Buy Low and Sell High*

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Abstract

In trading stocks investors naturally aspire to "buy low and sell high (BLSH)". This paper formalizes the notion of BLSH by formulating stock buying/selling in terms of four optimal stopping problems involving the global maximum and minimum of the stock prices over a given investment horizon. Assuming that the stock price process follows a geometric Brownian motion, all the four problems are solved and buying/selling strategies completely characterized via a free-boundary PDE approach.

Key words: Black–Scholes market, optimal stopping, stock goodness index, value function, free-boundary PDE (variational inequality)

1 Introduction

Assume that a discounted stock price, S_t, evolves according to

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

where constants $\mu \in (-\infty, +\infty)$ and $\sigma > 0$ are the excess rate of return and the volatility rate, respectively, and $\{B_t; t > 0\}$ is a standard 1-dimension Brownian motion on a filtered probability space $(\mathbb{S}, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with $B_0 = 0$ almost surely.

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We are interested in the following optimal decisions to buy or sell the stock over a given investment horizon $[0, T]$:

$$\text{Buying:} \qquad \min_{\tau \in \mathscr{T}} \mathbb{E}\left(\frac{S_\tau}{M_T}\right), \tag{1.1}$$

$$\min_{\tau \in \mathscr{T}} \mathbb{E}\left(\frac{S_\tau}{m_T}\right); \tag{1.2}$$

$$\text{Selling:} \qquad \max_{\tau \in \mathscr{T}} \mathbb{E}\left(\frac{S_\tau}{M_T}\right), \tag{1.3}$$

$$\max_{\tau \in \mathscr{T}} \mathbb{E}\left(\frac{S_\tau}{m_T}\right); \tag{1.4}$$

where $\mathbb{E}$ stands for the expectation, $\mathscr{T}$ is the set of all $\mathscr{F}_t$-stopping time $\tau \in [0, T]$, and M_T and m_T are respectively the global maximum and minimum of the stock price on $[0, T]$, namely,

$$\begin{cases} M_T = \max_{0 \le \nu \le T} S_\nu, \\ m_T = \min_{0 \le \nu \le T} S_\nu. \end{cases} \tag{1.5}$$

Some discussions on the motivations of the above problems are in order. When an investor trades a stock she naturally hopes to "buy low and sell high (BLSH)". Since one could never be able to "buy at the lowest and sell at the highest", there could be different interpretations on the maxim BLSH depending on the meanings of the "low" and the "high". Problems (1.1)–(1.4) make precise these in terms of optimal stopping (timing). Specifically, Problem (1.1) is equivalent to

$$\max_{\tau \in \mathscr{T}} \mathbb{E}\left(\frac{M_T - S_\tau}{M_T}\right),$$

i.e., the investor attempts to *maximize* the expected *relative* error between the buying price and the highest possible stock price by choosing a proper time to buy. This is motivated by a typical investment sentiment that an investor wants to stay away, as far as possible, from the highest price when she is buying. Similarly, Problem (1.2) is to *minimize* the expected relative error between the buying price and the lowest possible stock price. In the same spirit, Problem (1.3) (resp. (1.4)) is to minimize (resp. maximize) the expected relative error between the selling price and the highest (resp. lowest) possible stock price when selling. Therefore, Problems (1.1)–(1.4) capture BLSH from various perspectives.

Optimal stock trading involving relative error is first formulated and solved in Shiryaev, Xu and Zhou (2008), where Problem (1.3) is investigated using a purely probabilistic approach. There is, however, a case that remains unsolved in Shiryaev, Xu and Zhou (2008).[1] In this paper, we will take a PDE approach, which enables us to solve not only all the cases associated with Problem (1.3), but also all the other problems (1.1), (1.2) and (1.4) simultaneously. Moreover, we will derive completely the buying/selling regions for all the problems, leading to optimal *feedback* trading strategies, ones that would respond to all the scenarios in time and in (certain) well-defined states (rather than to the ones at $t = 0$ only).

[1]It is argued in Shiryaev, Xu and Zhou (2008) that this missing case is economically insignificant. The case is subsequently covered by du Toit and Peskir (2009) employing the same probabilistic approach.

The results derived from our models are quite intuitive and consistent with the common investment practice. For the two buying problems (1.1) and (1.2), our results dictate that one ought to buy immediately or never buy depending on whether the underlying stock is "good" or "bad" (which will be defined precisely). If, on the other hand, the stock is intermediate between good and bad, then one should buy if and only if either the current stock price is sufficiently cheap compared with the historical high (for Problem (1.1)) or it has sufficiently risen from the historical low (for Problem (1.2)). For the selling problems (1.3) and (1.4), one should never sell (i.e. hold until the final date) or immediately sell depending, again, on whether the stock is "good" or "bad" (which will however be defined differently from the buying problems). If the stock is in between good and bad, then one should sell if and only if either the current stock price is sufficiently close to the historical high (for Problem (1.3)) or it is sufficiently close to the historical low (for Problem (1.4)). In particular, at time $t = 0$ the stock price is trivially both historical high and low; hence the selling strategy would be to sell at $t = 0$, suggesting that this intermediate case is "bad" after all.

The remainder of the paper is organised as follows. In Section 2 we give mathematical preliminaries needed in solving the four problems, and in Section 3 we present the main results. Some concluding remarks are given in Setion 4, while the proofs are relegated to an appendix.

2 Preliminaries

As they stand Problems (1.1)–(1.4) are not standard optimal stopping time problems since they all involve the global maximum and minimum of a stochastic process which are not adapted. In this section we first turn these problems into standard ones, and then present the corresponding free-boundary PDEs for solving them. We do these in two sub-sections for buying and selling respectively.

