# Behavioral Portfolio Selection with Loss Control*

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**Abstract**

In this paper we formulate a continuous-time behavioral (*à la* cumulative prospect theory) portfolio selection model where the losses are constrained by a pre-specified upper bound. Economically the model is motivated by the previously proved fact that the losses occurring in a bad state of the world can be catastrophic for an unconstrained model. Mathematically solving the model boils down to solving a *concave* Choquet *minimization* problem with an additional upper bound. We derive the optimal solution explicitly for such a loss control model. The optimal terminal wealth profile is in general characterized by three pieces: the agent has gains in the good states of the world, gets a moderate, endogenously constant loss in the intermediate states, and suffers the maximal loss (which is the given bound for losses) in the bad states. Examples are given to illustrate the general results.

KEYWORDS: Cumulative prospect theory, portfolio choice, gains and losses, constraint, Choquet integral, quantile function

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# 1 Introduction

Study on continuous-time portfolio choice has so far predominantly centered around expected utility maximisation (EUM), following the seminal paper of Merton (1969). The underlying assumption of EUM is that decision makers are rational and risk averse when facing uncertainty. This assumption, however, has long been challenged by many observed and repeatable empirical patterns as well as paradoxes and puzzles such as the Allais paradox and the equity premium puzzle.

A number of alternative preference measures to expected utility have been proposed, notably in Yaari's "dual theory of choice" (Yaari 1987), Lopes' SP/A model (Lopes 1987), and Kahneman and Tversky's cumulative prospect theory (CPT; Kahneman and Tversky 1979, Tversky and Kahneman 1992). All these theories (where SP/A and CPT are regarded as instances of behavioural economics) involve subjective probability distortions (although they may have different economic interpretations), and CPT is the richest one in that it in addition incorporates a reference point and an $S$-shaped utility function which both have significant impact on the corresponding decision-making models and their solutions.

There have been burgeoning research interests in incorporating these new theories into portfolio choice; nonetheless these have been overwhelmingly limited to the single-period setting; see for example Benartzi and Thaler (1995), Lopes and Oden (1999), Shefrin and Statman (2000), De Giorgi and Post (2008), and He and Zhou (2009c). Little analytical treatment has been carried out for *dynamic*, especially continuous-time, asset allocation featuring behavioural criteria. Such a lack of study is because all the main mathematical approaches dealing with the conventional EUM models fail. In particular, probability distortions abolish virtually all the good properties associated with the standard additive probability and linear expectation. Moreover, in the CPT framework, the utility function is non-convex and non-concave, while traditionally the global convexity/concavity is a general prerequisite in solving an optimization problem.

Because of these difficulties, study on behavioural portfolio selection is still in its infancy in the mathematical finance community. To our best knowledge there are less than a handful of papers available in the literature that investigate continuous-time behavioural portfolio choice. Berkelaar, Kouwenberg and Post (2004) consider a very special two-piece S-shaped power utility function, and employ a convexification technique to tackle the non-convexity of the problem. However, the probability distortion is absent in that paper. Jin and Zhou (2008) develop a systematic approach to solving a model under CPT with a complete market and general Itô processes for asset prices, featuring both *S*-shaped utility functions and probability distortions. Their approach includes a divide-and-conquer procedure to separate the optimizations for gains and losses, a quantile formulation to deal with the probability distortions, and a technique to solve a concave Choquet minimization problem corresponding to the risk-seeking part. The optimal trading strategies derived therein behave markedly differently from those of the classical EUM models: they are gambling policies, betting on good states of the world while accepting a fixed, known loss in bad ones. This feature is indeed reminiscent of the trading pattern of many investors and, in particular, of hedge funds.

Recently, the quantile approach is further developed by He and Zhou (2009a) as a general machinery in formulating and solving portfolio selection with a very broad class of law-invariant performance criteria. In particular, models with Yaari's and SP/A criteria are solved explicitly in He and Zhou (2009a,b) respectively.

The model in Jin and Zhou (2008) is essentially unconstrained (save for the tameness requirement for admissible portfolios). Jin and Zhou (2009) prove that potential losses can be catastrophically large with a sufficiently strong agent greed (as reflected by a very high reference point). In other words, when one loses one can lose real big. A naturally arising problem is, therefore, to study a CPT model where the loss is *a priori* bounded by a given level, thereby the greed is contained, if *indirectly*, at a manageable level. This is clearly an economically sensible model (and very relevant to the current financial crisis) from a loss control or regulatory point of view.

This paper is to formulate and solve a CPT portfolio selection model with loss control.[1] The additional technical challenge, compared with Jin and Zhou (2008), is that we have to solve a *concave* Choquet *minimization* problem with an additional upper bound. The minima of a constrained concave functional usually lie on the boundary of the constrained set; hence the problem is essentially a combinatorial optimization problem in an infinite dimesion. The main mathematical contribution of this paper is to solve such a non-conventional problem explicitly. The final optimal terminal wealth, again derived in an explicit form, to the portfolio choice problem exhibits different (indeed richer) qualitative features than that of an unconstrained model: the agent has gains in the good states of the world, gets a moderate, endogenously *constant* loss in the intermediate states, and suffers the maximal loss (which is the given bound for losses) in the bad states.

The rest of this paper is arranged as follows. The portfolio selection model is formulated in Section 2, and a divide-and-conquer solution scheme adapted to the present model is highlighted in Section 3. Section 4 consists the main technical contribution of the paper, where we solve a general constrained Choquet minimization problem. Section 5 presents the optimal solution to the behavioral model. In Section 6 we give a number of concrete examples to illustrate the general results. Finally, Section 7 concludes. The proof of a technical lemma is put in an appendix.

[1] In Berkelaar, Kouwenberg and Post (2004), the terminal wealth is constrained to be nonnegative, implying a specific bound for losses. However as mentioned they do not consider probability distortions.

# 2 Formulation of the Model

We take the same market and agent preferences (and use exactly the same notation) as those in Jin and Zhou (2008). Consider a behavioral agent with an investment planning horizon $[0, T]$ and an initial endowment $x_0 > 0$, both exogenously fixed throughout, in an arbitrage-free and complete market[2]. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a standard filtered complete probability space representing the underlying uncertainty, along with a standard Brownian motion $\{W_t, t \geq 0\}$. Here the filtration $\mathcal{F}_t$ is generated by $W_t$, augmented with all $P$-null sets. The unique pricing kernel in this market is

$$\rho = \exp\left\{ -\int_0^T [r(s) + \theta(s)^2]ds + \theta(s)'dW(s) \right\},$$

where $r(t)$ is the risk-free interest rate process, and $\theta(t)$ is the market price of risk. Assume that $\rho$ is atomless, and denote by $F(\cdot)$ the probability distribution function of $\rho$ throughout this paper.

The agent risk preference is dictated by the CPT. Specifically, she has a reference point $B$ at the terminal time $T$, which is an $\mathcal{F}_T$-measurable contingent claim (random variable)[3] with $P(|B| < \infty) = 1$. The reference point $B$ determines whether a given terminal wealth position is a gain (excess over $B$) or a loss (shortfall from $B$). We always assume in this paper that the benchmark $B$ is lower bounded. The agent utility (value) function is $S$-shaped: $u(x) = u_+(x^+)\mathbf{1}_{x\geq 0}(x) - u_-(x^-)\mathbf{1}_{x<0}(x)$, where the superscripts $\pm$ denote the positive and negative parts of a real number, $u_+, u_-$ are *concave* functions on $\mathbf{R}^+$ with $u_\pm(0) = 0$. There are also subjective probability distortions on both gains and losses, which are captured by two nonlinear, nondecreasing functions $T_+, T_-$ from $[0, 1]$ onto $[0, 1]$, with $T_\pm(0) = 0, T_\pm(1) = 1$ and $T_\pm(p) > p$ when $p$ is close to $0$. The agent preference on a terminal cash flow $X$ is measured by the behavioural functional

$$V(X - B) = V_+((X - B)^+) - V_-((X - B)^-),$$

where $V_+(Y) = \int_0^{+\infty} T_+(P(u_+(Y) \geq y))dy, \quad V_-(Y) = \int_0^{+\infty} T_-(P(u_-(Y) \geq y))dy$. A portfolio selection model (without loss control) is therefore to solve the following optimization problem[4]

$$\begin{array}{ll} \text{Maximize} & V(X - B) \\ \text{subject to} & \begin{cases} E[\rho X] = x_0, \\ X \text{ is } \mathcal{F}_T\text{-measureable and lower bounded.} \end{cases} \end{array} \tag{1}$$

[2]We assume market completeness so as not to distract ourselves from the main issue of the paper, namely the loss control. Market incompleteness can be dealt with at least for markets with deterministic opportunity sets – see He and Zhou (2009a), Section 4.

[3]In Jin and Zhou (2008) it is assumed that $B = 0$ without loss of generality.

[4]If $X^*$ is an optimal solution to the problem, then the optimal portfolio is the one that replicates $X^*$, owing to the market completeness. An optimal terminal wealth profile is usually more revealing than the corresponding portfolio about the trading behaviors of the agent.

In the above, the lower bound of $X$ may depend on $X$ (i.e. the bounds are allowed to be different with different $X$). This requirement is due to the tameness of the admissible portfolios.

