}\sigma^2)^2 + 2\sigma^2 r}}{\sigma^2}. \tag{3.5}$$

The key to proving this theorem is to make use of the SP; see Appendix A.2.

3.2. Equivalent conditions under time-inconsistency. We now consider exactly the same stopping problem as the above time-consistent benchmark except that the exponential discount function is replaced by a WDF, namely, the cost functional is changed to

$$J(x; \tau) = \mathbb{E}\left[\int_0^\tau h(s)f(X_s)ds + h(\tau)K\Big|X_0 = x\right], \tag{3.6}$$

where h is a WDF with a weighting distribution F.

As in the case of exponential discounting, we need to impose the following regularity conditions on the parameters of the problem:

$$b < r,\ \forall r \in \text{supp}(F);\ \text{and}\ \max\left\{\int_0^\infty \frac{1}{r-b}dF(r), \int_0^\infty \frac{1}{r}dF(r), \int_0^\infty rdF(r)\right\} < \infty.$$

[6]In this formulation the final cost is assumed to be a constant lump sum K without loss of generality. In fact, by properly modifying the running cost, we are able to reduce the stopping problem with a final cost function g to one with a final cost being any given constant $K > 0$. To see this, applying Ito's formula to $e^{-rt}(g(X_t) - K)$, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\tau e^{-rs}f(X_s)ds + e^{-r\tau}g(X_\tau)\Big|X_0 = x\right] \\
=&\mathbb{E}\left[\int_0^\tau e^{-rs}f(X_s)ds + e^{-r\tau}K + e^{-r\tau}(g(X_\tau) - K)\Big|X_0 = x\right] \\
=&\mathbb{E}\left\{\int_0^\tau e^{-rs}f(X_s)ds + e^{-r\tau}K + \int_0^\tau e^{-rs}\hat{f}(X_s)ds\Big|X_0 = x\right\},
\end{aligned}$$

where $\hat{f}(x) := \frac{1}{2}\sigma^2x^2g_{xx}(x) + bxg_x(x) - r(g(x) - K)$. With $\tilde{f}(x) := f(x) + \hat{f}(x)$, the cost functional now becomes the one in problem (3.1) with running cost $\tilde{f}$. We certainly would need to impose conditions on g so that $\tilde{f}$ satisfies all the assumptions on f specified in this paragraph.

These conditions either hold automatically or reduce to the respective counterparts when the discount function degenerates into the exponential one. On the other hand, they hold valid with many genuine WDFs, including the generalized hyperbolic discount function (2.2) when $\gamma < \beta$ and the pseudo-exponential discount function.

We now attempt to use the SP principle to solve the Bellman system in Theorem 2.5 with the cost functional (3.6). We start by conjecturing that the equilibrium stopping region is $[x_*, \infty)$ for some $x_* > 0$. (As in the time-consistent case, x_* is called the triggering boundary or the stopping threshold.)

It follows from the Feynman–Kac formula that w in the Bellman system is given by

$$w(x;r) = \begin{cases} (K - L(x_*;r))\left(\frac{x}{x_*}\right)^{\alpha(r)} + L(x;r), & x < x_*, \\ K, & x \geq x_*, \end{cases}$$

where $L(x;r)$ is defined by (3.3) and $\alpha(r)$ by (3.5). Recall we have defined V by $\int_0^\infty w(x;r)dF(r)$, then we have

$$V(x) = \begin{cases} \int_0^\infty((K - L(x_*;r))(\frac{x}{x_*})^{\alpha(r)} + L(x;r))dF(r), & x < x_*, \\ K, & x \geq x_* \end{cases}$$

and

$$\hat{u}(x) = \begin{cases} 0 & x < x_*, \\ 1 & \text{otherwise.} \end{cases}$$

The SP applied to V (*not* to w) yields $V_x(x_*) = 0$, implying that x_* is the solution to the following algebraic equation in y

$$\int_0^\infty \left[\alpha(r)(K - L(y;r)) + L_x(y;r)y\right] dF(r) = 0. \tag{3.7}$$

Clearly, this equation is a generalization of its time-consistent counterpart, (3.4). The following proposition stipulates that it has a unique solution.

Proposition 3.2. *Equation (3.7) admits a unique solution in* $(0, \infty)$.

Proof. Following the same lines of proof of Proposition 3.1 (Appendix A.2), we have that $Q(x) := \int_0^\infty(\alpha(r)(K - L(x;r)) + L_x(x;r)x)dF(r)$ is strictly decreasing in $x > 0$, with $Q(0) > 0$ and $Q(\infty) < 0$. This completes the proof. ∎

Proposition 3.2 indicates that following the conventional SP line of argument does indeed give rise to a *candidate* solution to the Bellman system, even under time inconsistency. Whether this candidate solution indeed solves the Bellman system in Theorem 2.5 and hence the corresponding stopping rule $\hat{u}$ solves the equilibrium stopping problem boils down to the validity of an additional condition, as shown in the following result.

