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Path-Dependent and Randomized Strategies in Barberis' Casino Gambling Model

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We consider the dynamic casino gambling model initially proposed by Barberis (2012) and study the optimal stopping strategy of a pre-committing gambler with cumulative prospect theory (CPT) preferences. We illustrate how the strategies computed in Barberis (2012) can be strictly improved by reviewing the betting history or by tossing an independent coin, and we explain that the improvement generated by using randomized strategies results from the lack of quasi-convexity of CPT preferences. Moreover, we show that any path-dependent strategy is equivalent to a randomization of path-independent strategies.

Key words: casino gambling; cumulative prospect theory; path-dependence; randomized strategies; quasi-convexity; optimal stopping

1. Introduction

Barberis (2012) formulated and studied a casino gambling model, in which a gambler comes to a casino at time 0 and is repeatedly offered a bet with an equal chance to win or lose \$1. The gambler decides when to stop playing. Clearly, the answer depends on the gambler's preferences. In Barberis (2012), the gambler's risk preferences are represented by the cumulative prospect theory

(CPT) of Tversky and Kahneman (1992). In this theory, individuals' preferences are determined by an S-shaped utility function and two inverse-S shaped probability weighting functions. The latter effectively overweight the tails of a distribution, so a gambler with CPT preferences overweights very large gains of small probability and thus may decide to play in the casino.

A crucial contribution in Barberis (2012) lies in showing that the optimal strategy of a gambler with CPT preferences, a descriptive model for individuals' preferences, is consistent with several commonly observed gambling behaviors such as the popularity of casino gambling and the implementation of gain-exit and loss-exit strategies. However, the setting of Barberis (2012) was restrictive as it assumed that the gambler can only choose among simple (path-independent) strategies.

We find that when allowed to take path-dependent strategies (i.e. strategies which may depend on the betting history), the gambler may *strictly* prefer these strategies over path-independent ones. Moreover, by tossing an independent (possibly biased) coin at some points in time, the gambler may further *strictly* improve his preference value. Our findings are consistent with empirical observations of the use of path-dependent strategies such as the *house money effect* documented by Thaler and Johnson (1990) and the use of random devices by individuals to aid their choices in various contexts (Dwenger et al. 2013, Agranov and Ortoleva 2015).

We also study, at a theoretical level, the issue of why the gambler strictly prefers path-dependent strategies over path-independent ones and prefers randomized strategies over nonrandomized ones. First, we show that tossing independent coins at some points while following a path-independent strategy can be regarded as a fully randomized strategy: a random selection from the set of path-independent strategies. It is well known in the decision analysis literature that individuals do not prefer fully randomized strategies over nonrandomized ones if their preferences are quasi-convex. CPT preferences, however, are not quasi-convex, so randomized strategies can be strictly preferred in the casino gambling model.

Second, we prove that any path-dependent strategy is equivalent to a randomization of path-independent strategies. Consequently, the gambler in the casino problem strictly prefers path-dependent strategies for the same reason why he prefers randomized strategies. It also follows that it is always enough to consider randomized, path-independent strategies since the more complex randomized *and* path-dependent strategies cannot further improve the gambler's preference value.

In the present paper, we focus on the optimal strategy for sophisticated agents with pre-commitment, who are able to commit themselves in the future to the strategy that is set up at time 0 through some commitment device. This allows us to concentrate on studying the optimal behavior resulting from CPT preferences. Moreover, such sophisticated agents with pre-commitment are themselves an important type of agents, and many gamblers belong to this type with the aid of some commitment devices; see the discussion in Barberis (2012, p. 49). In addition, understanding the behaviors of these agents are key to studying those of any other types of agents; see Barberis (2012), Ebert and Strack (2015), Henderson et al. (2014) and He et al. (2016).

2. Comparison of Different Types of Strategies in the Casino Gambling Model

2.1. Model

We consider the casino gambling model proposed by Barberis (2012). At time 0, a gambler is offered a fair bet, e.g., an idealized black or red bet on a roulette wheel: win or lose one dollar with equal probability. If he decides to play, the outcome of the bet is played out at time 1, at which time he either wins or loses one dollar. The gambler is then offered the same bet and he can again choose to play or not, and so forth until a terminal time T or the time he declines to play and exits the casino, whichever comes first.

