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OPTIMAL STOPPING UNDER PROBABILITY DISTORTION*

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We formulate an optimal stopping problem for a geometric Brownian motion where the probability scale is distorted by a general nonlinear function. The problem is inherently time inconsistent due to the Choquet integration involved. We develop a new approach, based on a reformulation of the problem where one optimally chooses the probability distribution or quantile function of the stopped state. An optimal stopping time can then be recovered from the obtained distribution/quantile function, either in a straightforward way for several important cases or in general via the Skorokhod embedding. This approach enables us to solve the problem in a fairly general manner with different shapes of the payoff and probability distortion functions. We also discuss economical interpretations of the results. In particular, we justify several liquidation strategies widely adopted in stock trading, including those of "buy and hold", "cut loss or take profit", "cut loss and let profit run", and "sell on a percentage of historical high".

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1. Introduction. Many experimental evidences show that people tend to inflate, intentionally or unintentionally, small probabilities. Here we present two simplified examples. We write a random variable (*prospect*) $X = (x_i, p_i; i = 1, 2, \cdots, m)$ if $X = x_i$ with probability p_i, and write $X \succ Y$ if prospect X is preferred than prospect Y. Then it is a general observation that $(\$5000, 0.001; \$0, 0.999) \succ (\$5, 1)$ although the two prospects have the same mean. One of the explanations is that people usually exaggerate the small probability associated with a big payoff (so people buy lotteries). On the other hand, it is common that $(-\$5, 1) \succ (-\$5000, 0.001; \$0, 0.999)$, indicating an inflation of the small probability in respect of a big loss (so people buy insurances).

Probability distortion (or weighting) is one of the building blocks of a number of modern behavioral economics theories including Kahneman and Tversky's cumulative prospect theory (CPT; [18, 32]) and Lopes' SP/A theory [19]. Yaari's dual theory of choice [35] uses probability distortion as a substitute for expected utility in describing people's risk preferences. Probability distortion has also been extensively investigated in the insurance literature; see, e.g., [33, 34, 6].

In this paper we introduce and study optimal stopping of a geometric Brownian motion when probability scale is distorted. To our best knowledge such a problem has not been formally formulated nor attacked before. Due to the probability distortion, the payoff functional of the stopping problem is evaluated via the so-called Choquet integration, a type of nonlinear expectation. We are interested in developing a general approach to solving the problem, and in understanding whether and how the probability distortion changes optimal stopping strategies.

There have been well-developed approaches in solving classical optimal stopping without probability distortion, including those of probability (martingale) and PDE (dynamic programming or variational inequality). We refer to [10] and [28] for classical accounts of the theory. These approaches are based crucially on the time-consistency of the underlying problem. In [22] and [26], the authors study optimal stopping problems under Knightian uncertainty (or ambiguity), involving essentially a different type of nonlinear expectation in their payoff functionals. However, both papers *assume* upfront that time-consistency (or, equivalently, the so-called rectangularity) is kept intact, which enables the applicability of the classical approaches. Henderson ([14]) investigates the disposition effect in stock selling through an optimal stopping with S-shaped payoff functions (motivated by Kahneman and Tversky's CPT); however, since there is no probability distortion involved she is again able to apply the martingale theory to solve the problem.

In the presence of probability distortion, however, the fundamental time-consistency structure is lost, to which the traditional martingale or dynamic programming approaches fail to apply. This is the major challenge arising from probability distortion in optimal stopping. Barberis ([2]) studies optimal exit strategies in casino gambling with CPT preferences (including probability distortion) in a discrete-time setting. He highlights the inherent time-inconsistency issue of the problem, and obtains only numerical solutions via exhaustive enumeration.

In this paper we develop a new approach to overcome the difficulties resulting from the (probability) distortion including the time inconsistency. An important technical ingredient in our approach is the *Skorokhod embedding*. Skorokhod ([30]) introduced and solved the following problem: Given a standard Brownian motion B_t and a probability measure m with 0 mean and finite second moment, find an integrable stopping time τ such that the distribution of B_τ is m. Since then there have been great number of variants, generalizations and applications of the Skorokhod embedding problem; see [21] for a recent survey.