Jin and Zhou (2009) carry out an asymptotic analysis on the optimal solutions of the above problem, obtained in Jin and Zhou (2008), when (the present value of) $B$ approaches infinity, for the case when $u_+(x) = x^\alpha, u_-(x) = kx^\beta$ with $0 < \alpha \leq \beta < 1$ and $\rho$ is lognormal. Their main results show that the losses, once they occur, will grow to $+\infty$ when the present value of $B$ goes to $+\infty$. In other words, potential losses may go catastrophic if the reference point is set excessively high. Motivated by this result, we consider a constrained portfolio selection model where the loss is almost surely capped by a pre-specified constant $L$, leading to the following problem

$$\begin{array}{ll} \text{Maximize} & V(X-B) \\ \text{subject to} & \begin{cases} E[\rho X] = x_0, \quad X \geq B - L \text{ a.s..} \\ X \text{ is } \mathcal{F}_T\text{-measureable.} \end{cases} \end{array} \tag{2}$$

This problem is equivalent to

$$\begin{array}{ll} \text{Maximize} & V(X) \\ \text{subject to} & \begin{cases} E[\rho X] = x_0, \ X \geq -L \text{ a.s.,} \\ X \text{ is } \mathcal{F}_T\text{-measureable,} \end{cases} \end{array} \tag{3}$$

where, with an abuse of notation, $X$ represents the terminal gain or loss of the portfolio (i.e, $X$ should have been $X - B$ as in (2)), and $x_0$ now denotes the deviation of the initial endowment from the present value of $B$. Henceforth (3) will be the problem under investigation.

We remark that Berkelaar et al. (2004) study a similar problem where $L = 0$, in the absence of probability distortions. Probability distortion is an integral part of the CPT, and its presence reflects the agent risk preferences in a dimension different from the utility/value functions. As a result, it will greatly change the agent trading behaviours. Among other things, Jin and Zhou (2008) point out that the unconstrained problem is ill-posed (i.e. the prospect value is unbounded) if there is no probability distortion on losses. Significant qualitative changes are expected for the constrained model as well. On the other hand, probability distortions pose great technical difficulties for analytically treating the portfolio choice models.

To conclude this section we address the issue of model well-posedness for Problem (3). In general a maximization problem is ill-posed if its supremium is infinite; otherwise it is called well-posed. It was shown in Jin and Zhou (2008) that an unconstrained behavioural model could be easily ill-posed. The reason is that, at a conceptual level, one could take a huge leverage (since the potential losses are not capped) to bet for enormous gains, and if the utility on gains overrides the disutility on losses then one ends up with an arbitrarily large prospective value. Now, if the losses are a priori contained, so is the corresponding leverage. Thus our model (3) is more likely to be well-posed so long as the gain part is well-posed.

To make it precise, consider a maximization problem involving the Choquet integral:

$$\begin{aligned} &\text{Maximize } V_+(X) = \int_0^{+\infty} T_+(P(u_+(X) > y))dy \\ &\text{subject to } E[\rho X] = x_0,\ X \geq 0 \text{ a.s.}, \\ &\qquad\qquad X \text{ is } \mathcal{F}_T\text{-measureable.} \end{aligned} \tag{4}$$

Denote $R_{u_+}(x) := -\frac{xu''_+(x)}{u'_+(x)}$, $x > 0$, the Arrow-Pratt index of relative risk aversion of the utility function $u_+(\cdot)$ for gains.

**Theorem 1.** *Assume that* $\liminf_{x\to\infty} R_{u_+}(x) > 0$. *Then Problem (3) is well-posed for any initial capital* $x_0 > 0$ *under any one of the following conditions:*

(i) *Problem (4) is well-posed for any initial capital* $x_0 > 0$.

(ii) $E\left[u_+\left(u'_+\left(\frac{\rho}{T'_+(F(\rho))}\right)\right)T'_+(F(\rho))\right] < +\infty$.

(iii) $E\left[u_+\left(u'_+\left(\frac{\lambda\rho}{T'_+(F(\rho))}\right)\right)T'_+(F(\rho))\right] < +\infty$ *for some* $\lambda > 0$.

**Proof:** From Jin and Zhou (2008), p. 418, it follows that Problem (4) is well-posed if and only if the following problem is well-posed:

$$\begin{aligned} &\text{Maximize} \quad \tilde{v}_1(g) := E[u_+(g(Z))T'_+(1-Z)] \\ &\text{subject to} \quad E[g(Z)F^{-1}(1-Z)] = x_0,\ g \in \Gamma, \end{aligned} \tag{5}$$

where $Z := 1 - F(\rho) \sim U(0,1)$, a uniform random variable on interval $[0,1)$ and

$$\Gamma := \{g : [0,1) \mapsto \mathbf{R}^+ \text{is nondecreasing, left continuous, with } g(0) = 0\}. \tag{6}$$

Define a new probability measure $\tilde{P}$ whose expectation $\tilde{E}(X) := E[T'_+(1-Z)X]$, and $\zeta := \frac{F^{-1}(1-Z)}{T'_+(1-Z)} \equiv \frac{\rho}{T'_+(F(\rho))}$. Then $\zeta > 0$ a.s.. Rewrite Problem (5) in terms of the probability measure $\tilde{P}$ as follows

$$\begin{aligned} &\text{Maximize} \quad \bar{v}_1(g) := \tilde{E}[u_+(g(Z))] \\ &\text{subject to} \quad \tilde{E}[\zeta g(Z)] = x_0, \quad g \in \Gamma. \end{aligned} \tag{7}$$

Now take the following enlarged set of $\Gamma$:

$$\tilde{\Gamma} := \{g : [0,1) \mapsto \mathbf{R}^+ \text{is left continuous, with } g(0) = 0\},$$

and consider Problem (7) with $\Gamma$ replaced by $\tilde{\Gamma}$. The resulting problem has a larger feasible set, but it is well-posed under (ii) or (iii) according to Jin, Xu and Zhou (2008), Theorem 5.4. Hence (5), or (4), is well-posed under (ii) or (iii).

It remains to show that Problem (3) is well-posed for any $x_0 > 0$ under (i). Indeed, if $X$ is a feasible solution of (3), then $E[\rho X^+] = E[\rho(X + X^-)] \leq E[\rho X] + LE[\rho] < \infty$, indicating that $X^+$ is a feasible solution to (4) with an initial capital $\tilde{x}_0 \leq x_0 + LE[\rho]$. Denote $\mu(a) := \sup_{E[\rho X]=a,\ X\geq 0,\ X \text{ is } \mathcal{F}_T-\text{measureable}} V_+(X)$, which is a nondecreasing function in $a > 0$. If (4) is well-posed for any initial capital $a > 0$, then $V(X) = V_+(X^+) - V_-(X^-) \leq V_+(X^+) \leq \mu(x_0 + LE[\rho]) < +\infty$, implying that (3) is also well-posed for $x_0 > 0$. $\square$

## 3 Solution Scheme: Divide and Conquer

We use the same divide-and-conquer approach developed in Jin and Zhou (2008) to solve (3), and highlight the main difference and difficulty resulting from the additional loss constraint. First, split Problem (3) into the following positive part and negative part problems:

*Positive Part Problem*: A problem with parameters $(A, x_+)$:

$$\begin{array}{ll} \text{Maximize} & V_+(X) = \int_0^{+\infty} T_+(P(u_+(X) > y))dy \\ \text{subject to} & E[\rho X] = x_+, \quad X \geq 0 \text{ a.s.}, \quad X = 0 \text{ a.s. on } A^C, \end{array} \tag{8}$$

where $x_+ \geq x_0^+$ $(\geq 0)$ and $A \in \mathcal{F}_T$ with $P(A) \leq 1$ are given. We define the optimal value of Problem (8), denoted $v_+(A, x_+)$, in the following way. If $P(A) > 0$, in which case the feasible set of (8) is non-empty ($X = x_+\mathbf{1}_A/\rho P(A)$ is a feasible solution satisfying all the constraints), then $v_+(A, x_+)$ is defined to be the supremum of (8). If $P(A) = 0$ and $x_+ = 0$, then (8) has only one feasible solution $X = 0$ a.s. and $v_+(A, x_+) := 0$. If $P(A) = 0$ and $x_+ > 0$, then (8) has no feasible solution, where we define $v_+(A, x) := -\infty$.

*Negative Part Problem*: A problem with parameters $(A, x_+)$:

$$\begin{array}{ll} \text{Minimize} & V_-(X) = \int_0^{+\infty} T_-(P(u_-(X) > y))dy \\ \text{subject to} & E[\rho X] = x_+ - x_0,\ 0 \leq X \leq L \text{ a.s.},\ X = 0 \text{ a.s. on } A, \end{array} \tag{9}$$

where $x_+ \geq x_0^+$ and $A \in \mathcal{F}_T$ with $P(A) \leq 1$ are given. We define the optimal value $v_-(A, x_+)$ of Problem (9) as follows. If $LE[\rho 1_{A^c}] \geq x_+ - x_0$, Problem (9) admits a feasible solution and $v_-(A, x_+)$ is the infimum value of (9). If $LE[\rho 1_{A^c}] < x_+ - x_0$, then (9) has no feasible solution, in which case we define $v_-(A, x_+) = +\infty$.