Theorem 3.3. *Assume that* $\alpha(r)[\alpha(r) - 1][K - L(x_*;r)]$ *is increasing in* $r \in$ *supp*(F)*, and let* x_* *be the unique solution to* (3.7)*. Then the triplet* $(V, w, \hat{u})$ *solves the Bellman system in Theorem 2.5 and in particular* $\hat{u}$ *is an equilibrium stopping rule if and only if*

$$f(x_*) \geq \int_0^\infty r dF(r)K, \tag{3.8}$$

and

$$\int_0^\infty \alpha(r)[\alpha(r)-1][K-L(x_*;r)]dF(r)+\int_0^\infty x_*^2 L_{xx}(x_*;r)dF(r)\le 0. \tag{3.9}$$

As the proof to the above theorem is lengthy, we defer it to Appendix A.3. The above theorem presents characterizing conditions (on the model primitives) for the SP to work for stopping problems with general WDFs. These conditions are satisfied automatically in the classical time-consistent case, but not in the time-inconsistent case in general. We will demonstrate this with a classical real options problem in the next subsection.

3.3. A real options problem: failure of SP and nonexistence of equilibrium. In this subsection we consider a special case of the model studied in the previous subsection, which is a time-inconsistent counterpart of the well-studied (time-consistent) problem of real options. Such a problem can be used to model, among others, when to start a new project or to abandon an ongoing project; see [7] for a systematic account on the classical real options theory.

The problem is to minimize

$$\mathbb{E}\left[\int_0^\tau h(s)X_s ds + h(\tau)K \Big| X_0 = x\right] \tag{3.10}$$

by choosing $\tau\in\mathcal{T}$, where $X=\{X_t\}_{t\ge 0}$ is governed by

$$dX_t = \sigma X_t dW_t. \tag{3.11}$$

We now apply Theorem 3.3 to this problem, and see what the equivalent conditions (3.8) and (3.9) boil down to.

First of all,

$$L(x;r)=\mathbb{E}[\int_0^\infty e^{-rt}X_t dt \Big| X_0=x]=\frac{x}{r}.$$

Hence

$$w(x;r)=\begin{cases} \left(K-\frac{x_*}{r}\right)\left(\frac{x}{x_*}\right)^{\alpha(r)}+\frac{x}{r}, & x<x_*,\\ K, & x\ge x_*,\end{cases}$$

$$V(x)=\begin{cases} \int_0^\infty \left(K-\frac{x_*}{r}\right)\left(\frac{x}{x_*}\right)^{\alpha(r)}dF(r)+\int_0^\infty \frac{x}{r}dF(r), & x<x_*,\\ K, & x\ge x_*,\end{cases}$$

and

$$\hat{u}(x)=\begin{cases} 0 & x<x_*,\\ 1 & \text{otherwise},\end{cases}$$

where

$$\alpha(r)=\frac{\frac{1}{2}\sigma^2+\sqrt{\frac{1}{4}\sigma^4+2\sigma^2 r}}{\sigma^2}. \tag{3.12}$$

Moreover, it follows from (3.7) that x_* is the solution to the following equation in y:

$$\int_0^\infty \left(K - \frac{y}{r}\right)\alpha(r)dF(r) + \int_0^\infty \frac{y}{r}dF(r) = 0.$$

Thus

$$x_* = \frac{\displaystyle\int_0^\infty \alpha(r)dF(r)}{\displaystyle\int_0^\infty \frac{\alpha(r)-1}{r}dF(r)}K. \tag{3.13}$$

Next, it is easy to verify that $\alpha(r)[\alpha(r)-1][K-L(x_*;r)] = \frac{2}{\sigma^2}(Kr-x_*)$; hence it is an increasing function in $r \geq 0$. Moreover, substituting the explicit representation of x_* in (3.13) into (3.8) and (3.9) we find that the latter two inequalities are both identical to the following single inequality

$$\int_0^\infty \alpha(r)dF(r) \geq \int_0^\infty rdF(r)\int_0^\infty \frac{\alpha(r)-1}{r}dF(r). \tag{3.14}$$

Remark 3.4. Here we assume that the geometric Brownian motion is driftless without loss of generality. Indeed, for a drifted geometric Brownian motion

$$dX_t = bX_tdt + \sigma X_tdW_t, \quad x > 0,$$

a completely analogous analysis shows that the inequality (3.14) needs to be revised to

$$\int_0^\infty \alpha(r)dF(r) \geq \int_0^\infty (r-b)dF(r)\int_0^\infty \frac{\alpha(r)-1}{r-b}dF(r),$$

where $\alpha(r) = \frac{-(b-\frac{1}{2}\sigma^2)+\sqrt{(b-\frac{1}{2}\sigma^2)^2+2\sigma^2 r}}{\sigma^2}$.

We have proved the following.

PROPOSITION 3.5. *The triplet $(V, w, \hat{u})$ solves the Bellman system of the real options problem if and only if (3.14) holds.*

Inequality (3.14) is a critical condition on the model primitives we must verify before we can be sure that the solution constructed through the SP is indeed an equilibrium solution to the time-inconsistent real options problem. It is immediate to see that the strict inequality of (3.14) is satisfied *trivially* when the distribution function F is degenerate corresponding to the classical time-consistent case with an exponential discount function. In this case, x_* defined by (3.13) coincides with the stopping threshold derived in subsection 3.1. This reconciles with the time-consistent setting.