2.1 Buying Problems

We start with the buying problem (1.1). Denote by M_t the running maximum stock price over $[0,t]$, i.e., $M_t = \max_{0 \le \nu \le t} S_\nu$. Then, we have

$$\begin{aligned}
\mathbb{E}\left(\frac{S_\tau}{M_T}\right) &= \mathbb{E}\left(\frac{S_\tau}{\max\{M_\tau, \max_{\tau\le s\le T} S_s\}}\right) = \mathbb{E}\left(\max\left\{\frac{M_\tau}{S_\tau}, \max_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1} \\
&= \mathbb{E}\left[\mathbb{E}\left(\max\left\{\frac{M_\tau}{S_\tau}, \max_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1} \mid \mathscr{F}_\tau\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\min\left\{e^{-x}, e^{-\max_{t\le s\le T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma B_{(s-t)}\right\}}\right\} \mid \log\frac{M_\tau}{S_\tau} = x, \tau = t\right]\right] \\
&= \mathbb{E}\left[\Psi(\log\frac{M_\tau}{S_\tau}, \tau)\right] \qquad (2.1)
\end{aligned}$$

where

$$\Psi(x,t)=\mathbb{E}\left[\min\left\{e^{-x},e^{-\max_{t\le s\le T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma B_{(s-t)}\right\}}\right\}\right],\forall (x,t)\in\Omega^+,$$

and $\Omega^+=(0,+\infty)\times[0,T)$.

The expression of $\Psi(x,t)$ is as follows: [cf. Shiryaev, Xu and Zhou (2008)]

$$\Psi(x,t)=\begin{cases}\frac{3\sigma^2-2\mu}{2(\sigma^2-\mu)}e^{(\sigma^2-\mu)(T-t)}\Phi(d_1)+e^{-x}\Phi(d_2)+\frac{\sigma^2}{2(\mu-\sigma^2)}e^{\frac{2(\mu-\sigma^2)x}{\sigma^2}}\Phi(d_3) & \text{if } \mu\neq\sigma^2,\\ e^{-x}\Phi(d_2)+(1+x+\frac{\sigma^2(T-t)}{2})\Phi(d_3)-\sigma\sqrt{\frac{T-t}{2\pi}}e^{-\frac{d_3^2}{2}} & \text{if } \mu=\sigma^2,\end{cases}\tag{2.2}$$

where $d_1=\frac{-x+(\mu-\frac{3}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2=\frac{x-(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_3=\frac{-x-(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $\Phi(\cdot)=\int_{-\infty}^{\cdot}\frac{1}{\sqrt{2\pi}}e^{-\frac{s^2}{2}}ds$.

Equation (2.1) implies that (1.1) is equivalent to a standard optimal stopping problem with a terminal payoff Ψ and an underlying adapted state process

$$X_t=\log\frac{M_t}{S_t},\ \ X_0=0.$$

In view of the dynamic programming approach, we need to consider the following problem

$$V(x,t)\doteq\min_{0\le\tau\le T-t}\mathbb{E}_{t,x}\left(\Psi(X_{\tau+t},\tau+t)\right),\tag{2.3}$$

where $X_t=x$ under $\mathbb{P}_{t,x}$ with $(x,t)\in\Omega^+$ given and fixed. Obviously, the original problem is $V(0,0)=\min_{0\le\tau\le T}\mathbb{E}\left(\frac{S_\tau}{M_T}\right)$.

It is a standard exercise to show that $V(\cdot,\cdot)$, the value function, satisfies the following free-boundary PDE (also known as the variational inequalities)

$$\begin{cases}\max\{\mathscr{L}V,\ V-\Psi\}=0,\ \text{in }\Omega^+,\\ V_x(0,t)=0,\ V(x,T)=\Psi(x,T),\end{cases}\tag{2.4}$$

where the operator $\mathscr{L}$ is defined by

$$\mathscr{L}=-\partial_t-\frac{\sigma^2}{2}\partial_{xx}-(\frac{1}{2}\sigma^2-\mu)\partial_x.\tag{2.5}$$

Therefore, the buying region for Model (1.1) is

$$BR=\left\{(x,t)\in\widetilde{\Omega^+}:\ V(x,t)=\Psi(x,t)\ \right\},\tag{2.6}$$

where $\widetilde{\Omega^+}=\Omega^+\cup\{x=0\}$.

Let us now turn to the alternative buying model (1.2). Denote by m_t the running minimum stock price over $[0,t]$, i.e., $m_t=\min_{0\le\nu\le t}S_\nu$. Then, we have

$$\begin{aligned}\mathbb{E}\left(\frac{S_\tau}{m_T}\right)&=\mathbb{E}\left(\frac{S_\tau}{\min\{m_\tau,\min_{\tau\le s\le T}S_s\}}\right)=\mathbb{E}\left(\min\left\{\frac{m_\tau}{S_\tau},\min_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1}\\&=\mathbb{E}\left[\mathbb{E}\left(\min\left\{\frac{m_\tau}{S_\tau},\min_{\tau\le s\le T}\frac{S_s}{S_\tau}\right\}\right)^{-1}\mid\mathscr{F}_\tau\right]\\&=\mathbb{E}\left[\mathbb{E}\left[\max\left\{e^{-x},e^{-\min_{t\le s\le T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma B_{(s-t)}\right\}}\right\}\mid\log\frac{m_\tau}{S_\tau}=x,\tau=t\right]\right]\\&=\mathbb{E}\left[\psi(\log\frac{m_\tau}{S_\tau},\tau)\right]\end{aligned}$$

where

$$\psi(x,t) = \mathbb{E}\left[\max\left\{e^{-x}, e^{-\min_{t\leq s\leq T}\left\{(\mu-\frac{\sigma^2}{2})(s-t)+\sigma B_{(s-t)}\right\}}\right\}\right], \forall (x,t)\in\Omega^-,$$

and $\Omega^- = (-\infty, 0)\times[0,T)$.

Similar to (2.2), we can find the expression of $\psi(x,t)$, $\forall (x,t)\in\Omega^-$, as follows

$$\psi(x,t) = \begin{cases} \frac{3\sigma^2-2\mu}{2(\sigma^2-\mu)} e^{(\sigma^2-\mu)(T-t)}\Phi(d_1') + e^{-x}\Phi(d_2') + \frac{\sigma^2}{2(\mu-\sigma^2)} e^{\frac{2(\mu-\sigma^2)x}{\sigma^2}}\Phi(d_3') & \text{if } \mu\neq\sigma^2, \\ e^{-x}\Phi(d_2') + (1+x+\frac{\sigma^2(T-t)}{2})\Phi(d_3') - \sigma\sqrt{\frac{T-t}{2\pi}} e^{-\frac{(d_3')^2}{2}} & \text{if } \mu=\sigma^2, \end{cases} \tag{2.7}$$

where $d_1' = \frac{x-(\mu-\frac{3}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_2' = \frac{-x+(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$, $d_3' = \frac{x+(\mu-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$.