Suppose the stochastic process to be stopped is $\{S_t, t \geqslant 0\}$. The key idea in solving our "distorted" optimal stopping consists of first determining the probability distribution of an optimally stopped state, S_{τ^*}, and then recovering an optimal stopping τ^*, either in an obvious way for several important cases or in general via the Skorokhod embedding. The first part is inspired by the observation that the payoff functional, even though evaluated under the distorted probability, still depends only on the distribution function of the stopped state S_τ; so one can take the distribution function – instead of the stopping time – as the decision variable in solving the optimal stopping problem. The resulting problem is said to have a *distribution formulation*. In some cases it is more convenient to consider the quantile function – the left-continuous inverse of the distribution function – as the decision variable, based on which we have the *quantile formulation*.[1] To summarize, our original problem can be generally solved by a three-step procedure. The first step is to rewrite the problem in a distribution or quantile formulation, the second one is to solve the resulting distribution/quantile optimization problem, and the last one is to derive an optimal stopping from the optimal distribution/quantile function.

[1] Quantile formulation has been introduced and developed in the context of financial portfolio selection involving probability distortion; see [27, 7] and [5] for earlier works. Jin and Zhou ([16]) employ the formulation to solve a continuous-time portfolio selection model with the behavioral CPT preferences. The quantile formulation has recently been further developed in [13] into a general paradigm of solving non-expected utility maximization models.

The remainder of the paper is organized as follows. In Section 2, we formulate the optimal stopping problem under probability distortion, and then transfer the problem into one where the underlying process is a martingale. In Section 3 we present the distribution and quantile formulations of the original problem. In Sections 4–6 we solve the problem respectively for different shapes[2] of the probability distortion and the payoff functions. We also discuss financial/economical implications of the derived results, and compare our results with the case when there is no probability distortion. In particular, we justify several liquidation strategies widely adopted in stock trading. We finally conclude this paper in Section 7. Some technical proofs are placed in an Appendix.

2. Optimal Stopping Formulation.

2.1. *The problem.* Consider a stochastic process, $\{P_t, t \geqslant 0\}$, that follows a geometric Brownian motion (GBM)

$$\mathrm{d}P_t = \mu P_t \,\mathrm{d}t + \sigma P_t \,\mathrm{d}B_t, \quad P_0 > 0, \tag{2.1}$$

where μ and $\sigma > 0$ are real constants, and $\{B_t, t \geqslant 0\}$ is a standard one-dimensional Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$. In many discussions below $\{P_t, t \geqslant 0\}$ will be interpreted as the price process of an asset.

Let $\mathcal{T}$ be the set of all $\{\mathcal{F}_t\}_{t\geqslant 0}$-stopping times τ with $\mathbf{P}(\tau < +\infty) = 1$. A decision-maker (agent) chooses $\tau \in \mathcal{T}$ to stop the process and obtain a payoff $U(P_\tau)$, where $U(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a given non-decreasing, continuous function. The agent distorts the probability scale with a distortion (weighting) function $w(\cdot) : [0, 1] \mapsto [0, 1]$, which is a strictly increasing, absolutely continuous function with $w(0) = 0$ and $w(1) = 1$. The agent's target is to maximize her "distorted" mean payoff functional:

$$\text{Maximize} \quad J(\tau) := \int_0^\infty w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x \tag{2.2}$$

over $\tau \in \mathcal{T}$. In probabilistic terms the above criterion (2.2) is a nonlinear expectation, called the *Choquet expectation* or *Choquet integral*, of the random payoff $U(P_\tau)$ under the *capacity* $w(P(\cdot))$. Note that when $U(P_\tau)$ is a

[2]Throughout this paper the term "shape" mainly refers to the property of a function related to piecewise convexity and concavity. A function is called S-shaped (respectively reverse S-shaped) if it includes two pieces, with the left piece being convex (respectively concave) and the right one concave (respectively convex). These shapes all have economical interpretations related to risk preferences.

discrete random variable, (2.2) agrees with that in the CPT ([32]). So our criterion is a natural generalization of the CPT value function covering both continuous and discrete payoffs.