Once the preceding two problems are solved, we then "conquer" the original problem by finding the best $(A, x_+)$ in yet another optimization problem:

$$\begin{array}{ll} \text{Maximize} & v_+(A, x_+) - v_-(A, x_+) \\ \text{subject to} & A \in \mathcal{F}_T,\ x_+ \geq x_0^+, \end{array} \tag{10}$$

Compared with Jin and Zhou (2008), the positive part problem (8) here is unaffected by the loss constraint $L$; and a more general version of it has been solved already in Jin and Zhou (2008), Appendix C. The only difference here lies in the negative part problem (9) where there is an additional upper bound constraint on $X$. This, certainly, is due to the loss control in the original model. Using the quantile formulation developed in Jin and Zhou (2008) and He and Zhou (2009a), we will be able to turn (9) into a *minimization* problem of a *concave* functional. The solution thus must lie on the boundary of the budget constraint set. Via a delicate analysis, Jin and Zhou (2008), Appendix D, have solved this (unconventional) problem when $L$ is absent. The additional upper constraint $L$ introduces significant difficulties in solving this combinatorial-type optimization problem in an infinite dimensional space. The main technical contribution of the present paper is the tackling of the difficulty arising from this additional constraint in solving (9), which in turn leads to significantly different qualitative features in the final solution. Problem (9) will be solved in the next section.

Finally, let us remark that as in Jin and Zhou (2008), Theorem 5.1, one needs only to solve the three problems (8)–(10) with $A = \{\rho \le c\}$ for some $c \in [\underline{\rho}, \bar{\rho}]$ where $\underline{\rho}$ and $\bar{\rho}$ are the essential lower bound and upper bound of $\rho$ respectively. The essential reason for this is that the optimal $A$ is the set when the agent ends up with gains, and the optimal terminal wealth must be anti-comonotonic with $\rho$; see He and Zhou (2009a) for a detailed discusson on this point.

## 4 Solving the Negative Part Problem

We solve the negative part problem (9) in this section. For ease of exposition we consider first a general constrained utility *minimization* problem involving the Choquet integral:

$$\begin{array}{ll} \text{Minimize} & V_2(X) := \displaystyle\int_0^{+\infty} T(P(u(X) > y))dy \\ \text{subject to} & E[\rho X] = a,\ 0 \le X \le L \text{ a.s.}, \end{array} \tag{11}$$

where $\rho$ is the pricing kernel defined earlier, with no atom and the distribution function $F(\cdot)$, $a > 0$ (the case $a = 0$ is trivial), $T : [0,1] \mapsto [0,1]$ is a strictly increasing, differentiable function with $T(0) = 0,\ T(1) = 1$, and $u(\cdot)$ is strictly increasing, concave with $u(0) = 0$. When $E[\rho] \ge a/L$, Problem (11) admits a feasible solution, and hence has a finite optimal value.

The exactly same argument used in Jin and Zhou (2008), Appendix D, reveals that the optimal solution $X^*$ to (11) must be in the form of $G^{-1}(F(\rho))$ for some probability distribution function[5] $G(\cdot)$, where $Z = F(\rho) \sim U(0,1)$. Hence, (11) reduces to the following problem seeking the optimal $G(\cdot)$:

[5]There are left-continuous version and right-continuous version in defining an inverse function. The above result holds true with both versions. In this paper, we use the right-continuous version for inverses of monotone functions, i.e. $G^{-1}(y) := \inf\{x : G(x) > y\}$.

$$
\begin{array}{ll}
\text{Minimize} & v_2(G) := \int_0^{+\infty} T\left(P(u(G^{-1}(Z)) > y)\right) dy \\
\text{subject to} & \begin{cases} E[F_\rho^{-1}(Z)G^{-1}(Z)] = a, \\ G \text{ is the distribution function of a random variable} \\ \quad \text{taking values in } [0, L] \text{ a.s.} \end{cases}
\end{array}
\tag{12}
$$

The following result formalizes the equivalence between Problems (11) and (12), which can be proved in exactly the same way as in Jin and Zhou (2008), Proposition D.1.

**Proposition 1.** *If $G^*$ is optimal for (12), then $X^* := (G^*)^{-1}(Z)$ is optimal for (11). Conversely, if $X^*$ is optimal for (11), then its distribution function $G^*$ is optimal for (12) and $X^* = (G^*)^{-1}(Z)$, a.s..*

A simple integration by parts shows (see Jin and Zhou (2008), Appendix D) that $v_2(G) = E[u(G^{-1}(Z))T'(1-Z)]$. Denoting $g = G^{-1}$, Problem (12) can be rewritten as

$$
\begin{array}{ll}
\text{Minimize} & \bar{v}_2(g) := E[u(g(Z))T'(1-Z)] \\
\text{subject to} & E[g(Z)F_\rho^{-1}(Z)] = a,\ g \in \Gamma,\ g(t) \le L\ \forall t,
\end{array}
\tag{13}
$$

where $\Gamma$ is the set of quantile functions (inverses of probability distribution functions) of all the nonnegative random variables, which was defined earlier in (6).

The above procedure of changing decision variables from a random cash flow $X$ to its quantile function $g$ is called a *quantile formulation*. The idea of this formulation was around in Carlier and Dana (2006), fully exploited in Jin and Zhou (2008) to solve the behavioral portfolio model and later systematically developed in He and Zhou (2009a) for a general non-expected utility model. The advantage of this formulation is that it overcomes the difficulty caused by the nonlinear expectation (involved in many problems including those with probability distortions) and reduces the problem to one with the usual linear expectation.

Notice that (13) is to *minimize* a *concave* functional over some constraint set of the quantile functions; hence the solutions must lie on the boundary of the constraint set. Due to the presence of the additional constraint $g(t) \le L\ \forall t$, one needs to find those boundary points carefully.

Define $b_0 := \sup\{b \ge 0 : E[L1_{Z>b}F_\rho^{-1}(Z)] \ge a\}$ where we convent $\sup\emptyset := -\infty$. If (13) admits a feasible solution, then clearly $b_0 \in [0, 1)$.

The following result is crucial, which dictates the form of any possible optimal solution to (13).

**Proposition 2.** *Assume $u(\cdot)$ is strictly concave. Then the optimal solution for Problem (13), if it exists, must be in the form $g(t) = q(b_1, b_2)1_{[b_1,\, b_2)}(t) + L1_{[b_2,\, 1)}(t)$, $t \in [0, 1)$, with $b_1,\ b_2$ satisfying $0 \le b_1 < b_0 \le b_2 \le 1$, and $q(b_1, b_2) := \frac{a - LE[F_\rho^{-1}(Z)1_{[b_2,1)}(Z)]}{E[F_\rho^{-1}(Z)1_{[b_1,b_2)}(Z)]} < L$.*

**Proof**: Let $g$ be an optimal solution to (13). Clearly $g \not\equiv 0$. We first show that there are no $0 < t_1 < t_2 < 1$ such that $0 < g(t_1) < g(t_2) < L$. Indeed, if there are such $t_1$ and $t_2$, then we consider two cases.

*Case 1:* $g(1-) = g(t_2)$. In this case $g$ is constant on $[t_2, 1)$. For $\beta \in (0,1)$ define

$$g_1^\beta(t) := \begin{cases} \beta g(t), & 0 \le t < t_1, \\ \beta g(t_1) + \frac{\gamma(\beta)g(t_2) - \beta g(t_1)}{g(t_2) - g(t_1)}(g(t) - g(t_1)), & t_1 \le t < t_2, \\ \gamma(\beta) g(t_2), & t_2 < t < 1, \end{cases} \tag{14}$$

where $\gamma(\beta)$ is uniquely determined by $E[g_1^\beta(Z)F_\rho^{-1}(Z)] = a$. If $\gamma(\beta) \le 1$, then it is easy to show that $g_1^\beta(t) \le g(t)\ \forall t \in [t_1, 1]$ and hence $E[g_1^\beta(Z)F_\rho^{-1}(Z)] < E[g(Z)F_\rho^{-1}(Z)] = a$ which is a contradiction. So $\gamma(\beta) > 1$. A similar argument shows that $\lim\limits_{\beta \uparrow 1} \gamma(\beta) = 1$. Choose $\beta^* \in (0,1)$ appropriately such that $1 < \frac{\gamma(\beta^*)g(t_2) - \beta^* g(t_1)}{g(t_2) - g(t_1)} < 2$ and $g_1^{\beta^*}(t_2) = \gamma(\beta^*)g(t_2) \le L$.

*Case 2:* $g(1-) > g(t_2)$. Then for $\beta \in (0,1)$ define

$$g_1^\beta(t) := \begin{cases} \beta g(t), & 0 \le t < t_1, \\ \beta g(t_1) + \frac{\gamma(\beta)g(t_2) - \beta g(t_1)}{g(t_2) - g(t_1)}(g(t) - g(t_1)), & t_1 \le t < t_2, \\ \gamma(\beta) g(t_2) + (g(t) - g(t_2))\frac{g(1-) - \gamma(\beta)g(t_2)}{g(1-) - g(t_2)}, & t_2 \le t < 1, \end{cases} \tag{15}$$

where $\gamma(\beta)$, again, is defined by $E[g_1^\beta(Z)F_\rho^{-1}(Z)] = a$. Similarly, $\gamma(\beta) > 1$ and $\lim\limits_{\beta \uparrow 1} \gamma(\beta) = 1$. Choose $\beta^* \in (0,1)$ appropriately such that $1 < \frac{\gamma(\beta^*)g(t_2) - \beta^* g(t_1)}{g(t_2) - g(t_1)} < 2$ and $0 < \frac{g(1-) - \gamma(\beta^*)g(t_2)}{g(1-) - g(t_2)} < 2$.