Thus, we define the associated value function as

$$v(x,t) = \min_{0\leq\tau\leq T-t} \mathbb{E}_{t,x}\left(\psi(X_{\tau+t}, \tau+t)\right),$$

where $X_t = \log\frac{m_t}{S_t} = x$ under $\mathbb{P}_{t,x}$ with $(x,t)\in\Omega^-$ given and fixed. The variational inequalities that $v(x,t)$ satisfies are given as follows

$$\begin{cases} \max\{\mathscr{L}v,\ v-\psi\} = 0, \ \text{in } \Omega^-, \\ v_x(0,t) = 0,\ v(x,T) = \psi(x,T), \end{cases} \tag{2.8}$$

where $\mathscr{L}$ is defined by (2.5).

The buying region for Model (1.2) is, therefore

$$BR = \left\{(x,t)\in\widetilde{\Omega^-}:\ v(x,t)=\psi(x,t)\ \right\}, \tag{2.9}$$

where $\widetilde{\Omega^-} = \Omega^-\cup\{x=0\}$.

2.2 Selling Problems

We now consider the selling problems. In a similar manner, we introduce the value function associated with problem (1.3) as follows:

$$U(x,t) \doteq \max_{0\leq\tau\leq T-t} \mathbb{E}\left(\Psi(X_{\tau+t}, \tau+t)\right), \forall (x,t)\in\Omega^+.$$

It is also easy to see that U satisfies

$$\begin{cases} \min\{\mathscr{L}U, U-\Psi\} = 0, \ \text{in } \Omega^+, \\ U_x(0,t) = 0,\ U(x,T) = \Psi(x,T), \end{cases} \tag{2.10}$$

where $\mathscr{L}$ and Ψ are as given in (2.5) and (2.2) respectively. The corresponding selling region is

$$SR = \left\{(x,t)\in\widetilde{\Omega^+}:\ U(x,t)=\Psi(x,t)\right\}. \tag{2.11}$$

For Problem (1.4), we introduce the value function

$$u(x,t) \doteq \max_{0\le\tau\le T-t} \mathbb{E}\left(\psi(X_{\tau+t},\tau+t)\right), \forall (x,t)\in\Omega^-.$$

It is easy to see that u satisfies

$$\begin{cases} \min\{\mathscr{L}u, u-\psi\}=0, \text{ in } \Omega^-, \\ u_x(0,t)=0, \ u(x,T)=\psi(x,T), \end{cases} \tag{2.12}$$

where $\mathscr{L}$ and ψ are as given in (2.5) and (2.7) respectively. The corresponding selling region is as follows:

$$SR=\left\{(x,t)\in\widetilde{\Omega^-}: \ u(x,t)=\psi(x,t)\right\}. \tag{2.13}$$

3 Optimal Buying and Selling Strategies

In this section we present the main results of our paper. Again, we divide the section into two sub-sections dealing with the buying and selling decisions respectively.

The following "goodness index" of the stock is crucial in defining whether the stock is "good", "bad" or "intermediate" for the four problems under consideration:

$$\alpha=\frac{\mu}{\sigma^2}.$$

3.1 Buying Strategies

The following result characterizes the optimal buying strategy for Problem (1.1).

Theorem 3.1 *(Optimal Buying Strategies against the Highest Price) Let BR be the buying region as defined in (2.6).*

i) If $\alpha\le 0$, then $BR=\emptyset$;

ii) If $0<\alpha<1$, then there is a monotonically decreasing boundary $x_b^*(t):[0,T)\to(0,+\infty)$ such that

$$BR=\{(x,t)\in\Omega^+ : x\ge x_b^*(t), 0\le t<T\}. \tag{3.1}$$

Moreover, $\lim_{t\to T^-} x_b^*(t)=0$, and

$$\lim_{T-t\to\infty} x_b^*(t)=\begin{cases} \frac{8(1-\alpha)}{(3-2\alpha)(2\alpha-1)} & \text{if } \frac{1}{2}<\alpha<1, \\ +\infty & \text{if } 0<\alpha\le\frac{1}{2}; \end{cases} \tag{3.2}$$

iii) If $\alpha\ge 1$, then $BR=\widetilde{\Omega^+}$.

We place the proof in Appendix B.

So, if the buying criterion is to stay away as much as possible from the highest price, then one should never buy if the stock is "bad" ($\alpha\le 0$), and immediately buy if the stock is "good" ($\alpha\ge 1$). If the stock is somewhat between good and bad ($0<\alpha<1$), then one should buy as soon as the ratio between the historical high and the current stock price, M_t/S_t, exceeds certain time-dependent level (or equivalently the current stock price is sufficiently – depending on when the time is – away in proportion from the historical high). Moreover, in this case when the terminal date is sufficiently near one should always buy.

The other buying problem (1.2) is solved in the following theorem.

Theorem 3.2 *(Optimal Buying Strategies against the Lowest Price) Let BR be the buying region as defined in (2.9).*

i) If $\alpha \leq 0$, then $BR = \emptyset$;

ii) If $0 < \alpha < 1$, then there is a monotonically increasing boundary $x_b^*(t) : [0,T) \to (-\infty, 0)$ such that

$$BR = \{(x,t) \in \Omega^- : x \leq x_b^*(t), 0 \leq t < T\}. \tag{3.3}$$

Moreover, $\lim_{t\to T^-} x_b^*(t) = 0$, and

$$\lim_{T-t\to\infty} x_b^*(t) = -\infty.$$

iii) If $\alpha \geq 1$, then $BR = \widetilde{\Omega^-}$.

We place the proof in Appendix C.

The above results suggest that, if the buying criterion is to buy at a price as close to the lowest price as possible, then one should never buy if the stock is "bad" ($\alpha \leq 0$), and immediately buy if the stock is "good" ($\alpha \geq 1$). If the stock is intermediate between good and bad ($0 < \alpha < 1$), then one should buy as soon as the ratio between the current price and the historical low, S_t/m_t, exceeds certain time-dependent level (or the current stock price is sufficiently away in proportion from the historical low). Moreover, in this case when the terminal date is sufficiently near one should always buy. On the other hand, one tend not to buy if the duration of the investment horizon is exceedingly long.