Another important point to note is that here the underlying process P_t is independent of the probability distortion. In the context of stock trading, this means that the agent is a "small investor"; so her preference only affects her own stopping strategies – but not the asset dynamics. How probability distortions of market participants might collectively affect the asset price is a significant open problem and is certainly beyond the scope of this paper.

If there is no probability distortion, i.e., $w(x) \equiv x$, then the objective functional (2.2) is nothing else than the expected payoff appearing in a standard optimal stopping problem:

$$J(\tau) = \int_0^\infty w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x = \int_0^\infty \mathbf{P}(U(P_\tau) > x)\,\mathrm{d}x = \mathbf{E}[U(P_\tau)].$$

Hence the problem considered in this paper is that of a "distorted" optimal stopping in the sense that the probability scale is distorted. As with the classical optimal stopping there can be many applications of our formulation. For instance, the following problem falls into our formulation: An agent with CPT preferences needs to determine the time of exercising a perpetual American option written on an asset whose discounted price process follows (2.1), whereas the option pays $U(P)$ at the exercises price P.[3] Our problem can also be interpreted simply as an investor hoping to determine the best selling time of a stock that she is holding, and $U(\cdot)$ in this case is a utility function of the proceeds of the liquidation. A yet another example of our formulation is the so-called irreversible investment where the objective is to determine the best time to carry out an investment project (see, e.g., [8, 22]).

2.2. *Transformation.* For subsequent analysis we need to transform problem (2.2) into one where the underlying process is a martingale. To proceed let us first study the simple case when $\mu = \frac{1}{2}\sigma^2$. Indeed, in this case $P_t = P_0 e^{\sigma B_t}$. Let

$$\tau_x = \inf\left\{t \geqslant 0 : B_t = \sigma^{-1}\ln(x/P_0)\right\}, \quad \forall\, x \in (0, +\infty).$$

[3]We assume in this paper that $U(\cdot)$ is non-decreasing. Although some payoff functions may be non-increasing, such as that of a put option, the case of a non-increasing $U(\cdot)$ can be dealt with in exactly the same way as with the non-decreasing counterpart to be presented in this paper. On the other hand, we do not assume $U(\cdot)$ to be smooth or strictly increasing so as to accommodate call-option type of payoffs.

Then $\tau_x \in \mathcal{T}$, $P_{\tau_x} = x$ almost surely, and $J(\tau_x) = U(x)$, $\forall\, x \in (0, +\infty)$. However, for any $\tau \in \mathcal{T}$,

$$J(\tau) = \int_0^\infty w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x = \int_0^{\bar{U}} w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x$$
$$\leqslant \int_0^{\bar{U}} w(1)\,\mathrm{d}x = \bar{U} = \sup_{x>0} J(\tau_x),$$

where $\bar{U} := \sup_{x>0} U(x)$. This shows that the optimal value of problem (2.2) is $\bar{U}$, and that an optimal stopping time, if it ever exists, is of the form τ_x. Moreover, if there exists at least one $x^* > 0$ such that $U(x^*) = \bar{U}$, then τ_{x^*} is an optimal stopping time. If, on the other hand, $U(y) < \bar{U}$ for every $y > 0$, then for any stopping time $\tau \in \mathcal{T}$, we have $U(P_\tau) < \bar{U}$. Therefore, noting that $w(\cdot)$ is strictly increasing,

$$J(\tau) = \int_0^\infty w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x < \int_0^\infty w(\mathbf{P}(\bar{U} > x))\,\mathrm{d}x = \bar{U},$$

which means that the optimal value is not achievable by any stopping time. However, $\lim_{n\to\infty} J(\tau_{x_n}) = \sup_{\tau\in\mathcal{T}} J(\tau)$ for any sequence $\{x_n > 0 : n = 1, 2, \cdots\}$ satisfying $\lim_{n\to\infty} U(x_n) = \bar{U}$.