In either of the above two cases, define

$$g_2^{\beta^*} = 2g - g_1^{\beta^*}.$$

Henceforth we write $g_1$ and $g_2$ as shorthands respectively for $g_1^{\beta^*}$ and $g_2^{\beta^*}$. By the constructions it is easy to check that $g_1,\ g_2 \in \Gamma$, $g_1(t) \le L,\ g_2(t) \le L,\ \forall t \in [0,1)$. As $g(\cdot),\ g_1(\cdot),\ g_2(\cdot)$ are left continuous functions and $g_1(t_2) > g(t_2) > g_2(t_2)$, there exists $\delta > 0$ such that $\delta < t_2 - t_1$ and $g_1(t) > g(t) > g_2(t), \forall t \in [t_2 - \delta, t_2]$. The strict concavity of $u(\cdot)$ implies that $\bar{v}_2(g) > [\bar{v}_2(g_1) + \bar{v}_2(g_2)]/2$; so either $\bar{v}_2(g) > \bar{v}_2(g_1)$ or $\bar{v}_2(g) > \bar{v}_2(g_2)$ holds, which contradicts the optimality of $g(\cdot)$.

Denote $b_2 = \min(\inf\{t > 0 : g(t) = L\},\ 1)$, $b_1 := \max(\sup\{t > 0 : g(t) < g(b_2-)\},\ 0)$. The analysis above shows that $0 \le b_1 < b_0 \le b_2 \le 1$ and $g(t) = k1_{[b_1,\ b_2)}(t) + L1_{[b_2,\ 1)}(t)$ for some $k \in \mathbb{R}^+$. The feasibility condition $E[g(Z)F_\rho^{-1}(Z)] = a$ implies

$$k := q(b_1, b_2) = \frac{a - LE[F_\rho^{-1}(Z)1_{[b_2,1)}(Z)]}{E[F_\rho^{-1}(Z)1_{[b_1,b_2)}(Z)]} < L,$$

where the last inequality is due to the fact that $b_1 < b_0$. $\square$

In view of the preceding result, we need only to determine the "best" $b_1$ and $b_2$ by solving the following two-dimensional optimization problem:

$$\begin{array}{ll} \text{Minimize} & \bar{v}_2(b_1, b_2) := E[u(g(Z))T'(1-Z)], \\ \text{subject to} & g(t) = q(b_1,\ b_2)1_{[b_1,\ b_2)}(t) + L1_{[b_2,\ 1)}(t),\ 0 \le b_1 < b_0 \le b_2 \le 1. \end{array} \tag{16}$$

**Proposition 3.** *Problems (13) and (16) have the same infimum values.*

**Proof**: Denote by $\alpha$ and $\beta$ the infimum values of (13) and (16) respectively. Obviously, $\alpha \le \beta$. If the opposite inequality is false, there exists a feasible solution $g$ of (13) such that $\bar{v}_2(g) < \beta$.

Denote $f(\cdot) := F_\rho^{-1}(\cdot)$ for notational convenience. For each $n \ge 1,\ 0 \le k \le 2^n - 1$, define $\tilde{a}(n,k) := \frac{\int_{(k-1)/2^n}^{k/2^n} f(t)g(t)dt}{\int_{(k-1)/2^n}^{k/2^n} f(t)dt}$. Then $g(\frac{k-1}{2^n}) \le \tilde{a}(n,k) \le g(\frac{k}{2^n})$, and it is easy to check that

$$g_n(t) := \sum_{k=1}^{2^n} \tilde{a}(n,k) 1_{[\frac{k-1}{2^n}, \frac{k}{2^n})}(t)$$

is a feasible solution of Problem (13). Since $g_n \to g$ a.s. and $|g_n(t)| \le L$, we have $\bar{v}_2(g_n) \to \bar{v}_2(g)$. Hence, there exists $n \in \mathbb{N}$ such that $\bar{v}_2(g_n) < \beta$. Now $g_n$ is right continuous, nondecreasing and piece-wise constant; so, after combining the adjacent terms with an identical function value, we can rewrite $g_n$ as

$$g_n(t) = \sum_{k=1}^{l} a(k) 1_{[t_k,\ t_{k+1})}(t) + \sum_{k=l+1}^{l+m} a(k) 1_{[t_k,\ t_{k+1})}(t),$$

where $l \ge 1$ satisfying $t_l \le b_0 < t_{l+1}$, $m \ge 0$, $0 \le t_1 < \cdots < t_{l+m+1} = 1$, and $0 < a(1) < \cdots < a(m+l) \le L$. By virtue of Lemma 1 below, we have the following representation

$$g_n(t) = \sum_{i=1}^{l} \sum_{j=l}^{l+m+1} \lambda_{ij} J_{(t_i,\ t_j)}(t),$$

where

$$J_{(t_i,\ t_j)}(t) = q(t_i,\ t_j) 1_{[t_i,\ t_j)}(t) + L1_{[t_j,\ 1)}(t),$$

and $\lambda_{ij},\ 1 \le i \le l,\ l+1 \le j \le l+m+1$ are nonnegative constants satisfying $\sum_{i=1}^{l}\sum_{j=l+1}^{l+m+1} \lambda_{ij} = 1$. In other words, $g_n$ is a convex combination of $\{J_{(t_i,\ t_j)} : 1 \le i \le j,\ l+1 \le j \le l+m+1\}$. However, $u$ is concave; so

$$\beta > \bar{v}_2(g_n) \ge \sum_{i=1}^{l} \sum_{j=l+1}^{l+m+1} \lambda_{ij} \bar{v}_2(J_{(t_i,\ t_j)}).$$

Hence there exists $(i,j)$ such that $\bar{v}_2(J_{(t_i,\ t_j)}) < \beta$, which contradicts the fact that $\beta$ is the infimum of (16). $\square$

**Lemma 1.** *If $g \in \Gamma$ satisfying $E[g(Z)F_\rho^{-1}(Z)] = a$ with*

$$g(t) = \sum_{k=1}^{l} a(k)1_{[t_k,t_{k+1})}(t) + \sum_{k=l+1}^{l+m} a(k)1_{[t_k,t_{k+1})}(t),$$

*where $l \geq 1$, $t_l \leq b_0 < t_{l+1}$, $m \geq 0$, $0 \leq t_1 < \cdots < t_{m+l+1} = 1$, and $0 < a(1) < \cdots < a(m+l) \leq L$, then $g$ is a convex combination of $\{J_{(t_i,\, t_j)} : 1 \leq i \leq l,\ l+1 \leq j \leq l+m+1\}$ on $[0,1)$, namely,*

$$g = \sum_{i=1}^{l} \sum_{j=l+1}^{l+m+1} \lambda_{ij} J_{(t_i,\, t_j)}$$

*for some constants $\{\lambda_{ij} \geq 0 : 1 \leq i \leq l,\ l+1 \leq j \leq l+m+1\}$ satisfying $\sum\limits_{i=1}^{l} \sum\limits_{j=l}^{l+m+1} \lambda_{ij} = 1$.*

**Proof**: We apply induction on $m$. If $m = 0$, then, with $a(0) := 0$, we have

$$\begin{aligned} g(t) &= \sum_{k=1}^{l} a(k)1_{[t_k,\, t_{k+1})}(t) \\ &= \sum_{k=1}^{l} (a(k) - a(k-1))1_{[t_k,\, 1)}(t) \\ &= \sum_{k=1}^{l} \lambda_{k,\, l+1} J_{[t_k,\, 1)}(t), \end{aligned}$$

where $\lambda_{k,l+1} := \frac{a(k)-a(k-1)}{q(t_k,1)} \geq 0$. Since

$$\begin{aligned} a \equiv \int_0^1 g(t)f(t)dt &= \sum_{i=1}^{l} \lambda_{i,l+1} \int_0^1 J_{(t_i,1)}(t)f(t)dt \\ &= \sum_{i=1}^{l} \lambda_{i,l+1} a, \end{aligned}$$

we get $\sum\limits_{i=1}^{l} \lambda_{i,l+1} = 1$.

Now assuming the lemma is true for $m-1$ and any $l \geq 1$, we are to prove it is also true for $m$, where $m \in \boldsymbol{N}$.

Let $\tilde{\lambda}_{1,l+m} := \min(\frac{a(1)}{q(t_1,t_{l+m})}, \frac{a(l+m)-a(l+m-1)}{L-q(t_1,t_{l+m})})$. Obviously $\tilde{\lambda}_{1,l+m} \geq 0$. We claim $\tilde{\lambda}_{1,l+m} \leq 1$. If it is not true, then $a(1) > q(t_1, t_{l+m})$, $a(l+m) > a(l+m-1) + L - q(t_1,\ t_{l+m})$; and hence

$$\begin{aligned} g(t) &> q(t_1,\ t_{l+m})1_{[t_1,\, t_{l+m})} + (L - q(t_1, t_{l+m}) + a(l+m-1))1_{[t_{l+m},\, t_{l+m+1})} \\ &> q(t_1,\ t_{l+m})1_{[t_1,\, t_{l+m})} + L1_{[t_{l+m},\, t_{l+m+1})} \\ &= J_{(t_1,\, t_{l+m})}(t), \end{aligned}$$

which contradicts the feasibility of $g$.