The condition (3.14) may hold for some non-exponential discount functions. Consider a generalized hyperbolic discount function

$$h(t) = \frac{1}{(1+\gamma t)^{\frac{\beta}{\gamma}}} \equiv \int_0^\infty e^{-rt}\frac{r^{\frac{\beta}{\gamma}-1}e^{-\frac{r}{\gamma}}}{\gamma^{\frac{\beta}{\gamma}}\Gamma(\frac{\beta}{\gamma})}dr, \quad \gamma > 0, \beta > 0.$$

We assume that $\gamma < \beta \le \frac{\sigma^2}{2}$. Noting that $\alpha(r) - 1 = -\frac{1}{2} + \frac{\sqrt{\frac{1}{4}\sigma^4 + 2\sigma^2 r}}{\sigma^2}$ is a concave function in r, we have

$$\alpha(r) - 1 \le (\alpha(r) - 1)'|_{r=0} r + \alpha(0) - 1 = \frac{2}{\sigma^2} r.$$

Moreover, it is easy to see that

$$\int_0^\infty r dF(r) = \beta \text{ and } \alpha(r) \ge 1.$$

Therefore,

$$\int_0^\infty \frac{\alpha(r) - 1}{r} dF(r) \int_0^\infty r dF(r) \le \beta \frac{2}{\sigma^2} \le 1 \le \int_0^\infty \alpha(r) dF(r)$$

which is (3.14). So, in this case the SP works and the stopping threshold x_* is given by

$$x_* = \frac{\displaystyle\int_0^\infty \alpha(r) \frac{r^{\frac{\beta}{\gamma}-1} e^{-\frac{r}{\gamma}}}{\gamma^{\frac{\beta}{\gamma}} \Gamma(\frac{\beta}{\gamma})} dr}{\displaystyle\int_0^\infty \frac{\alpha(r) - 1}{r} \frac{r^{\frac{\beta}{\gamma}-1} e^{-\frac{r}{\gamma}}}{\gamma^{\frac{\beta}{\gamma}} \Gamma(\frac{\beta}{\gamma})} dr} K.$$

However, it is also possible that (3.14) fails, which is the case even with the simplest class of non-exponential WDFs – the pseudo-exponential discount functions. To see this, let $h(t) = \delta e^{-rt} + (1-\delta) e^{-(r+\lambda)t}, 0 < \delta < 1, r > 0, \lambda > 0$. It is straightforward to obtain that

$$\int_0^\infty \alpha(r) dF(r) = \delta\alpha(r) + (1 - \delta)\alpha(r + \lambda)$$

and

$$\int_0^\infty r dF(r) \int_0^\infty \frac{\alpha(r) - 1}{r} dF(r) > (1 - \delta)(r + \lambda)\delta \left(\frac{\alpha(r) - 1}{r} \right).$$

Since $(1-\delta)(r+\lambda)\delta(\frac{\alpha(r)-1}{r})$ grows faster than $\delta\alpha(r) + (1-\delta)\alpha(r+\lambda)$ when λ becomes large, we conclude that (3.14) is violated when r, δ are fixed and λ is sufficiently large.

What we have discussed so far shows that the solution constructed through the SP does not solve the time-inconsistent real options problem whenever inequality (3.14) fails. A natural question in this case is whether there might exist equilibrium solutions that cannot be obtained by the SP or even by the Bellman system. The answer is resoundingly negative.

PROPOSITION 3.6. *For the real options problem* (3.10)–(3.11)*, if* (3.14) *does not hold, then no equilibrium stopping rule exists.*

Proof. We prove by contradiction. Suppose $\hat{u}$ is an equilibrium stopping rule. We first note that $\mathcal{C}_{\hat{u}} \equiv \{x > 0 : \hat{u}(x) = 0\} \ne (0, \infty)$; otherwise $\hat{u} \equiv 0$, leading to $J(x; \tau_{\hat{u}}) = \int_0^\infty L(x; r) dF(r) = \int_0^\infty \frac{x}{r} dF(r)$, and hence $J(x; \tau_{\hat{u}}) \to \infty$ as $x \to \infty$ contradicting Lemma A.2 in Appendix A.2.

Define $x_* = \inf\{x : x \in \bar{\mathcal{S}}_{\hat{u}}\}$. It follows from Lemma A.3 in Appendix A.2 that $x_* \in (0, \infty)$. A standard argument then leads to

$$J(x;\tau_{\hat{u}}) = \int_0^\infty \left(K - \frac{x_*}{r}\right)\left(\frac{x}{x_*}\right)^{\alpha(r)} dF(r) + \int_0^\infty \frac{x}{r} dF(r), \quad x \in (0, x_*].$$