Given that the case when $\mu = \frac{1}{2}\sigma^2$ has been completely solved, henceforth we only consider the case when $\mu \neq \frac{1}{2}\sigma^2$. We now convert problem (2.2) into an equivalent one. Let

$$\beta := \frac{-2\mu + \sigma^2}{\sigma^2} \neq 0, \quad S_t := P_t^\beta, \quad u(x) := U(x^{1/\beta}), \quad \forall\, x \in (0, +\infty). \tag{2.3}$$

Then Itô's rule gives

$$\mathrm{d}S_t = \beta\sigma S_t\,\mathrm{d}B_t, \quad S_0 = P_0^\beta := s > 0. \tag{2.4}$$

Now we can rewrite problem (2.2) as

$$\text{Maximize} \quad J(\tau) = \int_0^\infty w(\mathbf{P}(U(P_\tau) > x))\,\mathrm{d}x = \int_0^\infty w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x \tag{2.5}$$

over $\tau \in \mathcal{T}$, where the new, auxiliary process S_t follows (2.4), and the new payoff function $u(\cdot)$ is defined in (2.3). In the remainder of this paper we will mainly consider the objective functional in (2.5) instead of that in (2.2).

The advantage of this transformation is that S_t is now a martingale, which enables us to apply the Skorokhod theorem later on. Interestingly, $u(\cdot)$ may now have a completely different shape than $U(\cdot)$, depending on the value of β.

2.3. *Examples.* We now discuss several popular payoff functions $U(\cdot)$ as examples; these examples will also serve as benchmarks for illustrating the main results of this paper.

Let us start with the example of a call option written on an underlying asset whose discounted price process follows (1). The payoff function is $U(x) = (x - K)^+$ for some $K > 0$. Then $u(x) = (x^\beta - K)^+$. If $\beta < 0$ or equivalently $\frac{\mu}{\sigma^2} > 0.5$, the underlying asset is "good"[4]. In this case $u(\cdot)$ is non-increasing and convex. If $0 < \beta \leqslant 1$ or $0 \leqslant \frac{\mu}{\sigma^2} < 0.5$, the asset is between good and bad, and $u(\cdot)$ is non-decreasing and S-shaped. If $\beta > 1$ or $\mu < 0$, the asset is "bad", and $u(\cdot)$ is non-decreasing and convex.

Now take a power function $U(x) = \frac{1}{\gamma}x^\gamma$, $\gamma \in (0,1)$. Then $u(x) = \frac{1}{\gamma}x^{\gamma/\beta}$ is strictly decreasing and convex if $\beta < 0$, strictly increasing and convex if $0 < \beta \leqslant \gamma$ (i.e. the asset is "not so bad" in respect of the original payoff/utility function), and strictly increasing and concave if $\beta > \gamma$ (the asset is sufficiently bad).

For a log utility function $U(x) = \ln(x+1)$, $u(x) = \ln(x^{1/\beta} + 1)$ is strictly decreasing if $\beta < 0$, strictly increasing and S-shaped if $0 < \beta < 1$, and strictly increasing and concave if $\beta \geqslant 1$.

For an exponential utility function $U(x) = 1 - e^{-\alpha x}$, $\alpha > 0$, $u(x) = 1 - e^{-\alpha x^{1/\beta}}$ is strictly decreasing if $\beta < 0$, strictly increasing and concave if $0 < \beta < 1$, and strictly increasing and S-shaped if $\beta \geqslant 1$.

Next, let us take an S-shaped piecewise power function $U(x) = (x/k)^{\alpha_1}\mathbf{1}_{(0,k]}(x) + (x/k)^{\alpha_2}\mathbf{1}_{(k,\infty)}(x)$, where $\alpha_1 \geqslant 1 \geqslant \alpha_2 > 0$, $k > 0$. Then $u(x) = x^{\alpha_2/\beta}k^{-\alpha_2}\mathbf{1}_{(0,k^\beta]} + x^{\alpha_1/\beta}k^{-\alpha_1}\mathbf{1}_{(k^\beta,\infty)}$ is strictly decreasing if $\beta < 0$, strictly increasing and piecewise convex if $0 < \beta < \alpha_2$, strictly increasing and S-shaped if $\alpha_2 \leqslant \beta \leqslant \alpha_1$, and strictly increasing and piecewise concave if $\beta > \alpha_1$.