Define

$$\tilde{g} := \frac{g - \tilde{\lambda}_{1,l+m} J_{(t_1, t_{l+m})}}{1 - \tilde{\lambda}_{1,l+m}}. \tag{17}$$

It is not a difficult exercise to check that $\tilde{g} \in \Gamma$ and $\tilde{g}(t) \leq L\ \forall t$. Rewrite (17) as $g = \tilde{\lambda}_{1,l+m} J_{(t_1, t_{l+m})} + (1 - \tilde{\lambda}_{1,l+m})\tilde{g}$.

If $\tilde{\lambda}_{1,l+m} = \frac{a(l+m) - a(l+m-1)}{L - q(t_1, t_{l+m})}$, then it is not hard to see that $\tilde{g}(t_{l+m}-) = \tilde{g}(t_{l+m})$, i.e.

$$\tilde{g}(t) = \sum_{k=1}^{l} \tilde{a}(k) 1_{[t_k,\ t_{k+1})}(t) + \sum_{k=l+1}^{l+m-1} \tilde{a}(k) 1_{[t_k,\ t_{k+1})}(t),$$

with the same $l, t_j, m$ and some (possibly) different $0 < \tilde{a}(1) < \cdots < \tilde{a}(l+m-1) \leq L$. So $\tilde{g}$ has only $m-1$ steps on $(b_0, 1)$ in this expression and hence is a convex combination of $\{J_{(t_i,\ t_j)}\}$ by the induction hypothesis. As a result $g$ is a convex combination of $\{J_{(t_i,\ t_j)}\}$, which completes the proof.

If, on the other hand, $\tilde{\lambda}_{1,l+m} = \frac{a(1)}{q(t_1, t_{l+m})}$, then it is easy to check that $\tilde{g}(t) = 0,\ \forall t \in [t_1,\ t_2)$, and $\tilde{g}$ can be written as

$$\tilde{g}(t) = \sum_{k=2}^{l} \tilde{a}(k) 1_{[t_k, t_{k+1})}(t) + \sum_{k=l+1}^{l+m-2} \tilde{a}(k) 1_{[t_k, t_{k+1})}(t) + \tilde{a}(l+m-1) 1_{[t_{l+m-1}, 1)},$$

with the same $l,\ t_j,\ m$ and some $0 < \tilde{a}(2) < \cdots < \tilde{a}(l+m-1) \leq L$. In this expression, $\tilde{g}$ has $l-1$ steps on $[0, b_0)$ and $m$ steps on $[b_0, 1)$. Repeat the previous procedure starting with defining a yet another new function via (17) where $g$ is replaced by $\tilde{g}$ on the right hand side. We claim that after $s(m) \leq l$ times, we get

$$\begin{aligned} g =& \tilde{\lambda}_{1,l+m} J_{(t_1, t_{l+m})} + (1 - \tilde{\lambda}_{1,l+m}) \tilde{\lambda}_{2,l+m} J_{(t_2, t_{l+m})} + \cdots \\ &+ \prod_{k=1}^{s_m - 1} (1 - \tilde{\lambda}_{k,l+m}) \tilde{\lambda}_{s(m),l+m} J_{(t_{s(m)}, t_{l+m})} + \prod_{k=1}^{s(m)} (1 - \tilde{\lambda}_{k,l+m}) \tilde{g}^{m,s(m)}, \end{aligned}$$

where $\tilde{g}^{m,s(m)}$ has only $m-1$ steps on $(b_0, 1)$ and hence is a convex combination of $\{J_{(t_i,\ t_j)}\}$ by the induction hypothesis, which would then lead to the desired conclusion. If, on the other hand, the above claim is false, then after $l$ times, we get

$$g = \tilde{\lambda}_{1,l+m} J_{(t_1, t_{l+m})} + \cdots + \prod_{k=1}^{l-1} (1 - \tilde{\lambda}_{k,l+m}) \tilde{\lambda}_{l,l+m} J_{(t_l, t_{l+m})} + \prod_{k=1}^{l} (1 - \tilde{\lambda}_{k,l+m}) \tilde{g}^{m,l}, \tag{18}$$

where $\tilde{g}^{m,l} \in \Gamma$, $\tilde{g}^{m,l}(t) \leq L$ and $\tilde{g}^{m,l}$ is zero on $[0,\ t_{l+1}]$. By the definition of $b_0$ and the inequality $t_{l+1} > b_0$ we have

$$\begin{aligned}
& E[g(Z)F_\rho^{-1}(Z)] \\
= \ & a\tilde{\lambda}_{1,l+m} + \cdots + a\prod_{k=1}^{l-1}(1-\tilde{\lambda}_{k,l+m})\tilde{\lambda}_{l,l+m} + \prod_{k=1}^{l}(1-\tilde{\lambda}_{k,l+m})E\left[\tilde{g}^{m,l}(Z)F_\rho^{-1}(Z)\right] \\
< \ & a\tilde{\lambda}_{1,l+m} + \cdots + a\prod_{k=1}^{l-1}(1-\tilde{\lambda}_{k,l+m})\tilde{\lambda}_{l,l+m} + a\prod_{k=1}^{l}(1-\tilde{\lambda}_{k,l+m}) \\
= \ & a,
\end{aligned}$$

which contradicts the feasibility of $g$. $\square$

Define $c_0 := F_\rho^{-1}(b_0)$. The following theorem gives a complete solution to Problem (11).

**Theorem 2.** *Problem (11) and (16) have the same infimum values. If, in addition, $u(\cdot)$ is strictly concave, then (11) admits an optimal solution if and only if the following optimization problem in $(c_1, c_2)$:*

$$\min_{\underline{\rho}\le c_1 < c_0 \le c_2 \le \bar{\rho}} \quad \left\{ u\left(\frac{a - LE(\rho 1_{\rho > c_2})}{E(\rho 1_{c_1 < \rho \le c_2})}\right)[T(P(\rho > c_1)) - T(P(\rho > c_2))] \right. \\ \left. + u(L)T(P(\rho > c_2)) \right\} \tag{19}$$

*admits an optimal solution $(c_1^*, c_2^*)$, in which case the optimal solution to (11) is*

$$X^* = \frac{a - LE[\rho 1_{\rho > c_2^*}]}{E[\rho 1_{c_1^* < \rho \le c_2^*}]} 1_{c_1^* < \rho \le c_2^*} + L 1_{\rho > c_2^*}. \tag{20}$$

**Proof**: The first conclusion follows from Proposition 3. For the second conclusion, substituting $g(t) = q(b_1, b_2)1_{[b_1,b_2)}(t) + L1_{[b_2,1)}(t)$ into the objective function of (16) and changing the variables by $c_i = F^{-1}(b_i)$, $i = 1, 2$, we obtain the objective function of (19). Finally, the optimal solution (20) follows from the the fact that $X^* = g^*(F(\rho))$, a.s., where $g^*$ is the optimal quantile function corresponding to the optimal solution $(c_1^*, c_2^*)$ to (19). $\square$

In summary, we have obtained an explicit optimal solution to Problem (11), in terms of the optimal solution to a two-dimensional mathematical program (19).

We are now in the position to solve the negative part problem (9) thanks to the general solution obtained above. As in Jin and Zhou (2008) we will solve (9) when $A = \{\rho \le c\}$. For any $x_+ \ge x_0^+$, define $c_0(x_+) := \sup\{d \ge \underline{\rho} : E[L\rho 1_{\rho > d}] \ge x_+ - x_0\}$. Only when $c \le c_0(x_+)$, (9) with parameters $(\{\rho \le c\}, x_+)$ has a feasible solution.

**Theorem 3.** *Assume that $u_-(\cdot)$ is strictly positive. Given $A = \{\omega : \rho \le c\}$ with $\underline{\rho} \le c \le c_0(x_+)$ and $x_+ \ge x_0^+$.*

(i) *If $x_+ = x_0$ and $c = c_0(x_+) \equiv \bar{\rho}$, then the optimal solution of (9) is $X^* = 0$ and $v_-(c, x_+) = 0$.*

(ii) *Otherwise, we have*

$$v_-(c,x_+) = \inf_{c\le c_1<c_0(x_+)\le c_2\le\bar\rho} \Big\{u_-\left(\frac{x_+-x_0-LE(\rho 1_{\rho>c_2})}{E(\rho 1_{c_1<\rho\le c_2})}\right)\Big[T(P(\rho>c_1)) - T(P(\rho>c_2))\Big] + u(L)T(P(\rho>c_2))\Big\}. \tag{21}$$

*Moreover, Problem (9) with parameters* $(\{\rho\le c\},x_+)$ *admits an optimal solution* $X^*$ *if and only if the minimization problem on the right hand side of (21) admits an optimal solution* $(c_1^*,c_2^*)$*, in which case*

$$X^* = \frac{x_+-x_0-LE[\rho 1_{\rho>c_2^*}]}{E[\rho 1_{c_1^*<\rho\le c_2^*}]} 1_{c_1^*<\rho\le c_2^*} + L1_{\rho>c_2^*}$$

.

