Thou Shalt Buy and Hold*

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Abstract

An investor holding a stock needs to decide when to sell it over a given investment horizon. It is tempting to think that she should sell at the maximum price over the entire horizon, which is however impossible to achieve. A close yet realistic goal is to sell the stock at a time when the expected relative error between the selling price and the aforementioned maximum price is minimized. This problem is investigated for a Black–Scholes market. A stock "goodness index" α, defined to be the ratio between the excess return rate and the squared volatility rate, is employed to measure the quality of the stock. It is shown that when the stock is good enough, or to be precise when $\alpha \geq \frac{1}{2}$, the optimal strategy is to hold on to the stock selling only at the end of the horizon. Moreover, the resulting expected relative error diminishes to zero when α goes to infinity. On the other hand, one should sell the stock immediately if $\alpha < \frac{1}{2}$. These results justify the widely accepted financial wisdom that one should buy and hold a stock – if it is good, that is.

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1 Introduction

A conventional and widely accepted investment wisdom is the so-called buy-and-hold rule, i.e., one should buy a good stock and leave it alone for a long time. It is based on the empirical observation that in the long run investing in a good company gives good rate of return while the volatility – which is short-term in nature – is insignificant. One of the main theoretical foundations of the buy-and-hold rule is the efficient market hypothesis (EMH): if the market is efficient and the price is right at any given time, then there is no point to trade (or to sell). So you sell a stock because something happens to *you*, not because something happens to the *market*. Another argument is based on transaction costs, including bid/offer spread, brokerage, capital gains tax, and so on. Warren Buffett, who clearly does not believe in EMH because his primary investment philosophy is to find under-priced stocks, is a buy-and-hold advocate belonging to the transaction cost school. Barber and Odean (2000) examine the issue from the behavioural finance point of view. They argue that due to over-confidence investors tend to trade excessively. An elaborative empirical study on US household investors reveals that active trading has led to poor performance; hence "trading is hazardous to your wealth".

Perverse enough, to our best knowledge no well-established dynamic investment model[1] has generally produced the *pure* buy-and-hold rule.[2] When transaction costs are ignored, the famous Merton portfolio (Samuelson 1969, Merton 1971) via utility maximization requires trading continuously so as to keep a constant proportion in wealth among stocks. The continuous-time Markowitz model (Richardson 1989, Zhou and Li 2000) or the recent behavioural portfolio selection model (Jin and Zhou 2008) both lead to similar continuously rebalancing optimal portfolios, even in the Black–Scholes case where there is only one stock available. When transaction costs are present, trading is significantly reduced; in fact in this case there are buying, selling and no-trade regions and an optimal strategy is to move the wealth process to the no-trade region as soon as possible and stay there as far as possible (Davis, Panas and Zariphopoulou 1993, Liu and Loewenstein 2002, Dai, Xu and Zhou 2008). However, such a strategy in general is still not exactly buy-and-hold.

Consider an innocent individual investor who has no knowledge whatsoever about utility maximization or mean–variance theories. He just bought a stock at the beginning

[1] Clearly buy-and-hold applies nontrivially only to a dynamic model. For a single-period model, every strategy is buy and hold.

[2] See also the concluding section of this paper and the introduction section of Liu and Loewenstein (2002) for an overview of the literature on this.

of a year and for some reason (e.g. to invest for a specific event or an obligation) must sell it in one year's time. Therefore he needs to decide the best time to sell within the year. What would be his criterion to evaluate the "best timing"? A simple and naïve, yet natural and perhaps dominating one for many small investors, would be "to sell higher". But how high is *higher*? It would be ideal if the stock could be sold exactly at the maximum price over the entire year; unfortunately this is mission impossible because the timing of the maximum price would be known only at the end of the year. Then how about if he tries to sell at the price *closest* to the maximum? This sounds sensible, but one needs to define precisely the meaning of the "closeness". An immediate measure would be the absolute difference – the *absolute* error – (or any power of the difference) between the selling price and the maximum price, except that such a measure would fail to capture the scale of the prices themselves. The scale will be taken care of if we consider the *relative* error. In conclusion, a natural criterion for the investor is to minimize the expected relative error between the selling price and the maximum price.

Motivated by the above, in this paper we formulate and investigate the stock selling problem for a Black–Scholes market. Although the relative error criterion involves the maximum stock price over the entire investment horizon (and hence is not adapted), we are able to transform the problem into a standard optimal stopping problem with a terminal payoff and an adapted state process. We introduce a stock goodness index α, which is defined to be the ratio between the excess return rate and the squared volatility rate,[3] to measure the quality of the stock. Our main result is that one should sell at the end of the horizon if the goodness index of the stock is greater than or equal to $\frac{1}{2}$, and sell immediately if the index is non-positive.[4] The implication of this result is two-fold: 1) being a "good" or "bad" stock is precisely characterized by a critical level of the goodness index, and 2) the optimal selling strategy for a *good* stock is a pure buy-and-hold one. Moreover, the optimal relative errors for both good and bad stocks are explicitly derived, based on which sensitivity analysis on the market parameters is carried out. In particular, it is shown that the expected relative error when one buys and holds diminshes to zero if the stock goodness index tends to infinity. This implies that for a sufficiently good stock the buy-and-sell rule *almost* realizes selling at the ultimate maximum price.

It is interesting to note that the buy-and-hold rule has long been believed to be the antithesis of the market timing: the notion that one can enter the market on the lows and sell on the highs does not work for small investors at least; so it is better to just buy

[3]Clearly the goodness index is intimately related to the Sharpe ratio – in fact it is the Sharpe ratio further normalized by the volatility rate. It turns out that this index plays a more prominent role in the particular problem considered in this paper.

[4]Indeed a stronger result holds that one should sell immediately if the index is less than $\frac{1}{2}$; see the concluding section.

and hold. Our model and result, however, show that market timing (attempting to sell higher) indeed *leads to* buy-and-hold!

The initial impetus to investigating this model was the combination of two papers by Shiryaev (2002) and Li and Zhou (2006) which at the first glance seemed to be rather unrelated. The former investigates the quickest detection problem for a change of market parameters, whereas the latter reveals the high chance of a Markowitz mean–variance strategy hitting the expected return target. The chemistry between the two papers has nonetheless emerged, somewhat unexpectedly, leading to the formulation and solution of the present stock selling problem. The mathematical analysis of the underlying optimal stopping problem is related to Graversen, Peskir, and Shiryaev (2001), which appears to be the first paper of its kind where the problem is to stop a Brownian motion so as to minimize the squared error from the maximum.

The remainder of the paper is organized as follows. The stock selling model is formulated and the main results presented in section 2. Section 3 is devoted to transforming the model into a standard optimal stopping problem. In sections 4 and 5 optimality for various cases is proved using two different approaches, while in section 6 the optimal relative errors are derived. Section 7 concludes with final remarks. Some technical details are relegated to an appendix.

2 A Stock Selling Problem

Consider a Black–Scholes economy where there is a stock with an appreciation rate a and a volatility rate $\sigma > 0$, along with a saving account with a continuously compounding interest rate r. The *discounted* stock price process, P_t, follows

$$dP_t = (a - r)P_t dt + \sigma P_t dB_t, \quad P_0 = 1 \tag{1}$$

on a standard filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$, where $B_.$ is a standard Brownian motion with $B_0 = 0$ under $\mathbf{P}$. Here, $\{\mathcal{F}_t\}_{t\geq 0}$ is the $\mathbf{P}$-augmentation of the filtration generated by $B_.$. We can alternatively write

$$P_t = e^{\mu t + \sigma B_t} \tag{2}$$

where $\mu = a - r - \frac{1}{2}\sigma^2$. Define the running maximum price process

$$M_t = \max_{0\leqslant s\leqslant t} P_s, \quad t \geq 0 \tag{3}$$

and a goodness index of the stock

$$\alpha = \frac{a - r}{\sigma^2}. \tag{4}$$

An investor buys a share of the stock at time 0, and for some reason she must sell the stock by a pre-specified date $T > 0$. The question is to determine the "best" time to sell. In general we could define the following criterion for the optimal selling problem:

$$\text{Maximize} \quad \mathbf{E}\left[U\left(\frac{P_\tau}{M_T}\right)\right] \tag{5}$$

over $\tau \in \mathcal{T}$, the set of all $\mathcal{F}_t$-stopping time $\tau \in [0,T]$, where U is certain "utility" function. In this paper we investigate the special case when U is linear. This is a particularly interesting case because it is equivalent to the following

$$\min_{\tau\in\mathcal{T}} \mathbf{E}\left[\frac{M_T - P_\tau}{M_T}\right], \tag{6}$$

which means that the investor wishes to minimize the expected *relative error* between the (discounted) selling price and the (discounted) maximum price (over the entire horizon $[0,T]$).