Comparing Theorems 3.1 and 3.2, we find that the two buying models (1.1) and (1.2), albeit different in formulation, produce quite similar trading behaviours. The only difference lies in the (endogenous) criteria (those in terms of M_t/S_t and S_t/m_t) to be used to trigger buying for the intermediate case $0 < \alpha < 1$.

3.2 Selling Problems

The first selling problem (1.3) is solved in the following theorem.

Theorem 3.3 *(Optimal Selling Strategies against the Highest Price) Let SR be the optimal selling region as defined in (2.11).*

i) If $\alpha > \frac{1}{2}$, then $SR = \emptyset$.

ii) If $\alpha = \frac{1}{2}$, then $SR = \{x = 0\}$.

iii) If $0 < \alpha < \frac{1}{2}$, then $\{x = 0\} \subset SR$. Moreover, there exists a boundary $x_s^*(t) : [0,T) \to [0,+\infty)$ such that

$$SR = \left\{(x,t) \in \widetilde{\Omega^+} : x \leq x_s^*(t)\right\}, \tag{3.4}$$

and $x_s^*(t) \leq x_b^*(t) < \infty$, where $x_b^*(t)$ is defined in (3.1).

iv) If $\alpha \leq 0$, then $SR = \widetilde{\Omega^+}$.

The proof is placed in Appendix D.

The above results indicate that, apart from the definitely holding case $\alpha > \frac{1}{2}$ and the immediately selling case $\alpha \leq 0$, there is an intermediate case $0 < \alpha \leq \frac{1}{2}$ where one should sell only when M_t/S_t is sufficiently small. However, at $t = 0$ this quantity is automatically the smallest; hence one should also sell at $t = 0$ if $0 < \alpha \leq \frac{1}{2}$. This therefore fills the gap case $0 < \alpha \leq \frac{1}{2}$ missing in Shiryaev, Xu and Zhou (2008).

On the other hand, extensive numerical results have shown consistently that $x_s^*(t)$ is monotonically decreasing and $x_s^*(t) > 0$ when $0 < \alpha < \frac{1}{2}$, although we are yet to be able to establish these analytically. Moveover, $\lim_{T-t\to\infty} x_s^*(t) = \frac{1}{2\alpha-1} \log\left(1 - (2\alpha - 1)^2\right)$, which can be shown as (3.2), is confirmed by our numerical results.

The following is on Problem (1.4).

Theorem 3.4 *(Optimal Selling Strategies against the Lowest Price) Let SR be the optimal selling region as defined in (2.13).*

i) If $\alpha > \frac{1}{2}$, then $SR = \emptyset$.

ii) If $\alpha = \frac{1}{2}$, then $SR = \{x = 0\}$.

iii) If $0 < \alpha < \frac{1}{2}$, then $\{x = 0\} \subset SR$. Moreover, there exists a boundary $x_s^*(t) : [0, T) \to (-\infty, 0]$ such that

$$SR = \left\{ (x, t) \in \widetilde{\Omega^-} : x \geq x_s^*(t) \right\}, \tag{3.5}$$

and $x_s^*(t) \geq x_b^*(t) > -\infty$, where $x_b^*(t)$ is defined in (3.3).

iv) If $\alpha \leq 0$, then $SR = \widetilde{\Omega^-}$.

The proof is the same as that for Theorem 3.3. We omit it here.

The trading behaviour derived from this model is virtually the same as the other selling model at time $t = 0$.

Incidentally, numerical results show that $x_s^*(t)$ is monotonically increasing. However, it is an open problem to establish $\lim_{T-t\to\infty} x_s^*(t)$ since the solution to the stationary problem is not unique.

3.3 Concluding Remarks

In this paper four stock buying/selling problems are formulated as optimal stopping problems so as to capture the investment motto "buy low and sell high". The free boundary PDE approach, as opposed to the probabilistic approach taken by Shiryaev, Xu and Zhou (2008), is employed to solve all the problems thoroughly. The optimal trading strategies derived are simple and consistent with the normal investment behaviors. For buying problems, apart from the straightforward extreme cases (depending on the quality of the underlying stock) where one should always or never buy, one ought to buy so long as the stock price has declined sufficiently from the historical high or risen sufficiently from the historical low. For selling problems, optimal strategies exhibit similar (or indeed opposite) pattens.

The paper certainly (or at least we hope to) suggest more problems than solutions. An immediate question would be to extend the geometric Brownian stock price to more complex and realistic processes.

Appendix: Proofs

A Some Transformations

Before proving the results in the paper, let us present some transformations, first introduced by Dai and Zhong (2008), which play a critical role in our analysis:

$$F(x,t) = \log\left(\Psi(x,t)\right), \ \overline{V}(x,t) = \log\left(\frac{V(x,t)}{\Psi(x,t)}\right), \ \text{and } \overline{U}(x,t) = \log\left(\frac{U(x,t)}{\Psi(x,t)}\right), \quad \text{(A.1)}$$

and introduce two lemmas which are useful for both buying and selling cases. Without loss of generality, we assume that $\sigma = 1$, as one could make a change of time $\bar{t} = \sigma^2 t$ if otherwise. Thus, we use α instead of μ in the following.

A direct calculation ([cf. Shiryaev, Xu and Zhou (2008)]) shows that $\Psi(x,t)$ satisfies

$$\begin{cases} \mathscr{L}\Psi = \Psi_x + (1-\alpha)\,\Psi, \text{ in } \Omega^+, \\ \Psi_x(0,t) = 0, \Psi(x,T) = e^{-x}. \end{cases} \quad \text{(A.2)}$$

Then, $F(x,t)$ satisfies

$$\begin{cases} -F_t - \frac{1}{2}(F_{xx} + F_x^2) + (\alpha - \frac{3}{2})F_x + (\alpha - 1) = 0, \text{ in } \Omega^+, \\ F_x(0,t) = 0, \ F(x,T) = -x. \end{cases} \quad \text{(A.3)}$$

Accordingly, (2.4) and (2.10) reduce to

$$\begin{cases} \max\{\mathscr{L}_0\,\overline{V} + F_x - (\alpha - 1), \ \overline{V}\} = 0, \ \text{ in } \Omega^+, \\ \overline{V}_x(0,t) = 0, \ \overline{V}(x,T) = 0, \end{cases} \quad \text{(A.4)}$$

and

$$\begin{cases} \min\{\mathscr{L}_0\,\overline{U} + F_x - (\alpha - 1), \ \overline{U}\} = 0, \ \text{ in } \Omega^+, \\ \overline{U}_x(0,t) = 0, \ \overline{U}(x,T) = 0, \end{cases} \quad \text{(A.5)}$$

respectively, where $\mathscr{L}_0 = -\partial_t - \frac{1}{2}\left[\partial_{xx} + (\partial_x)^2 + 2F_x\partial_x\right] + (\alpha - \frac{1}{2})\partial_x$.