Because $J(x;\tau_{\hat{u}}) \le K$ (Lemma A.2) and $J(x_*;\tau_{\hat{u}}) = K$, we have $J_x(x_*-;\tau_{\hat{u}}) \ge 0$, i.e.,

$$\int_0^\infty \left(K - \frac{x_*}{r}\right)\alpha(r)\frac{1}{x_*} dF(r) + \int_0^\infty \frac{1}{r} dF(r) \ge 0,$$

which in turn gives

$$x_* \le \frac{\int_0^\infty \alpha(r) dF(r)}{\int_0^\infty \frac{\alpha(r)-1}{r} dF(r)} K.$$

Combining with the failure of condition (3.14), we derive

$$x_* < \int_0^\infty r dF(r),$$

which contradicts Lemma A.3. This completes the proof. ◻

The above is a stronger result. It suggests that for the problem to have any equilibrium stopping rule at all (not necessarily the one obtainable by the SP principle), condition (3.14) *must* hold. So, when it comes to a time-inconsistent stopping problem with non-exponential discounting, it is highly likely that no equilibrium stopping rule exists, even if the SP principle does generate a “solution”, or even if the time-consistent counterpart (in which everything else is identical except the discount function) is indeed solvable by the SP. Applying these conclusions to the pseudo-exponential discount functions discussed above, we deduce that there is no equilibrium stopping when λ is sufficiently large.

Having said this, a logical conclusion from Proposition 3.5 and Proposition 3.6 is that when equilibria do exist, one of them must be a solution generated by the SP. In general if there exists an equilibrium then there may be multiple ones; see, for example, [24] and [13] for multiple equilibria in time-inconsistent control problems. In this case, the SP can only generate *one* of them, but not necessarily *all* of them. (This statement is true even for a classical time-consistent stopping problem.) So, after all, the SP is still a useful, proper method for time-inconsistent problems; we can simply apply it to generate a candidate solution. If the solution is an equilibrium (which we must verify), then we have found one (but not necessarily other equilibria); if it is not an equilibrium, then we know there is no equilibrium at all. All these conclusions, however, are drawn on the special real option problem in this subsection. It remains an interesting problem to extend them to more general settings.

Remark 3.7. A result of non-existence of equilibrium stopping is presented in [5], Theorem 4.6, for the following mean–variance stopping problem:

$$\max_{\tau \in \mathcal{T}} \mathbb{E}[X_\tau - \gamma \mathrm{Var}(X_\tau)], \tag{3.15}$$

where

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \ X_0 = x \ge 0, \tag{3.16}$$

with $\gamma > 0$ and $\sigma^2/4 < \mu < \sigma^2/2$. The main differences between that result and the ones presented in this paper are as follows. First, mean–variance and non-exponential discounting are two different *sources* of time-inconsistency. While the definition of an equilibrium can be the same,[7] the approaches to characterize and derive it can be remarkably different. Likewise, a result that is valid for one problem is not automatically valid for the other. That is why in the literature the two problems have been studied separately. Second, and indeed more importantly, in the present paper we have painstakingly demonstrated that *the reason for the negative results in our setting is the time-inconsistency*: they occur when we simply change the exponential discounting to a non-exponential one *while keeping everything else the same*. We presented the time-consistent benchmark (Subsection 3.1) for comparison purpose: the benchmark case is time-consistent, and can be solved completely by SP (see Appendix A.2). However, the SP may fail for its time-inconsistent counterpart. In contrast, in [5], time-inconsistency does not seem to be responsible for the failure of SP nor for the non-existence of an equilibrium. Indeed, although no time-consistent benchmark is discussed in [5], because the variance term is what solely causes time-inconsistency, the natural benchmark is the case when the variance vanishes (i.e. $\gamma = 0$), in which case the Bellman equation is

$$\max\left\{\frac{1}{2}\sigma^2 x^2 V_{xx}(x) + \mu x V_x(x), x - V(x)\right\} = 0, \quad V(0) = 0. \tag{3.17}$$

The equation in the continuation region is

$$\frac{1}{2}\sigma^2 x^2 V_{xx}(x) + \mu x V_x(x) = 0, \quad V(0) = 0$$

whose solution is

$$V(x) = Cx^\lambda$$

where $\lambda = \frac{\sigma^2 - 2\mu}{\sigma^2} \in (0, 1/2)$. Now, the smooth pasting at a free boundary a yields

$$Ca^\lambda = a, \quad C\lambda a^{\lambda-1} = 1.$$

The two equations imply that $\lambda = 1$ which is a contradiction. This means the SP fails even for the time-consistent case, suggesting it is *not* time-inconsistency that causes the negative result in [5].[8]

4. Conclusions. While the SP principle has been widely used to study time-inconsistent stopping problems, our results indicate the risk of using this principle on such problems. We have shown that the SP principle solves the time-inconsistent problem if and only if certain inequalities are satisfied.

By a simple model of the classical real options problem, we have found that these inequalities may be violated even for simple and commonly used non-exponential discount functions. When the SP principle fails, we have shown the intra-personal equilibrium does not exist. The nonexistence result and the failure of the SP principle

[7] The definition of equilibrium in [5] is from the state (space), instead of time, point of view, due to the time-homogeneity of the model. This definition is inspired by [10], as noted in [5], Remark 2.7.