Before we solve (6), let us consider (5) with $U(x) = \log x$, $x > 0$. This problem turns out to have a simple solution. To see this, let $\sigma = 1$ for simplicity and write

$$B_t^\mu := \mu t + B_t, \quad S_t^\mu := \max_{0\leqslant s\leqslant t} B_s^\mu. \tag{7}$$

Then,

$$R^* := \sup_{\tau\in\mathcal{T}} \mathbf{E}\left[\log\left(\frac{P_\tau}{M_T}\right)\right] = \sup_{\tau\in\mathcal{T}} \mathbf{E}(\mu\tau - S_T^\mu).$$

Thus, the optimal stopping is, trivially,

$$\begin{aligned} \tau^* &= \begin{cases} T, & \text{if } \mu > 0 \\ \text{any time between } [0,T], & \text{if } \mu = 0 \\ 0, & \text{if } \mu < 0 \end{cases} \\ &= \begin{cases} T, & \text{if } \alpha > \frac{1}{2} \\ \text{any time between } [0,T], & \text{if } \alpha = \frac{1}{2} \\ 0, & \text{if } \alpha < \frac{1}{2} \end{cases} \end{aligned} \tag{8}$$

So the solution is of a simple bang-bang structure (stop either at the beginning or at the end).[5] Now, if U is linear (which is the problem we would like to solve in this paper), the above argument fails, and it is not clear what an optimal stopping time might be. In this paper, via rather involved probabilistic analysis, we shall solve the cases when $\alpha \geq \frac{1}{2}$ and $\alpha \leq 0$ and show that the optimal solution possesses similar bang-bang structure. As a by-product, Problem (5) has bang-bang solution for any power utility function

[5]Incidentally, once may also consider a "dual" problem $R_* := \inf_{\tau\in\mathcal{T}} \mathbf{E}\left[\log\left(\frac{M_T}{P_\tau}\right)\right] = \inf_{\tau\in\mathcal{T}} \mathbf{E}(S_T^\mu - \mu\tau) = -R^*$. This problem has the same bang-bang solution (8).

$U(x) = x^\gamma$ because by modifying appropriately the drift and volatility values the problem is mathematically equivalent to the one with a linear utility function.

Let $\Phi(\cdot)$ denote the probability distribution function of a standard normal random variable. The main results of the paper are as follows.

Theorem 2.1 (i) *If $\alpha \geq \frac{1}{2}$, then $\tau^* = T$ is the unique optimal selling time to Problem (6) when $\alpha > \frac{1}{2}$, and either $\tau^* = T$ or $\tau^* = 0$ is optimal when $\alpha = \frac{1}{2}$. Moreover, the optimal expected relative error is given by*

$$r^*(\alpha,\sigma) = 1 - \left(1 - \frac{1}{2\alpha}\right)\Phi\left(\left(\alpha - \frac{1}{2}\right)\sigma\sqrt{T}\right) - \left(1 + \frac{1}{2\alpha}\right)e^{\alpha\sigma^2 T}\Phi\left(-\left(\alpha + \frac{1}{2}\right)\sigma\sqrt{T}\right). \tag{9}$$

Furthermore, $r^*(\alpha,\sigma)$ decreases in α and increases in σ, and

$$0 \leq r^*(\alpha,\sigma) < \frac{1}{2\alpha} \quad \forall(\alpha,\sigma) \in \left[\frac{1}{2},\infty\right) \times [0,\infty). \tag{10}$$

(ii) *If $\alpha \leq 0$, then $\tau^* = 0$ is the unique optimal selling time to Problem (6). Moreover, the optimal relative error is given by*

$$r^*(\alpha,\sigma) = 1 - \frac{2\alpha - 1}{2(\alpha - 1)}\Phi\left(-\left(\alpha - \frac{1}{2}\right)\sigma\sqrt{T}\right) - \frac{2\alpha - 3}{2(\alpha - 1)}e^{(1-\alpha)\sigma^2 T}\Phi\left(\left(\alpha - \frac{3}{2}\right)\sigma\sqrt{T}\right). \tag{11}$$

So when the stock goodness index $\alpha \geq \frac{1}{2}$, one should hold on to the stock, selling only at T. This in turn implies that the stock must be a good one. The better the stock (as measured by α) the smaller the relative error, the latter being subject to an upper bound that is inversely proportional to α. In particular, the error diminishes to zero when α goes to infinity. This suggests that the buy-and-hold rule almost realizes selling at the maximum price if the stock is sufficiently good. On the other hand, if $\alpha \leq 0$, then one should sell the stock immediately or short sell if possible. This is a bad stock the investor ought to get rid of as soon as possible.

It is interesting to examine our results applied to some real data. We take the one in Mehra and Prescott (1985) based on S&P 500 (1889-1978). The estimated parameters are the following (both are annual figures): $a - r = 6.18\%$ and $\sigma = 16.67\%$. In this case $\alpha = 2.2239 > 0.5$ (by large margin)! Moreover, if we take $T = 1$ (year), then it follows from (9) that $r^*(\alpha,\sigma) = 10.15\%$. This means that if you buy and hold an S&P 500 index fund for one year then statistically you are expected to achieve almost 90% of the maximum possible return.[6]

[6] One might argue that it is not reasonable to model an index such as S&P 500 as a geometric Brownian motion, and hence the results in this paper may not apply. However, the figures $a - r = 6.18\%$ and $\sigma = 16.67\%$ do appear very *plausible* for a typical good stock. On the other hand, we expect that our analysis and results in this paper extend to the case of a market index which is nothing else than a linear combination of geometric Brownian motions.

3 An Equivalent Problem

To prove the part of the optimal selling times stated in Theorem 2.1 we assume without loss of generality that $\sigma = 1$, since by a change of time one can write $\sigma B_t = \tilde{B}_{\sigma^2 t}$ where $\tilde{B}_t$ is a standard Browian motion.

Recall the notation (7). Problem (6) is equivalent to (with $\sigma = 1$) the following stopping problem

$$\sup_{\tau\in\mathcal{T}} \mathbf{E}\left[\frac{e^{B^\mu_\tau}}{e^{S^\mu_T}}\right]. \tag{12}$$

The above is not a standard optimal stopping problem, because the term $e^{S^\mu_T}$ is not $\mathcal{F}_t$-adapted. To get around, for any stopping time $\tau \in \mathcal{T}$, we have

$$\begin{aligned}
\mathbf{E}\left[\frac{e^{B^\mu_\tau}}{e^{S^\mu_T}}\right] &= \mathbf{E}\left[\frac{e^{B^\mu_\tau}}{\max\{e^{S^\mu_\tau}, e^{\max_{\tau\leqslant t\leqslant T} B^\mu_t}\}}\right] = \mathbf{E}\left[\min\left\{e^{-(S^\mu_\tau - B^\mu_\tau)}, e^{-\max\limits_{\tau\leqslant t\leqslant T}(B^\mu_t - B^\mu_\tau)}\right\}\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\min\left\{e^{-(S^\mu_\tau - B^\mu_\tau)}, e^{-\max\limits_{\tau\leqslant t\leqslant T}(B^\mu_t - B^\mu_\tau)}\right\}\middle|\mathcal{F}_\tau\right]\right] \\
&= \mathbf{E}\left[\mathbf{E}\left[\min\left\{e^{-x}, e^{-S^\mu_{T-\tau}}\right\}\right]\Big|_{x=S^\mu_\tau - B^\mu_\tau}\right] \\
&= \mathbf{E}[G(\tau, S^\mu_\tau - B^\mu_\tau)],
\end{aligned} \tag{13}$$

where

$$G(t,x) := \mathbf{E}\left[\min\left\{e^{-x}, e^{-S^\mu_{T-t}}\right\}\right] > 0, \quad (t,x) \in [0,T]\times[0,\infty). \tag{14}$$

Direct computations show (for details see Appendix A) that when $\mu \neq 1/2$,

$$\begin{aligned}
G(t,x) = \; & \frac{2(\mu-1)}{2\mu-1} e^{-(\mu-1/2)(T-t)} \Phi\left(\frac{-x+(\mu-1)(T-t)}{\sqrt{T-t}}\right) \\
& + \frac{1}{2\mu-1} e^{-(1-2\mu)x} \Phi\left(\frac{-x-\mu(T-t)}{\sqrt{T-t}}\right) + e^{-x}\Phi\left(\frac{x-\mu(T-t)}{\sqrt{T-t}}\right);
\end{aligned} \tag{15}$$

and when $\mu = 1/2$,

$$\begin{aligned}
G(t,x) = \; & (1 + x + (T-t)/2)\Phi\left(\frac{-x-(T-t)/2}{\sqrt{T-t}}\right) \\
& - \sqrt{\frac{T-t}{2\pi}} e^{-\frac{(x+(T-t)/2)^2}{2(T-t)}} + e^{-x}\Phi\left(\frac{x-(T-t)/2}{\sqrt{T-t}}\right).
\end{aligned} \tag{16}$$

Equation (13) implies that (12) is actually a standard optimal stopping problem with a terminal payoff G and an underlying (adapted) state process

$$X_t = S^\mu_t - B^\mu_t, \quad X_0 = 0,$$

the so-called "drawdown process".