We call a random time chosen by the gambler to exit the casino a *strategy*. As in Barberis (2012), we assume CPT preferences for the gambler; so the decision criterion is to maximize the CPT value of his wealth at the time when he leaves the casino. More precisely, the gambler first computes his gain and loss X relative to some reference point and evaluates X by

$$V(X) := \int_0^{\infty} u(x)d[-w_+(1-F_X(x))] + \int_{-\infty}^{0} u(x)d[w_-(F_X(x))], \tag{1}$$

where $F_X(\cdot)$ is the cumulative distribution function (CDF) of X. The function $u(\cdot)$, which is strictly increasing, is called the *utility function* (or value function) and $w_{\pm}(\cdot)$, two strictly increasing mappings from $[0,1]$ onto $[0,1]$, are *probability weighting* (or *distortion*) *functions* on gains and losses, respectively. Following Barberis (2012), we use the following parametric forms proposed by Tversky and Kahneman (1992) for the utility and probability weighting functions:

$$u(x) = \begin{cases} x^{\alpha_+} & \text{for } x \geq 0 \\ -\lambda(-x)^{\alpha_-} & \text{for } x < 0, \end{cases} \quad \text{and} \quad w_\pm(p) = \frac{p^{\delta_\pm}}{(p^{\delta_\pm} + (1-p)^{\delta_\pm})^{1/\delta_\pm}}, \tag{2}$$

where $\lambda \geq 1$, $\alpha_\pm \in (0,1]$, and $\delta_\pm \in [0.28, 1]$. Such u is S-shaped and $w_\pm$ are inverse-S-shaped, and thus they are able to describe the fourfold pattern of individuals' choice under risk that cannot be explained by the classical expected utility theory (EUT); see Tversky and Kahneman (1992).

In the casino gambling problem, as the bet is assumed to be fair, the cumulative gain or loss of the gambler, while he continues to play, is a standard symmetric random walk $S_n, n \geq 0$, on $\mathbb{Z}$, the set of integers. We further assume that the gambler uses his initial wealth as the reference point, so he perceives S_n as his cumulative gain or loss after n bets. As a result, for any exit time τ, the gambler's CPT preference value of his gain and loss at the exit time is

$$\begin{aligned} V(S_\tau) = &\sum_{n=1}^{T} u(n)\Big(w_+(\mathbb{P}(S_\tau \geq n)) - w_+(\mathbb{P}(S_\tau > n))\Big) \\ &+ \sum_{n=1}^{T} u(-n)\Big(w_-(\mathbb{P}(S_\tau \leq -n)) - w_-(\mathbb{P}(S_\tau < -n))\Big), \end{aligned} \tag{3}$$

with the convention that $+\infty - \infty = -\infty$, so that $V(S_\tau)$ is always well-defined. The gambler's problem at time 0 is to find an exit time τ in a set of admissible strategies which maximizes $V(S_\tau)$.

2.2. Comparison of Strategies

Consider a 6-period horizon, $\alpha = 0.9$, $\delta = 0.4$, and $\lambda = 2.25$, which are empirically reasonable parameter values; see for instance Tversky and Kahneman (1992). Barberis (2012) consider *path-independent* strategies only, in which for any $t \geq 0$, $\{\tau = t\}$ (conditioning on $\{\tau \geq t\}$) is determined by (t, S_t), i.e., by the agent's cumulative gain and loss at time t only. The left-panel of Figure 1 shows the optimal path-independent strategy: the nodes in the recombining binomial tree stand for the cumulative gain and loss at different times t provided that the agent has not exited the casino yet, and the pairs of numbers above each node are (t, S_t). Black nodes stand for "stop" and white nodes stand for "continue", so the gambler exits the casino at the first time when his gain/loss process hits one of the black nodes. Barberis (2012) computed the CPT preference value of the optimal path-independent strategy to be $V = 0.250440$.

Next, suppose the gambler can take *path-dependent* strategies, in which for any $t \geq 0$, $\{\tau = t\}$ is determined by the information set $\mathcal{F}_t = \sigma(S_u : u \leq t)$, i.e., by the whole history of the gambler's cumulative gains and losses in the casino up to time t. The middle panel of Figure 1 presents the optimal path-dependent strategy: black nodes stand for "stop", white nodes stand for "continue", and at the half-black-half-white node the gambler stops if the previous two nodes the path has gone through were $(3, 3)$ and $(4, 2)$; otherwise he continues. The decision whether to stop or not at the half-black-half-white node depends not only on the cumulative gain and loss at this node but also on the cumulative gains and losses in the past. The CPT preference value of the optimal path-dependent strategy is $V = 0.250693$, which is a strict improvement from that of the optimal path-independent strategy.[1]

Finally, we consider a strategy as presented in the right panel of Figure 1: Black nodes stand for "stop", white nodes stand for "continue", and each grey node means that a coin is tossed at that node independently from the bet outcomes and the gambler stops if and only if the coin turns up tails. The coins at the grey nodes $(5, 3)$ and $(5, 1)$ are tossed independently with the probabilities of the coin turning up tails to be $1/32$ and $1/2$, respectively. The CPT preference value of such a strategy is $V = 0.250702$, which is a strict improvement from that of the optimal path-independent strategy and even from that of the optimal path-dependent strategy.

