Continuous-Time Portfolio Selection under Ambiguity*

Hanqing Jin† and Xun Yu Zhou‡

August 27, 2014

Abstract

In a financial market, the appreciation rates of stocks are statistically difficult to estimate, and typically only some confidence intervals in which the rates reside can be estimated. In this paper we study continuous-time portfolio selection under ambiguity, in the sense that the appreciation rates are only known to be in a certain convex closed set and the portfolios are allowed to be based on only the historical stocks prices. We formulate the problem in both the expected utility and the mean–variance frameworks, and derive robust portfolios explicitly for both models.

KEYWORDS: Portfolio selection, continuous time, ambiguity, expected utility, mean–variance, Sharpe ratio, robust portolio.

1 Introduction

Portfolio selection has been studied in great width and depth since Markowitz [16] proposed the seminal mean–variance model. Different types of portfolio selection models have been

*Jin owes thanks for aid to a startup grant when he started this work in the National University of Singapore, and Zhou acknowledges financial support from a start-up fund at the University of Oxford and a research fund from the Oxford-Man Institute of Quantitative Finance.

†Mathematical Institute and Nomura Centre for Mathematical Finance, and Oxford–Man Institute of Quantitative Finance, The University of Oxford, Andrew Wiles Building Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, England. Email: <jinh@maths.ox.ac.uk>.

‡Mathematical Institute and Nomura Centre for Mathematical Finance, and Oxford–Man Institute of Quantitative Finance, The University of Oxford, Andrew Wiles Building Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, England. Email: <zhouxy@maths.ox.ac.uk>.

formulated and studied in the last half century, including the expected utility maximization model and some generalization of the mean–variance model.

In all those models built for portfolio selection, several parameters of the market, such as the stocks appreciation rates and volatility rates, must be specified in order to implement the optimal portfolios derived from the models. Typically, one estimates the parameters values from the historical data of the asset prices using statistical techniques. But it is also well known that these estimations are prone to (sometimes significant) errors. In particular, estimating the appreciation rates of risk assets is notoriously difficult, which is known as the mean-blur problem (see, e.g., [11]).

On the other hand, optimal portfolios obtained from portfolio selection models are typically sensitive to the parameters. As a result, an "optimal portfolio" may perform badly if the underlying parameters are not sufficiently accurate. A model with large errors in parameters may therefore be completely meaningless from a practical viewpoint.

Research has been carried out in two directions to address the issue of possible poor performance of optimal solutions arising from "wrong" parameters. One is to improve the accuracy of the market parameters estimation by calibration techniques or some learning rules from new arriving data. For example, Chopra et al. [5] proposed a James–Stein estimator for the appreciation rates. Klein and Bawa [14] among others suggested to revise the estimation of the market parameters by Baysian updating. Yet, none of these improvements could rule out the estimation errors.

The other direction is to model the problem in a way to reduce the sensitivity of key market parameters. Lakner [15] formulated an expected utility maximization with the information exclusively generated by the the price process of investment, where the appreciation rates are not observable. Specifically, the appreciation rates are modeled by random processes, implying some distributional information, on which the optimal portfolios depend. This dependence is less sensitive than that involving the *values* of the appreciation rates.

Robust modeling is the most notable one in the second direction, introduced in the context of portfolio selection to resolve the issue with uncertain (or ambiguous) market parameters. A robust model is built to maximize the performance under the worst scenario, and the resulting optimal solution is expected to be the most insensitive to the ambiguity in parameters.

In a single-period financial market, Goldfarb and Iyangar [7] studied a mean–variance type portfolio selection with uncertainty in the probability distributions of risky returns. Therein the problem is formulated to maximize the minimal expected return and minimize the maximal variance, and the optimal solution is searched by some robust optimization techniques from Ben-Tal and Nemirovski [1] and Boyd et al. [2].

For the portfolio selection problem in continuous time financial markets, most existing works have considered the uncertainty on the subjective probability measure, which is modeled by a convex set of multiple priors. Gilboa and Schmeidler [6] put forward an axiomatic system for an atemporal preference represented by a multi-prior expected utility. Chen and Epstein [4] studied continuous-time recursive utility with multiple priors, and then applied it to the robust optimal consumption problem previously investigated by Hansen and Sargent [9] and Skiadas [17]. More relevant to the present paper is Gundel [8], in which maximization of expected utility on terminal wealth with the multiple priors was studied for complete/incomplete markets, and some abstract duality result was obtained.

In this paper, we study continuous-time portfolio selection under ambiguity, in the sense that the appreciation rates are only known to be in a certain convex closed set; namely, the model ambiguity lies in the values of the parameters rather than in the subjective priors. With this setup we aim to derive explicit, rather than abstract, results. An important consideration is that, since now the driving Brownian motion can no longer be inferred from the historical share prices due to the uncertainty in the appreciation rates, an admissible portfolio must be restricted to be adapted to the stocks price process. This restriction, in turn, poses significant technical difficulties.

We model our robust problem in both the expected utility and the mean–variance frameworks. While it is rather standard to formulate the former, it is not straightforward with the latter in a continuous-time market because the worst scenarios may not be the same for mean and variance. We get around by considering the Sharpe ratio of the terminal wealth as the single objective and discuss how it is related to the original mean–variance problem.

The two models formulated are both min-max problems. If the minimization and maximization can be interchanged, then these problems would reduce to classical ones without ambiguity. However, the interchangeability usually requires convexity/concavity in the underlying decision variables, which are absent in our problems. We employ rather *ad hoc* approach to solve both models analytically and explicitly.

The remainder of the paper is organized as follows. In Section 2 we specify the underlying continuous-time market. Sections 3 and 4 are respectively devoted to the formulations and solutions of robust expected utility maximization and mean–variance models. Concluding remarks are made in Section 5.

2 Market Setting

In this paper T is a fixed terminal time and $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed filtered complete probability space on which is defined a standard $\mathcal{F}_t$-adapted m-dimensional Brownian motion $W(t) \equiv (W^1(t), \cdots, W^m(t))'$ with $W(0) = 0$. It is assumed that $\mathcal{F}_t \equiv \mathcal{F}_t^W \triangleq \sigma\{W(s) : 0 \leq s \leq t\} \bigvee \mathcal{N}(\mathbb{P})$. Throughout the paper $A^\top$ denotes the transpose of a matrix A.

We define a continuous-time financial market following Karatzas and Shreve [13]. In the market there are $m+1$ assets being traded continuously. One of the assets is a bank account whose price process $S_0(t)$ is subject to the following equation:

$$dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T]; \quad S_0(0) = s_0 > 0, \tag{1}$$

where the interest rate $r(\cdot)$ is generally a scalar-valued, uniformly bounded $\mathcal{F}_t$-adapted measurable stochastic process. The other m assets are stocks, whose price processes $S_i(t)$, $i = 1, \cdots, m$, satisfy the following stochastic differential equation (SDE):

$$dS_i(t) = S_i(t)[\bar{b}_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)], \quad t \in [0, T]; \quad S_i(0) = s_i > 0, \tag{2}$$

where $\bar{b}_i(\cdot)$ and $\sigma_{ij}(\cdot)$, the appreciation and volatility rates respectively, are scalar-valued, $\mathcal{F}_t$-adapted measurable stochastic processes with

$$\int_0^T \left[\sum_{i=1}^{m} |\bar{b}_i(t)| + \sum_{i,j=1}^{m} |\sigma_{ij}(t)|^2 \right] dt < +\infty, \text{ a.s..}$$

Set the excess rate of return vector process

$$\bar{B}(t) \triangleq (\bar{b}_1(t) - r(t), \cdots, \bar{b}_m(t) - r(t))^\top,$$

and define the volatility matrix process $\sigma(t) \triangleq (\sigma_{ij}(t))_{m\times m}$.