[8] Interestingly, Theorem 4.6 in [5] also states that when $0 < \mu \le \sigma^2/4$, the SP works for the time-inconsistent problem (where $\gamma > 0$) and an equilibrium is explicitly found, whereas the SP fails for the time-consistent benchmark (where $\gamma = 0$) based on exactly the same argument as above. So in this case time-inconsistency actually *helps* the SP work, quite contrary to the finding in our paper.

suggest that it is imperative that the techniques for conventional optimal stopping problems be used more carefully when extended to solving time-inconsistent stopping problems.

Appendix A. Proofs.

A.1. Proof of Theorem 2.5. For the stopping time $\tau^{\epsilon,a}$, if $a = 1$, then $J(x;\tau^{\epsilon,a}) = g(x)$. The Bellman equation (2.11) implies that $g(x) \geq V(x) \equiv J(x;\tau_{\hat{u}})$. This yields (2.9).

If $a = 0$, then

$$
\begin{aligned}
J(x;\tau^{\epsilon,a}) = \mathbb{E}&\left[\int_0^{\epsilon} h(s)f(X_s)ds\Big|X_0 = x\right] \\
&+\mathbb{E}\left[\int_{\epsilon}^{\tau^{\epsilon,a}} (h(s)-h(s-\epsilon))f(X_s)ds\Big|X_0 = x\right] \\
&+\mathbb{E}[(h(\tau^{\epsilon,a})-h(\tau^{\epsilon,a}-\epsilon))g(X_{\tau^{\epsilon,a}})|X_t = x] + \mathbb{E}[V(X_\epsilon)|X_0 = x]
\end{aligned}
$$

It follows from the weighted form of $h(t)$ that

$$
\begin{aligned}
J(x;\tau^{\epsilon,a}) = \mathbb{E}&\left[\int_0^{\epsilon} h(s)f(X_s)ds\Big|X_0 = x\right] \\
&+\mathbb{E}\left[\int_{\epsilon}^{\tau^{\epsilon,a}}\int_0^{\infty} e^{-r(s-\epsilon)}(e^{-\epsilon r}-1)dF(r)f(X_s)ds\Big|X_0 = x\right] \\
&+\mathbb{E}\left[\int_0^{\infty} e^{-r(\tau^{\epsilon,a}-\epsilon)}(e^{-\epsilon r}-1)dF(r)g(X_{\tau^{\epsilon,a}})\Big|X_0 = x\right] + \mathbb{E}[V(X_\epsilon)|X_0 = x]
\end{aligned}
$$

Combine the second and the third terms in the above representation, then we have that

$$
\begin{aligned}
J(x;\tau^{\epsilon,a}) = \mathbb{E}&\left[\int_0^{\epsilon} h(s)f(X_s)ds\Big|X_0 = x\right] + \mathbb{E}\left[\int_0^{\infty}(e^{-\epsilon r}-1)w(X_\epsilon;r)dF(r)\Big|X_0 = x\right] \\
&+\mathbb{E}[V(X_\epsilon)|X_0 = x]. \qquad \text{(A.1)}
\end{aligned}
$$

Define $\tau_n = \inf\{s \geq 0 : \sigma(X_s)V_x(X_s) > n\} \wedge \epsilon$. Then it follows from Ito's formula ([25]) that

$$
\mathbb{E}\left[V(X_{\tau_n})|X_0 = x\right] = \mathbb{E}\left[\int_0^{\tau_n}(\frac{1}{2}\sigma^2(X_s)V_{xx}(X_s) + b(X_s)V_x(X_s))ds\Big|X_0 = x\right] + V(x).
$$

By (2.11), we conclude

$$
\begin{aligned}
\mathbb{E}\left[V(X_{\tau_n})|X_0 = x\right] &= \mathbb{E}\left[\int_0^{\tau_n}(\frac{1}{2}\sigma^2(X_s)V_{xx}(X_s) + b(X_s)V_x(X_s))ds\Big|X_0 = x\right] + V(x) \\
&\geq \mathbb{E}\left[\int_0^{\tau_n}(-f(X_s) + \int_0^{\infty} rw(X_s;r)dF(r))ds\Big|X_0 = x\right] + V(x).
\end{aligned}
$$

Note that conditions (2.6) and (2.7) ensure that $-f(x) + \int_0^{\infty} rw(x;r)dF(r)$ has polynomial growth, i.e., there exist $C > 0, m \geq 1$ such that

$$
\left| -f(x) + \int_0^{\infty} rw(x;r)dF(r)\right| \leq C(|x|^m + 1),
$$

which leads to

$$\sup_{0\le t\le\epsilon}\Big|-f(X_s)+\int_0^\infty rw(X_s;r)dF(r)\Big|\le C(\sup_{0\le t\le\epsilon}|X_t|^m+1).$$

Moreover, under condition (2.4), it follows from standard SDE theory (see, for example, Chapter 1 of [35]) that equation (2.3) admits a unique strong solution X satisfying

$$\mathbb{E}[\sup_{0\le t\le\epsilon}|X_t|^m|X_0=x]\le K_\epsilon(|x|^m+1)$$

with $K_\epsilon>0$.