In view of the dynamic programming approach we consider the following problem

$$V(t,x) = \sup_{\tau\in\mathcal{T}_{T-t}} \mathbf{E}_{t,x}\left[G(t+\tau, X_{t+\tau})\right], \tag{17}$$

where $X_t = x$ under $\mathbf{P}_{t,x}$ with $(t,x) \in [0,T] \times [0,\infty)$ given and fixed, and $\mathcal{T}_s$ in general denotes the set of all $\mathcal{F}_t$-stopping times $\tau \in [0,s]$ for $s > 0$. The original problem is certainly

$$V(0,0) = \sup_{\tau \in \mathcal{T}} \mathbf{E}\left[\frac{e^{B^\mu_\tau}}{e^{S^\mu_T}}\right].$$

It is well known that V satisfies the following dynamic programming equation (or variational inequalities)

$$\min\{-\mathcal{L}V, V - G\} = 0, \quad (t,x) \in [0,T) \times (0,\infty) \tag{18}$$

$$V(T,x) = G(T,x), \quad x \in (0,\infty) \tag{19}$$

$$V_x(t,0+) = 0, \quad t \in [0,T) \tag{20}$$

where the operator $\mathcal{L}$ is defined by

$$(\mathcal{L}f)(t,x) = f_t(t,x) - \mu f_x(t,x) + \frac{1}{2} f_{xx}(t,x). \tag{21}$$

The holding region is therefore

$$C = \{(t,x) \in [0,T] \times [0,\infty) : V(t,x) > G(t,x)\}, \tag{22}$$

while the selling region is

$$D = \{(t,x) \in [0,T] \times [0,\infty) : V(t,x) = G(t,x)\}. \tag{23}$$

An optimal selling time is

$$\tau^* = \inf\{t \in [0,T] : (t, S^\mu_t - B^\mu_t) \in D\}. \tag{24}$$

So the problem boils down to finding and/or analyzing V.

Noting that $B^\mu_\cdot$ has stationary independent increments and $X_\cdot$ is a Markovian process, we may rewrite

$$V(t,x) = \sup_{0 \leqslant \tau \leqslant T-t} \mathbf{E}\left[G(t+\tau, X^x_\tau)\right] \tag{25}$$

where $X_\cdot$ under $\mathbf{P}$ is explicitly given as

$$X^x_t = x \vee S^\mu_t - B^\mu_t, \quad t \geq 0. \tag{26}$$

4 Optimal Selling Times When $\alpha \geq 1$ and $\alpha \leq 0$

In this section we derive optimal selling times for the cases when $\alpha \geq 1$ and $\alpha \leq 0$ respectively, via an approach that turns the problem with the terminal payoff to one with a *running* payoff. This approach is standard in the optimal stopping literature; see, e.g., Peskir and Shiryaev (2006). To proceed, note that

$$X^x_\cdot \overset{\text{law}}{=\!=} |Y_\cdot|, \tag{27}$$

where $Y_\cdot$ is the unique strong solution to the SDE

$$\begin{cases} \mathrm{d}Y_t = -\mu \operatorname{sign}(Y_t)\,\mathrm{d}t + \mathrm{d}B_t \\ Y_0 = x. \end{cases} \tag{28}$$

The process $Y_\cdot$ has an infinitesimal generator $\mathcal{L}$ defined by (21). By Itô–Tanaka's formula, we have

$$|Y_s| = x - \mu \int_0^s I(Y_u \neq 0)\,\mathrm{d}u + \int_0^s \operatorname{sign}(Y_u) I(Y_u \neq 0)\,\mathrm{d}B_u + \ell^0_s(Y), \tag{29}$$

where $\ell^0_s(Y)$ is the local time of $Y_\cdot$ at 0. So

$$\begin{aligned} G(t+s, |Y_s|) =& G(t,x) + \int_0^s \mathcal{L}G(t+u, |Y_u|)\,\mathrm{d}u + \int_0^s G_x(t+u, |Y_u|)\operatorname{sign}(Y_u)\,\mathrm{d}B_u \\ &+ \int_0^s G_x(t+u, |Y_u|)\,\mathrm{d}\ell^0_u(Y) \\ =& G(t,x) + \int_0^s H(t+u, |Y_u|)\,\mathrm{d}u + M_s, \end{aligned}$$

where we have used the fact that $G_x(t, 0+) = 0$ for $0 \leqslant t \leqslant T$ (details in Appendix C), and H and $M_\cdot$ are defined respectively by

$$\begin{aligned} H(t,x) &:= \mathcal{L}G(t,x) \equiv G_t(t,x) - \mu G_x(t,x) + \frac{1}{2} G_{xx}(t,x), \\ M_s &= \int_0^s G_x(t+u, |Y_u|)\operatorname{sign}(Y_u)\,\mathrm{d}B_u. \end{aligned}$$

Due to the definition (14) of G and (32) below, we have $-1 \leqslant -G \leqslant G_x \leqslant 0$; so $M_\cdot$ is a martingale, and Problem (25) can be expressed as

$$\begin{aligned} V(t,x) =& \sup_{0 \leqslant \tau \leqslant T-t} \mathbf{E}\left[G(t+\tau, X^x_\tau)\right] \\ =& \sup_{0 \leqslant \tau \leqslant T-t} \mathbf{E}\left[G(t+\tau, |Y_\tau|)\right] \\ =& G(t,x) + \sup_{0 \leqslant \tau \leqslant T-t} \mathbf{E}\left[\int_0^\tau H(t+u, |Y_u|)\,\mathrm{d}u\right] \\ =& G(t,x) + \sup_{0 \leqslant \tau \leqslant T-t} \mathbf{E}\left[\int_0^\tau H(t+u, X^x_u)\,\mathrm{d}u\right]. \end{aligned} \tag{30}$$

A lengthy calculation (see Appendix B) shows that

$$H(t,x) = (\mu - 1/2)G(t,x) - G_x(t,x). \tag{31}$$

Now, if $\alpha \geq 1$, then $\mu \equiv a - r - \frac{1}{2} \geq \frac{1}{2}$ (recall we are assuming that $\sigma = 1$). In this case, noting that $G_x \leqslant 0$ by the monotonicity of G in x, we have

$$H(t,x) = (\mu - 1/2)G(t,x) - G_x(t,x) \geqslant 0 \quad \forall (t,x) \in [0,T] \times [0,\infty),$$

and the inequality is strict if $\alpha > 1$. This shows that an optimal stopping time to Problem (30) is $\tau^* = T - t$, and it is the unique optimal solution if $\alpha > 1$. In particular, the conclusion applies to the original problem (6) with $\tau^* = T$.

On the other hand, if $\alpha \leq 0$, then $\mu \leq -\frac{1}{2}$. Noting that

$$e^x G(t,x) = e^x \, \mathbf{E}\left[\min\{e^{-x}, e^{-S^{\mu}_{T-t}}\}\right] = \mathbf{E}\left[\min\{1, e^{-S^{\mu}_{T-t}+x}\}\right]$$

is strictly increasing with respect to x, we have

$$\frac{\partial(e^x G(t,x))}{\partial x} > 0, \qquad \text{or} \quad G_x(t,x) + G(t,x) > 0. \tag{32}$$

Thus

$$H(t,x) = (\mu - 1/2)G(t,x) - G_x(t,x) = (\mu + 1/2)G(t,x) - (G(t,x) + G_x(t,x)) < 0.$$

This indicates that the unique optimal stopping time to Problem (30) is $\tau^* = 0$ and $V(t,x) = G(t,x)$. In particular, the conclusion is valid for the original problem (6).

5 Optimal Selling Time When $\frac{1}{2} \leq \alpha < 1$

While the approach in the previous section is rather direct, we have yet to cover the case when $\frac{1}{2} \leq \alpha < 1$ for which we now use a different technique to tackle.[7]

Proposition 5.1 *If $\alpha \geq \frac{1}{2}$, then*

$$V(t,x) > G(t,x) \quad \forall t \in [0,T), \quad x \geq 0 \tag{33}$$

whenever $\alpha > \frac{1}{2}$, and

$$V(t,x) > G(t,x) \quad \forall t \in [0,T), \quad x > 0 \tag{34}$$

whenever $\alpha = \frac{1}{2}$.

[7] As will be seen shortly, the case $\alpha \geq 1$, which has been already solved in the previous section, can also be solved by the approach presented in this section.

Proof. Note that $\alpha \geq \frac{1}{2}$ is equivalent to $\mu \geq 0$. Also as before we assume herein that $\sigma = 1$ without loss of generality. The proof consists of several steps.