In this paper we assume that both the interest rate $r(\cdot)$ and the volatility rate $\sigma(\cdot)$ are known and deterministic. A key feature, however, is that the true appreciation rate process, $\bar{b}_1(\cdot), \cdots, \bar{b}_m(\cdot))$, is random and *unobservable*. This is motivated by the notorious mean-blur problem in estimating the appreciate rate, i.e., it is impossible to estimate $\bar{b}_i(\cdot)$ to within a workable accuracy using pure statistical method based on historical data of the share prices. Hence we have ambiguity (or Knightian uncertainty) in the appreciation rates. Specifically, we model the ambiguity by assuming that at any time t, only a range of the appreciation rates can be estimated, in which the true appreciation rates reside. Notice that we only know the range of possible appreciation rates, and no other information is available on their distributions. This is the key difference between this paper and Lakner [15].

As a result of the above ambiguity, investors can only observe the price processes of all the assets traded in the market (and therefore the values of their portfolios), and they make decisions based only on this information. In particular, they are not able to infer the information about the underlying Bronian motion by the asset processes. Hence, an admissible portfolio is $\mathcal{F}_t^S$-adapted, where $\mathcal{F}_t^S = \sigma(S(s) : 0 \le s \le t)$. As noted, $\mathcal{F}_t^S \subset \mathcal{F}_t^W$ and the inclusion is possibly strict. Hereafter we say a random process is observable if it is $\mathcal{F}_t^S$-adapted.

Given $\mathcal{B}(\cdot)$, a deterministic, measurable and closed-convex-set-valued function on $[0, T]$, we make the following assumptions throughout this paper:

Assumption 2.1 (i) $r(\cdot)$ is deterministic and $\int_0^T |r(s)|ds < +\infty$.

(ii) $\sigma(\cdot)$ is deterministic, and $\exists \delta > 0$ such that $\sigma(t)\sigma'(t) > \delta I_m$ $\forall t \in [0, T]$.

(iii) $\bar{B}(\cdot)$ is measurable and $\mathcal{F}_t^W$-adapted, and $\bar{B}(t) \in \mathcal{B}(t)$ $\forall t \in [0, T]$, a.s..

(iv) $0 \notin \mathcal{B}(t)$ for a.e. $t \in [0, T]$.

We will use the following notations in this paper:

- The *true* risk premium process $\bar{\theta}(t) \stackrel{\triangle}{=} \sigma(t)^{-1}\bar{B}(t)$, which is random and not observable;
- The ambiguity set of all possible risk premium processes:

$$\Theta \stackrel{\triangle}{=} \{\sigma(\cdot)^{-1}B(\cdot) : B(\cdot) \text{ is measurable, } \mathcal{F}_t^W\text{-adapted, } B(t) \in \mathcal{B}(t) \ \ \forall t \in [0, T]\};$$

- The *minimal* possible risk premium $|\theta^*(t)|^2 = \min_{B\in\mathcal{B}(t)} |\sigma(t)^{-1}B|^2$; clearly $\theta^*(\cdot)$ is deterministic;

- For any $\theta \in \Theta$,

$$\rho_\theta(t) \triangleq e^{-\int_0^t [r(s)+\frac{|\theta(s)|^2}{2}]ds-\int_0^t \theta(s)^\top dW(s)}, \quad \rho^*_\theta(t) \triangleq e^{-\int_0^t [r(s)-\frac{|\theta^*(s)|^2}{2}+\theta(s)^\top\theta^*(s)]ds-\int_0^t \theta^*(s)^\top dW(s)}.$$

An investor's portfolio is an $\mathcal{F}_t^S$-adapted measurable process, $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_m(\cdot))^\top$, where $\pi_i(\cdot)$ is the amount of money invested in the i^{th} risky asset at time t. For technical reasons, we also require an admissible portfolio to satisfy $\int_0^T |\sigma(t)^\top \pi(t)|^2 dt < +\infty$ a.s.. We denote by $\Pi(S)$ the set of all admissible portfolios, i.e.,

$$\Pi(S) \triangleq \{\pi(\cdot) : \pi(\cdot) \text{ is an } \mathcal{F}_t^S\text{-adapted process valued in } \mathbb{R}^m, \int_0^T |\sigma(t)^\top\pi(t)|^2 dt < +\infty \text{a.s.}\}.$$

It is well known that, under the self-financing operation during the investment time period, the wealth process $x(\cdot;\pi)$ with an initial wealth x_0 and a portfolio $\pi \equiv \pi(\cdot)$ satisfies the *wealth equation*:

$$dx(t;\pi) = [r(t)x(t;\pi) + \pi(t)^\top\sigma(t)\bar\theta(t)]dt + \pi(t)^\top\sigma(t)dW(t), \quad x(0) = x_0.$$

Denote $W_{\bar\theta}(t) \triangleq W(t) + \int_0^t \bar\theta(s)ds$. Then we can rewrite the wealth equation as

$$dx(t;\pi) = r(t)x(t;\pi)dt + \pi(t)^\top\sigma(t)dW_{\bar\theta}(t), \quad x(0) = x_0. \tag{3}$$

We conclude this section by a useful lemma.

Lemma 2.1 *We have $\mathcal{F}_t^S = \mathcal{F}_t^{W_{\bar\theta}}$.*

Proof: The stock price equation (2) can be rewritten as

$$dS_i(t) = S_i(t)[r(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW^j_{\bar\theta}(t)], \quad t\in[0,T]; \quad S_i(0) = s_i > 0. \tag{4}$$

Since both $r(\cdot)$ and $\sigma(\cdot)$ are deterministic and $\sigma(t)$ is invertible per Assumption 2.1, the conclusion follows readily. *Q.E.D.*

3 Expected Utility Model under Ambiguity

3.1 Model Formulation

Consider an agent with an initial wealth $x_0 > 0$ and a utility function $u(\cdot)$ applied on the terminal wealth $x(T)$ at time T. We assume $u(\cdot)$ is a classical utility function, which is strictly concave, strictly increasing and satisfies $u'(0+) = +\infty$ and $u'(+\infty) = 0$. In the classical expected utility model, the objective of the agent is to find an optimal portfolio π^* such that the expected utility of the terminal wealth is maximized.