Finally, for a general non-decreasing function $U(\cdot)$, $u(x) = U(x^{1/\beta})$ is non-increasing if $\beta < 0$, and non-decreasing if $\beta > 0$.

2.4. *Solution to a trivial case.* While solving (2.5) in general requires a new approach, which will be developed in the subsequent sections, in this subsection we present the solution to a mathematically (almost) trivial yet economically significant case.

THEOREM 2.1. *If $u(\cdot)$ is non-increasing, then problem* (2.5) *has the optimal value $u(0+)$ and*

$$\lim_{T\to+\infty} J(T) = \sup_{\tau\in\mathcal{T}} J(\tau). \tag{2.6}$$

[4] In [29], $\frac{\mu}{\sigma^2}$ is termed the "goodness index" of an asset.

Moreover, if $u(\ell) = u(0+)$ *for some* $\ell > 0$*, then* $\tau_\ell := \inf\{t \geqslant 0 : S_t \leqslant \ell\}$ *is an optimal stopping time for problem* (2.5)*. If* $u(\ell) < u(0+)$ *for every* $\ell > 0$*, then* (2.5) *has no optimal solution.*

A proof can be found in Appendix A. We remark that $u(\cdot)$ is not required to be even continuous in the proof.

Identity (2.6) suggests that the supremum of the payoff functional can be achieved by *not* stopping at all, if $u(\cdot)$ is non-increasing. There is an interesting economical interpretation of the above result in the context of asset selling. In all the examples presented in Section 2.3, the case of $u(\cdot)$ being non-increasing corresponds to $\beta < 0$, namely the underlying asset being good. Moreover, in all but the last general example, it holds that $u(0+) > u(\ell)$ for all $\ell > 0$. Theorem 2.1 then indicates that one should not sell at any price level, or one should hold the asset perpetually. This is indeed consistent with the traditional investment wisdom that one should "buy and hold a good asset".[5]

3. Distribution/Quantile Formulation. In view of Theorem 2.1, henceforth we consider only the case when $u(\cdot)$ is non-decreasing. Let us specify the standing assumption we impose from this point on.

Assumption 1. $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is non-decreasing, absolutely continuous with $u(0) = 0$; $w(\cdot) : [0,1] \mapsto [0,1]$ is strictly increasing, absolutely continuous with $w(0) = 0$ and $w(1) = 1$.

Note that $u(0) = 0$ is just for simplicity, as one may consider $\bar{u}(\cdot) = u(\cdot) - u(0)$ if $u(0) \neq 0$.

Throughout this paper, for any non-decreasing function $f : \mathbb{R}^+ \mapsto [0,1]$, we denote by $f^{-1} : [0,1) \mapsto \mathbb{R}^+$ the left-continuous inverse function of f, which is defined by

$$f^{-1}(x) := \inf\{y \in \mathbb{R}^+ : f(y) \geqslant x\}, \quad x \in [0,1).$$

Clearly f^{-1} is non-decreasing and left continuous. We say $F : \mathbb{R}^+ \mapsto [0,1]$ is a cumulative distribution function (CDF) if $F(0) = 0$, $F(+\infty) \equiv \lim_{x\to+\infty} F(x) = 1$ and F is non-decreasing and *cádlág*. We call $G : [0,1) \mapsto \mathbb{R}^+$ a quantile

[5] In [29], a similar result is derived, albeit for a different asset selling model where the time horizon is finite, probability distortion is absent, and the objective is to minimize the relative error between the selling price and the all-time-high price.

function if $G(0) = 0$, $G(x) > 0\ \forall x \in (0,1)$, G is non-decreasing and left continuous.[6]

Now define the following distribution set $\mathcal{D}$ and quantile set $\mathcal{Q}$:

$$\mathcal{D} := \left\{ F : \mathbb{R}^+ \mapsto [0,1] \;\middle|\; F \text{ is the CDF of } S_\tau, \text{ for some } \tau \in \mathcal{T} \right\},$$
$$\mathcal{Q} := \left\{ G : [0,1) \mapsto \mathbb{R}^+ \;\middle|\; G = F^{-1} \text{ for some } F \in \mathcal{D} \right\}.$$