**Proof**: (i) is trivial. Given Theorem 2 for the general constrained Choquet minimization problem, (ii) can be proved in exactly the same way as that in proving Theorem 7.1, Jin and Zhou (2008). $\square$

# 5 Solution of the Behavioral Model

Now that we have solved the negative part problem (9), and the positive part problem (8) is solved exactly as in Jin and Zhou (2008), Theorem 5.1. So what remains to do is completely parallel to that in Jin and Zhou (2008).

First of all, in view of Theorem 3, we only need to consider the following problem in lieu of (10), where $(c,c_2,x_+)$ are the decision variables:

$$\begin{array}{ll} \text{Maximize} & v_+(c,x_+) - u_-\left(\frac{x_+-x_0-LE[\rho 1_{\rho>c_2}]}{E[\rho 1_{c<\rho\le c_2}]}\right)[T_-(P(\rho>c)) - T_-(P(\rho>c_2))] \\ & - u_-(L)T_-(P(\rho>c_2)), \\ \text{subject to} & \underline{\rho}\le c\le c_0(x_+)\le c_2\le\bar\rho,\ x_0^+\le x_+\le x_0+LE[\rho], \\ & x_+=0 \text{ when } c=0. \end{array} \tag{22}$$

Here we convent that $\frac{x_+-x_0-LE[\rho 1_{\rho>c_2}]}{E[\rho 1_{c<\rho\le c_2}]} := 0$ when $c=c_2$.

**Theorem 4.** *Assume that* $u_-(\cdot)$ *is strictly concave. We have the following conclusions:*

(i) *If* $X^*$ *is optimal to Problem (3), then* $c^* := F^{-1}(P(X^*>0))$, $c_2^* = F^{-1}(P(X^*> -L))$, $x_+^* := E[\rho(X^*)^+]$*, where* $F$ *is the distribution function of* $\rho$*, are optimal to Problem (22). Moreover,* $\{\omega : X^*\ge 0\}$ *and* $\{\omega:\rho\le c^*\}$ *are identical up to a zero probability set, and* $(X^*)^- = \frac{x_+-x_0-LE[\rho 1_{\rho>c_2^*}]}{E[\rho 1_{c^*<\rho\le c_2^*}]}1_{c^*<\rho\le c_2^*} + L1_{\rho>c_2^*}$.

(ii) *If $(c^*, c_2^*, x_+^*)$ is optimal to Problem (22) and $X_+^*$ is optimal to Problem (8) with parameters $(\{\rho \leq c^*\}, x_+^*)$, then the optimal solution to (3) can be represented as*

$$X^* = X_+^* 1_{\rho \leq c^*} - \frac{x_+ - x_0 - LE[\rho 1_{\rho > c_2^*}]}{E[\rho 1_{c^* < \rho \leq c_2^*}]} 1_{c^* < \rho \leq c_2^*} - L 1_{\rho > c_2^*}. \tag{23}$$

**Proof**: The proof is the same as that for Jin and Zhou (2008), Theorem 4.1. □

The optimal terminal wealth profile (23) is obtained explicitly, which depends on the solution to a three-dimensional mathematical program (22). In some cases the solutions can be derived analytically; see the next section for examples. Most importantly though, the expression (23) reveals interesting qualitative features different from that without loss control, obtained in Jin and Zhou (2008). The future world is divided into *three*, instead of two, classes of states: the good, the moderately bad, and the bad. In the good states ($\rho \leq c^*$), the agent obtains a gain $X_+^*$, a random variable which generally takes different values in different good states. In the moderately bad states ($c^* < \rho \leq c_2^*$), there is a *constant*, moderate loss $\frac{x_+ - x_0 - LE[\rho 1_{\rho > c_2^*}]}{E[\rho 1_{c^* < \rho \leq c_2^*}]}$. In a bad state, the agent incurs the maximum loss $L$. The corresponding trading strategy is still gambling: the agent sells two contingent claims, corresponding to the two lower classes of states, in order to raise funds and gamble on the high states of the world. So the agent will still take leverage if her reference point is high, but she will be more cautious in doing so – by differentiating the loss states and controlling (indirectly) the leverage level.

# 6 An Example with Two-Piece Power Utility

We take the same benchmark example as in Jin and Zhou (2008), Section 9, except now that we have an additional explicit bound for losses. The pricing kernel is lognormal with $\underline{\rho} = 0$, $\bar{\rho} = +\infty$, and the utility function is the one proposed by Tversky and Kahneman (1992), namely, $u_+(x) = x^\alpha$, $u_-(x) = k_- x^\alpha$, where $0 < \alpha < 1$ and $k_- > 0$. For simplicity, we take the interest rate as 0 and so $E[\rho] = 1$. The positive part problem (8) in this example (where $A = \{\rho \leq c\}$) has been solved explicitly by Jin and Zhou (2008) with the following results:

$$X_+^*(c, x_+) = \frac{x_+}{\varphi(c)} \left( \frac{T_+'(F(\rho))}{\rho} \right)^{\frac{1}{1-\alpha}} 1_{\rho \leq c}, \ 0 < c \leq \infty, \ x_+ \geq x_0^+, \tag{24}$$

and

$$v_+(c, x_+) = \varphi(c)^{1-\alpha} x_+^\alpha, \ 0 < c \leq +\infty, \ x_+ \geq x_0^+ \tag{25}$$

where

$$\varphi(c) := E\left[ \left( \frac{T_+'(F(\rho))}{\rho} \right)^{1/(1-\alpha)} \rho 1_{\rho \leq c} \right] > 0, \ 0 < c \leq +\infty.$$

Let

$$\tilde{\varphi}(c) = \begin{cases} \varphi(c), & \text{if } 0 < c \le +\infty, \\ 0, & \text{if } c = 0. \end{cases}$$

Then Problem (22) specializes to

$$\begin{aligned} \text{Maximize} \quad & v(c, c_2, x_+) \\ =& \tilde{\varphi}(c)^{1-\alpha} x_+^{\alpha} - k_- \left( \frac{x_+ - x_0 - LE[\rho 1_{\rho > c_2}]}{E[\rho 1_{c < \rho \le c_2}]} \right)^{\alpha} [T_-(P(\rho > c)) - T_-(P(\rho > c_2))] \\ & - k_- L^{\alpha} T_-(P(\rho > c_2)), \\ \text{subject to} \quad & \begin{cases} 0 \le c \le c_2 \le +\infty, \ \max\{x_0^+, x_0 + LE[\rho 1_{\rho > c_2}]\} \le x_+ \le x_0 + LE[\rho 1_{\rho > c}], \\ x_+ = 0 \text{ when } c = 0. \end{cases} \end{aligned} \tag{26}$$

Note that in the above, the original inequality constraint $c \le c_0(x_+) \le c_2$ is replaced by $x_0 + LE[\rho 1_{\rho > c_2}] \le x_+ \le x_0 + LE[\rho 1_{\rho > c}]$.

Denote $g_1(c_2) := P(\rho > c_2)$, $g_2(c_2) := E[\rho 1_{\rho > c_2}]$. Then define $g(x) := g_2(g_1^{-1}(x))$ and $h(x) := T_-(g^{-1}(x))$, $x \ge 0$. It is easy to verify that $g(\cdot)$ is concave and $h(\cdot)$ is nondecreasing satisfying $h(0) = 0$, $h(1) = 1$, and $T_-(x) = h(g(x))$.

**Theorem 5.** *If $h(\cdot)$ is convex, then the optimal solution to Problem (3) is*

$$X^* = \frac{x_0 + LE[\rho 1_{\rho > c_2^*}]}{\varphi(c_2^*)} \left( \frac{T'_+(F(\rho))}{\rho} \right)^{\frac{1}{1-\alpha}} 1_{\rho \le c_2^*} - L 1_{\rho > c_2^*} \tag{27}$$

*where $c_2^*$ (possibly infinite) is the optimal solution to*

$$\begin{aligned} \textit{Maximize} \quad & \tilde{\varphi}(c_2)^{1-\alpha} (x_0 + LE[\rho 1_{\rho > c_2}])^{\alpha} - k_- L^{\alpha} T_-(P(\rho > c_2)), \\ \textit{subject to} \quad & 0 \le c_2 \le +\infty, \ \ x_0 + LE[\rho 1_{\rho > c_2}] \ge x_0^+. \end{aligned} \tag{28}$$

**Proof**: Define

$$\begin{aligned} s(c, c_2) : \ &= \frac{T_-(P(\rho > c)) - T_-(P(\rho > c_2))}{E[\rho 1_{c < \rho \le c_2}]} \\ &= \frac{h(E[\rho 1_{\rho > c}]) - h(E[\rho 1_{\rho > c_2}])}{E[\rho 1_{\rho > c}] - E[\rho 1_{\rho > c_2}]}, \ \ 0 \le c \le c_2 \le +\infty, \end{aligned}$$

where we use the convention that $s(c, c_2) := 0$ when $c = c_2$. Then $s(c, c_2)$ is a nonincreasing function of $c$, by virtue of the convexity of $h(\cdot)$. Rewrite

$$\begin{aligned} & v(c, c_2, x_+) \\ = \ & \tilde{\varphi}(c)^{1-\alpha} x_+^{\alpha} - k_- \left[ (x_+ - x_0 - LE[\rho 1_{\rho > c_2}]) s(c, c_2) \right]^{\alpha} [T_-(P(\rho > c)) - T_-(P(\rho > c_2))]^{1-\alpha} \\ & - k_- L^{\alpha} T_-(P(\rho > c_2)). \end{aligned}$$

We see that $v(c, c_2, x_+)$ is nondecreasing in $c \in [0, c_2]$. Therefore at optimum $c = c_2$, $x_+ = x_0 + LE[\rho 1_{\rho > c_2}]$, and Problem (26) reduces to (28), which is a simple one-dimensional optimization problem in $c_2 \in [0, +\infty]$. The expression (27) then follows from the general one (23). $\square$

It is interesting to note that Berkelaar, Kouwenberg and Post (2004) derive an optimal wealth profile similar to (27): the agent gets either a gain or the maximum loss, depending on the states of the world. However, there is no probability distortion present in Berkelaar, Kouwenberg and Post (2004). Here, we have shown that for almost arbitrary probability distortions $T_+$ on gains and a large class of distortions $T_-$ on losses (so long as the corresponding $h(\cdot)$ is convex), the same qualitative trading behaviors of an behavioral agent prevail.