When there is ambiguity in the appreciation rates (or equivalently in the risk premium $\bar{\theta}$), it is common in the literature to formulate a min-max problem in which the agent is to maximize the expected utility in the worst case. Mathematically, we have the following *robust* expected utility maximization problem:

$$
\begin{array}{ll}
\text{Maximize} & \min_{\theta\in\Theta} \mathbb{E}[u(x(T;\pi)|_{\bar{\theta}=\theta}))] \\
\text{s.t.} & \left\{ \begin{array}{l} (x(\cdot;\pi),\pi) \text{ satisfies equation (3)}, \\ \pi \in \Pi(S). \end{array} \right.
\end{array}
\tag{5}
$$

This is a challenging problem caused by the uncertainty in appreciation rates. First of all, for any given admissible portfolio the distribution of the terminal wealth is unknown. In particular, there is no market completeness whatsoever under such a setting. So the classical martingale method can not be applied directly. Furthermore, we cannot fit the problem into those studied in Gundel [8] because the risk-neutral probability measure in our case is unknown. Second, Problem (5) is a max-min problem. If we could interchange the maximization and the minimization, then the classical expected utility maximization might be applied. However, the objective function here does not bear clear convexity/concavativity; so it is not clear how to apply the general min-max theorem in functional analysis. Finally, even if we could swap maximization and minimization, the constraint that the portfolio be $\mathcal{F}_t^S$-adapted, instead of being $\mathcal{F}_t^W$-adapted, would still pose a technical difficulty.

The main idea of overcoming these difficulties is to consider an *auxilary* problem in which the $\mathcal{F}_t^S$-adaptiveness constraint is relaxed to $\mathcal{F}_t^W$-adaptiveness:

$$\begin{array}{ll} \text{Maximize} & \min_{\theta\in\Theta}\mathbb{E}[u(x(T;\pi)|_{\bar{\theta}=\theta})] \\ \text{s.t.} & \begin{cases} (x(\cdot;\pi),\pi) \text{ satisfies equation (3) }, \\ \pi\in\Pi(W), \end{cases} \end{array} \tag{6}$$

where

$$\Pi(W) \triangleq \{\pi(\cdot) : \pi \text{ is an } \mathcal{F}_t^W\text{-adapted process valued in } \mathbb{R}^m, \int_0^T |\sigma(t)^\top\pi(t)|^2 dt < +\infty \text{a.s.}\}.$$

It is easy to see that the optimal objective of Problem (6) is no less than that of Problem (5). If, on the other hand, an optimal portfolio for Problem (6) *happens to* be $\mathcal{F}_t^S$-adapted, then it is also optimal for Problem (5). Finally, we will show that maximization and minimization in Problem (6) can indeed be swapped, and therefore the problem solved.

3.2 Optimal Robust Portfolios

In classical expected utility maximization, some additional assumptions on the utility function is necessary to ensure the well-posedness of the problem and the existence of optimal portfolios. Some general and recent results are provided in [12]. In this paper, we take the following assumptions following [12].

ASSUMPTION 3.1 *The utility function $u(\cdot)$ satisfies*

$$\mathbb{E}[u((u')^{-1}(\rho_{\theta^*}(T)))] < +\infty$$

and one of the following three condition:

(i) $\liminf_{x\to+\infty}(-\frac{xu''(x)}{u'(x)}) > 0$.

(ii) $\limsup_{x\to+\infty}\frac{u'(kx)}{u'(x)} < 1$ *for some* $k > 1$.

(iii) $\limsup_{x\to 0}\frac{(u')^{-1}(\lambda x)}{(u')^{-1}(x)} < +\infty$ *for some* $\lambda \in (0,1)$.

For any given and fixed possible version of the risk premium process $\theta \in \Theta$, define $J(\theta)$ as the optimal value of the problem:

$$\begin{array}{ll} \text{Maximize} & \mathbb{E}[u(x(T;\pi)|_{\bar{\theta}=\theta})] \\ \text{s.t.} & \begin{cases} (x(\cdot\,;\pi),\pi) \text{ satisfies equation (3) }, \\ \pi\in\Pi(W). \end{cases} \end{array} \tag{7}$$

This is a classical expected utility portfolio selection problem *assuming* that the true risk premium process $\bar{\theta} = \theta$. Denote by $I(x) = (u')^{-1}(x)$, the inverse function of $u'(\cdot)$. By Theorem 5.4 in [12], the optimal terminal wealth for Problem (7) is $I(\lambda_\theta \rho_\theta(T))$, where $\lambda_\theta > 0$ is the Lagrange multiplier satisfying the algebraic equation $\mathbb{E}[I(\lambda_\theta \rho_\theta(T))\rho_\theta(T)] = x_0$.

PROPOSITION 3.1 *For any $\theta \in \Theta$, $J(\theta^*) \le J(\theta)$.*

Proof: For any $\theta \in \Theta$, define a new probability measure by $\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} = e^{-\int_0^T |\theta(s)|^2/2ds - \int_0^T \theta(s)'dW(s)}$ with $\mathbb{E}_\theta[\cdot] \triangleq \mathbb{E}^{\mathbb{Q}_\theta}[\cdot]$ and $\lambda^* \triangleq \lambda_{\theta^*}$. Then $W_\theta(t) = W(t) + \int_0^t \theta(s)ds$ is a standard Brownian motion under $\mathbb{Q}_\theta$. Recall the definition of θ^*, we have, with the convexity of $\mathcal{B}(t)$, that

$$\theta^*(t)^\top \theta(t) \ge |\theta^*(t)|^2, \quad \forall t \in [0,T], \quad \text{a.s..}$$

Consequently,

$$\begin{aligned}
& \mathbb{E}[I(\lambda^* \rho_{\theta^*}(T))\rho_\theta(T)] \\
= \; & \mathbb{E}[e^{-\int_0^T r(s)ds} I(\lambda^* e^{-\int_0^T [r(s)+|\theta^*(s)|^2/2]ds - \int_0^T \theta^*(s)^\top dW(s)}) e^{-\int_0^T |\theta(s)|^2/2ds - \int_0^T \theta(s)^\top dW(s)}] \\
= \; & \mathbb{E}_\theta[e^{-\int_0^T r(s)ds} I(\lambda^* e^{\int_0^T \theta^*(s)^\top \theta(s)ds} e^{-\int_0^T [r(s)+|\theta^*(s)|^2/2]ds - \int_0^T \theta^*(s)^\top dW_\theta(s)})] \\
\le \; & \mathbb{E}_\theta[e^{-\int_0^T r(s)ds} I(\lambda^* e^{\int_0^T |\theta^*(s)|^2 ds} e^{-\int_0^T [r(s)+|\theta^*(s)|^2/2]ds - \int_0^T \theta^*(s)^\top dW_\theta(s)})] \\
= \; & E_{\theta^*}[e^{-\int_0^T r(s)ds} I(\lambda^* e^{\int_0^T |\theta^*(s)|^2 ds} e^{-\int_0^T [r(s)+|\theta^*(s)|^2/2]ds - \int_0^T \theta^*(s)' dW_{\theta^*}(s)})] \\
= \; & \mathbb{E}[I(\lambda^* \rho_{\theta^*}(T))\rho_{\theta^*}(T)] = x_0,
\end{aligned}$$

where the second last equality is because $\theta^*(\cdot)$ and $r(\cdot)$ are both deterministic. Thanks to the positivity of $I(\cdot)$, we can find $k > 1$ such that $\mathbb{E}[kI(\lambda^* \rho_{\theta^*}(T))\rho_\theta(T)] = x_0$, which makes $kI(\lambda^* \rho_{\theta^*}(T))$ a feasible solution for Problem (7) with parameter θ. Therefore,

$$J(\theta) \ge \mathbb{E}[u(kI(\lambda^* \rho_{\theta^*}(T))] \ge \mathbb{E}[u(I(\lambda^* \rho_{\theta^*}(T)))] = J(\theta^*).$$

Q.E.D.