Step 1. We first consider the case when $\mu = 0$. In this case, we are to show that

$$\mathbf{E}[G(T, X_T^x)] > G(0, x) \qquad \forall x > 0, \quad \text{and } \mathbf{E}[G(T, X_T^x)] = G(0, x) \ \text{ for } x = 0 \tag{35}$$

where $X_\cdot^x$ is defined by (26). To this end, taking $\sigma = 1$ and $\mu = 0$ in the general expression of $\mathbf{E}[G(T, X_T^x)] \equiv \mathbf{E}[e^{-X_T^x}]$ (derived in Appendix D) we have

$$\mathbf{E}[G(T, X_T^x)] = e^{T/2-x}\Phi\left(\frac{x-T}{\sqrt{T}}\right) + e^{T/2+x}\Phi\left(\frac{-x-T}{\sqrt{T}}\right).$$

On the other hand, it follows from (15) with $\mu = 0$ that

$$G(0, x) = 2e^{T/2}\Phi\left(\frac{-x-T}{\sqrt{T}}\right) - e^{-x}\Phi\left(\frac{-x}{\sqrt{T}}\right) + e^{-x}\Phi\left(\frac{x}{\sqrt{T}}\right).$$

Define

$$\begin{aligned} g(x) :=& e^x \left(\mathbf{E}[G(T, X_T^x)] - G(0, x)\right) \\ =& e^{T/2}\Phi\left(\frac{x-T}{\sqrt{T}}\right) + e^{T/2+2x}\Phi\left(\frac{-x-T}{\sqrt{T}}\right) \\ & - 2e^{T/2+x}\Phi\left(\frac{-x-T}{\sqrt{T}}\right) + \Phi\left(\frac{-x}{\sqrt{T}}\right) - \Phi\left(\frac{x}{\sqrt{T}}\right), \quad x \geq 0. \end{aligned}$$

Then

$$g'(x) = 2e^{T/2+x}(e^x - 1)\Phi\left(\frac{-x-T}{\sqrt{T}}\right) > 0 \quad \forall x > 0.$$

However, it is straightforward to check that $g(0) = 0$, proving (35).

Step 2. In this step we show that when $\mu > 0$,

$$\mathbf{E}[G(T, X_T^x)] > G(0, x) \qquad \forall x \geq 0. \tag{36}$$

Indeed, by definitions

$$\mathbf{E}[G(T, X_T^x)] = \mathbf{E}\left[e^{-(x \vee S_T^\mu - B_T^\mu)}\right], \quad G(0, x) = \mathbf{E}\left[e^{-x \vee S_T^\mu}\right]. \tag{37}$$

Applying Girsanov's theorem, we have

$$\begin{aligned} \mathbf{E}[G(T, X_T^x)] - G(0, x) &= \mathbf{E}\left[e^{-(x \vee S_T^\mu - B_T^\mu)} - e^{-x \vee S_T^\mu}\right] \\ &= \mathbf{E}\left[e^{-x \vee S_T^\mu}\left(e^{B_T^\mu} - 1\right)\right] \\ &= \mathbf{E}^{\mathbf{Q}}\left[e^{-x \vee S_T^{\mathbf{Q}}}\left(e^{B_T^{\mathbf{Q}}} - 1\right)e^{-\frac{1}{2}\mu^2 T + \mu B_T^{\mathbf{Q}}}\right] \\ &= \mathbf{E}\left[e^{-x \vee S_T}\left(e^{B_T} - 1\right)e^{-\frac{1}{2}\mu^2 T + \mu B_T}\right], \end{aligned}$$

where $B^{\mathbf{Q}} \equiv B_t^{\mu} = \mu t + B_t$ is a standard Brownian motion under probability $\mathbf{Q}$, with $\mathrm{d}\mathbf{Q} = e^{-\frac{1}{2}\mu^2 T - \mu B_T}\,\mathrm{d}\mathbf{P}$. This leads to

$$e^{\frac{1}{2}\mu^2 T}\left(\mathbf{E}[G(T, X_T^x)] - G(0,x)\right) = \mathbf{E}\left[e^{-x \vee S_T}\left(e^{B_T} - 1\right) e^{\mu B_T}\right].$$

Hence

$$\frac{\partial}{\partial \mu}\left(e^{\frac{1}{2}\mu^2 T}\left(\mathbf{E}[G(T, X_T^x)] - G(0,x)\right)\right) = \mathbf{E}\left[e^{-x \vee S_T}\left(e^{B_T} - 1\right) B_T e^{\mu B_T}\right] > 0 \quad \forall \mu > 0. \tag{38}$$

The desired inequality (36) then follows from (35).

Step 3. By the arbitrariness of $T > 0$ and $x \geq 0$ in the strict inequality (36), we can prove in exactly the same way that

$$\mathbf{E}_{t,x}[G(T, X_T)] > G(t,x) \qquad \forall t \in [0,T), \;\; x \geq 0 \tag{39}$$

where $X_t = x$ under $\mathbf{P}_{t,x}$, provided that $\mu > 0$. Thus (33) follows from the fact that $V(t,x) \geq \mathbf{E}_{t,x}[G(T, X_T)]$. Similarly, (34) is implied by (35).

Now we return to the proof of Theorem 2.1. If $\mu > 0$ (or $\alpha > \frac{1}{2}$), then the unique optimality of $\tau^* = T$ follows immediately from the preceding proposition, in view of the definition of the holding region (22).

If $\mu = 0$ (or $\alpha = \frac{1}{2}$), by the arbitrariness of $T > 0$ and $x \geq 0$ in (35), we can prove in exactly the same way that

$$\mathbf{E}_{t,x}[G(T, X_T)] > G(t,x) \qquad \forall x > 0, \quad \text{and } \mathbf{E}_{t,x}[G(T, X_T)] = G(t,x) \;\text{ for } x = 0. \tag{40}$$

Then, for any $\tau \in \mathcal{T}$, we have

$$\mathbf{E}[G(T, X_T^x)|\mathcal{F}_\tau] \geqslant G(\tau, X_\tau^x) \;\; \forall x \geq 0, \tag{41}$$

or

$$\mathbf{E}[G(T, X_T^x)] \geqslant \mathbf{E}[G(\tau, X_\tau^x)] \;\; \forall x \geq 0. \tag{42}$$

Since $\tau \in \mathcal{T}$ is arbitrary in the above, we conclude

$$V(0,x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}[G(\tau, X_\tau^x)] = \mathbf{E}[G(T, X_T^x)] \;\; \forall x \geq 0. \tag{43}$$

In particular, applying (35), we obtain

$$V(0,0) = \mathbf{E}[G(T, X_T^0)] = G(0,0). \tag{44}$$

This implies that both $\tau^* = T$ and $\tau^* = 0$ are optimal.[8]

[8] In exactly the same way we can show that $V(t,0) = G(t,0)\;\; \forall t \in [0,T]$, meaning that when $\alpha = \frac{1}{2}$ one should either sell at the end or sell whenever the drawdown state $x = 0$ (in particular at time 0).

6 Optimal Relative Errors

Based on the proved results so far, a stock with the goodness index $\alpha \geq \frac{1}{2}$ is a *good* stock because one should hold on to it until the end, and one with $\alpha \leq 0$ is a *bad* one since one should sell it immediately. In this section we complete the proof of Theorem 2.1 by deriving the optimal expected relative errors for both a good and a bad stock. Since we need to investigate the sensitivity of the optimal relative errors in σ, we allow σ to vary (instead of assuming $\sigma = 1$) throughout this section.

6.1 Good stock: $\alpha \geq \frac{1}{2}$

Recall

$$P_t = e^{(a-r-\frac{1}{2}\sigma^2)t+\sigma B_t}, \quad M_t = \max_{0 \leqslant u \leqslant t} P_u.$$

The joint probability density function of (P_t, M_t) is given by

$$f(t,s,m) = \frac{2}{\sigma^3\sqrt{2\pi t^3}} \frac{\ln(m^2/s)}{sm} \exp\left(-\frac{\ln^2(m^2/s)}{2\sigma^2 t} + \frac{\beta}{\sigma}\ln(s) - \frac{1}{2}\beta^2 t\right),$$

where $0 < s \leqslant m$, $m \geqslant 1$, and

$$\beta := \frac{a-r}{\sigma} - \frac{\sigma}{2} \equiv \left(\alpha - \frac{1}{2}\right)\sigma; \tag{45}$$

see e.g. [7] p. 368, or [12] p.402.[9] Now, we compute, for any $t > 0$ and $0 < y < 1$:

$$\begin{aligned}
\mathbf{P}\left(\frac{P_t}{M_t} > y\right) &= \int_y^\infty \int_{s\vee 1}^{s/y} f(t,s,m)\,\mathrm{d}m\,\mathrm{d}s \\
&= \int_y^\infty \int_{s\vee 1}^{s/y} \frac{2}{\sigma^3\sqrt{2\pi t^3}} \frac{\ln(m^2/s)}{sm} \exp\left(-\frac{\ln^2(m^2/s)}{2\sigma^2 t} + \frac{\beta}{\sigma}\ln(s) - \frac{1}{2}\beta^2 t\right) \mathrm{d}m\,\mathrm{d}s \\
&= \frac{2}{\sigma^3\sqrt{2\pi t^3}} e^{-\frac{1}{2}\beta^2 t} \int_y^\infty s^{\beta/\sigma - 1} \int_{s\vee 1}^{s/y} \frac{\ln(m^2/s)}{m} \exp\left(-\frac{\ln^2(m^2/s)}{2\sigma^2 t}\right) \mathrm{d}m\,\mathrm{d}s \\
&= \frac{2}{\sigma^3\sqrt{2\pi t^3}} e^{-\frac{1}{2}\beta^2 t} \int_y^\infty s^{\beta/\sigma - 1} \int_{s\vee 1}^{s/y} -\frac{\sigma^2 t}{2}\,\mathrm{d}\left[\exp\left(-\frac{\ln^2(m^2/s)}{2\sigma^2 t}\right)\right] \mathrm{d}s \\
&= \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2}\beta^2 t} \int_y^\infty s^{\beta/\sigma - 1} \left(\exp\left(-\frac{\ln^2((s\vee 1)^2/s)}{2\sigma^2 t}\right) - \exp\left(-\frac{\ln^2(s/y^2)}{2\sigma^2 t}\right)\right) \mathrm{d}s \\
&= \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2}\beta^2 t} \left[\int_y^\infty s^{\beta/\sigma - 1} \exp\left(-\frac{\ln^2(s)}{2\sigma^2 t}\right) \mathrm{d}s - y^{2\beta/\sigma} \int_{1/y}^\infty s^{\beta/\sigma - 1} \exp\left(-\frac{\ln^2(s)}{2\sigma^2 t}\right) \mathrm{d}s\right] \\
&= \frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2}\beta^2 t} \left[\int_{\ln(y)}^\infty \exp\left(-\frac{u^2}{2\sigma^2 t} + \frac{\beta}{\sigma}u\right) \mathrm{d}u - y^{2\beta/\sigma} \int_{-\ln(y)}^\infty \exp\left(-\frac{u^2}{2\sigma^2 t} + \frac{\beta}{\sigma}u\right) \mathrm{d}u\right] \\
&= \Phi\left(\frac{\beta\sigma t - \ln(y)}{\sigma\sqrt{t}}\right) - y^{2\beta/\sigma}\Phi\left(\frac{\beta\sigma t + \ln(y)}{\sigma\sqrt{t}}\right).
\end{aligned}$$

[9] There is a typo in [12] p.402: β there should be $r/\sigma - \sigma/2$ instead of $r/\sigma + \sigma/2$.

Consequently,

$$
\begin{aligned}
\mathbf{E}\left[\frac{P_t}{M_t}\right] &= \int_0^1 \mathbf{P}\left(\frac{P_t}{M_t} > y\right) \mathrm{d}y \\
&= \int_0^1 \left[\Phi\left(\frac{\beta\sigma t - \ln(y)}{\sigma\sqrt{t}}\right) - y^{2\beta/\sigma}\Phi\left(\frac{\beta\sigma t + \ln(y)}{\sigma\sqrt{t}}\right)\right] \mathrm{d}y \\
&= \int_0^1 \int_{-\infty}^{\frac{\beta\sigma t - \ln(y)}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,\mathrm{d}u\,\mathrm{d}y - \int_0^1 y^{2\beta/\sigma} \int_{-\infty}^{\frac{\beta\sigma t + \ln(y)}{\sigma\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,\mathrm{d}u\,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_0^{(e^{\beta\sigma t - u\sigma\sqrt{t}})\wedge 1} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,\mathrm{d}y\,\mathrm{d}u - \int_{-\infty}^{\beta\sqrt{t}} \int_{e^{u\sigma\sqrt{t}-\beta\sigma t}}^1 \frac{1}{\sqrt{2\pi}} e^{-u^2/2} y^{2\beta/\sigma}\,\mathrm{d}y\,\mathrm{d}u \\
&= \int_{-\infty}^{\beta\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,\mathrm{d}u + \int_{\beta\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2+\beta\sigma t - u\sigma\sqrt{t}}\,\mathrm{d}u \\
&\quad - \frac{1}{2\beta/\sigma + 1} \int_{-\infty}^{\beta\sqrt{t}} \left(1 - e^{(2\beta/\sigma+1)(u\sigma\sqrt{t}-\beta\sigma t)}\right) \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\,\mathrm{d}u \\
&= \frac{2\beta/\sigma}{2\beta/\sigma+1}\Phi\left(\beta\sqrt{t}\right) + \frac{2\beta/\sigma+2}{2\beta/\sigma+1} e^{(\beta+\sigma/2)\sigma t}\Phi\left(-(\beta+\sigma)\sqrt{t}\right).
\end{aligned}
$$

Next we prove that $\mathbf{E}\left[\frac{P_t}{M_t}\right]$ strictly decreases in $t > 0$. Indeed, if $\beta + \sigma > 0$, then by Lemma E.1,

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\mathbf{E}\left[\frac{P_t}{M_t}\right]\right) &= -\frac{\sigma}{\sqrt{2\pi t}} e^{-\beta^2 t/2} + (\beta+\sigma)\sigma e^{(\beta+\sigma/2)\sigma t}\Phi\left(-(\beta+\sigma)\sqrt{t}\right) \\
&< -\frac{\sigma}{\sqrt{2\pi t}} e^{-\beta^2 t/2} + (\beta+\sigma)\sigma e^{(\beta+\sigma/2)\sigma t} \frac{1}{(\sigma+\beta)\sqrt{2\pi t}} e^{-(\beta+\sigma)^2 t/2} = 0.
\end{aligned}
$$

If $\beta + \sigma \leqslant 0$, then

$$
\frac{\partial}{\partial t}\left(\mathbf{E}\left[\frac{P_t}{M_t}\right]\right) = -\frac{\sigma}{\sqrt{2\pi t}} e^{-\beta^2 t/2} + (\beta+\sigma)\sigma e^{(\beta+\sigma/2)\sigma t}\Phi\left(-(\beta+\sigma)\sqrt{t}\right) < 0.
$$

This establishes the strict monotonicity in t. On the other hand, when $\beta \geq 0$, straightforward computation leads to

$$
\lim_{t\to 0+} \mathbf{E}\left[\frac{P_t}{M_t}\right] = 1, \quad \lim_{t\to +\infty} \mathbf{E}\left[\frac{P_t}{M_t}\right] = \frac{2\beta/\sigma}{2\beta/\sigma+1} \equiv 1 - \frac{1}{2\alpha}.
$$

Hence, when $\beta \geq 0$ (or $\alpha \geq \frac{1}{2}$), we have the following

$$
0 \leq \mathbf{E}\left[\frac{M_t - P_t}{M_t}\right] < \frac{1}{2\alpha} \quad \forall t > 0, \quad \alpha \geq \frac{1}{2}. \tag{46}
$$

Moreover, if $\beta \geqslant 0$, then

$$
\begin{aligned}
\frac{\partial}{\partial\beta}\mathbf{P}\left(\frac{P_t}{M_t} > y\right) &= \frac{\partial}{\partial\beta}\left[\Phi\left(\frac{\beta\sigma t - \ln(y)}{\sigma\sqrt{t}}\right) - y^{2\beta/\sigma}\Phi\left(\frac{\beta\sigma t + \ln(y)}{\sigma\sqrt{t}}\right)\right] \\
&= -\frac{2\ln(y)}{\sigma} y^{2\beta/\sigma}\Phi\left(\frac{\beta\sigma t + \ln(y)}{\sigma\sqrt{t}}\right) \geqslant 0 \quad \forall y \in (0,1).
\end{aligned}
$$

As a result

$$\frac{\partial}{\partial\beta}\mathbf{E}\left[\frac{P_t}{M_t}\right] = \frac{\partial}{\partial\beta}\int_0^1 \mathbf{P}\left(\frac{P_t}{M_t} > y\right)\mathrm{d}y \geqslant 0.$$

Meanwhile

$$\begin{aligned}\frac{\partial}{\partial\sigma}\mathbf{P}\left(\frac{P_t}{M_t} > y\right) =& \frac{\partial}{\partial\sigma}\left[\Phi\left(\frac{\beta\sigma t - \ln(y)}{\sigma\sqrt{t}}\right) - y^{2\beta/\sigma}\Phi\left(\frac{\beta\sigma t + \ln(y)}{\sigma\sqrt{t}}\right)\right]\\ =& \frac{2\ln(y)}{\sigma^2\sqrt{2\pi t}}e^{-\frac{(\beta\sigma t - \ln(y))^2}{2\sigma^2 t}} + \frac{2\beta\ln(y)}{\sigma^2}y^{2\beta/\sigma}\Phi\left(\frac{\beta\sigma t + \ln(y)}{\sigma\sqrt{t}}\right) \leqslant 0.\end{aligned}$$

Hence

$$\frac{\partial}{\partial\sigma}\mathbf{E}\left[\frac{P_t}{M_t}\right] = \frac{\partial}{\partial\sigma}\int_0^1 \mathbf{P}\left(\frac{P_t}{M_t} > y\right)\mathrm{d}y \leqslant 0.$$

Returning to Problem (6) we have proved that when $\alpha \geq \frac{1}{2}$ an optimal selling time is $\tau^* = T$. Hence the corresponding optimal relative error is $r^*(\alpha,\sigma) = 1 - \mathbf{E}\left[\frac{P_T}{M_T}\right]$. Noting (45) and (46) we complete the proof of Theorem 2.1-(i).