Our finding that the gambler strictly prefers to use path-dependent strategies and to toss coins in the casino gambling model is not specific to the choice of probability weighting functions. For instance, we have obtained the same results by using the probability weighting function proposed by Prelec (1998), i.e., $w_{\pm}(p) = e^{-\gamma(-\ln p)^{\delta}}$, with $\gamma = 0.5$ and $\delta = 0.4$.

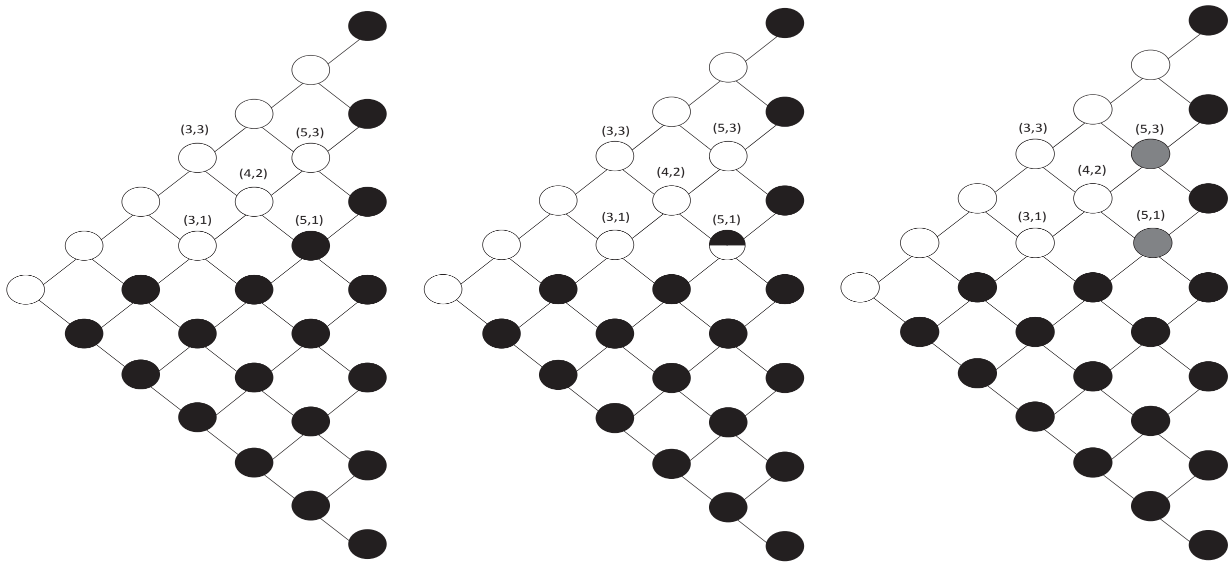

Figure 1 Optimal path-independent strategy (left panel), optimal path-dependent strategy (middle panel), and a randomized, path-independent strategy (right panel) of the casino gambler.

3. Why Path-Dependent and Randomized Strategies Outperform

In what follows, we explain theoretically, via a general optimal stopping problem, the reasons underlying the observed strict preferences for randomization and for path-dependence in the casino gambling model.

3.1. Three Types of Strategies

We consider a general discrete-time Markov chain $X = \{X_t\}_{t\in[0,T]}$.[2] Here and henceforth by $t \in [0,T]$ we mean $t = 0, 1, \ldots, T$. Without loss of generality, we assume that X_t takes values in $\mathbb{Z}$ and that $X_0 = 0$. We suppose an agent wants to choose a stopping time $\tau \le T$ to maximize his preference value of X_τ.

Denote by $\mathcal{A}_M$ and $\mathcal{A}_D$ the set of path-independent stopping times and the set of path-dependent stopping times, respectively, i.e., $\mathcal{A}_D$ is the set of $\{\mathcal{F}_t\}_{t\ge 0}$-stopping times, where $\mathcal{F}_t = \sigma(X_u, u \le t)$, and $\mathcal{A}_M$ is the set of τ's in $\mathcal{A}_D$ such that for each $t \ge 0$, conditional on $\{\tau \ge t\}$, $\{\tau = t\}$ depends on (t, S_t) only. Clearly, $\mathcal{A}_M$ and $\mathcal{A}_D$ are the two types of strategies considered in the left and middle panels of Figure 1.