LEMMA 3.1. *For any $\tau \in \mathcal{T}$, we have*

$$J(\tau) = J_D(F) := \int_0^\infty w\,(1 - F(x))\, u'(x)\,\mathrm{d}x, \tag{3.1}$$

$$J(\tau) = J_Q(G) := \int_0^1 u\,(G(x))\, w'(1-x)\,\mathrm{d}x, \tag{3.2}$$

where F and G are the CDF and the quantile function of S_τ, respectively. Moreover,

$$\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{F \in \mathcal{D}} J_D(F) = \sup_{G \in \mathcal{Q}} J_Q(G). \tag{3.3}$$

A proof is relegated to Appendix B.

We name (3.1) and (3.2) as the *distribution formulation* and the *quantile formulation* of problem (2.5), respectively.

We observe certain symmetry – or rather duality – between the distribution formulation (3.1) and the quantile formulation (3.2). In particular, $w(\cdot)$ and $u(\cdot)$ play symmetric roles in the two formulations.[7] The availability of two formulations enables us to choose a convenient one in solving the original stopping problem (2.5), depending on the shape of $w(\cdot)$ and $u(\cdot)$. For instance, if $u(\cdot)$ is known to be concave or convex (while $w(\cdot)$ is arbitrary) , then it might be advantageous to choose the quantile formulation (3.2).

Next, we need to characterize the sets $\mathcal{D}$ and $\mathcal{Q}$ more explicitly for the second step – solving the distribution/quantile optimization problem.

Let $F \in \mathcal{D}$, namely F is the CDF of S_τ for some $\tau \in \mathcal{T}$. Since S_t is a nonnegative martingale, optional sampling theorem and Fatou's lemma yield, necessarily, $\int_0^\infty (1 - F(x))\,\mathrm{d}x \equiv \mathbf{E}[S_\tau] \leqslant s$. It turns out that this inequality, $\int_0^\infty (1 - F(x))\,\mathrm{d}x \leqslant s$, is not only necessary but also sufficient for F to belong to $\mathcal{D}$.

[6] Note that in this paper the underlying process S_t is strictly positive at any time; hence we need to consider only the CDF and quantile function of strictly positive random variables.

[7] Indeed, in the context of utility theory both a probability distortion function and a utility function describe an investor's preference towards risk - they do play some dual roles; see [35].

LEMMA 3.2. *We have the following expressions of the distribution set $\mathcal{D}$ and quantile set $\mathcal{Q}$:*

$$\mathcal{D} = \left\{ F : \mathbb{R}^+ \mapsto [0,1] \,\middle|\, F \text{ is a CDF and } \int_0^\infty (1 - F(x))\,\mathrm{d}x \leqslant s \right\},$$
$$\mathcal{Q} = \left\{ G : [0,1) \mapsto \mathbb{R}^+ \,\middle|\, G \text{ is a quantile function and } \int_0^1 G(x)\,\mathrm{d}x \leqslant s \right\}.$$

PROOF. First assume $\beta > 0$. We write $S_t = s\exp\left(-\frac{1}{2}\beta^2\sigma^2 t + \beta\sigma B_t\right) \equiv s\exp\left(\beta\sigma\tilde{B}_t\right)$, where $\tilde{B}_t := B_t - \frac{1}{2}\beta\sigma t$ is a drifted Brownian motion with a *negative* drift. Denote by F_X the CDF of a random variable X. For any $\tau \in \mathcal{T}$, we have

$$F_{\tilde{B}_\tau}(x) = \mathbf{P}(\tilde{B}_\tau \leqslant x) = \mathbf{P}\left(S_\tau \leqslant se^{\beta\sigma x}\right) = F_{S_\tau}(se^{\beta\sigma x}).$$