Next, we are going to construct an $\mathcal{F}_t^S$-adapted portfolio π^* that achieves the maximal expected utility in (7) when $\bar{\theta} = \theta^*$, i.e., $x(T; \pi^*)|_{\bar{\theta}=\theta^*} = I(\lambda^* \rho_{\theta^*}(T))$. To this end, define f as the solution to the following PDE in the range $(t,y) \in [0,T] \times (0,+\infty)$:

$$\begin{cases} f_t(t,y) + \frac{1}{2}|\theta^*(t)|^2 y^2 f_{yy}(t,y) + \left(|\theta^*(t)|^2 - r(t)\right) y f_y(t,y) = r(t) f(t,y), \\ f(T,y) = I(\lambda^* y). \end{cases} \tag{8}$$

The above equation is indeed the Black–Scholes PDE in replicating the contingent claim $I(\lambda^* \rho_{\theta^*}(T))$ in the market with $\bar{\theta} \equiv \theta^*$; hence it admits a unique solution, which is strictly decreasing in y. Furthermore, the replicating wealth process is $f(t, \rho_{\theta^*}(t))$; in particular, $f(0,1) = x_0$.

Define

$$x^*(t) \triangleq f(t, \rho^*_{\bar{\theta}}(t)), \text{ and } \pi^*(t) \triangleq -\rho^*_{\bar{\theta}}(t) f_y(t, \rho^*_{\bar{\theta}}(t))(\sigma(t)^\top)^{-1}\theta^*(t). \tag{9}$$

According to its definition, $\rho^*_{\bar{\theta}}(\cdot)$ is $\mathcal{F}_t^{W_{\bar{\theta}}} \equiv \mathcal{F}_t^S$-adapted, so is $\pi^*(\cdot)$. Furthermore, it follows from Itô's formula that

$$dx^*(t) = r(t)x^*(t)dt + \pi^*(t)^\top \sigma(t) dW_{\bar{\theta}}(t), \quad x^*(T) = I(\lambda^* \rho^*_{\bar{\theta}}(T)),$$

which means $x^*(\cdot)$ is the wealth process associated with the admissible portfolio $\pi^*(\cdot)$, i.e. $x^*(t) = x(t;\pi^*)$. Furthermore, π^* replicates $I(\lambda^* \rho^*_{\bar{\theta}}(T))$ with initial wealth $x^*(0) = f(0,1) = x_0$. Hence $(\pi^*(\cdot), x^*(\cdot))$ is an admissible portfolio-wealth pair with initial wealth x_0 and terminal wealth $I(\lambda^* \rho^*_{\bar{\theta}}(T))$.

Moreover, noting $\rho^*_{\theta^*} \equiv \rho_{\theta^*}$, we have $x(T;\pi^*)|_{\bar{\theta}=\theta^*} = I(\lambda^* \rho_{\theta^*}(T)) = I(\lambda^* \rho^*_{\theta^*}(T))$.

PROPOSITION 3.2 *We have $x(T;\pi^*) \geq x(T;\pi^*)|_{\bar{\theta}=\theta^*}$, and the equality holds if and only if $\bar{\theta} = \theta^*$.*

Proof: Since $x(T;\pi^*) = I(\lambda^* \rho^*_{\bar{\theta}}(T))$, $x(T;\pi^*)|_{\bar{\theta}=\theta^*} = I(\lambda^* \rho^*_{\theta^*}(T))$, and $I(\cdot)$ is strictly decreasing, we only need to show that $\rho^*_{\bar{\theta}}(T) \leq \rho^*_{\theta^*}(T)$. Indeed,

$$\begin{aligned}
\rho^*_{\bar{\theta}}(T) &= e^{-\int_0^T (r(s) - |\theta^*(s)|^2/2)ds - \int_0^T \theta^*(s)^\top dW_{\bar{\theta}}(s)} \\
&= e^{-\int_0^T r(s)ds} e^{\int_0^T (|\theta^*(s)|^2/2 - \theta^*(s)^\top \bar{\theta}(s))ds - \int_0^T \theta^*(s)^\top dW(s)} \\
&\leq e^{-\int_0^T (r(s) - |\theta^*(s)|^2/2)ds - \int_0^T \theta^*(s)^\top dW_{\theta^*}(s)} = \rho^*_{\theta^*}(T),
\end{aligned}$$

and the equality holds if and only if $\bar{\theta} = \theta^*$. *Q.E.D.*

The strategy π^* is in fact the optimal strategy of the classical expected utility problem (7) *as if* the underlying true risk premium is $\bar{\theta} = \theta^*$. When one applies this strategy while the *actual* risk premium is not θ^*, there is a minimum guaranteed terminal wealth, which is $x(T;\pi^*)|_{\bar{\theta}=\theta^*}$, as stipulated by the above proposition. In other words, $x(T;\pi^*)|_{\bar{\theta}=\theta^*} =$

$I(\lambda^* \rho^*_{\theta^*}(T))$ is the most pessimistic outcome however the true risk premium might be, under the strategy π^*.

Here comes the main result of this section.

Theorem 3.1 *Let π^* be the admissible portfolio with wealth process x^* as defined by (9). Then*

(i) (θ^*, π^*) *is a saddle point of Problem (5), namely,*

$$\max_{\pi\in\Pi(S)} \min_{\theta\in\Theta} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] = \min_{\theta\in\Theta} \max_{\pi\in\Pi(S)} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] = \mathbb{E}[u(x(T;\pi^*)|_{\bar\theta=\theta^*})]. \tag{10}$$

(ii) π^* *is an optimal portfolio for Problem (5).*

Proof: We first show that

$$\max_{\pi\in\Pi(W)} \min_{\theta\in\Theta} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] = \min_{\theta\in\Theta} \max_{\pi\in\Pi(W)} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] = \mathbb{E}[u(x(T;\pi^*)|_{\bar\theta=\theta^*})]. \tag{11}$$

Indeed,

$$\begin{aligned} & \min\nolimits_{\theta\in\Theta} \max\nolimits_{\pi\in\Pi(W)} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] \\ = \; & \min\nolimits_{\theta\in\Theta} J(\theta) = J(\theta^*) = E[u(x(T;\pi^*)|_{\bar\theta=\theta^*})] \\ = \; & \min\nolimits_{\theta\in\Theta} E[u(x(T;\pi^*)|_{\bar\theta=\theta})] \le \max\nolimits_{\pi\in\Pi(W)} \; \min\nolimits_{\theta\in\Theta} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})]. \end{aligned}$$

The other inequality is trivial. Hence (11) is valid. Since π^* is $\mathcal{F}^S_t$-adapted, the above yields that π^* is an optimal portfolio for Problem (5). Moreover,

$$\begin{aligned} & \max\nolimits_{\pi\in\Pi(S)} \; \min\nolimits_{\theta\in\Theta} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] \\ = \; & \max\nolimits_{\pi\in\Pi(W)} \; \min\nolimits_{\theta\in\Theta} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] \\ = \; & \min\nolimits_{\theta\in\Theta} \max\nolimits_{\pi\in\Pi(W)} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] \\ \ge \; & \min\nolimits_{\theta\in\Theta} \max\nolimits_{\pi\in\Pi(S)} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})] \\ \ge \; & \max\nolimits_{\pi\in\Pi(S)} \; \min\nolimits_{\theta\in\Theta} \mathbb{E}[u(x(T;\pi)|_{\bar\theta=\theta})]. \end{aligned}$$

This establishes (10). *Q.E.D.*

REMARK 3.1 So the optimal robust portfolio is obtained by solving the optimal portfolio of a classical utility maximization problem as if the market price of risk $\bar\theta$ is the minimal one, θ^*, within the ambiguity set. A key step in establishing this result is to show such a portfolio is $\mathcal{F}^S_t$-adapted, hence implementable even in the presence of ambiguity.