To formally define the type of strategies considered in the right panel of Figure 1, consider a family of 0-1 random variables $\xi_{t,x}$, $t\in[0,T]$, $x\in\mathbb{Z}$ that are independent of $\{X_t\}$ and are mutually independent. Each $\xi_{t,x}$ stands for a coin toss at time t when $X_t=x$ with $\xi_{t,x}=0$ standing for tails and $\xi_{t,x}=1$ standing for heads. For such family of coin tosses, we can consider the strategy of stopping at the first time when a coin toss turns up tails, i.e.,

$$\tau := \inf\{t\in[0,T]|\xi_{t,X_t}=0\}. \tag{4}$$

Note that in this case, the information that is used to decide whether to stop or not includes not only the Markov process $\{X_t\}$ but also the coin toss outcomes in the history; i.e., the information at t becomes $\mathcal{G}_t := \sigma(X_u, \xi_{u,X_u}, u\le t)$. Clearly, τ in (4) is a stopping time with respect to this enlarged information flow. However, τ defined by (4) is *path-independent* in that $\{\tau=t\}$ depends only on X_t and ξ_{t,X_t} (conditioning on $\{\tau\ge t\}$). Let us stress that we do not specify the distribution of these random coins, i.e., the value of $r_{t,x} := \mathbb{P}(\xi_{t,x}=0)=1-\mathbb{P}(\xi_{t,x}=1)$; these numbers are determined by the agent as *part* of his stopping strategy. We denote by $\mathcal{A}_C$ the set of such *randomized, path-independent* strategies generated by tossing coins, i.e.,

$$\begin{aligned}\mathcal{A}_C := \{\tau|\tau \text{ is defined as in (4), for some } \{\xi_{t,x}\}_{t\in[0,T],x\in\mathbb{Z}} \text{ that are independent 0-1}\\ \text{random variables and are independent of } \{X_t\}_{t\in[0,T]}.\}\end{aligned}$$

Note that $\mathcal{A}_M\subset\mathcal{A}_C$ since non-randomized path-independent strategies correspond simply to the case in which $r_{t,x}\in\{0,1\}$. In addition, in the casino gambling problem, the strategy showed in the right panel of Figure 1 belongs to $\mathcal{A}_C$. However, $\mathcal{A}_C$ and $\mathcal{A}_D$ do not contain each other.

3.2. Preference for Randomization

A fully randomized strategy is implemented by randomly choosing from a given set of non-randomized stopping times, e.g., from $\mathcal{A}_M$, using a random device. Such a device is represented by a random mapping σ from Ω to $\mathcal{A}_M$ and is independent of the Markovian process $\{X_t\}_{t\ge0}$. Suppose the outcome of σ is a stopping time $\tau\in\mathcal{A}_M$; the agent then chooses τ as his strategy. To visualize this, imagine an uneven die with a possibly infinite but countable number of faces, each corresponding to a strategy in $\mathcal{A}_M$.

We observe that any $\tilde{\tau} \in \mathcal{A}_C$, i.e., $\tilde{\tau} = \inf\{t \in [0,T] | \xi_{t,X_t} = 0\}$ for a given set of mutually independent 0-1 random variables $\{\xi_{t,x}\}_{t\in[0,T],x\in\mathbb{Z}}$ that are independent of $\{X_t\}_{t\in[0,T]}$, can be viewed as a special case of full randomization. Indeed, for any $\mathbf{a} := \{a_{t,x}\}_{t\in[0,T],x\in\mathbb{Z}}$ such that $a_{t,x} \in \{0,1\}$, define $\tau_{\mathbf{a}} := \inf\{t \in [0,T] | a_{t,X_t} = 0\}$. Then, $\tau_{\mathbf{a}} \in \mathcal{A}_M$. Let A be a random variable taking values in sequences $\mathbf{a}$ given by $A = \{\xi_{t,x}(\omega)\}_{t\in[0,T],x\in\mathbb{Z}}$ and define $\sigma := \tau_A$. Then, σ is a random mapping taking values in $\mathcal{A}_M$ and because A is independent of $\{X_t\}_{t\in[0,T]}$, σ is a fully randomized strategy and, clearly, $\sigma = \tilde{\tau}$. Finally, by an analogous argument, we note that any randomized path-dependent strategy, i.e. a strategy where coin tosses are independent of $\{X_t\}_{t\in[0,T]}$ but the relative chance of tails may depend on past path of X, can be seen as a full randomization σ taking values in $\mathcal{A}_D$.