6.2 Bad stock: $\alpha \leq 0$

First for $y \geqslant 1$:

$$\begin{aligned}\mathbf{P}(M_t < y) = \mathbf{P}\left(\max_{0\leq s\leq t}(\beta s + B_s) < \frac{1}{\sigma}\ln(y)\right)\\ = \Phi\left(\frac{\ln(y) - \beta\sigma t}{\sigma\sqrt{t}}\right) - y^{2\beta/\sigma}\Phi\left(\frac{-\ln(y) - \beta\sigma t}{\sigma\sqrt{t}}\right).\end{aligned}$$

Notice that $\beta \neq \frac{\sigma}{2}$ when $\alpha \leq 0$. Thus

$$\begin{aligned}
&\mathbf{E}\left[\frac{1}{M_t}\right] = \int_0^1 \mathbf{P}\left(\frac{1}{M_t} > y\right) \mathrm{d}y = \int_0^1 \mathbf{P}\left(M_t < \frac{1}{y}\right) \mathrm{d}y \\
=&\frac{1}{\sigma\sqrt{2\pi t}} \int_0^1 \left[\int_{-\infty}^{-\ln(y)} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u - y^{-2\beta/\sigma} \int_{-\infty}^{\ln(y)} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u\right] \mathrm{d}y \\
=&\frac{1}{\sigma\sqrt{2\pi t}} \int_0^1 \int_{-\infty}^{-\ln(y)} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u\, \mathrm{d}y - \frac{1}{\sigma\sqrt{2\pi t}} \int_0^1 y^{-2\beta/\sigma} \int_{-\infty}^{\ln(y)} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u\, \mathrm{d}y \\
=&\frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{\infty} \int_0^{e^{-u}\wedge 1} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}y\, \mathrm{d}u - \frac{1}{\sigma\sqrt{2\pi t}} \int_{-\infty}^{0} \int_{e^u}^1 y^{-2\beta/\sigma} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}y\, \mathrm{d}u \\
=&\frac{1}{\sigma\sqrt{2\pi t}} \left[\int_{-\infty}^{0} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u + \int_0^{\infty} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t} - u\right) \mathrm{d}u\right] \\
&- \frac{1}{1-2\beta/\sigma} \frac{1}{\sigma\sqrt{2\pi t}} \left[\int_{-\infty}^{0} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u - \int_{-\infty}^{0} \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t} + (1-2\beta/\sigma)u\right) \mathrm{d}u\right] \\
=&\frac{-2\beta/\sigma}{1-2\beta/\sigma} \Phi\left(-\beta\sqrt{t}\right) + \frac{2-2\beta/\sigma}{1-2\beta/\sigma} e^{(\sigma/2-\beta)\sigma t} \Phi\left((\beta-\sigma)\sqrt{t}\right).
\end{aligned}$$

Now, for Problem (6) the optimal relative error is

$$r^*(\alpha,\sigma) = \mathbf{E}\left[\frac{M_T - P_0}{M_T}\right] = 1 - \mathbf{E}\left[\frac{1}{M_T}\right],$$

which is (11) after some easy manipulations.

7 Conclusions

This paper formulates a finite horizon stock selling model as one minimzing the expected relative error between the selling price and the maximum price over the horizon. It is shown that one should hold on to the stock until the end if the stock goodness index is no less than $\frac{1}{2}$, while one should sell immediately if the index is no greater than 0. Our results justify the classical maxim that one should buy and hold a good stock. Indeed, the very fact that our model is able to produce the buy-and-hold rule suggests in turn that the criterion proposed in this paper may be a sensible one that warrants further investigations in more general settings.

It is intriguing to compare our result to that of Merton (1971). Merton's portfolio in the Black–Scholes setting without consumption, for a utility function $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ ($\gamma > 0$), stipulates that the stock-to-wealth ratio should be kept as $\frac{\alpha}{\gamma}$ where α is exactly our stock goodness index (also called the Merton line). Hence Merton's strategy degenerates to buy-and-hold only in a very *exceptional* case when $\gamma = \alpha$, which requires a coordination between the risk attitude of the agent and the market opportunities. Of course, such a requirement is impractical because all the parameters are estimates prone to (sometimes large) errors.

This brings about another advantage of our model. Unlike with other standard portfolio selection models (including Merton's) our optimal solutions are *insensitive* to the market parameters. Indeed, our definition of a good (as well as a bad) stock involves a *range* of the parameters, instead of specific values for them. As demonstrated by the S&P 500 example the criterion that $\alpha \geq \frac{1}{2}$ is satisfied by a large margin which would accommodate sufficient level of errors. In general statistical terms, verifying whether $\alpha \geq \frac{1}{2}$ is much easier than estimating the value of α itself. Hence the notorious mean–blur problem is hardly an issue in our model, especially for very good stocks.

Finally, one may have by now noticed that the case when the goodness index is between 0 and $\frac{1}{2}$ has been left unsolved in this paper. This gap, $0 < \alpha < \frac{1}{2}$, can be filled by a partial differential equation (PDE) approach, and the result is that one should sell immediately if $\alpha < \frac{1}{2}$.[10] For the pure PDE argument we refer to a companion work, Dai, Jin, Zhong and Zhou (2008), where a buying decision is incorporated in addition to the selling one. We have chosen to present the probabilistic approach here – at the cost of having a gap – for two reasons. First, it was indeed the approach we had liked and employed since the very beginning of our research. Second, it is our view that, while solving the case $0 < \alpha < \frac{1}{2}$ may be mathematically interesting for the sake of completeness, it is not as significant and interesting financially. The reason is that the definition of a good stock $\alpha \geq \frac{1}{2}$ (or that the excess return rate is greater than or equal to half of the squared volatility) is so generous that it covers many of the stocks commonly perceived as "good"[11]. Indeed, as is shown in section 2 the S&P 500 has a goodness index greater than $\frac{1}{2}$ by a *large* margin. With a large set of "good" stocks (as per our definition) available, it is less interesting to consider stocks outside of this set.

[10] In fact, the cases solved in this paper could also be treated by the PDE approach.

[11] Let us take the excess return rate to be 6.18% p.a., a very modest one for a typical good stock, which was estimated to be the equity premium based on S&P 500 data in Mehra and Prescott (1985). Then any stock whose annual volatility is less than 35.19% (!) will be qualified as a good stock according to our definition.

Appendix

A Function G

Here we derive the explicit expression of the function G, defined by

$$
\begin{aligned}
G(t,x) &= \mathbf{E}\left[\min\left\{e^{-x}, e^{-S^{\mu}_{T-t}}\right\}\right] \\
&= \int_x^{\infty} e^{-z}\, \mathrm{d}\mathbf{P}\left(S^{\mu}_{T-t} \leqslant z\right) + e^{-x}\,\mathbf{P}\left(S^{\mu}_{T-t} \leqslant x\right), \quad 0 \le t \le T,\ x \ge 0.
\end{aligned}
$$

Noting

$$
\mathbf{P}\left(S^{\mu}_{T-t} \leqslant z\right) = \Phi\left(\frac{z-\mu(T-t)}{\sqrt{T-t}}\right) - e^{2\mu z}\Phi\left(\frac{-z-\mu(T-t)}{\sqrt{T-t}}\right),
$$

we have

$$
\begin{aligned}
\int_x^{\infty} e^{-z}\,\mathrm{d}\Phi\left(\frac{z-\mu(T-t)}{\sqrt{T-t}}\right) &= \int_x^{\infty} e^{-z}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(z-\mu(T-t))^2}{2(T-t)}}\,\mathrm{d}z \\
&= e^{-(\mu-1/2)(T-t)}\Phi\left(\frac{-x+(\mu-1)(T-t)}{\sqrt{T-t}}\right).
\end{aligned}
$$