As a result, BR (2.6) and SR (2.11) can be rewritten as

$$BR = \{(x,t) \in \widetilde{\Omega^+} : \overline{V} = 0\} \text{ and } SR = \{(x,t) \in \widetilde{\Omega^+} : \overline{U} = 0\}.$$

Lemma A.1 *Let $\overline{V}(x,t)$ and $\overline{U}(x,t)$ be the solutions to Problems (A.4) and (A.5), respectively. Then,*

$$\overline{V}(x,t) < \overline{U}(x,t) \ \textit{in } \Omega^+.$$

Proof. It is easy to see that $\mathscr{L}_0\overline{V} \le \mathscr{L}_0\overline{U}$ in Ω^+. Applying the strong maximum principle gives the result. □

Lemma A.2 *Suppose $F(x,t)$ is the solution to (A.3). Then $F(x,t)$ has the following properties:*

i) $-1 < F_x(x,t) < 0, \forall\ (x,t) \in \Omega^+$;
ii) $F_{xx}(x,t) \le 0, \forall\ (x,t) \in \Omega^+$;
iii) $F_{xt}(x,t) \le 0, \forall\ (x,t) \in \Omega^+$;
iv) $F_x(x,t;\alpha+\delta) \le F_x(x,t;\alpha) + \delta, \forall\ \delta > 0,\ (x,t) \in \Omega^+$;
v) $\lim_{x\to\infty} F_x(x,t) = -1, \forall\ t \in [0,T)$.

Proof. Denote $\widetilde{F}(x,t) \doteq F_x(x,t)$, and $F^{xx}(x,t) \doteq F_{xx}(x,t)$. It is easy to verify that $\widetilde{F}$ and F^{xx} satisfy

$$\begin{cases} -\widetilde{F}_t - \frac{1}{2}(\widetilde{F}_{xx} + 2\widetilde{F}\widetilde{F}_x) + (\alpha - \frac{3}{2})\widetilde{F}_x = 0, \text{ in } \Omega^+, \\ \widetilde{F}(0,t) = 0,\ \ \widetilde{F}(x,T) = -1, \end{cases}$$

and

$$\begin{cases} -F^{xx}_t - \frac{1}{2}(F^{xx}_{xx} + 2F_x F^{xx}_x + 2(F^{xx})^2) + (\alpha - \frac{3}{2})F^{xx}_x = 0, \text{ in } \Omega^+, \\ F^{xx}(0,t) \le 0,\ \ F^{xx}(x,T) = 0, \end{cases}$$

respectively. By virtue of the (strong) maximum principle, we obtain parts i) and ii).

To show part iii), let us define $F_x^{-\delta}(x,t) \doteq F_x(x,t-\delta)$. It suffices to show $G(x,t) \doteq F_x^{-\delta}(x,t) - F_x(x,t) \ge 0$, $\forall \delta \ge 0$. It is easy to verify that $G(x,t)$ satisfies

$$\begin{cases} -G_t - \frac{1}{2}\left(G_{xx} + 2F_x^{-\delta}G_x + 2F_{xx}G\right) + (\alpha - \frac{3}{2})G_x = 0, \text{ in } (-\infty,0)\times[\delta,T), \\ G(0,t) = 0,\ \ G(x,T) = F_x(x,T-\delta) + 1 \ge 0. \end{cases}$$

Thanks to the minimum principle, we see $G(x,t) \ge 0$, in $(-\infty,0]\times[\delta,T)$, which results in $F_{xt}(x,t) \le 0$.

Next we prove part iv). Denote $\widetilde{F}^\delta(x,t) \doteq F_x(x,t;\alpha+\delta)$ and $\widehat{F}(x,t) \doteq F_x(x,t;\alpha) + \delta$. Let $P = \widetilde{F}^\delta - \widehat{F}$. Then, it suffices to show $P < 0$ in Ω^+. It is easy to check that $\widetilde{F}^\delta(x,t)$ and $\widehat{F}(x,t)$ satisfy

$$\begin{cases} -\widetilde{F}^\delta_t - \frac{1}{2}(\widetilde{F}^\delta_{xx} + 2\widetilde{F}^\delta\widetilde{F}^\delta_x) + (\alpha + \delta - \frac{3}{2})\widetilde{F}^\delta_x = 0, \text{ in } \Omega^+, \\ \widetilde{F}^\delta(0,t) = 0,\ \ \widetilde{F}^\delta(x,T) = -1 \end{cases} \tag{A.6}$$

and

$$\begin{cases} -\widehat{F}_t - \frac{1}{2}(\widehat{F}_{xx} + 2\widehat{F}\widehat{F}_x) + (\alpha + \delta - \frac{3}{2})\widehat{F}_x = 0, \text{ in } \Omega^+, \\ \widehat{F}(0,t) = \delta,\ \ \widehat{F}(x,T) = -1 + \delta, \end{cases} \tag{A.7}$$

respectively. Subtracting (A.7) from (A.6), we obtain

$$\begin{cases} -P_t - \frac{1}{2}(P_{xx} + \widehat{F}P_x + \widetilde{F}^\delta_x P) + (\alpha + \delta - \frac{3}{2})P_x = 0, \text{ in } \Omega^+, \\ P(0,t) = -\delta,\ \ P(x,T) = -\delta. \end{cases}$$

Applying the maximum principle gives the desired result.