Remark 3.2 Gundel [8] studied the following model

$$\begin{array}{ll} \text{Maximize} & \min_{\mathbb{Q}\in\mathcal{Q}} \hat{\mathbb{E}}[u(X)] \\ \text{s.t.} & \hat{\mathbb{E}}[X] = x_0, \end{array} \tag{12}$$

where $\hat{\mathbb{E}}$ is the expectation under the risk neutral probability measure $\hat{\mathbb{P}}$ and $\mathcal{Q}$ is the set of probability measure $\mathbb{Q}$ representing the model ambiguity. Gundel [8] showed that $X^* = I(\lambda\frac{d\mathbb{Q}^*}{d\hat{\mathbb{P}}})$, where $\mathbb{Q}^*$ is the solution to the optimization problem $\min_{\mathbb{Q}\in\mathcal{Q}} \hat{\mathbb{E}}[v(\lambda\frac{d\mathbb{Q}}{d\hat{\mathbb{P}}})]$, with v being the convex conjugate of u. How to replicate X^* and in what class of strategies were not a concern in [8]. Moreover, it is unclear to us how to “embed” our model into the abstract formulation of [8], without assuming that the true risk premium process is $\mathcal{F}_t^S$-adapted.

4 Mean–Variance Model under Ambiguity

4.1 Model Formulation

A classical mean–variance model with certain market parameters is formulated as

$$\begin{array}{ll} \text{Minimize} & \text{Var}(x(T;\pi)) \\ \text{s.t.} & \begin{cases} \mathbb{E}[x(T;\pi)] = z, \\ (x(\cdot\,;\pi),\pi) \text{ satisfies equation (3)}, \\ \pi \in \Pi(W), \end{cases} \end{array} \tag{13}$$

where $\bar{\theta}$ is known and $z \geq x_0 e^{\int_0^T r(s)ds}$ is given.

When the market parameters are uncertain, so are both $\text{Var}(x(T;\pi))$ and $\mathbb{E}[x(T;\pi)]$. At first glance, a natural choice of the model in such a case is to consider the worst scenario for the two objectives (i.e., mean and variance) separatively:

$$\begin{array}{ll} \text{Minimize} & \max_{\theta\in\Theta} \text{Var}(x(T;\pi)|_{\bar{\theta}=\theta}) \\ \text{s.t.} & \begin{cases} \min_{\theta\in\Theta} \mathbb{E}[x(T;\pi)|_{\bar{\theta}=\theta}] = z, \\ (x(\cdot\,;\pi),\pi) \text{ satisfies equation (3)}, \\ \pi \in \Pi(S). \end{cases} \end{array} \tag{14}$$

A major drawback of the above model, however, is that the worst scenarios, i.e., θ’s, for the two objectives are generally different. In the real word, only one scenario is realized; hence it will be more reasonable to aggregate the two objectives in certain way and consider only one "worst" scenario.

It is known that the efficient frontier for the classical continuous-time mean–variance problem (13) is a straight line in the mean–standard deviation plane, and the efficient portfolios are those with the maximal Sharpe ratio (see, e.g., [18, 3]). In other words, the classical problem (13) is equivalent to that of maximizing the Sharpe ratio.

With this observation, we now formulate the mean–variance model with uncertain market parameters as follows:

$$\begin{array}{ll} \text{Maximize} & \min_{\theta\in\Theta} \frac{\mathbb{E}[x(T;\pi)|_{\bar\theta=\theta}]-x_0 e^{\int_0^T r(s)ds}}{\mathrm{Std}(x(T;\pi)|_{\bar\theta=\theta})} \\ \text{s.t.} & \left\{\begin{array}{l} (x(\cdot\,;\pi),\pi) \text{ satisfies equation (3) }, \\ \pi \in \Pi(S), \\ \min_{\theta\in\Theta} \mathbb{E}[x(T;\pi)|_{\bar\theta=\theta}] > x_0 e^{\int_0^T r(s)ds}, \end{array}\right. \end{array} \tag{15}$$

where (and henceforth) $\mathrm{Std}(X)$ stands for the standard deviation of a random variable X. Note in the above the last constraint is just to ensure that the portfolio performs better than the risk-free account even in the worst state of nature, which is consistent with the classical model.

The following proposition clarifies the connection between Problem (15) and the mean-variance efficiency in terms of the Sharpe ratio.

Proposition 4.1 *Suppose π^* is an optimal portfolio for Problem (15). Then for any $z > x_0 e^{\int_0^T r(s)ds}$, there is a constant $k > 0$ such that the portfolio $k\pi^*$ satisfies the constraint $\min_{\theta\in\Theta} \mathbb{E}[x(T;k\pi^*)|_{\bar\theta=\theta}] = z$, while maximizing the worst Sharpe ratio, i.e., for any feasible portfolio $\hat\pi$ for problem (15),*

$$\min_{\theta\in\Theta} \frac{\mathbb{E}[x(T;k\pi^*)|_{\bar\theta=\theta}]-x_0 e^{\int_0^T r(s)ds}}{\mathrm{Std}(x(T;k\pi^*)|_{\bar\theta=\theta})} \ge \min_{\theta\in\Theta} \frac{\mathbb{E}[x(T;\hat\pi)|_{\bar\theta=\theta}]-x_0 e^{\int_0^T r(s)ds}}{\mathrm{Std}(x(T;\hat\pi)|_{\bar\theta=\theta})}.$$

Proof: Denote by $V(\pi)$ the Sharpe ratio of an admissible portfolio π, and $\hat x(t;\pi) = x(t;\pi)e^{-\int_0^t r(s)ds}$, $\hat\pi(t) = \pi(t)e^{-\int_0^t r(s)ds}$. Then

$$d\hat x(t;\pi) = \hat\pi(t)\bar B(t)dt + \hat\pi(t)\sigma(t)dW(t),\ \hat x(0;\pi) = x_0,$$

and hence

$$V(\pi) = \frac{\mathbb{E}[x(T;\pi)]-x_0 e^{\int_0^T r(s)ds}}{\mathrm{Std}(x(T;\pi))} = \frac{\mathbb{E}[\hat x(T;\pi)]-x_0}{\mathrm{Std}(\hat x(T;\pi))}$$

$$= \frac{\mathbb{E}[\int_0^T e^{-\int_0^t r(s)ds}\pi(t)^\top \sigma(t)\bar{\theta}(t)dt]}{\mathrm{Std}\left(\int_0^T e^{-\int_0^t r(s)ds}\pi(t)^\top \sigma(t)\bar{\theta}(t)dt + \int_0^T e^{-\int_0^t r(s)ds}\pi(t)^\top \sigma(t)dW(t)\right)}.$$

As a result, $V(k\pi) = V(\pi)$ for any $k > 0$.