Assume for now that $\mu \neq 1/2$. Then

$$
\begin{aligned}
&\int_x^{\infty} e^{-z}\,\mathrm{d}\left[e^{2\mu z}\Phi\left(\frac{-z-\mu(T-t)}{\sqrt{T-t}}\right)\right] \\
=&\int_x^{\infty} 2\mu e^{-(1-2\mu)z}\Phi\left(\frac{-z-\mu(T-t)}{\sqrt{T-t}}\right)\mathrm{d}z + \int_x^{\infty} e^{-(1-2\mu)z}\,\mathrm{d}\Phi\left(\frac{-z-\mu(T-t)}{\sqrt{T-t}}\right) \\
=&-\frac{2\mu}{2\mu-1}e^{-(1-2\mu)x}\Phi\left(\frac{-x-\mu(T-t)}{\sqrt{T-t}}\right) - \frac{2\mu}{2\mu-1}\int_x^{\infty} e^{-(1-2\mu)z}\,\mathrm{d}\Phi\left(\frac{-z-\mu(T-t)}{\sqrt{T-t}}\right) \\
&+\int_x^{\infty} e^{-(1-2\mu)z}\,\mathrm{d}\Phi\left(\frac{-z-\mu(T-t)}{\sqrt{T-t}}\right) \\
=&-\frac{2\mu}{2\mu-1}e^{-(1-2\mu)x}\Phi\left(\frac{-x-\mu(T-t)}{\sqrt{T-t}}\right) + \frac{1}{2\mu-1}e^{-(\mu-1/2)(T-t)}\Phi\left(\frac{-x+(\mu-1)(T-t)}{\sqrt{T-t}}\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
G(t,x) =& \int_x^{\infty} e^{-z}\,\mathrm{d}\mathbf{P}\left(S^{\mu}_{T-t} \leqslant z\right) + e^{-x}\,\mathbf{P}\left(S^{\mu}_{T-t} \leqslant x\right) \\
=& \int_x^{\infty} e^{-z}\,\mathrm{d}\Phi\left(\frac{z-\mu(T-t)}{\sqrt{T-t}}\right) - \int_x^{\infty} e^{-z}\,\mathrm{d}\left[e^{2\mu z}\Phi\left(\frac{-z-\mu(T-t)}{\sqrt{T-t}}\right)\right] \\
&+ e^{-x}\Phi\left(\frac{x-\mu(T-t)}{\sqrt{T-t}}\right) - e^{-(1-2\mu)x}\Phi\left(\frac{-x-\mu(T-t)}{\sqrt{T-t}}\right) \\
=&\frac{2(\mu-1)}{2\mu-1}e^{-(\mu-1/2)(T-t)}\Phi\left(\frac{-x+(\mu-1)(T-t)}{\sqrt{T-t}}\right) \\
&+ \frac{1}{2\mu-1}e^{-(1-2\mu)x}\Phi\left(\frac{-x-\mu(T-t)}{\sqrt{T-t}}\right) + e^{-x}\Phi\left(\frac{x-\mu(T-t)}{\sqrt{T-t}}\right).
\end{aligned}
$$

This is (15). The case when $\mu = 1/2$ can be dealt with similarly leading to (16). Denoting by $G(t,x;\mu)$ to highlight the dependence on μ it is not hard to verify that, in fact,

$$\lim_{\mu\to 1/2} G(t,x;\mu) = G(t,x;1/2).$$

B Equations for H

We prove (31) for the case when $\mu \neq 1/2$, the other case $\mu = 1/2$ being similar. Write

$$\begin{aligned}
G(t,x) =& \frac{2(\mu-1)}{2\mu-1} e^{-(\mu-1/2)(T-t)} \Phi\left(\frac{-x+(\mu-1)(T-t)}{\sqrt{T-t}}\right) \\
&+ \frac{1}{2\mu-1} e^{-(1-2\mu)x} \Phi\left(\frac{-x-\mu(T-t)}{\sqrt{T-t}}\right) + e^{-x}\Phi\left(\frac{x-\mu(T-t)}{\sqrt{T-t}}\right) \\
:=& \frac{2(\mu-1)}{2\mu-1} A(t,x) + \frac{1}{2\mu-1} B(t,x) + C(t,x).
\end{aligned}$$

Then

$$\begin{aligned}
A_x(t,x) =& e^{-(\mu-\frac{1}{2})(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-(\mu-1)(T-t))^2}{2(T-t)}} (-(T-t)^{-1/2}), \\
A_{xx}(t,x) =& e^{-(\mu-\frac{1}{2})(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-(\mu-1)(T-t))^2}{2(T-t)}} (-(T-t)^{-1/2}) (-\frac{2(x-(\mu-1)(T-t))}{2(T-t)}) \\
=& A_x(t,x)(-x(T-t)^{-1} + \mu - 1), \\
A_t(t,x) =& (\mu-\frac{1}{2})A(t,x) + e^{-(\mu-\frac{1}{2})(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-(\mu-1)(T-t))^2}{2(T-t)}} (-(T-t)^{-1/2}) \frac{1}{2}(x(T-t)^{-1} + \mu - 1)) \\
=& (\mu-\frac{1}{2})A(t,x) + A_x(t,x)\frac{1}{2}(x(T-t)^{-1} + \mu - 1)).
\end{aligned}$$

Hence

$$\begin{aligned}
& A_t(t,x) - \mu A_x(t,x) + \frac{1}{2}A_{xx}(t,x) \\
=& (\mu-\frac{1}{2})A(t,x) + A_x(t,x)\frac{1}{2}(x(T-t)^{-1} + \mu - 1)) - \mu A_x(t,x) + \frac{1}{2}A_x(t,x)(-x(T-t)^{-1} + \mu - 1) \\
=& (\mu-\frac{1}{2})A(t,x) - A_x(t,x).
\end{aligned}$$

Next,

$$
\begin{aligned}
B_x(t,x) =&(2\mu-1)B(t,x) + e^{-(1-2\mu)x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+\mu(T-t))^2}{2(T-t)}}(-(T-t)^{-1/2}),\\
B_t(t,x) =&e^{-(1-2\mu)x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+\mu(T-t))^2}{2(T-t)}}(T-t)^{-1/2}\frac{1}{2}(-x(T-t)^{-1}+\mu)\\
=&((2\mu-1)B(t,x)-B_x(t,x))\frac{1}{2}(-x(T-t)^{-1}+\mu),\\
B_{xx}(t,x) =&(2\mu-1)B_x(t,x) + e^{-(1-2\mu)x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+\mu(T-t))^2}{2(T-t)}}(-(T-t)^{-1/2})(-\frac{2(x+\mu(T-t))}{2(T-t)})\\
&+(2\mu-1)e^{-(1-2\mu)x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x+\mu(T-t))^2}{2(T-t)}}(-(T-t)^{-1/2})\\
=&(2\mu-1)B_x(t,x)+(B_x(t,x)-(2\mu-1)B(t,x))(-x(T-t)^{-1}-\mu)\\
&+(2\mu-1)(B_x(t,x)-(2\mu-1)B(t,x))\\
=&(-x(T-t)^{-1}+3\mu-2)B_x(t,x)+(2\mu-1)(x(T-t)^{-1}-\mu+1)B(t,x),
\end{aligned}
$$

leading to

$$
\begin{aligned}
&B_t(t,x)-\mu B_x(t,x)+\frac{1}{2}B_{xx}(t,x)\\
=&((2\mu-1)B(t,x)-B_x(t,x))\frac{1}{2}(-x(T-t)^{-1}+\mu)-\mu B_x(t,x)\\
&+\frac{1}{2}((-x(T-t)^{-1}+3\mu-2)B_x(t,x)+(2\mu-1)(x(T-t)^{-1}-\mu+1)B(t,x))\\
=&(\mu-\frac{1}{2})B(t,x)-B_x(t,x).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
C_x(t,x) =&-C(t,x)+e^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu(T-t))^2}{2(T-t)}}(T-t)^{-1/2},\\
C_{xx}(t,x) =&-C_x(t,x)+e^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu(T-t))^2}{2(T-t)}}(T-t)^{-1/2}(-\frac{2(x-\mu(T-t))}{2(T-t)})\\
&-e^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu(T-t))^2}{2(T-t)}}(T-t)^{-1/2}\\
=&-C_x(t,x)+(C_x(t,x)+C(t,x))(-x(T-t)^{-1}+\mu-1),\\
C_t(t,x) =&\frac{1}{2}e^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu(T-t))^2}{2(T-t)}}(T-t)^{-1/2}(x(T-t)^{-1}+\mu)\\
=&\frac{1}{2}(C_x(t,x)+C(t,x))(x(T-t)^{-1}+\mu).
\end{aligned}
$$

So

$$
\begin{aligned}
C_t(t,x)-\mu C_x(t,x)+\frac{1}{2}C_{xx}(t,x) =&\frac{1}{2}(C_x(t,x)+C(t,x))(x(T-t)^{-1}+\mu)-\mu C_x(t,x)\\
&+\frac{1}{2}(-C_x(t,x)+(C_x(t,x)+C(t,x))(-x(T-t)^{-1}+\mu-1))\\
=&(\mu-\frac{1}{2})C(t,x)-C_x(t,x).
\end{aligned}
$$

Since G is a linear combination of A, B, and C we conclude (31).

C Proof of $G_x(t, 0+) = 0$.