Now, suppose the agent's preference for X_τ is law-invariant, i.e., the preference value of X_τ is $\mathcal{V}(F_{X_\tau})$ for some functional $\mathcal{V}$ on distributions. We say $\mathcal{V}$ is *quasi-convex* if

$$\mathcal{V}(pF_1 + (1-p)F_2) \le \max\{\mathcal{V}(F_1), \mathcal{V}(F_2)\}, \quad \forall F_1, F_2, \forall p \in [0,1]. \tag{5}$$

If, further, $\mathcal{V}$ is continuous, then we can conclude that[3]

$$\mathcal{V}\left(\sum_{i=1}^{+\infty} p_i F_i\right) \le \max_{i\ge 1} \mathcal{V}(F_i), \quad \forall F_i, \forall p_i \ge 0, i \ge 1, \text{ with } \sum_{i=1}^{\infty} p_i = 1.$$

However, the state distribution at a randomized stopping time is a convex combination of the state distributions at nonrandomized stopping times. To see this, suppose σ takes values τ_i's with probabilities p_i's, respectively. Then,

$$\begin{aligned} F_{X_\sigma}(x) = \mathbb{P}(X_\sigma \le x) = \mathbb{E}\left[\mathbb{P}(X_\sigma \le x|\sigma)\right] &= \sum_i p_i \mathbb{P}(X_{\tau_i} \le x | \sigma = \tau_i) \\ &= \sum_i p_i \mathbb{P}(X_{\tau_i} \le x) = \sum_i p_i F_{X_{\tau_i}}(x). \end{aligned}$$

Hence, we can conclude that the agent dislikes any type of randomization if his preference representation $\mathcal{V}$ is quasi-convex.

It has been noted in the literature that individuals with quasi-convex preferences dislike full randomization; see for instance Machina (1985), Camerer and Ho (1994), Wakker (1994), and Blavatskyy (2006).[4] It is easy to see that a sufficient condition for quasi-convexity is *betweenness*: for any two distributions F_1 and F_2 such that $\mathcal{V}(F_1) \geq \mathcal{V}(F_2)$ and any $p \in (0,1)$, $\mathcal{V}(F_1) \geq \mathcal{V}(pF_1 + (1-p)F_2) \geq \mathcal{V}(F_2)$. Betweenness is further implied by *independence*: for any distributions F_1, F_2, G and any $p \in (0,1)$, if $\mathcal{V}(F_1) \geq \mathcal{V}(F_2)$, then $\mathcal{V}(pF_1 + (1-p)G) \geq \mathcal{V}(pF_2 + (1-p)G)$.[5] Because expected utility (EU) theory is underlined by the independence axiom, agents with EU preferences dislike randomization. Similarly, agents with preferences satisfying betweenness or quasi-convexity also dislike randomization.

However, empirically individuals have been observed to violate quasi-convexity and, as a result, to prefer randomization; see for instance Camerer and Ho (1994), Blavatskyy (2006), and the references therein. This is reflected in descriptive models of preferences which involve probability weighting, such as rank-dependent utility (Quiggin 1982), prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992), and the dual theory of choice (Yaari 1987). These preferences are in general not quasi-convex (see for instance Camerer and Ho (1994)) and hence may exhibit a strict preference for randomization, as shown in our casino gambling model.

Finally, let us emphasize that a randomized, path-independent strategy that is obtained by simply tossing coins dynamically is easier to implement and simpler than a fully randomized strategy. Although it is well understood in the literature that an agent with quasi-convex preferences dislike fully randomized strategies, it is unclear whether a randomized, path-independent strategy will be strictly preferred in a particular model with non-quasi-convex preferences such as CPT preferences. We contribute to the existing literature by showing that it is indeed the case in the casino gambling model with reasonable parameter values, i.e., with CPT preferences that are empirically descriptive of individuals' behavior under risk.

3.3. Preference for Path-Dependent Strategies

It is well known that if the agent's preferences are represented by expected utility, i.e., $\mathcal{V}(F) = \int u(x)dF(x)$ for some utility function u, then the agent's optimal stopping time may be taken to be path-independent. This result is a consequence of dynamic programming. For dynamic decision problems with non-EU preferences, however, dynamic programming may not hold if the problems are time inconsistent, so it is unclear whether the optimal stopping time is still path-independent in this case. In the following, we show that any path-dependent strategy can be viewed as a randomization of path-independent strategies, so it follows from the conclusion of the previous subsection that the agent does not strictly prefer path-dependent strategies if his preference for X_τ is quasi-convex.