Now, π^* is an optimal portfolio for Problem (15). Denote

$$z_0 = \min_{\theta\in\Theta} \mathbb{E}[x(T;\pi^*)|_{\bar{\theta}=\theta}] > x_0 e^{\int_0^T r(s)ds}; \quad v = \min_{\theta\in\Theta} V(\pi^*)|_{\bar{\theta}=\theta}.$$

Then

$$\begin{aligned}\min_{\theta\in\Theta} \mathbb{E}[x(T;k\pi^*)|_{\bar{\theta}=\theta}] &= \min_{\theta\in\Theta} \mathbb{E}[x(T;k\pi^*)|_{\bar{\theta}=\theta} - x_0 e^{\int_0^T r(s)ds}] + x_0 e^{\int_0^T r(s)ds} \\ &= k \min_{\theta\in\Theta} \mathbb{E}[x(T;\pi^*)|_{\bar{\theta}=\theta} - x_0 e^{\int_0^T r(s)ds}] + x_0 e^{\int_0^T r(s)ds}.\end{aligned}$$

Taking $k = \frac{z - x_0 e^{\int_0^T r(s)ds}}{z_0 - x_0 e^{\int_0^T r(s)ds}} > 0$, we get $\min_{\theta\in\Theta} \mathbb{E}[x(T;\pi^*)|_{\bar{\theta}=\theta}] = z$. Finally, by the fact that $V(k\pi^*) = V(\pi^*)$, we have $\min_{\theta\in\Theta} V(k\pi^*)|_{\bar{\theta}=\theta} \equiv \min_{\theta\in\Theta} V(\pi^*)|_{\bar{\theta}=\theta} = v$, which is the maximized worst Sharpe ratio. *Q.E.D.*

4.2 Efficient Robust Portfolios

Although economically we have shown that Problem (15) is intimately related to the classical mean–variance problem, mathematically the former is much more difficult to solve than the latter. Due to the absence of convexity and monotonicity of the Sharpe ratio in θ and π, we are unable to apply most of the existing techniques in portfolio selection theory.

To start, we henceforth suppose $x_0 = 1, r(\cdot) \equiv 0, \sigma(\cdot) = I$ (and hence $\bar{B}(\cdot) \equiv \bar{\theta}(\cdot)$) without loss of generality, as otherwise we could consider $\hat{x}(t;\pi) = x(t;\pi)/(x_0 e^{\int_0^T r(t)dt}), \hat{\pi}(t) = \sigma(t)^\top \pi(t)/(x_0 e^{\int_0^T r(t)dt})$ instead. With this simplification Problem (15) now becomes

$$\begin{array}{ll}\text{Maximize} & \min_{\theta\in\Theta} \frac{\mathbb{E}[x(T;\pi)|_{\bar{\theta}=\theta}]-1}{\mathrm{Std}(x(T;\pi)|_{\bar{\theta}=\theta})} \\ \text{s.t.} & \begin{cases} dx(t;\pi) = \pi(t)^\top \bar{\theta}(t)dt + \pi(t)^\top dW(t), \quad x(0;\pi) = 1, \\ \pi \in \Pi(S), \\ \min_{\theta\in\Theta} \mathbb{E}[x(T;\pi)|_{\bar{\theta}=\theta}] > 1. \end{cases}\end{array} \tag{16}$$

Define $y(t;\pi) = x(t;\pi) - 1$. Then the problem is further equivalent to

$$
\begin{array}{ll}
\text{Minimize} & \max_{\theta\in\Theta} \; f(\theta,\pi) \stackrel{\triangle}{=} \frac{\mathrm{Var}(y(T;\pi)|_{\bar\theta=\theta})}{(\mathbb{E}[y(T;\pi)|_{\bar\theta=\theta}])^2} \\
\text{s.t.} & \begin{cases} dy(t;\pi) = \pi(t)^\top \bar\theta(t)dt + \pi(t)^\top dW(t), \quad y(0;\pi) = 0, \\ \pi \in \Pi(S), \\ \min_{\theta\in\Theta} \mathbb{E}[y(T;\pi)|_{\bar\theta=\theta}] > 0. \end{cases}
\end{array}
\tag{17}
$$

In the rest of this section, we focus on studying Problem (17).

Lemma 4.1 *It holds that $f(\theta,\pi) \geq \frac{1}{\mathbb{E}[\rho_\theta(T)^2]-1}$ for any $\theta \in \Theta$ and any π satisfying the constraints in (17). Moreover, if θ is deterministic, then there exists a portfolio $\pi \in \Pi(S)$ such that the equality holds.*

Proof: For any given $\theta \in \Theta$ and π satisfying the constraints in (17), let $z = \mathbb{E}[y(T;\pi)|_{\bar\theta=\theta}] > 0$. Consider the following classical mean–variance problem:

$$
\begin{array}{ll}
\text{Minimize} & \mathbb{E}[x(T)^2] \\
\text{s.t.} & \begin{cases} dx(t) = \pi(t)^\top \theta(t)dt + \pi(t)^\top dW(t), \quad x(0) = 0, \\ \pi \in \Pi(W), \\ \mathbb{E}[x(T)] = z. \end{cases}
\end{array}
\tag{18}
$$

This problem has been studied extensively in the literature (see, e.g., [18, 3]), with the optimal portfolio in feedback form $\pi^*(t) = (\lambda - x^*(t))\theta(t)$ and the optimal terminal wealth being $x^*(T) = \lambda - \mu\rho_\theta(T)$, where (λ,μ) is determined by

$$\lambda - \mu = z, \;\; \lambda - \mu\mathbb{E}[\rho_\theta(T)^2] = 0.$$

It follows that $\mu = \frac{z}{\mathbb{E}[\rho_\theta(T)^2]-1}, \lambda = \mu\mathbb{E}[\rho_\theta(T)^2]$, and the corresponding optimal mean and variance are respectively

$$\mathbb{E}[x^*(T)] = z, \;\; \mathrm{Var}(x^*(T)) = \mu^2(\mathbb{E}[\rho_\theta(T)^2] - 1) = \frac{z^2}{\mathbb{E}[\rho_\theta(T)^2] - 1}.$$

Hence $\frac{\mathrm{Var}(x^*(T))}{(\mathbb{E}[x^*(T)])^2} = \frac{1}{\mathbb{E}[\rho_\theta(T)^2]-1}$ which by definition is less or equal than $f(\theta,\pi)$.

Now, we apply the same feedback *law* as the above π^* to the *original* market (i.e., the one where the risk premium is $\bar\theta$) as follows:

$$\pi_\theta(t) = (\lambda - y_\theta(t))\theta(t).$$

Then the corresponding wealth process y_θ satisfies

$$dy_\theta(t) = (\lambda - y_\theta(t))\theta(t)^\top dW_{\bar{\theta}(t)}.$$

If θ is deterministic, then y_θ is $\mathcal{F}_t^{W_{\bar{\theta}}} \equiv \mathcal{F}_t^S$-adapted, and so is π_θ. On the other hand, recall that x^* satisfies

$$dx^*(t) = (\lambda - x^*(t))\theta(t)^\top dW_{\theta(t)}.$$

Thus $y_\theta(T)$ and $x^*(T)$ have the same probability distribution due to the uniqueness in law of the above SDE. However, $x^*(T)$ achieves the above lower bound, and hence so does $y_\theta(T)$. *Q.E.D.*

Corollary 4.1 *The optimal value of problem (17) is no less than* $\frac{1}{e^{\int_0^T |\theta^*(s)|^2 ds} - 1}$.

