Erratum to "Behavioral Portfolio Selection in Continuous Time"*

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Abstract

We fill a gap in the proof of a (rather critical) lemma, Lemma B.1, in Jin and Zhou [Mathematical Finance, Vol. 18 (2008), pp. 385–426]. We also correct a couple of other minor errors in the same paper.

Keywords. portfolio selection, continuous time, cumulative prospect theory, behavioral criterion, S-shaped function, probability distortion

In our paper "Behavioural Portfolio Selection in Continuous Time", *Mathematical Finance*, Vol. 18, No. 3, pp. 385–426, July 2008, a central idea for overcoming the difficulty arising from probability distortions is to change the decision variable from the (terminal) cash flow – which is a random variable – to its quantile function. The idea is based on the following reasoning: if the preference measure (to be minimized or maximized) in the underlying model is law-invariant (which is inherently true for the behavioural model under prospect theory, as well as for many other models), then one could freely swap around the cash flows so long as their distributions are the same. Therefore, in order to find an optimal cash flow one

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only needs to search among those that maximize or minimize – depending on whether the performance measure is to be minimized or maximized – the costs of the same distributional classes. This leads to the following optimization problems (Problems (B1) and (B2) in Jin and Zhou 2008):

$$\begin{aligned} &\text{Maximize}_X \quad E[XY] \\ &\text{subject to} \quad P(X \le x) = G(x) \;\; \forall x \in \mathbb{R}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\text{Minimize}_X \quad E[XY] \\ &\text{subject to} \quad P(X \le x) = G(x) \;\; \forall x \in \mathbb{R}, \end{aligned} \tag{2}$$

where $Y > 0$ a.s. and G is a probability distribution function[1], both given.

These problems were first considered by Dybvig (1988) for a finite probability space.[2] Theorem B.1 in Jin and Zhou (2008) gives complete, explicit solutions to the above two problems for general probability spaces. These solutions form the foundation of the quantile formulations in solving the positive- and negative-part problems in Jin and Zhou (2008). The quantile-based optimization is proposed by Schied (2004, 2005) to solve a class of convex, robust portfolio selection problems, and employed by Dana (2005) and Carlier and Dana (2006) to study calculus of variations problems with law-invariant concave criteria. Recently, the quantile approach is systematically developed by He and Zhou (2009) into a general paradigm in solving non-expected, non-convex/concave utility maximization models, including both neoclassical and behavioral ones.

In Jin and Zhou (2008), Theorem B.1 is proved based upon Lemma B.1. However, there is a gap in the proof of the latter. The proof of Lemma B.1-(i) implicitly assumes that $h(0) > -\infty$ (or in the case of Lemma B.1-(ii), $h(0) < +\infty$). However, when applying Lemma B.1 to prove Theorem B.1, it is likely that those assumptions are invalid. In particular, in the proof of Theorem B.1-(ii), the nonincreasing function h is taken as $h(x) = G^{-1}(1 - F(x))$, where F is the probability distribution function of Y while G that of X. So it is likely that $h(0) = G^{-1}(1) = +\infty$.

Although this is only a technical gap in Jin and Zhou (2008), the results are so important that an erratum is justified. We shall fill the gap by taking into consideration the possibility that $h(0) = -\infty$ in Lemma B.1-(i) (or equivalently $h(0) = +\infty$ in Lemma B.1-(ii)).

Henceforth we write $X_1 \sim X_2$ if the two random variables X_1 and X_2 have the same distribution. The following is a re-statement of Lemma B.1 along with its proof.

[1] In Jin and Zhou (2008), G is given satisfying $G(0) = 0$. This assumption is however not needed.

[2] Despite its title, Dybvig (1988) does not formulate or solve any specific class of portfolio choice problems per se. Instead, it is concerned with the *dual problem* of portfolio choice, namely, to characterise the lowest cost of any given terminal distribution. We were not aware of Dybvig's result while we were working on Jin and Zhou (2008).

LEMMA 1 *Given a random variable $Y > 0$ a.s. with $EY < +\infty$.*

(i) *Let $h(\cdot)$ be a nondecreasing function on $[0, +\infty)$. If $X \sim h(Y)$, then $E[XY] \le E[h(Y)Y]$. On the other hand, if $-\infty < E[XY] = E[h(Y)Y] < +\infty$, then $X \in [h(Y-), h(Y+)]$* a.s..

(ii) *Let $h(\cdot)$ be a nonincreasing function on $[0, +\infty)$. If $X \sim h(Y)$, then $E[XY] \ge E[h(Y)Y]$. On the other hand, if $-\infty < E[XY] = E[h(Y)Y] < +\infty$, then $X \in [h(Y+), h(Y-)]$* a.s..