PROPOSITION 1. *For any $\tau\in\mathcal{A}_D$, there exists $\tilde\tau\in\mathcal{A}_C$ such that (X_τ,τ) has the same distribution as $(X_{\tilde\tau},\tilde\tau)$. More generally, for any fully randomized path-dependent strategy σ, there exists $\tilde\tau\in\mathcal{A}_C$ such that (X_σ,σ) has the same distribution as $(X_{\tilde\tau},\tilde\tau)$.*

Proof. For any $\tau\in\mathcal{A}_D$, let $r(t,x)=\mathbb{P}(\tau=t|X_t=x,\tau\ge t)$, $t\in[0,T], x\in\mathbb{Z}$. Take mutually independent random variables $\{\xi_{t,x}\}_{t\in[0,T],x\in\mathbb{Z}}$, which are also independent of $\{X_t\}$, such that $\mathbb{P}(\xi_{t,x}=0)=r(t,x)=1-\mathbb{P}(\xi_{t,x}=1)$. Define $\tilde\tau=\inf\{t\in[0,T]|\xi_{t,X_t}=0\}\in\mathcal{A}_C$. We will show that $(X_{\tilde\tau},\tilde\tau)$ is identically distributed as (X_τ,τ), i.e., for any $s\in[0,T], x\in\mathbb{Z}$,

$$\mathbb{P}(X_\tau=x,\tau=s)=\mathbb{P}(X_{\tilde\tau}=x,\tilde\tau=s). \tag{6}$$

We prove this by mathematical induction.

We first show that (6) is true for $s=0$. Indeed, for any $x\in\mathbb{Z}$, we have

$$\begin{aligned}\mathbb{P}(X_\tau=x,\tau=0)&=\mathbb{P}(X_0=x,\tau=0,\tau\ge 0)=\mathbb{P}(X_0=x,\tau\ge 0)\mathbb{P}(\tau=0|X_0=x,\tau\ge 0)\\&=\mathbb{P}(X_0=x)r(0,x)=\mathbb{P}(X_0=x)\mathbb{P}(\xi_{0,x}=0)\\&=\mathbb{P}(X_0=x,\xi_{0,x}=0)=\mathbb{P}(X_0=x,\tilde\tau=0)=\mathbb{P}(X_{\tilde\tau}=x,\tilde\tau=0),\end{aligned}$$

where the fifth equality is due to the independence of $\xi_{0,x}$ and $\{X_t\}$ and the sixth equality follows from the definition of $\tilde\tau$.

Next, we suppose that (6) is true for $s\le t$ and show that it is also true for $s=t+1$. First, note that $\{X_t\}$ is Markovian with respect both to the filtration generated by itself and to the enlarged filtration $\{\mathcal{G}_t\}$. Furthermore, τ and $\tilde{\tau}$ are stopping times with respect to these two filtrations, respectively. As a result, for any $s < t$, given X_s, events $\{\tau = s\}$ and $\{\tilde{\tau} = s\}$ are independent of X_t. Then, we have

$$\begin{aligned}
\mathbb{P}(X_t = x, \tau \le t) &= \sum_{s\le t}\sum_{y} \mathbb{P}(X_t = x | X_s = y, \tau = s)\mathbb{P}(X_s = y, \tau = s) \\
&= \sum_{s\le t}\sum_{y} \mathbb{P}(X_t = x | X_s = y)\mathbb{P}(X_s = y, \tau = s) \\
&= \sum_{s\le t}\sum_{y} \mathbb{P}(X_t = x | X_s = y)\mathbb{P}(X_s = y, \tilde{\tau} = s) \qquad (7)\\
&= \sum_{s\le t}\sum_{y} \mathbb{P}(X_t = x | X_s = y, \tilde{\tau} = s)\mathbb{P}(X_s = y, \tilde{\tau} = s) \\
&= \mathbb{P}(X_t = x, \tilde{\tau} \le t),
\end{aligned}$$

where the third equality holds by the inductive hypothesis. Consequently,

$$\begin{aligned}
&\mathbb{P}(X_\tau = x, \tau = t+1) \\
&\quad = \mathbb{P}(\tau = t+1 | X_{t+1} = x, \tau \ge t+1)\mathbb{P}(X_{t+1} = x, \tau \ge t+1) \\
&\quad = r(t+1, x)\sum_{y} \mathbb{P}(X_{t+1} = x, X_t = y, \tau \ge t+1) \\
&\quad = r(t+1, x)\sum_{y} \mathbb{P}(X_{t+1} = x | X_t = y, \tau \ge t+1)\mathbb{P}(X_t = y, \tau \ge t+1) \\
&\quad = r(t+1, x)\sum_{y} \mathbb{P}(X_{t+1} = x | X_t = y)\mathbb{P}(X_t = y, \tau \ge t+1) \\
&\quad = r(t+1, x)\sum_{y} \mathbb{P}(X_{t+1} = x | X_t = y)\mathbb{P}(X_t = y, \tilde{\tau} \ge t+1) \\
&\quad = r(t+1, x)\sum_{y} \mathbb{P}(X_{t+1} = x | X_t = y, \tilde{\tau} \ge t+1)\mathbb{P}(X_t = y, \tilde{\tau} \ge t+1) \\
&\quad = r(t+1, x)\sum_{y} \mathbb{P}(X_{t+1} = x, X_t = y, \tilde{\tau} \ge t+1) \\
&\quad = \mathbb{P}(\tilde{\tau} = t+1 | X_{t+1} = x, \tilde{\tau} \ge t+1)\mathbb{P}(X_{t+1} = x, \tilde{\tau} \ge t+1) \\
&\quad = \mathbb{P}(X_{\tilde{\tau}} = x, \tilde{\tau} = t+1),
\end{aligned}$$