Then letting $n\to\infty$, we conclude by the dominated convergence theorem that

$$\mathbb{E}\left[V(X_\epsilon)|X_0=x\right]\ge\mathbb{E}\left[\int_0^\epsilon(-f(X_s)+\int_0^\infty rw(X_s;r)dF(r))ds\Big|X_0=x\right]+V(x).$$

Consequently,

$$\begin{aligned}
&\liminf_{\epsilon\to0+}\frac{J(x;\tau^{\epsilon,a})-J(x;\tau_{\hat u})}{\epsilon}\\
&\ge\liminf_{\epsilon\to0+}\mathbb{E}\left[\int_0^\epsilon h(s)f(X_s)ds\Big|X_0=x\right]+\mathbb{E}\left[\int_0^\infty\frac{e^{-\epsilon r}-1}{\epsilon}w(X_\epsilon;r)dF(r)\Big|X_0=x\right]\\
&+\liminf_{\epsilon\to0+}\frac{1}{\epsilon}\mathbb{E}\left[\int_0^\epsilon\int_0^\infty(rw(X_t;r)dF(r)-f(X_t))dt\Big|X_0=x\right].
\end{aligned}$$

The continuity of f and w along with the polynomial growth conditions (2.6) and (2.7) allow the use of the dominated convergence theorem, yielding

$$\begin{aligned}
&\liminf_{\epsilon\to0+}\mathbb{E}\left[\int_0^\epsilon h(s)f(X_s)ds\Big|X_0=x\right]+\mathbb{E}\left[\int_0^\infty\frac{e^{-\epsilon r}-1}{\epsilon}w(X_\epsilon;r)dF(r)\Big|X_0=x\right]\\
&+\liminf_{\epsilon\to0+}\frac{1}{\epsilon}\mathbb{E}\left[\int_0^\epsilon\int_0^\infty(rw(X_t;r)dF(r)-f(X_t))dt\Big|X_0=x\right]\\
&=f(x)-\int_0^\infty rw(x;r)dF(r)+\int_0^\infty(rw(x;r)-f(x))dF(r)\\
&=0.
\end{aligned}$$

This leads to

$$\liminf_{\epsilon\to0+}\frac{J(x;\tau^{\epsilon,a})-J(x;\tau_{\hat u})}{\epsilon}\ge0,$$

completing the proof.

A.2. Proof of Proposition 3.1. Let V^B be the value function of the optimal stopping problem. It follows from the standard argument (see, for example, Chapter 6 of [25]) that V^B is continuously differentiable and its first-order derivative is absolutely continuous. Moreover, V^B solves the following Bellman equation

$$\min\left\{\frac{1}{2}\sigma^2x^2V^B_{xx}(x)+bxV^B_x(x)+f(x)-rV^B(x),K-V^B(x)\right\}=0. \tag{A.2}$$

Define the continuation region $\mathcal{C}^B = \{x > 0 : V^B(x) < K\}$ and the stopping region $\mathcal{S}^B = \{x > 0 : V^B(x) = K\}$.

We claim that $\mathcal{S}^B \neq (0, \infty)$. If not, then $V^B \equiv K$. Thus $\frac{1}{2}\sigma^2 x^2 V^B_{xx}(x) + bxV^B_x(x) + f(x) - rV^B(x) < 0$ whenever $x \in \{x > 0 : f(x) - rK < 0\}$. However, since $f(0) < rK$, the continuity of f implies $\{x > 0 : f(x) - rK < 0\} \neq \emptyset$. This contradicts the Bellman equation (A.2).

We now show that $\mathcal{C}^b \neq (0, \infty)$. If it is false, then we have $V^B(x) = L(x; r)$, with L defined by (3.3). Since f is increasing and bounded from below by 0, we have

$$V^B(\infty) \equiv \lim_{x\to\infty} V^B(x) = \int_0^\infty \int_0^\infty \lim_{x\to\infty} f(yx)e^{-rs}G(y, s)dyds.$$

The concavity of f yields $f(x) \geq xf_x(x) + f(0)$. It then follows from $\lim_{x\to\infty} xf_x(x) = \infty$ that $\lim_{x\to\infty} f(x) = \infty$, which yields that $V^B(\infty) = \infty$. This contradicts the fact that $V^B(x) \leq K$.

Next, since X is a geometric Brownian motion and f is increasing, it is clear that V is increasing too. Now, we derive the value of the triggering boundary, x_B, via the SP principle. Specifically, it follows from (A.2) that

$$V^b(x) = (K - L(x_B; r))(\frac{x}{x_B})^{\alpha(r)} + L(x; r), \quad x < x_B$$

$$V^b(x) = K, x \geq x_B,$$

where $\alpha(r)$ is defined by (3.5). Then the SP implies that $V^B_x(x_B) = 0$ which after some calculations yields that x_B is the solution of the equation (3.4).