Again we only prove for the case when $\mu \neq 1/2$. Following the calculation in the previous subsection, we have

$$\begin{aligned}
G_x(t,x) =& \frac{2(\mu-1)}{2\mu-1} A_x(t,x) + \frac{1}{2\mu-1} B_x(t,x) + C_x(t,x) \\
=& \frac{2(\mu-1)}{2\mu-1} e^{-(\mu-\frac{1}{2})(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-(\mu-1)(T-t))^2}{2(T-t)}} (-(T-t)^{-1/2}) \\
&+ \frac{1}{2\mu-1}((2\mu-1)B(t,x) + e^{-(1-2\mu)x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x+\mu(T-t))^2}{2(T-t)}} (-(T-t)^{-1/2})) \\
&- C(t,x) + e^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu(T-t))^2}{2(T-t)}} (T-t)^{-1/2} \\
=& B(t,x) - C(t,x).
\end{aligned}$$

Hence

$$G_x(t, 0+) = B(t, 0+) - C(t, 0+) = 0.$$

D Calculation of $\mathbf{E}\left[e^{-X_T^x}\right]$

We first study the distribution function of $X_t^x = x \vee S_t^\mu - B_t^\mu$ $\forall t > 0$, $x \geq 0$. For any $y \geq 0$,

$$
\begin{aligned}
&\mathbf{P}(x \vee S_t^\mu - B_t^\mu \leqslant y) = \mathbf{P}(e^{x \vee S_t^\mu} \leqslant e^{B_t^\mu + y}) \\
=&\mathbf{P}(e^x \vee e^{S_t^\mu} \leqslant e^{B_t^\mu} e^y) = \int_0^\infty \int_{s\vee 1}^{+\infty} \mathbf{1}_{\{e^x \vee m \leqslant s e^y\}} f(t,s,m)\, \mathrm{d}m\, \mathrm{d}s = \int_{e^{x-y}}^\infty \int_{s\vee 1}^{se^y} f(t,s,m)\, \mathrm{d}m\, \mathrm{d}s \\
=&\int_{e^{x-y}}^\infty \int_{s\vee 1}^{se^y} \frac{2}{\sigma^3\sqrt{2\pi t^3}} \frac{\ln(m^2/s)}{sm} \exp\left(-\frac{\ln^2(m^2/s)}{2\sigma^2 t} + \frac{\beta}{\sigma}\ln(s) - \frac{1}{2}\beta^2 t\right) \mathrm{d}m\, \mathrm{d}s \\
=&\frac{2}{\sigma^3\sqrt{2\pi t^3}} e^{-\frac{1}{2}\beta^2 t} \int_{e^{x-y}}^\infty s^{\beta/\sigma - 1} \int_{s\vee 1}^{se^y} \frac{\ln(m^2/s)}{m} \exp\left(-\frac{\ln^2(m^2/s)}{2\sigma^2 t}\right) \mathrm{d}m\, \mathrm{d}s \\
=&\frac{2}{\sigma^3\sqrt{2\pi t^3}} e^{-\frac{1}{2}\beta^2 t} \int_{e^{x-y}}^\infty s^{\beta/\sigma - 1} \int_{s\vee 1}^{se^y} -\frac{\sigma^2 t}{2}\, \mathrm{d}\left[\exp\left(-\frac{\ln^2(m^2/s)}{2\sigma^2 t}\right)\right] \mathrm{d}s \\
=&\frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2}\beta^2 t} \int_{e^{x-y}}^\infty s^{\beta/\sigma - 1} \left(\exp\left(-\frac{\ln^2((s\vee 1)^2/s)}{2\sigma^2 t}\right) - \exp\left(-\frac{\ln^2(se^{2y})}{2\sigma^2 t}\right)\right) \mathrm{d}s \\
=&\frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2}\beta^2 t} \left[\int_{e^{x-y}}^\infty s^{\beta/\sigma - 1} \exp\left(-\frac{\ln^2(s)}{2\sigma^2 t}\right) \mathrm{d}s - \int_{e^{x-y}}^\infty s^{\beta/\sigma - 1} \exp\left(-\frac{\ln^2(se^{2y})}{2\sigma^2 t}\right) \mathrm{d}s\right] \\
=&\frac{1}{\sigma\sqrt{2\pi t}} e^{-\frac{1}{2}\beta^2 t} \left[\int_{x-y}^\infty e^{u\beta/\sigma} \exp\left(-\frac{u^2}{2\sigma^2 t}\right) \mathrm{d}u - e^{-2y\beta/\sigma} \int_{x+y}^\infty e^{u\beta/\sigma} \exp\left(-\frac{u^2}{2\sigma^2 t}\right) \mathrm{d}u\right] \\
=&\frac{1}{\sigma\sqrt{2\pi t}} \left[\int_{x-y}^\infty \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u - e^{-2y\beta/\sigma} \int_{x+y}^\infty \exp\left(-\frac{(u-\beta\sigma t)^2}{2\sigma^2 t}\right) \mathrm{d}u\right] \\
=&\Phi\left(-\frac{x-y-\beta\sigma t}{\sigma\sqrt{t}}\right) - e^{-2y\beta/\sigma}\Phi\left(-\frac{x+y-\beta\sigma t}{\sigma\sqrt{t}}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbf{E}\left[e^{-X_T^x}\right] = \int_0^\infty e^{-y}\,\mathrm{d}\mathbf{P}(X_T^{0,x} \leqslant y)\\
&= \int_0^\infty e^{-y}\,\mathrm{d}\left[\Phi\left(-\frac{x-y-\beta\sigma T}{\sigma\sqrt{T}}\right) - e^{-2y\beta/\sigma}\Phi\left(-\frac{x+y-\beta\sigma T}{\sigma\sqrt{T}}\right)\right]\\
&=\frac{1}{\sigma\sqrt{2\pi t}}\int_0^\infty \exp\left(-\frac{(x-y-\beta\sigma T)^2+2\sigma^2 Ty}{2\sigma^2 T}\right)\mathrm{d}y\\
&\quad - \int_0^\infty e^{-y}e^{-2y\beta/\sigma}\,\mathrm{d}\left[\Phi\left(-\frac{x+y-\beta\sigma T}{\sigma\sqrt{T}}\right)\right] - \int_0^\infty e^{-y}\Phi\left(-\frac{x+y-\beta\sigma T}{\sigma\sqrt{T}}\right)\mathrm{d}\left[e^{-2y\beta/\sigma}\right]\\
&=e^{-(x-\beta\sigma T-\frac{1}{2}\sigma^2 T)}\Phi\left(\frac{x-\beta\sigma T-\sigma^2 T}{\sigma\sqrt{T}}\right) - \int_0^\infty e^{-(1+2\beta/\sigma)y}\,\mathrm{d}\left[\Phi\left(-\frac{x+y-\beta\sigma T}{\sigma\sqrt{T}}\right)\right]\\
&\quad - \frac{2\beta/\sigma}{1+2\beta/\sigma}\int_0^\infty \Phi\left(-\frac{x+y-\beta\sigma T}{\sigma\sqrt{T}}\right)\mathrm{d}\left[e^{-(1+2\beta/\sigma)y}\right]\\
&=e^{-(x-\beta\sigma T-\frac{1}{2}\sigma^2 T)}\Phi\left(\frac{x-\beta\sigma T-\sigma^2 T}{\sigma\sqrt{T}}\right) + \frac{2\beta/\sigma}{1+2\beta/\sigma}\Phi\left(-\frac{x-\beta\sigma T}{\sigma\sqrt{T}}\right)\\
&\quad - \frac{1}{1+2\beta/\sigma}\int_0^\infty e^{-(1+2\beta/\sigma)y}\,\mathrm{d}\left[\Phi\left(-\frac{x+y-\beta\sigma T}{\sigma\sqrt{T}}\right)\right]\\
&=e^{-(x-\beta\sigma T-\frac{1}{2}\sigma^2 T)}\Phi\left(\frac{x-\beta\sigma T-\sigma^2 T}{\sigma\sqrt{T}}\right) + \frac{2\beta/\sigma}{1+2\beta/\sigma}\Phi\left(-\frac{x-\beta\sigma T}{\sigma\sqrt{T}}\right)\\
&\quad + \frac{1}{1+2\beta/\sigma}e^{(1+2\beta/\sigma)(x+\frac{1}{2}\sigma^2 T)}\Phi\left(-\frac{x+\beta\sigma T+\sigma^2 T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

Now suppose $\sigma = 1$, then $\beta = r - 1/2 = \mu$.

$$
\begin{aligned}
\mathbf{E}\left[e^{-X_T^x}\right] =& e^{(\mu+1/2)T-x}\Phi\left(\frac{x-(\mu+1)T}{\sqrt{T}}\right) + \frac{2\mu}{2\mu+1}\Phi\left(\frac{-x+\mu T}{\sqrt{T}}\right)\\
&+ \frac{1}{2\mu+1}e^{(2\mu+1)(x+T/2)}\Phi\left(\frac{-x-(\mu+1)T}{\sqrt{T}}\right).
\end{aligned}
$$

E A Lemma

Lemma E.1 *If $x > 0$, then*

$$
\frac{x}{1+x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} < \Phi(-x) < \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.
$$

Proof. This is evident by

$$
\Phi(-x) = \int_{-\infty}^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,\mathrm{d}t < \int_{-\infty}^{-x}\frac{t}{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,\mathrm{d}t = \frac{1}{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},
$$

and

$$
\begin{aligned}
\Phi(-x) &= \int_{-\infty}^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,\mathrm{d}t > \int_{-\infty}^{-x}\frac{x^2}{t^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,\mathrm{d}t\\
&=\frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} - x^2\int_{-\infty}^{-x}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,\mathrm{d}t = \frac{x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} - x^2\Phi(-x).
\end{aligned}
$$

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