We now demonstrate why, at least in theory, a large class of distortion functions $T_-(\cdot)$, be they concave, convex or even inverse $S$-shaped, will lead to $h(\cdot)$ being convex. First, take $T_-(x) = g(x)$ which is a concave distortion. Then $h(x) = x$, which is trivially convex. Next, take $T_-(x) = x^2$, a convex function. Then $h(x) = [g^{-1}(x)]^2$ is again convex. Finally, we can construct a reverse $S$-shaped $T_-(\cdot)$ so that the corresponding $h(\cdot)$ is convex. The construction is based on a "reversed" procedure, namely, we start with a proper convex $h(\cdot)$ and then end with an $S$-shaped $T_-(\cdot)$. Specifically, for a given constant $c_0 \in (0, 1)$, let $\delta \in \left(0, \frac{2c_0}{2c_0 g(c_0) + g'(c_0)(1-c_0^2)}\right]$, and

$$h(x) := \begin{cases} \delta x, & x < g(c_0), \\ 1 - \frac{1-\delta g(c_0)}{1-c_0^2}(1 - g^{-1}(x)^2), & x \geq g(c_0). \end{cases}$$

Then

$$h'(x) = \begin{cases} \delta, & x < g(c_0), \\ \frac{2(1-\delta g(c_0))}{1-c_0^2} \frac{g^{-1}(x)}{g'(g^{-1}(x))}, & x > g(c_0), \end{cases}$$

which is a nondecreasing function. Thus $h(\cdot)$ is convex. Accordingly,

$$T_-(x) = h(g(x)) = \begin{cases} \delta g(x), & x < c_0, \\ 1 - \frac{1-\delta g(c_0)}{1-c_0^2}(1 - x^2), & x \geq c_0, \end{cases}$$

which is concave on $[0, c_0]$ and convex on $[c_0, 1]$, hence reverse $S$-shaped.

So, Theorem 5 covers a large class of loss distortions. However, the convexity assumption on $h(\cdot)$ may be violated in many other cases. Here let us take an example. Let $T_-(x) = g(x)^\beta$, where $\beta \in (0, 1)$. In this case $h(x) = x^\beta$ is a concave function. To solve this problem, first note that (26) now reduces to

$$\text{Maximize} \qquad v(c,c_2,x_+)=\tilde{\varphi}(c)^{1-\alpha}x_+^{\alpha}$$
$$-k_-\left(\frac{x_+-x_0-LE[\rho 1_{\rho>c_2}]}{E[\rho 1_{c<\rho\le c_2}]}\right)^{\alpha}\left[(E[\rho 1_{\rho>c}])^{\beta}-(E[\rho 1_{\rho>c_2}])^{\beta}\right]-k_-L^{\alpha}(E[\rho 1_{\rho>c_2}])^{\beta},$$
$$\text{subject to } \begin{cases}0\le c\le c_2\le+\infty,\ \max\{x_0^+,x_0+LE[\rho 1_{\rho>c_2}]\}\le x_+\le x_0+LE[\rho 1_{\rho>c}],\\ x_+=0 \text{ when } c=0.\end{cases}$$
(29)

Denote $y_2:=(E[\rho 1_{\rho>c_2}])^{\beta}$ as a new variable in lieu of $c_2$. By the constraints in (29), $y_2$ takes value in $[0,\left(\frac{x_+-x_0}{L}\right)^{\beta}]$. Now, the following function

$$\hat{v}(y_2):=\tilde{\varphi}(c)^{1-\alpha}x_+^{\alpha}-k_-\left(\frac{x_+-x_0-Ly_2^{1/\beta}}{E[\rho 1_{\rho>c}]-y_2^{1/\beta}}\right)^{\alpha}(E[\rho 1_{\rho>c}]^{\beta}-y_2)-k_-L^{\alpha}y_2 \quad (30)$$

is a *convex* function in $y_2\in[0,\left(\frac{x_+-x_0}{L}\right)^{\beta}]$ (see Appendix for a proof); so its *maximum* is achieved either at $y_2=0$ or $y_2=\left(\frac{x_+-x_0}{L}\right)^{\beta}$. Simple calculations shows

$$\hat{v}(0)-\hat{v}\left(\left(\frac{x_+-x_0}{L}\right)^{\beta}\right)\begin{cases}>0, & \text{if } \beta<\alpha,\\ <0, & \text{if } \beta>\alpha,\\ =0, & \text{if } \beta=\alpha.\end{cases} \quad (31)$$

Hence the optimal $c_2^*$ for (29) is, taking into consideration the constraints:

$$c_2^*=\begin{cases}+\infty, & \text{if } \beta<\alpha,\\ c_0(x_+), & \text{if } \beta>\alpha,\\ +\infty \text{ or } c_0(x_+), & \text{if } \beta=\alpha,\end{cases}$$

where we recall that $c_0(x_+)$ is such that $\int_{c_0(x_+)}^{+\infty}xdF(x)=\frac{x_+-x_0}{L}$.

The final solutions to the underlying behavioral portfolio choice problem (3) depend on the relation between $\alpha$ and $\beta$.

*Case when* $\beta\ge\alpha$: in this case it is necessary that $c_2=\bar{c}(x_+)$ at optimum; so (29) is rewritten as

$$\text{Maximize} \qquad v(c,c_2,x_+)=\tilde{\varphi}(c)^{1-\alpha}x_+^{\alpha}-k_-L^{\alpha}E[\rho 1_{\rho>c_2}]^{\beta},$$
$$\text{subject to } \begin{cases}0\le c\le c_2\le+\infty,\ x_+=x_0+LE[\rho 1_{\rho>c_2}]\ge 0,\\ x_+=0 \text{ when } c=0,\ x_+=x_0 \text{ when } c=+\infty.\end{cases} \quad (32)$$

By the monotonicity of $\tilde{\varphi}(c)$, it is obvious that at optimum $c^*=c_2^*$ and $x_+^*=x_0+LE[\rho 1_{\rho>c_2^*}]$. The optimal solution of Problem (3) is

$$X^*=\frac{x_0+LE[\rho 1_{\rho>c_2^*}]}{\varphi(c_2^*)}\left(\frac{T_+'(F(\rho))}{\rho}\right)^{\frac{1}{1-\alpha}}1_{\rho\le c_2^*}-L1_{\rho>c_2^*}. \quad (33)$$

Interestingly, the above is exactly the same as (27).

*Case when* $\beta < \alpha$: in this case optimal $c_2^* = +\infty$ and Problem (29) is

$$\begin{aligned} &\text{Maximize} \quad v(c, c_2, x_+) = \tilde{\varphi}(c)^{1-\alpha} x_+^\alpha - k_-(x_+ - x_0)^\alpha E[\rho 1_{\rho > c}]^{\beta-\alpha}, \\ &\text{subject to} \begin{cases} 0 \le c \le +\infty, \ x_0^+ \le x_+ \le x_0 + LE[\rho 1_{\rho>c}], \\ x_+ = 0 \text{ when } c = 0, \ x_+ = x_0 \text{ when } c = +\infty. \end{cases} \end{aligned} \tag{34}$$

Let $(x_+^*, \ c^*)$ be the optimal solution of (34), then

$$X^* = \frac{x_+}{\varphi(c_2^*)} \left( \frac{T'_+(F(\rho))}{\rho} \right)^{\frac{1}{1-\alpha}} 1_{\rho \le c^*} - \frac{x_+ - x_0}{E[\rho 1_{\rho > c^*}]} 1_{\rho > c^*}. \tag{35}$$

solves (3). Note that in this case, although qualitatively the solution is similar to that of (27) or (33) (namely, the terminal wealth has a two-piece structure), there is an intriguing difference. In (35), the lower loss bound $L$ is never realised; only moderat losses are incurred in bad states.