To prove the unique existence of the solution of (3.4), define $Q(x) := \alpha(r)(K - L(x; r)) + L_x(x; r)x$. Then $Q_x(x) = (-\alpha(r)+1)L_x(x; r) + L_{xx}(x; r)x$. As L is strictly increasing and concave and $\alpha(r) > 1$, we deduce that Q is strictly decreasing. It remains to show $Q(0) > 0$ and $Q(\infty) < 0$. It is easy to see that $Q(0) = \alpha(r)(K - L(0; r)) = \alpha(r)(K - \frac{f(0)}{r}) > 0$ and $Q(x) = \alpha(r)(K - L(0; r) - \int_0^x L_x(s; r)ds) + L_x(x; r)x$. Since L is concave, we have $\int_0^x L_x(s; r)ds \geq xL_x(x; r)$. Thus $Q(x) \leq \alpha(r)(K - L(0; r)) + (-\alpha(r) + 1)xL_x(x; r)$. Recalling that $\lim_{x\to\infty} xL_x(x; r) = \infty$ and $\alpha(r) > 1$, we have $Q(\infty) = -\infty$. This completes the proof.

A.3. Proof of Theorem 3.3. We need to present a series of lemmas before giving a proof of Theorem 3.3.

Lemma A.1. *Given a stopping rule u and a discount rate $r > 0$, the function $E(x; \tau_u, r) := \mathbb{E}[\int_0^{\tau_u} e^{-rt} f(X_t)dt + e^{-r\tau_u}K | X_0 = x]$ is continuous in $x \in (0, \infty)$.*

Proof. We prove the right continuity of $E(\cdot; \tau_u, r)$ at a given $x_0 > 0$; the left continuity can be discussed in the same way.

If there exists $\delta > 0$ such that $(x_0, x_0 + \delta) \in \mathcal{S}_u$, then the right continuity of $E(\cdot; \tau_u, r)$ at x_0 is obtained immediately. If there exists $\delta > 0$ such that $(x_0, x_0 + \delta) \in \mathcal{C}_u$, then it follows from the Feynman-Kac formula that $E(\cdot; \tau_u, r)$ is the solution to the differential equation $\frac{1}{2}\sigma^2 x^2 E_{xx} + bxE_x - rE + f = 0$ on $(x_0, x_0 + \delta)$. This in particular implies that $E(\cdot; \tau_u, r) \in C^2((x_0, x_0 + \delta)) \cap C([x_0, x_0 + \delta])$ due to the regularity of f and the coefficients of the differential equations; hence the right continuity of $E(\cdot; \tau_u, r)$ at x_0.

Otherwise, we first assume that $f(x_0) \geq rK$ and consider the set $\mathcal{C}_u \cap (x_0, \infty)$. Since it is an open set, we have $\mathcal{C}_u \cap (x_0, \infty) = \cup_{n\geq 1}(a_n, b_n)$, where $a_n, b_n \in \bar{\mathcal{S}}_u, \forall n \geq 1$. It is then easy to see that x_0 is an accumulation point of $\{a_n\}_{n\geq 1}$ and hence $x_0 \in \bar{\mathcal{S}}_u$.

Define $I(x) := E(x;\tau_u, r) - K$ for $x \in (a_n, b_n)$. It is easy to see that I solves the following differential equation

$$\frac{1}{2}\sigma^2 x^2 I_{xx}(x) + bxI_x(x) - rI(x) + f(x) - rK = 0. \tag{A.3}$$

with the boundary conditions

$$I(a_n) = I(b_n) = 0.$$

Consider an auxiliary function H that solves the following differential equation

$$\frac{1}{2}\sigma^2 x^2 H_{xx}(x) + bxH_x(x) - rH(x) + f(x) - rK = 0,$$

with the boundary conditions

$$H(x_0) = H(b_1) = 0.$$

Since $f(x) > rK$ on (x_0, ∞), the comparison principle shows that $H(x) \geq 0, \forall x \in [x_0, b_1]$. Applying the comparison principle again on any $(a_n, b_n) \cap (x_0, b_1), \forall n \in \mathbb{N}^+$, we have $0 \leq I(x) \leq H(x)$. Noting that $H(x) \to H(x_0) = 0$ as $x \to x_0+$, we conclude that $I(\cdot)$ is right continuous at x_0 and so is $E(\cdot;\tau_u, r)$.

For the case $f(x_0) < rK$, a similar argument applies. Indeed, consider an auxiliary function H_1 satisfying the differential equation (A.3) on $(x_0, f^{-1}(rK))$ with the boundary condition $H_1(x_0) = H_1(f^{-1}(rK)) = 0$. The comparison principle yields that $H_1(x) \leq I(x) \leq 0$ on $(a_n, b_n) \cap (x_0, f^{-1}(rK)), \forall n \in \mathbb{N}^+$. The right continuity of $I(\cdot)$ and $E(\cdot;\tau_u, r)$ then follows immediately. ∎

Lemma A.2. *If $\hat{u}$ is an equilibrium stopping rule, then $J(x;\tau_{\hat{u}}) \leq K$ $\forall x \in (0, \infty)$.*

Proof. If there exists $x_0 \in (0, \infty)$ such that $J(x_0;\tau_{\hat{u}}) > K$, then we have

$$\limsup_{\epsilon \to 0} \frac{J(x_0;\tau_{\hat{u}}) - J(x_0;\tau^{\epsilon,1})}{\epsilon} = \infty,$$

where $\tau^{\epsilon,1}$ is given by (2.10). This contradicts the definition of an equilibrium stopping rule. ∎