Proof: (i) First assume $h(0+) = 0$. Employing Lemma A.1 in Jin and Zhou (2008), together with the assumption that $X \sim h(Y)$, we have

$$\begin{aligned} E[XY] &\le E[\int_0^X h^{-1}(u)du] + E[\int_0^Y h(u)du] \\ &= E[\int_0^{h(Y)} h^{-1}(u)du] + E[\int_0^Y h(u)du] = E[h(Y)Y]. \end{aligned}$$

If the equality holds and $|E[h(Y)Y]| < +\infty$, then $X \in [h(Y-), h(Y+)]$ a.s..

For the more general case when $0 \ne h(0+) > -\infty$, we assume $E[h(Y)Y] < +\infty$ (for otherwise the inequality holds trivially). Define $\bar{h}(x) := h(x) - h(0+)$. Then

$$E[XY] = E[(X - h(0+))Y] + h(0+)EY \le E[\bar{h}(Y)Y] + h(0+)EY = E[h(Y)Y].$$

Moreover, the equality holds only if $X - h(0+) \in [\bar{h}(Y-), \bar{h}(Y+)]$, or $X \in [h(Y-), h(Y+)]$.

If $h(0+) = -\infty$, then for any integer $n \ge 1$, define $h_n(y) := h(y) \vee (-n)$. Then $h_n(Y) \sim X \vee (-n)$, and $h_n(0+) > -\infty$; so

$$E[(X \vee (-n))Y] \le E[h_n(Y)Y].$$

The inequality $E[XY] \le E[h(Y)Y]$ is trivial when $E[h(Y)Y] = +\infty$. So consider only the case when $E[h(Y)Y] < +\infty$, in which case $h_0(Y)Y = [h(Y)Y]^+$ and hence $E[h_0(Y)Y] < +\infty$. By monotone convergence theory, we have $\lim_{n\to+\infty} E[h_n(Y)Y] = E[h(Y)Y]$. Consequently, $E[XY] \le \lim_{n\to+\infty} E[(X \vee (-n))Y] \le \lim_{n\to+\infty} E[h_n(Y)Y] = E[h(Y)Y]$.

Next, assume $-\infty < E[XY] = Eh(Y)Y] < +\infty$. Fix any $y_0 > 0$ and let $k := h(y_0+)$. Define

$$\underline{h}_k(y) := h(y) \wedge k, \quad \bar{h}_k(y) := h(y) \vee k.$$

Clearly $\underline{h}_k(Y) \sim X \wedge k$ and $\bar{h}_k(Y) \sim X \vee k$. Hence

$$\begin{aligned} E[XY] &= E[(X \wedge k)Y] + E[(X \vee k)Y] - kEY \\ &\le E[\underline{h}_k(Y)Y] + E[\bar{h}_k(Y)Y] - kEY \\ &= E[h(Y)Y], \end{aligned}$$

and the equality holds only if $-\infty < E[\bar{h}_k(Y)Y] = E[(X \vee k)Y] < +\infty$. Since $\bar{h}_k(0+) > -\infty$, we can apply the proved result to conclude that $X \vee k \in [\bar{h}_k(Y-), \bar{h}_k(Y+)]$, which is valid for $k = h(y_0+)$ with any $y_0 > 0$.

Note that $k \to h(0+) = -\infty$ as $y_0 \to 0$, and $P(X > -\infty) = P(h(Y) > -\infty) = P(Y > 0) = 1$. Fixing $\omega \in \Omega_0$ where Ω_0 is a proper subset of Ω with full measure, we choose $y_0 > 0$ sufficiently small so that $Y(\omega) > y_0$ and $X(\omega) > k$. In this case $X(\omega) \vee k \in [\bar{h}_k(Y(\omega)-), \bar{h}_k(Y(\omega)+)]$ reduces to $X(\omega) \in [h(Y(\omega)-), h(Y(\omega)+)]$.

(ii) It is straightforward by applying the result in (i) to $-X$ and $-h(Y)$. *Q.E.D.*

It should be mentioned that the above lemma is closely related to the so-called Hardy–Littlewood's inequality, which appeared in the book Hardy and Littlewood (1952), p. 278, in an integral form.

Next, for the benefit of the reader, we reproduce Theorem B.1 of Jin and Zhou (2008) and its proof (with some slight modifications).