Although the examples so far all have optimal terminal wealth profiles with two – instead of three – pieces (one gain part and one loss part), it is easy to find a three-piece solution as indicated in the general result, Theorem 4. We use a numerical example. Take $x_0 = -1$, $L = 10$, $\beta = 0.85$, $\alpha = 0.88$, $k = 2.25$, and $\log(\rho) \sim N(-0.045, 0.09)$. The values of $\alpha$ and $k$ are taken from Kahneman and Tversky (1992), and the agent starts with a loss (due to a high reference point). The probability distortions are $T_+(x) = x$, and $T_-(x) = h(g(x))$ where

$$h(x) = \begin{cases} 0.5x, & 0 \le x \le \frac{-x_0}{2L}, \\ \frac{2L}{-x_0} \left( (-\frac{x_0}{L})^\beta (x + \frac{x_0}{2L}) + \frac{-x_0}{4L} (\frac{-x_0}{L} - x) \right), & \frac{-x_0}{2L} < x \le \frac{-x_0}{L}, \\ x^\beta, & \frac{-x_0}{L} < x \le 1. \end{cases} \tag{36}$$

Numerically solving the mathematical program (26) we obtain $c^* = 0.47834$, $c_2^* = 1.7126$, and $x_+^* = 0.0035334$. So

$$X^* = \begin{cases} 0.00049434 \rho^{-8.3333}, & \rho \le 0.47834, \\ -1.04153, & 0.47834 < \rho \le 1.7126, \\ -10, & \rho > 1.7126. \end{cases} \tag{37}$$

Figure 1 depicts the optimal terminal wealth $X^*$ as a function of the pricing kernel $\rho$. So when the market is good ($\rho \le 0.47834$), the agent gets substantial gains (and the gains soar as $\rho$ becomes smaller[6]). When the market is between good and bad ($0.47834 < \rho \le 1.7126$), the agent ends with a constant moderate loss $X^* = -1.04153$ (compare with the initial $x_0 = -1$). If the market is bad ($\rho > 1.7126$), the agent has the maximum loss $X^* = -10$ no matter how bad the market might be.

[6]For example, $X^* = 0.2306$ when $\rho = 0.47834$, $X^* = 6.7655$ when $\rho = 0.47834/1.5$, $X^* = 74.3788$ when $\rho = 0.47834/2$, and $X^* = 2182$ when $\rho = 0.47834/3$.

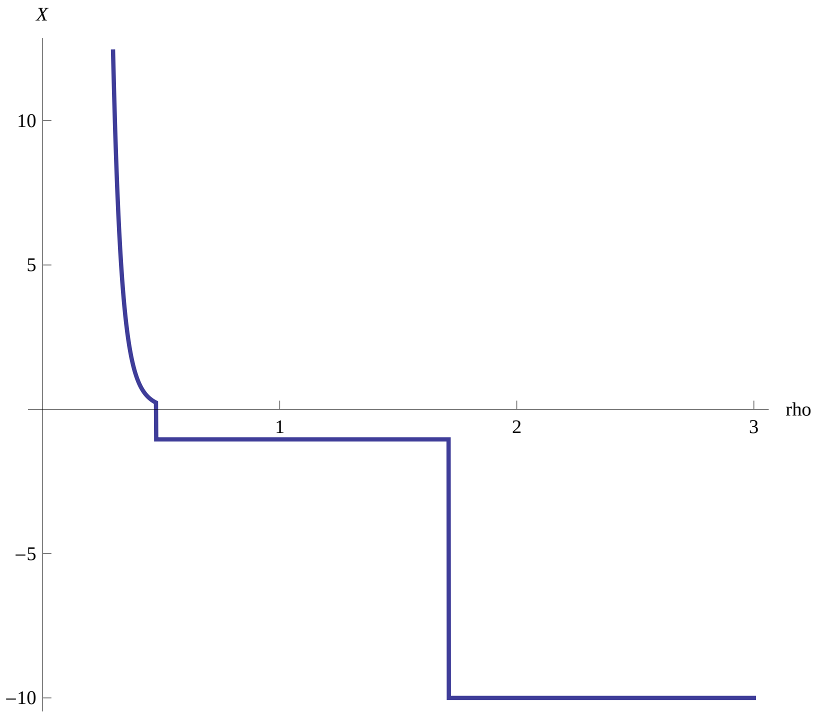


Figure 1: Three-Piece Solution

# 7 Concluding Remarks

Motivated by a companion paper Jin and Zhou (2009) which shows that a behavioral agent may suffer catastrophic losses if there is no constraint on potential losses and/or on leverage level, this paper investigates a behavioral CPT portfolio choice model where the losses are *a priori* constrained. The mathematical contribution of the paper lies in solving completely an associated, unconventional Choquet minimization problem with both upper and lower constraints, in an infinite dimensional space (the space of quantile functions). Economically, the final solution exhibits quite different trading behaviors compared with their unconstrained counterparts: while the agent is still gambling (on the good states of the world), she is more cautious in taking leverage so as to meet the regulation on losses. The paper demonstrates that *constrained* behavioral portfolio selection problems are both mathematically interesting (as well as challenging) and economically sensible.

# Appendix

## A Proof of The Convexity of $\hat{v}(y_2)$

In this appendix we prove that the function $\hat{v}$ defined in (30) is convex in $y_2$. Denote $y := E[\rho 1_{\rho>c}]$ and $\bar{y} := \frac{x_+ - x_0}{L}$. It suffices to prove that the function

$$I(y_2) := \left(\frac{\bar{y} - y_2^{1/\beta}}{y - y_2^{1/\beta}}\right)^{\alpha} (y^\beta - y_2) + y_2 \tag{38}$$

is concave in $y_2 \in [0, \bar{y}^\beta]$. Define $p(y_2) := \left(\frac{\bar{y}-y_2^{1/\beta}}{y-y_2^{1/\beta}}\right)^{\alpha}$ and $q(y_2) := \frac{\bar{y}-y_2^{1/\beta}}{y-y_2^{1/\beta}}$. Then we have the following results:

$$I'(y_2) = p'(y_2)(y^\beta - y_2) - p(y_2) + 1; \tag{39}$$

$$I''(y_2) = (y^\beta - y_2)p''(y_2) - 2p'(y_2); \tag{40}$$

$$\begin{aligned}
p'(y_2) &= \alpha q(y_2)^{\alpha-1} q'(y_2) = \alpha p(y_2) q'(y_2)/q(y_2);\\
p''(y_2) &= \alpha^2 p(y_2)\left(\frac{q'(y_2)}{q(y_2)}\right)^2 + \alpha p(y_2)\left(\frac{q'(y_2)}{q(y_2)}\right)';\\
\frac{q'(y_2)}{q(y_2)} &= -\frac{y_2^{1/\beta-1}}{\beta}\left(\frac{1}{\bar{y} - y_2^{1/\beta}} - \frac{1}{y - y_2^{1/\beta}}\right) < 0;\\
\left(\frac{q'(y_2)}{q(y_2)}\right)' &= \frac{q'(y_2)}{y_2 q(y_2)}\left(\frac{1}{\beta} - 1 + \frac{y_2^{1/\beta}}{\beta}\left(\frac{1}{\bar{y} - y_2^{1/\beta}} + \frac{1}{y - y_2^{1/\beta}}\right)\right).
\end{aligned}$$

Substituting the last four equalities into (40) and simplifying it gives

$$\begin{aligned}
&I''(y_2)\\
&= \alpha p(y_2)\frac{q'(y_2)}{q(y_2)}\left\{(y^\beta - y_2)\left[\alpha\frac{q'(y_2)}{q(y_2)} + \frac{1}{y_2}\left(\frac{1}{\beta} - 1 + \frac{y_2^{\frac{1}{\beta}}}{\beta}\left(\frac{1}{\bar{y} - y_2^{1/\beta}} + \frac{1}{y - y_2^{1/\beta}}\right)\right)\right] - 2\right\}\\
&\le \alpha p(y_2)\frac{q'(y_2)}{q(y_2)}\left\{(y^\beta - y_2)\left[\frac{q'(y_2)}{q(y_2)} + \frac{1}{y_2}\left(\frac{1}{\beta} - 1 + \frac{y_2^{\frac{1}{\beta}}}{\beta}\left(\frac{1}{\bar{y} - y_2^{1/\beta}} + \frac{1}{y - y_2^{1/\beta}}\right)\right)\right] - 2\right\}\\
&= \alpha p(y_2)\frac{q'(y_2)}{q(y_2)}\frac{(1-\beta)(y^{\beta+1} - y_2^{1+1/\beta}) + (1+\beta)y^\beta y_2(y_2^{1/\beta-1} - y^{1-\beta})}{\beta y_2(y - y_2^{1/\beta})}.
\end{aligned} \tag{41}$$

Denote $l(y_2) := (1-\beta)(y^{\beta+1} - y_2^{1+1/\beta}) + (1+\beta)y^\beta y_2(y_2^{1/\beta-1} - y^{1-\beta})$, $y_2 \in [0,\ y^\beta]$. It is easy to verify that $l(y^\beta) = 0$, $l'(y^\beta) = 0$, and $l''(y_2) = \frac{1-\beta^2}{\beta^2}(y^\beta - y_2)y_2^{1/\beta-2} > 0$ $\forall y_2 \in [0,\ y^\beta]$. It follows that $l'(y_2) < l'(y^\beta) = 0$, and hence $l(y_2) > 0$ for $\forall y_2 \in [0, \bar{y}^\beta] \subseteq [0,\ y^\beta]$. Since $\frac{q'(y_2)}{q(y_2)} < 0$, we have $I''(y_2) < 0$ $\forall y_2 \in [0, \bar{y}^\beta]$, which completes the proof. $\square$

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