Proof: We again omit $\pi \in \Pi(S)$ and $\theta \in \Theta$ in this proof. We have

$$\begin{aligned}
\min_\pi \max_\theta f(\theta, \pi) &\geq \max_\theta \min_\pi f(\theta, \pi) \\
&\geq \max_\theta \frac{1}{\mathbb{E}[\rho_\theta(T)^2] - 1} = \frac{1}{\min_\theta \mathbb{E}[\rho_\theta(T)^2] - 1} \\
&= \frac{1}{\min_\theta \mathbb{E}[e^{\int_0^T |\theta(s)|^2 ds} e^{-\int_0^T 2|\theta(s)|^2 ds - \int_0^T 2\theta(s)^\top dW(s)}] - 1} \\
&= \frac{1}{e^{\int_0^T |\theta^*(s)|^2 ds} - 1}.
\end{aligned}$$

This completes the proof. *Q.E.D.*

Before presenting the main result, we need to do some preparations.

Lemma 4.2 *Let $Y > 0$ be a given random variable without atom in its probability distribution. Then for any random variable X valued in $(0, 1]$, there exists $k \in [0, +\infty]$ such that $\mathbb{E}X = \mathbb{P}(Y > k)$ and*

$$\mathbb{E}[X^2 Y] \leq \mathbb{E}[Y \mathbf{1}_{Y>k}]$$

while the equality holds if and only if $X = \mathbf{1}_{Y>k}$ a.s..

Proof: Denote $z = \mathbb{E}X \in (0, 1]$. Since Y is atomless, there is $k \in [0, +\infty]$ such that $z = \mathbb{P}(Y > k)$. Furthermore

$$\begin{aligned}
\mathbb{E}[X^2 Y] &\leq \mathbb{E}[XY] = \mathbb{E}[X(Y-k)] + kz = \mathbb{E}[X(Y-k)\mathbf{1}_{Y>k}] + \mathbb{E}[X(Y-k)\mathbf{1}_{Y\leq k}] + kz \\
&\leq \mathbb{E}[(Y-k)\mathbf{1}_{Y>k}] + k\mathbb{E}\mathbf{1}_{Y>k} = \mathbb{E}[Y\mathbf{1}_{Y>k}],
\end{aligned}$$

and the equality holds if and only if $X = \mathbf{1}_{Y>k}$ *a.s.*. *Q.E.D.*

Lemma 4.3 *For any $\theta \in \Theta$,*

$$\frac{\mathrm{Var}(e^{-\int_0^T |\theta^*(s)|^2 ds}\rho_\theta^*(T))}{(1-\mathbb{E}[e^{-\int_0^T |\theta^*(s)|^2 ds}\rho_\theta^*(T)])^2} \leq \frac{1}{e^{\int_0^T |\theta^*(t)|^2 dt} - 1},$$

and the equality holds if and only if $\theta = \theta^*$.

Proof: Fix $\theta \in \Theta$. We denote $c \triangleq e^{\int_0^T |\theta^*(t)|^2 dt} > 1$, and

$$X \triangleq e^{-\int_0^T \theta^*(t)^\top(\theta(t)-\theta^*(t))dt}, \; Y \triangleq e^{-\int_0^T |\theta^*(t)|^2/2dt - \int_0^T \theta^*(t)^\top dW(t)}.$$

Then $X \in (0,1]$ and $XY = \rho_\theta^*(T)$. Define a new probability measure $\hat{\mathbb{P}}$ by $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = Y$ and $\hat{\mathbb{E}} \triangleq \mathbb{E}_{\hat{\mathbb{P}}}$. Then Y is a $\hat{\mathbb{P}}$-lognormal random variable with $\hat{\mathbb{E}}[Y] = c$, and

$$\mathbb{E}[c^{-1}\rho_\theta^*(T)] = c^{-1}\hat{\mathbb{E}}[X], \;\; \mathbb{E}[c^{-2}\rho_\theta^*(T)^2] = c^{-2}\hat{\mathbb{E}}[X^2 Y].$$

Hence we have

$$\frac{\mathrm{Var}(c^{-1}\rho_\theta^*(T))}{(1-\mathbb{E}[c^{-1}\rho_\theta^*(T)])^2} = \frac{c^{-2}\left(\hat{\mathbb{E}}[X^2Y] - (\hat{\mathbb{E}}[X])^2\right)}{(1-c^{-1}\hat{\mathbb{E}}[X])^2} = \frac{\hat{\mathbb{E}}[X^2Y] - (\hat{\mathbb{E}}[X])^2}{(c-\hat{\mathbb{E}}[X])^2}.$$

By Lemma 4.2, there exists $k_\theta \in [0,+\infty]$ such that $\hat{\mathbb{E}}[X] = \hat{\mathbb{P}}(Y > k_\theta)$ and $\hat{\mathbb{E}}[X^2Y] \leq \hat{\mathbb{E}}[Y\mathbf{1}_{Y>k_\theta}]$. Consequently,

$$\frac{\hat{\mathbb{E}}[X^2Y] - (\hat{\mathbb{E}}[X])^2}{(c-\hat{\mathbb{E}}[X])^2} \leq \frac{\hat{\mathbb{E}}[Y\mathbf{1}_{Y>k_\theta}] - (\hat{\mathbb{P}}(Y>k_\theta))^2}{(c-\hat{\mathbb{P}}(Y>k_\theta))^2}.$$

For any $k \in [0,+\infty]$, define $g(k) \triangleq (c-1)\left[\hat{\mathbb{E}}[Y\mathbf{1}_{Y>k}] - (\hat{\mathbb{P}}(Y>k))^2\right] - (c-\hat{\mathbb{P}}(Y>k))^2$. Then

$$\begin{aligned} g(k) &= (c-1)\hat{\mathbb{E}}[Y\mathbf{1}_{Y>k}] - c(\hat{\mathbb{P}}(Y>k))^2 + 2c\hat{\mathbb{P}}(Y>k) - c^2 \\ &= (c-1)\hat{\mathbb{E}}[Y\mathbf{1}_{Y>k}] - c(1-\hat{\mathbb{P}}(Y>k))^2 + c - c^2. \end{aligned}$$

It is now seen that $g(\cdot)$ is strictly decreasing. Moreover, it is easy to verify that $g(0) = 0$. Hence $g(k) \leq 0$ for $\forall k \geq 0$, and the equality holds if and only if $k = 0$ (or equivalently $X = \mathbf{1}_{Y>0} \equiv 1$ or $\theta = \theta^*$). *Q.E.D.*

Now we are ready to present the main result of this section.