where the fourth and sixth equalities hold because of the Markovian property of $\{X_t\}_{t\ge 0}$, the fifth follows from (7) and the eight holds by the definition of $\tilde{\tau}$. By mathematical induction, (6) holds for any s and x. Finally, we see that the above proof also applies to any fully randomized path-dependent strategy σ. $\square$

Proposition 1 shows that for any path-dependent stopping time τ, X_τ is identically distributed as $X_{\tilde{\tau}}$ for some $\tilde{\tau} \in \mathcal{A}_C$, which is a randomization of path-independent stopping times. This result has three implications for an agent whose preference for X_τ is represented by $\mathcal{V}(F_{X_\tau})$. First, the agent is indifferent between τ and $\tilde{\tau}$. Consequently, we must have $\sup_{\tau\in\mathcal{A}_C}\mathcal{V}(F_{X_\tau}) \geq \sup_{\tau\in\mathcal{A}_D}\mathcal{V}(F_{X_\tau})$, i.e., randomized, path-independent strategies always perform no worse than non-randomized path-dependent ones. The inequality may be *strict* as seen in our example in Section 2.2.

Second, as argued before, any randomized path-dependent strategy τ' can be seen as a fully randomized path-dependent strategy σ. In consequence, there also exists $\tilde{\tau} \in \mathcal{A}_C$ such that $X_{\tau'}$ has the same distribution as $X_{\tilde{\tau}}$. Therefore, using randomized path-dependent strategies cannot improve the gambler's preference value compared to simply using randomized but path-independent strategies. This explains why we consider only the latter in the casino gambling problem.

Third, if the agent's preference representation $\mathcal{V}$ is quasi-convex then it is optimal for him to use nonrandomized and path-independent strategies only. Indeed, we have already concluded that he dislikes any type of randomization. Further, by Proposition 1, any path-dependent strategy is equivalent to a randomization of path-independent strategies and is thus less preferred than some path-independent strategy. In the casino gambling problem, the gambler can improve his CPT value by considering path-dependent strategies only because CPT is not quasi-convex.

3.4. Discounting and Time-Dependent Preferences

For any full randomization σ taking possible values τ_i's with respective probabilities p_i's, we have

$$\mathbb{P}(\sigma \leq t, X_\sigma \leq x) = \mathbb{E}\left[\mathbb{P}(\sigma \leq t, X_\sigma \leq x|\sigma)\right] = \sum_{i=1} p_i \mathbb{P}(\tau_i \leq t, X_{\tau_i} \leq x|\sigma = \tau_i) = \sum_{i=1} p_i \mathbb{P}(\tau_i \leq t, X_{\tau_i} \leq x),$$

i.e., the joint distribution of (σ, X_σ) is a convex combination of the joint distributions of (τ_i, X_{τ_i}), $i \geq 1$. Furthermore, Proposition 1 shows that for any path-dependent stopping time τ (randomized or not), (τ, X_τ) is identically distributed as $(\tilde{\tau}, X_{\tilde{\tau}})$ for some randomized but path-independent strategy τ. Therefore, the conclusions in Sections 3.2 and 3.3 remain true if the agent's preferences are represented by a functional $\tilde{\mathcal{V}}$ of the joint distribution of (τ, X_τ). In particular, if $\tilde{\mathcal{V}}$ is quasi-convex, then the agent dislikes randomization and path-dependent strategies.

A special case of interest are agent's preferences in the form $\mathcal{V}(F_{H(\tau,X_\tau)})$, a functional of the distribution of $H(\tau, X_\tau)$ for some function H. If $\mathcal{V}(F_{H(\tau,X_\tau)})$ is quasi-convex in $F_{H(\tau,X_\tau)}$ then it is also quasi-convex in the joint distribution F_{τ,X_τ} and the agent dislikes randomization and path-dependent strategies. A simple example of function H is $H(t,x) = e^{-rt}x$, where r is a discount factor. Therefore, if the agent has law-invariant and quasi-convex preferences for the discounted value $e^{-r\tau}X_\tau$, he will choose only path-independent strategies.