To show part iv), note that

$$\Phi(-\frac{x}{a}) \sim O\left(\frac{1}{x}e^{-\frac{x^2}{2a^2}}\right), \text{ as } x \to +\infty, \forall a > 0.$$

It follows

$$\lim_{x\to+\infty} F_x(x,t) = \lim_{x\to+\infty} \frac{\Psi_x(x,t)}{\Psi(x,t)} = \lim_{x\to+\infty} \frac{-e^{-x}\Phi(d_2) + e^{2(\alpha-1)x}\Phi(d_3)}{\Psi(x,t)} = -1,$$

which completes the proof. □

Due to Lemma A.2 part ii) and iii), we can have the following proposition.

Proposition A.3 *The variational inequality problem (A.4) has a unique solution $\overline{V}(x,t) \in W_p^{2,1}(\Omega_N^+)$, $1 < p < +\infty$, where Ω_N^+ is any bounded set in Ω^+. Moreover, $\forall (x,t) \in \Omega^+$, we have*

i) $0 \le \overline{V}_x \le 1$;

ii) $\overline{V}_t \ge 0$;

iii) $\overline{V}(x,t;\alpha) \le \overline{V}(x,t;\alpha+\delta)$ *for* $\delta > 0$;

Proof. Using the penalized approach [cf. Friedman (1982)], it is not hard to show that (A.4) has a unique solution $\overline{V}(x,t) \in W_p^{2,1}(\Omega_N^+)$, $1 < p < +\infty$, where Ω_N^+ is any bounded set in Ω^+. To prove part i), we only need to confine to the noncoincidence set $\Lambda = \{(x,t) \in \Omega^+ : \overline{V} < 0\}$. Denote $w = \overline{V}_x$ and $\omega = \overline{V}_x - 1$, then w and ω satisfy

$$\begin{cases} -w_t - \frac{1}{2}(w_{xx} + 2ww_x + 2F_{xx}w + 2F_x w_x) + (\alpha - \frac{1}{2})w_x = -F_{xx}, & \text{in } \Lambda \\ w|_{\partial\Lambda} = 0, \end{cases}$$

and

$$\begin{cases} -\omega_t - \frac{1}{2}(\omega_{xx} + 2\omega\omega_x + 2F_{xx}\omega + 2F_x\omega_x) + (\alpha - \frac{3}{2})\omega_x = 0, \ \text{in } \Lambda \\ \omega|_{\partial\Lambda} = -1, \end{cases}$$

respectively. Since $-F_{xx} \ge 0$, one can deduce $w \ge 0$ and $\omega \le 0$ in Λ by the maximum principle, which is desired.

To show part ii), we denote $\widetilde{V}(x,t) = \overline{V}(x,t-\delta)$. It suffices to show $Q(x,t) \doteq \widetilde{V}(x,t) - \overline{V}(x,t) \le 0$ in Ω^+, $\forall\ \delta \ge 0$. If it is false, then

$$\Delta = \{(x,\tau) \in \Omega : Q(x,t) > 0\} \ne \emptyset.$$

It is easy to verify that $Q(x,t)$ satisfies

$$\begin{cases} -Q_t - \frac{1}{2}\left(Q_{xx} + (\widetilde{V}_x + \overline{V}_x)Q_x + 2F_xQ_x\right) + (\alpha - \frac{1}{2})Q_x \\ \qquad\qquad\qquad\qquad \le -\delta F_{xt}(\cdot,\cdot)(\widetilde{V}_x - 1) \le 0, \qquad \text{in } \Delta, \\ Q|_{\partial\Delta} = 0, \end{cases}$$

where we have used part iii) of Lemma A.2 and $\widetilde{V}_x \le 1$. Applying the maximum principle, we get $Q \le 0$ in Δ, which contradicts the definition of Δ.

At last, let us show part iii). If it is not true, then

$$\mathscr{O} = \{(x,t) \in \Omega : H(x,t) < 0\} \ne \emptyset,$$

where $H(x,t) = \overline{V}(x,t;\alpha+\delta) - \overline{V}(x,t)$. Denote $F_x^\delta(x,t) = F_x(x,t;\alpha+\delta)$. It can be verified that

$$\begin{cases} -H_t - \frac{1}{2}(H_{xx} + H_x^2 + 2\overline{V}_x H_x + 2F_x^\delta H_x) + (\alpha + \delta - \frac{1}{2})H_x \\ \qquad\qquad\qquad\qquad \ge -(F_x^\delta - F_x - \delta)(1 - \overline{V}_x) \quad \text{in } \mathscr{O}, \\ H|_{\partial\mathscr{O}} = 0. \end{cases}$$

By part iv) in Lemma A.2, $F_x^\delta < F_x + \delta$, which, together with $\overline{V}_x \le 1$, gives

$$-(F_x^\delta - F_x - \delta)(1 - \overline{V}_x) \ge 0.$$

Again applying the maximum principle, we get $H \ge 0$ in $\mathscr{O}$, which is a contradiction with the definition of $\mathscr{O}$. The proof is complete. $\square$

B Proof of Theorem 3.1

Proof of Theorem 3.1. According to part i) in Lemma A.2 and (A.4),

$$\mathscr{L}_0\overline{V} \leq -(F_x+1)+\alpha<0, \text{ for } \alpha \leq 0.$$

Applying the strong maximum principle, we infer $\overline{V}<0$ in Ω for $\alpha \leq 0$. Part i) then follows.

If $\alpha \geq 1$, part i) in Lemma A.2 leads to

$$F_x-(\alpha-1) \leq 0.$$

So, $\overline{V}=0$ is a solution to (A.4), which implies part iii).

It remains to show part ii). Since $\overline{V}_x \geq 0$, we can define a boundary

$$x_b^*(t)=\inf\{x\in(0,\infty) : \overline{V}(x,t)=0\}, \text{ for any } t\in[0,T).$$

Due to $\overline{V}_t \geq 0$, we infer that $x_b^*(t)$ is monotonically decreasing in t. Let us prove $x_b^*(t)> 0$ for all t. If not, then there exists a $t_0<T$, such that $x_b^*(t)=0$ for all $t\in[t_0,T)$. This leads to $\overline{V}(x,t)=0$, in $(0,+\infty)\times[t_0,T)$. By (A.4), we have $\mathscr{L}_0\,\overline{V}+F_x-(\alpha-1)\leq 0$ in $(0,+\infty)\times[t_0,T)$, namely,

$$F_x \leq \alpha-1 \text{ in } (0,+\infty)\times[t_0,T),$$

which is a contradiction with $F_x(0,t)=0, \forall t>0$. Further, it can be shown that $x_b^*(t)<\infty$ in terms of the standard argument of Brezis and Friedman (1976) [cf. also the proof of Lemma 4.2 in Dai, Kwok and Wu (2004)], where $\lim_{x\to+\infty}F_x(x,t)=-1$ will be used. In addition, due to the monotonicity of $x_b^*(t)$, we deduce that $\{x=0\}\notin BR$. So, (3.1) follows.