Lemma A.3. *If $\hat{u}$ is an equilibrium stopping rule, then we have $\{x > 0 : f(x) < \int_0^\infty r dF(r)K\} \subset \mathcal{C}_{\hat{u}}$.*

Proof. Suppose that there exists $x \in \{x > 0 : f(x) < \int_0^\infty r dF(r)K\} \cap \bar{\mathcal{S}}_{\hat{u}}$, then it follows from Lemma A.2 that $\mathbb{E}[J(X_t;\tau_{\hat{u}})|X_0 = x] \leq K$. Consider the stopping time $\tau^{\epsilon,0}$. Equation (A.1) and the fact that $J(x;\tau_{\hat{u}}) = K$ give

$$\begin{aligned}\frac{J(x;\tau^{\epsilon,0}) - J(x;\tau_{\hat{u}})}{\epsilon} \leq{}& \frac{1}{\epsilon}\mathbb{E}\left[\int_0^\epsilon h(s)f(X_s)ds \middle| X_0 = x\right] \\ &+ \mathbb{E}\left[\int_0^\infty \left(\frac{e^{-\epsilon r} - 1}{\epsilon}\right) w(X_\epsilon; r)dF(r) \middle| X_0 = x\right].\end{aligned}$$

As $w(\cdot, r)$ is continuous (Lemma A.1) and $w(x;r) = K$, we have

$$\liminf_{\epsilon \to 0} \frac{J(x;\tau^{\epsilon,0}) - J(x;\tau_{\hat{u}})}{\epsilon} \leq f(x) - \int_0^\infty rKdF(r) < 0.$$

This contradicts the definition of an equilibrium stopping rule. ∎

We now turn to the proof of Theorem 3.3. We begin with the sufficiency. To this end it suffices to show that $V(x) \le K, x \in (0, x_*)$ and $f(x) - \int_0^\infty r dF(r) K \ge 0, x \in (x_*, \infty)$.

We first show that $V_{xx} \le 0, x \in (0, x_*)$. By simple algebra, we have

$$V_{xx}(x) = \int_0^\infty \alpha(r)(\alpha(r)-1)(K - L(x;r))(\frac{x}{x_*})^{\alpha(r)} \frac{1}{x^2} dF(r) + \int_0^\infty L_{xx}(x;r) dF(r).$$

As L is concave, we only need to prove $\int_0^\infty \alpha(r)(\alpha(r)-1)(K-L(x;r))(\frac{x}{x_*})^{\alpha(r)} dF(r) \le 0$. It is easy to see that $(\frac{x}{x_*})^{\alpha(r)}$ is decreasing in r given that $\alpha(r)$ is increasing in r and $x < x_*$. Then the rearrangement inequality (e.g., Chapter 10 of [15]; [27]) yields[9]

$$\begin{aligned} & \int_0^\infty \alpha(r)(\alpha(r)-1)(K - L(x;r))(\frac{x}{x_*})^{\alpha(r)} dF(r) \\ \text{(A.4)} \qquad & \le \int_0^\infty \alpha(r)(\alpha(r)-1)(K - L(x;r)) dF(r) \int_0^\infty (\frac{x}{x_*})^{\alpha(r)} dF(r). \end{aligned}$$

Therefore it follows from (3.9) that $V_{xx}(x) \le 0,\ x \in (0, x_*)$. Now, $V_x(x_*) = 0$. Thus $V_x(x) \ge 0$ and consequently $V(x) \le K\ \forall x \in (0, x_*)$, due to $V(x_*) = K$.

Next, the inequality $f(x) - \int_0^\infty r dF(r) K\ \ \forall x \in (x_*, \infty)$ follows from f being increasing along with inequality (3.8). This completes the proof of the sufficiency.

We now turn to the necessity part. Since (3.8) is an immediate corollary of Lemma A.3, we only need to prove (3.9). Suppose (3.9) does not hold. Then by a simple calculation, we have

$$V_{xx}(x_*-) = \int_0^\infty \alpha(r)(\alpha(r)-1)(K - L(x_*;r)) \frac{1}{x_*^2} dF(r) + \int_0^\infty L_{xx}(x_*;r) dF(r) > 0.$$

However, $V_x(x_*) = 0$, implying that there exists $x_1 \in (0, x_*)$ such that $V_x(x) < 0$ on $x \in (x_1, x_*)$. Then it follows from $V(x_*) = K$ that $V(x) > K$ when $x \in (x_1, x_*)$, which contradicts Lemma A.2.

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[9]Inequality (A.4) can be read as

$$\mathrm{cov}(X, Y) \le 0,$$

with $X = \alpha(R)(\alpha(R)-1)(K - L(x;R))$ and $Y = (\frac{x}{x_*})^{\alpha(R)}$, where R is a random variable with distribution function F. Because of the monotonicity of X, Y in R, X and Y are anti-comonotonic. Then inequality (A.4) follows from the fact that the covariance of two anti-comonotonic random variables is non-positive.

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