4. Conclusion

This paper considers the dynamic casino gambling model with CPT preferences that was initially proposed by Barberis (2012). Our first contribution was to show that CPT, as a descriptive model for individuals' preferences, accounts for two types of gambling behavior, namely, use of path-dependent strategies and use of independent randomization for assisting ongoing decision making. Our second contribution was to show the improvement in performance brought by these strategies in the casino gambling problem is a consequence of lack of quasi-convexity of CPT preferences. Our third contribution was to show that any path-dependent strategy is equivalent to a randomized, path-independent strategy. Consequently, using randomized path-dependent strategies cannot improve the gambler's preference value compared to simply using randomized but path-independent strategies.

Note. Recently, and independently of our work, Henderson et al. (2014) observed that randomized strategies may be necessary for optimal gambling strategies. This observation emerged in the course of a conversation between one of those authors and two of the authors of the present paper at the SIAM Financial Mathematics meeting in November 2014 in Chicago. The other paper was subsequently posted on SSRN.

Endnotes

1. The optimal path-dependent strategy is not unique: the gambler can also choose to continue if the path leading to node $(5,1)$ goes through nodes $(3,3)$ and $(4,2)$ and stop if it goes through nodes $(3,1)$ and $(4,2)$.

2. The gains/losses process in the casino gambling problem is a symmetric random walk and thus a Markov chain.

3. Quasi-convexity implies $\mathcal{V}(\sum_{i=1}^{n} p_i F_i) \le \max_{1\le i\le n} \mathcal{V}(F_i)$ and the continuity is needed to pass with n to infinity.

4. In other words, among non-EU preferences only the ones without quasi-convexity can explain the preference for randomization. As summarized by Agranov and Ortoleva (2015), non-EU preferences fall in the category of models of deliberate randomization, and another two categories of models that can explain the preference for randomization are models of random utility and models of mistakes. The experimental results in Agranov and Ortoleva (2015) support the models of deliberate randomization.

5. Indeed, by setting G in the definition of the independence axiom to be F_1 and F_2, respectively, we immediately conclude that independence leads to betweenness.

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References

Agranov, Marina, Pietro Ortoleva. 2015. Stochastic choice and hedging. Tech. rep., Mimeo California Institute of Technology.

Barberis, Nicholas. 2012. A model of casino gambling. *Management Sci.* **58**(1) 35–51.

Blavatskyy, Pavlo R. 2006. Violations of betweenness or random errors? *Econ. Lett.* **91**(1) 34–38.

Camerer, Colin F., Teck-Hua Ho. 1994. Violations of the betweenness axiom and nonlinearity in probability. *J. Risk Uncertainty* **8**(2) 167–196.

Dwenger, Nadja, Dorothea Kübler, Georg Weizsacker. 2013. Flipping a coin: Theory and evidence. SSRN:2353282.

Ebert, Sebastian, Philipp Strack. 2015. Until the bitter end: on prospect theory in a dynamic context. *Amer. Econ. Rev.* **105**(4) 1618–1633.

He, Xue Dong, Sang Hu, Jan Obłój, Xun Yu Zhou. 2016. Optimal exit time from casino gambling: strategies of pre-committing and naive agents. SSRN:2684043.

Henderson, Vicky, David Hobson, Alex Tse. 2014. Randomized strategies and prospect theory in a dynamic context. SSRN: 2531457.

Hu, Sang. 2014. Optimal exist strategies of behavioral gamblers. Ph.D. thesis, The Chinese University of Hong Kong.

Kahneman, Daniel, Amos Tversky. 1979. Prospect theory: An analysis of decision under risk. *Econometrica* **47**(2) 263–291.

Machina, Mark J. 1985. Stochastic choice functions generated from deterministic preferences over lotteries. *Econ. J.* **95**(379) 575–594.

Prelec, Drazen. 1998. The probability weighting function. *Econometrica* **66**(3) 497–527.

Quiggin, John. 1982. A theory of anticipated utility. *J. Econ. Behav. Organ.* **3** 323–343.

Thaler, Richard H, Eric J Johnson. 1990. Gambling with the house money and trying to break even: The effects of prior outcomes on risky choice. *Management Sci.* **36**(6) 643–660.

Tversky, Amos, Daniel Kahneman. 1992. Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk Uncertainty* **5**(4) 297–323.

Wakker, Peter. 1994. Separating marginal utility and probabilistic risk aversion. *Theory Dec.* **36**(1) 1–44.

Yaari, Menahem E. 1987. The dual theory of choice under risk. *Econometrica* **55**(1) 95–115.

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