To show $\lim_{t\to T^-}x_b^*(t)=0$, let us assume the contrary, i.e., $\lim_{t\to T^-}x_b^*(t)=x_0>0$. Then, we have

$$\mathscr{L}_0\overline{V}+F_x-(\alpha-1)=0, \ \forall\ 0< \ x<x_0, 0\leq t<T,$$

which, combined with $\overline{V}(x,T)=0$ for all x, gives

$$\overline{V}_t|_{t=T}=F_x|_{t=0}-(\alpha-1)=-\alpha<0, \ \forall\ 0<x<x_0.$$

This conflicts with $\overline{V}_t \geq 0$.

At last, we need to prove (3.2), which leads us to consider a stationary problem of (2.4). Here, we prove the case of $\frac{1}{2}<\alpha<1$, while the case of $0<\alpha\leq\frac{1}{2}$ can be done similarly.

Noting that $\lim_{T-t\to+\infty}\Psi(x,t)=0$, if $\frac{1}{2}<\alpha<1$, it follows that $\lim_{T-t\to+\infty}V(x,t)=0$,

which is not desired. Thus, we define

$$\Psi^\infty(x)\doteq\lim_{t\to\infty}\sqrt{2\pi}(T-t)^{\frac{3}{2}}e^{\frac{(\alpha-\frac{1}{2})^2}{2}(T-t)}\Psi(x,t), \text{ and } V^\infty(x)\doteq\lim_{t\to\infty}\sqrt{2\pi}(T-t)^{\frac{3}{2}}e^{\frac{(\alpha-\frac{1}{2})^2}{2}(T-t)}V(x,t).$$

A direct calculation shows

$$\Psi^\infty(x)=\frac{2\left(x+\frac{2}{3-2\alpha}\right)}{\left(\alpha-\frac{1}{2}\right)^2\left(\frac{3}{2}-\alpha\right)}e^{-(\frac{3}{2}-\alpha)x}. \tag{B.1}$$

According to (2.4), we see $V^\infty(x)$ satisfies,

$$\begin{cases} -\frac{1}{2}V^\infty_{xx} + (\alpha - \frac{1}{2})V^\infty_x - \frac{1}{2}(\alpha - \frac{1}{2})^2 V^\infty = 0, \ 0 < x < x_\infty, \\ V^\infty_x(0) = 0, \ V^\infty(x_\infty) = \Psi^\infty(x_\infty), \quad V^\infty_x(x_\infty) = \Psi^\infty_x(x_\infty), \end{cases} \tag{B.2}$$

where x_∞ is the free boundary.

It is easy to check that

$$V^\infty(x) = \begin{cases} \frac{2}{(\alpha-\frac{1}{2})^3}(\frac{2}{2\alpha-1} - x)e^{\frac{2\alpha-1}{2}x - x_\infty}, & \text{if } 0 \le x < x_\infty, \\ \Psi^\infty(x), & \text{if } x \ge x_\infty, \end{cases}$$

satisfies (B.2), where $x_\infty = \frac{8(1-\alpha)}{(3-2\alpha)(2\alpha-1)}$. The proof is complete. $\square$

C Proof of Theorem 3.2

Proof of Theorem 3.2. The proof is similar to Theorem 3.1. The only difference is that now we have $\overline{v}_x \le 0$ instead of $0 \le \overline{V}_x \le 1$, where $\overline{v}(x,t) = \log \frac{v(x,t)}{\psi(x,t)}$. Thus, we can define the free boundary as

$$x^*_b(t) = \sup\{x \in (-\infty, 0) : \overline{v}(x,t) = 0\}, \ \text{for any } t \in [0,T). \tag{C.1}$$

The existence follows immediately. Now, we turn to the asymptotic behavior of $x^*_b(t)$.

Note that $\lim_{t\to+\infty} \psi(x,t) = +\infty$, if $0 < \alpha < 1$. So, we denote

$$\psi^\infty(x) \doteq \lim_{T-t\to\infty} e^{(\alpha-1)(T-t)}\psi(x,t) = \frac{3-2\alpha}{2(1-\alpha)}, \ \text{and } v^\infty(x) \doteq \lim_{T-t\to\infty} e^{(\alpha-1)(T-t)}v(x,t).$$

According to (2.4), it is easy to see $v^\infty(x)$ satisfies

$$\begin{cases} -\frac{1}{2}v^\infty_{xx} + (\alpha - \frac{1}{2})v^\infty_x - (\alpha - 1)v^\infty = 0, \ x_\infty < x < 0, \\ v^\infty_x(0) = 0, \ v^\infty_x(x_\infty) = 0, \quad v^\infty(x_\infty) = \frac{3-2\alpha}{2(1-\alpha)}, \end{cases} \tag{C.2}$$

where x_∞ is the free boundary.

The general solution to $(C.2)$ is

$$v^\infty(x) = Ae^x + Be^{2(\alpha-1)x}, \ x_\infty \le x \le 0. \tag{C.3}$$

Due to $v^\infty_x(0) = 0$, $(C.3)$ can be rewritten as

$$v^\infty(x) = A(e^x - \frac{1}{2(\alpha-1)}e^{2(\alpha-1)x}), \ x_\infty \le x \le 0, \tag{C.4}$$

which results in

$$v^\infty_x(x) = A(e^x - e^{2(\alpha-1)x}).$$

Due to $v^\infty_x(x_\infty) = 0$, we obtain $A = 0$ and $v^\infty(x) = 0$ for all x, which contradicts $v^\infty(x_\infty) = \frac{3-2\alpha}{2(1-\alpha)}$. Thus, there is no finite free boundary x_∞, i.e., $x_\infty = -\infty$. Since $\alpha < 1$, and $0 \le \lim_{x\to-\infty} v^\infty(x) \le \frac{3-2\alpha}{2(1-\alpha)}$, we deduce $A = 0$ by $(C.4)$, i.e., $v^\infty(x) \equiv 0 < \psi^\infty(x)$. The proof is complete. $\square$