Theorem 4.1 *For any $\gamma > 0$, define a feedback portfolio $\pi^*(t) = (\gamma - y(t;\pi^*))\theta^*(t)$. Then*

(i) (θ^*, π^*) *is a saddle point of problem (17), namely,*

$$\min_{\pi}\max_{\theta} f(\theta,\pi) = \max_{\theta}\min_{\pi} f(\theta,\pi) = f(\theta^*,\pi^*). \tag{19}$$

(ii) π^* *is an optimal portfolio for Problem (17).*

Proof: With a slight abuse of notation we use π^* to denote both the feedback strategy $\pi^*(t) = (\gamma - y(t;\pi^*))\theta^*(t)$ and the induced open-loop portfolio under $y(0) = 0$. Under π^* the corresponding wealth process satisfies

$$dy(t;\pi^*) = (\gamma - y(t;\pi^*))\theta^*(t)^\top[\bar{\theta}(t)dt + dW(t)], \quad y(0;\pi^*) = 0,$$

leading to

$$y(t;\pi^*) = \gamma\left(1 - e^{-\int_0^t [|\theta^*(s)|^2/2 + \bar{\theta}(s)^\top\theta^*(s)]ds - \int_0^t \theta^*(s)^\top dW(s)}\right) = \gamma\left(1 - e^{-\int_0^t |\theta^*(s)|^2 ds}\rho^*_{\bar{\theta}}(t)\right).$$

Therefore, given $\gamma > 0$, $\pi^*(\cdot)$ is feasible for Problem (17). On the other hand,

$$\begin{aligned}\mathbb{E}[y(T;\pi^*)] &= \gamma(1 - \mathbb{E}[e^{-\int_0^T |\theta^*(s)|^2 ds}\rho^*_{\bar{\theta}}(T)]),\\ \mathrm{Var}(y(T;\pi^*)) &= \gamma^2\mathrm{Var}(e^{-\int_0^T |\theta^*(s)|^2 ds}\rho^*_{\bar{\theta}}(T)).\end{aligned}$$

Hence

$$f(\theta,\pi^*) = \frac{\mathrm{Var}(y(T;\pi^*)|_{\bar{\theta}=\theta})}{(\mathbb{E}[y(T;\pi^*)|_{\bar{\theta}=\theta}])^2} = \frac{\mathrm{Var}(e^{-\int_0^T |\theta^*(s)|^2 ds}\rho^*_{\theta}(T))}{(1 - \mathbb{E}[e^{-\int_0^T |\theta^*(s)|^2 ds}\rho^*_{\theta}(T)])^2} \le \frac{1}{e^{\int_0^T |\theta^*(t)|^2 dt} - 1} = f(\theta^*,\pi^*),$$

where the inequality and the last equality are both due to Lemma 4.3.

Therefore, by virtue of Corollary 4.1,

$$\max_{\theta} f(\theta,\pi^*) = f(\theta^*,\pi^*) \le \min_{\pi}\max_{\theta} f(\theta,\pi),$$

implying the previous inequality is in fact an equality. It follows that π^* is an optimal solution for Problem (17).

Finally, consider the classical mean–variance model in which $\bar{\theta} = \theta^*$, we know (see the proof of Lemma 4.1) that $\min_\pi f(\theta^*,\pi) = f(\theta^*,\pi^*)$. Hence

$$f(\theta^*,\pi^*) = \min_{\pi} f(\theta^*,\pi) \le \max_{\theta}\min_{\pi} f(\theta^*,\pi) \le \min_{\pi}\max_{\theta} f(\theta^*,\pi) = f(\theta^*,\pi^*).$$

This completes the proof. *Q.E.D.*

Remark 4.1 The reason why we have infinitely many optimal portfolios – depending on the choice of the value of $\gamma > 0$ – is because the objective function is now the Sharpe ratio. Different γ leads to different levels of expected terminal wealth, however the Sharpe ratio remains constant.

Remark 4.2 Although the Sharpe ratio has neither convexity nor monotonicity, we have shown that, via ad hoc analysis, the order of maximization and minimization can be interchanged and the portfolio π^*, obtained through classical mean–variance theory assuming the underlying risk premium process is θ^*, is an optimal portfolio of our robust mean–variance problem.

5 Concluding Remarks

Motivated by the significant difficulty in the estimation of appreciation rates for risk assets, we have studied portfolio selection in a continuous-time financial market with uncertain or ambiguous appreciation rates. We have applied the idea of robust stochastic control to formulate portfolio selection problems in terms of the utility maximization and mean-variance respectively, and have solved the problems thoroughly.

The main contribution of our paper is that we limit our feasible trading strategies to be only dependent on the progressive information on share prices (but not on the underlying Brownian motion); this limitation is motivated by the fact that the true risk premium process is random and unobservable due to the ambiguity present. This feature in turn prohibits us from applying existing abstract results such as those of Gundel [8].

One of the key assumptions that the current analysis have rested on is that the interest rate and stocks volatility rates are deterministic functions of t. The analysis and results should remain the same if they are random but assumed to be adapted to the asset prices. However, it remains to be a significant open problem if they are adapted to the underlying Brownian motion.

References

[1] A. Ben-Tal, A. Nemirovski (1998): Robust convex optimization, *Math. Oepr. Res*, 23(4), pp. 796-805.

[2] S. Boyd, L. ElGhaoui, E. Feron and B. Balakrishnan (1994): Linear matrix inequalities in system and control theory, *SIAM*, Philadelphia, PA.

[3] T.R. Bielecki, H. Jin, S.R. Pliska and X.Y. Zhou (2005): Continuous-time mean–variance portfolio selection with bankruptcy prohibition, *Math. Finance* 15, pp. 213-244.

[4] Z. Chen, L. Epstein (2002): Ambiguit, risk, and asset returns in continuous time, *Econometrica*, 70, pp. 1403-1443.

[5] V.K. Chopra, W.T. Ziemba (1993): The effect of errors in means, variances and covariances on optimal portfoio choice, *J. Portfolio Management*, pp. 6-11

[6] I. Gilboa, D. Schmeidler (1989): Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, pp. 141-153.

[7] D. Goldfarb and G. Iyengar (2003): Robust portfoilo selection problems, *Mathmatics of Operationis Research*, 28, pp. 1-38.

[8] A. Gundel (2005): Robust utility maximization for complete and incomplete market models, *Finance Stochast.* 9, pp. 151-176.

[9] L.P. Hanse, T.J. Sargent (2001): Robust control and model uncertainty, *The American Economic Review*, 91, pp. 60-66.

[10] N. Ikeda and S. Watanabe (1981): *Stochastic Differential Equations and Difusiion Processes*, North-Holland, Amsterdam.

[11] J.E. Ingersoll (1987): *Theory of Financial Decision Making*, Rowman and Littlefield, Savage, MD.

[12] H. Jin, Z.Q. Xu and X.Y. Zhou (2008): A convex stochastic optimization problem arising from portfolio selection, *Math. Finance* 18, pp. 171-184.

[13] I. Karatzas and S.E. Shreve (1998): *Methods of Mathematical Finance*, Springer-Verlag, New York.

[14] R.W. Klein, V.S. Bawa (1976): The effect of estimation risk on optimal portfoio choice, *J. Financial Economics*, 3, pp. 215-231.

[15] P. Lakner (1995): Utility maximization with partial information, *Stochastic Processes and their Applications*, 56, pp. 247-273.

[16] H. Markowitz (1952): Portfolio selection, *J. Finance*, 7, pp. 77-91.

[17] C. Skiadas (2003): Robust control and recursive utility, *Finance Stochast.*, 7, pp. 475-489.

[18] X.Y. Zhou and D. Li (2000), Continuous time mean–variance portfolio selection: A stochastic LQ framework, *Appl. Math. Optim.*, 42, pp. 19-33.