THEOREM 1 *Assume that $Y > 0$ a.s. having no atom, with $EY < +\infty$.*

(i) *Define $X_1^* := G^{-1}(F(Y))$. Then $E[X_1^*Y] \geq E[XY]$ for any feasible solution X of Problem (1). If in addition $-\infty < E[X_1^*Y] < +\infty$, then X_1^* is the unique (in the sense of almost surely) optimal solution for (1).*

(ii) *Define $X_2^* := G^{-1}(1 - F(Y))$. Then $E[X_2^*Y] \leq E[XY]$ for any feasible solution X of Problem (2). If in addition $-\infty < E[X_2^*Y] < +\infty$, then X_2^* is the unique optimal solution for (2).*

Proof: First of all note that $Z := F(Y)$ follows uniform distribution on the (open or closed) unit interval.

(i) Define $h_1(x) := G^{-1}(F(x))$. Then $P\{h_1(Y) \leq x\} = P\{Z \leq G(x)\} = G(x)$, and $h_1(\cdot)$ is non-decreasing. By Lemma 1, $E[X_1^*Y] \geq E[XY]$ for any feasible solution X of Problem (1), where $X_1^* := h_1(Y)$. Furthermore, if $-\infty < E[X_1^*Y] < +\infty$, and there is X which is optimal for (1), then $E[XY] = E[X_1^*Y]$. By Lemma 1, $X \in [h_1(Y-), h_1(Y+)]$ a.s.. Since $h_1(\cdot)$ is non-decreasing, its set of discontinuous points is at most countable. However, Y admits no atom; hence $h_1(Y-) = h_1(Y+) = h_1(Y)$, a.s., which implies that $X = h_1(Y) = X_1^*$, a.s.. Therefore we have proved that X_1^* is the unique optimal solution for (1).

(ii) Define $h_2(x) := G^{-1}(1 - F(x))$. It is immediate that $P\{h_2(Y) \leq x\} = G(x)$, and $h_2(\cdot)$ is non-increasing. Applying Lemma 1 and a similar argument as in (i) we obtain the desired result. *Q.E.D.*

The only difference between Theorem 1 here and Theorem B.1 in Jin and Zhou (2008) is that we have an additional condition $E[X_1^*Y] > -\infty$ in Theorem 1-(i) (respectively $E[X_2^*Y] > -\infty$ in Theorem 1-(ii)). This is because here we no longer assume $G(0) = 0$; hence a random variable X with G as its distribution function is not necessarily nonnegative. Nevertheless, in the specific context of Jin and Zhou (2008) it is indeed true that $G(0) = 0$. Thus it holds automatically that $E[X_1^* Y] \geq 0$ and $E[X_2^* Y] \geq 0$.

There are two additional (minor) errors in Jin and Zhou (2008). In the proof of Theorem 9.2, the argument for Case (1) is incorrect, since $f''(x)$ is not always negative as claimed. The correct argument is as follows: If $k > 1$, then

$$f'(x) \begin{cases} > 0, \text{ if } x > \frac{-x_0}{k^{1/(1-\alpha)}-1} \\ = 0, \text{ if } x = \frac{-x_0}{k^{1/(1-\alpha)}-1} \\ < 0, \text{ if } x < \frac{-x_0}{k^{1/(1-\alpha)}-1}. \end{cases}$$

So $x^* = \frac{-x_0}{k^{1/(1-\alpha)}-1}$ is the only maximum point with the maximum value

$$f(x^*) = (x^*)^\alpha [1 - k(1 - x_0/x)^\alpha] = -(-x_0)^\alpha [k^{1/(1-\alpha)} - 1]^{1-\alpha}.$$

There is also a minor error in the proof of Theorem 9.2 for the case $\inf_{c>0} k(c) = 1$ and $k(c) > 1$ (page 411, line 13 from the bottom). The correct argument is the following:

$$\begin{aligned} \sup_{c>0, x_+ \geq x_0^+} v(c, x_+) &= -(-x_0)^\alpha \inf_{c>0} \{\varphi(c)^{1-\alpha} [k(c)^{1/(1-\alpha)} - 1]^{1-\alpha}\} \\ &\geq -(-x_0)^\alpha \inf_{c>0} \{\varphi(+\infty)^{1-\alpha} [k(c)^{1/(1-\alpha)} - 1]^{1-\alpha}\} \\ &= -(-x_0)^\alpha \varphi(+\infty)^{1-\alpha} [(\inf_{c>0} k(c))^{1/(1-\alpha)} - 1]^{1-\alpha}\} \\ &= 0. \end{aligned}$$

The subsequent reasoning in the original proof then follows through.

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