D Proof of Theorem 3.3

Similar to Theorem 3.1, we can deal with the cases of $\alpha \le 0$ and $\alpha \ge 1$ easily. However, the case of $0 < \alpha < 1$ is more challenging because we no longer have the monotonicity of $\overline{U}$ w.r.t. t. To overcome the difficulty, we introduce an auxiliary problem:

$$\begin{cases} \mathscr{L}_0\overline{U}^* + F_x - (\alpha - 1) = 0, \text{ in } \Omega^+, \\ \overline{U}^*_x(0,t) = 0, \overline{U}^*(x,T) = 0. \end{cases} \tag{D.1}$$

Lemma D.1 *Let $\overline{U}^*(x,t)$ be the solution to (D.1). Then for any $t \in [0,T)$,*

$$\begin{aligned} \overline{U}^*(0,t) &> 0 \text{ if } \alpha > \frac{1}{2}, \\ \overline{U}^*(0,t) &= 0 \text{ if } \alpha = \frac{1}{2}, \\ \overline{U}^*(0,t) &< 0 \text{ if } \alpha < \frac{1}{2}. \end{aligned}$$

Proof. $\overline{U}^*(x,t) = \log\left(\frac{U^*(x,t)}{\Psi(x,t)}\right)$ is the solution to (D.1), where $U^*(x,t)$ is the value function associated with a simple strategy: holding the stock until expiry T, i.e. $U^*(x,t) = \mathbb{E}\left(\frac{S_T}{M_T} \mid \log\frac{M_t}{S_t} = x\right)$. According to Shiryaev, Xu and Zhou (2008), Lemma D.1 automatically follows at $x = 0$. □

Proposition D.2 *Problem (A.5) has a unique solution $\overline{U}(x,t) \in W_p^{2,1}(\Omega_N^+)$, $1 < p < +\infty$, where Ω_N^+ is any bounded set in Ω^+. Moreover, for any $(x,t) \in \Omega^+$,*

i) $0 \le \overline{U}_x \le 1$;

ii) $\overline{U}(x,t;\alpha) \le \overline{U}(x,t;\alpha+\delta)$ for $\delta > 0$;

iii) $\overline{U}(x,t) = \overline{U}^*(x,t) > 0$ for $\alpha \ge \frac{1}{2}$. And, for any $t \in [0,T)$,

$$\overline{U}(0,t) > 0, \text{ if } \alpha > \frac{1}{2}, \tag{D.2}$$

$$\overline{U}(0,t) = 0, \text{ if } \alpha = \frac{1}{2}, \tag{D.3}$$

$$\overline{U}(0,t) = 0, \text{ if } \alpha < \frac{1}{2}. \tag{D.4}$$

Proof. The proofs of part i) and ii) are the same as that of Proposition A.3. Now let us prove part iii).

It is easy to see $\overline{U}^*_x(x,t) > 0$ in Ω^+ by virtue of $F_{xx} \le 0$ and the strong maximum principle. Combining with Lemma D.1, we infer $\overline{U}^*(x,t) > 0$ in Ω^+ when $\alpha \ge \frac{1}{2}$. So, $\overline{U}^*(x,t)$ must be the solution to (A.5), which yields $\overline{U}(x,t) = \overline{U}^*(x,t)$ for $\alpha \ge \frac{1}{2}$. Then (D.2) and (D.3) follow. To show (D.4), clearly we have $\overline{U}(0,t) \ge 0$. Thanks to part ii) and (D.3), we infer $\overline{U}(0,t) \le 0$ for $\alpha < \frac{1}{2}$, which leads to (D.4). This completes the proof. □

Now, we are going to prove Theorem 3.3.

Proof of Theorem 3.3. Part i) and ii) follow by part iii) of Proposition D.2. The proof of part iv) is similar to that of part i) in Theorem 3.1. Now let us prove part iii). Thanks to (D.4), we immediately get $\{x = 0\} \subset SR$. Combining with $\overline{U}_x \geq 0$, we can define

$$x_s^*(t) = \sup\{x \in [0, +\infty) : \overline{U}(x,t) = 0\}, \text{ for any } t \in [0, T).$$

We only need to show that $x_s^*(t) < \infty$. Let $x_b^*(t)$ be the free boundary as given in part ii) of Theorem 3.1. Due to Lemma A.1, we infer $x_s^*(t) \leq x_b^*(t)$, which, combined with $x_b^*(t) < \infty$, yields the desired result.

At last, let us prove $\lim_{T-t\to\infty} x_s^*(t) = \frac{1}{2\alpha-1}\log\left(1 - (2\alpha-1)^2\right)$. Again, consider the stationary problem of (2.10). Denote

$$\Psi^\infty(x) = \lim_{T-t\to\infty} \Psi(x,t) \text{ and } U^\infty(x) = \lim_{T-t\to\infty} U(x,t).$$

It is easy to see that $\Psi^\infty(x) = e^{-x} + \frac{1}{2(\alpha-1)}e^{2(\alpha-1)x}$, if $0 < \alpha < \frac{1}{2}$, and $U^\infty(x)$ satisfies

$$-\frac{1}{2}U_{xx}^\infty + (\alpha - \frac{1}{2})U_x^\infty = 0, \quad \forall x > x_\infty, \tag{D.5}$$

$$U_x^\infty(x_\infty) = \Psi_x^\infty(x_\infty), \qquad U^\infty(x_\infty) = \Psi^\infty(x_\infty), \tag{D.6}$$

where x_∞ is the free boundary.

The general solution to (D.5) is $U^\infty(x) = A + Be^{(2\alpha-1)x}$. Since $\lim_{x\to\infty} U^\infty(x) = 0$, we see $A = 0$. Thus, it is easy to get $U^\infty(x) = (e^{-x_\infty} + \frac{1}{2(\alpha-1)}e^{2(\alpha-1)x_\infty})e^{(2\alpha-1)(x-x_\infty)}$ from (D.6), where $x_\infty = \frac{1}{2\alpha-1}\log\left(1-(2\alpha-1)^2\right)$. The proof is complete. □

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