On the other hand, according to Theorem 2.1 in [11], a CDF F is the CDF of $\tilde{B}_\tau$ for some $\tau \in \mathcal{T}$ if and only if $\int_{-\infty}^\infty e^{\beta\sigma x}\,\mathrm{d}F(x) \leqslant 1$. So F is the CDF of S_τ for some $\tau \in \mathcal{T}$ if and only if it is a CDF and $\int_{-\infty}^\infty e^{\beta\sigma x}\,\mathrm{d}F(se^{\beta\sigma x}) \leqslant 1$, or $\int_0^\infty x\,\mathrm{d}F(x) \leqslant s$. The above is equivalent to $\int_0^\infty (1 - F(x))\,\mathrm{d}x \leqslant s$, or $\int_0^1 G(x)\,\mathrm{d}x \leqslant s$.

Now if $\beta < 0$, then write $S_t = s\exp\left(-\beta\sigma\tilde{B}_t\right)$, where $\tilde{B}_t := -B_t + \frac{1}{2}\beta\sigma t$ is still a drifted Brownian motion with a negative drift. The rest of the proof is exactly the same as above. This completes the proof. □

An important by-product of this lemma is that both the sets $\mathcal{D}$ and $\mathcal{Q}$ are convex.

Now we have reformulated our problem (2.5) into optimization problems maximizing (3.1) or (3.2) over a convex set $\mathcal{D}$ or $\mathcal{Q}$ respectively. In the following several sections, we will solve these problems with different shapes of the functions $u(\cdot)$ and $w(\cdot)$.

4. Convex $w(\cdot)$ or $u(\cdot)$.

In this section, we will solve problem (2.5) assuming that *either* $w(\cdot)$ *or* $u(\cdot)$ is convex.

4.1. *Convex $w(\cdot)$.*

Assume for now that $w(\cdot)$ is convex (whereas $u(\cdot)$, being non-decreasing, is allowed to have any shape). In this case the distribution formulation (3.1) is easier to study than its quantile counterpart (3.2), since the shape of $u(\cdot)$ is unknown in the latter. The distribution formulation in this case is to maximize a convex functional over a convex set.

Intuitively speaking, a maximum of (3.1) should be at "corners" of the constraint set, $\mathcal{D}$. We are going to establish that these corners must be step functions having at most two jumps.

For $n = 2, 3, \cdots$, define

$$\mathcal{S}_n := \left\{ F : F = \sum_{i=1}^{n-1} c_i \mathbf{1}_{[a_i, a_{i+1})} + \mathbf{1}_{[a_n, \infty)},\ 0 < c_i \leqslant c_{i+1} \leqslant 1,\ 0 < a_i \leqslant a_{i+1} \right\},$$

and $\mathcal{D}_n := \mathcal{S}_n \cap \mathcal{D}$. Clearly $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$ and $\mathcal{D}_n \subseteq \mathcal{D}_{n+1} \subseteq \mathcal{D}$, $n = 2, 3, \cdots$.

LEMMA 4.1. *If $w(\cdot)$ is convex, then*

$$\sup_{F \in \mathcal{D}} J_D(F) = \sup_{F \in \mathcal{D}_2} J_D(F).$$

A proof of this lemma is provided in Appendix C.

By virtue of Lemma 4.1, in maximizing (3.1) we need only to search over the set $\mathcal{D}_2$ or, equivalently, to find the best parameters a, b, and c in defining an element $F(x) = c\mathbf{1}_{[a,b)}(x) + \mathbf{1}_{[b,+\infty)}(x)$ in $\mathcal{D}_2$. This becomes a three-dimensional constrained optimization problem which is dramatically easier to solve than the original stopping problem. We present the results in the following theorem.

THEOREM 4.2. *If $w(\cdot)$ is convex, then*

$$\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{0 < a \leqslant s \leqslant b} \left[\left(1 - w\left(\frac{s-a}{b-a}\right)\right) u(a) + w\left(\frac{s-a}{b-a}\right) u(b) \right].$$

Moreover, if

$$(a^*, b^*) = \operatorname*{argmax}_{0 < a \leqslant s \leqslant b} \left[\left(1 - w\left(\frac{s-a}{b-a}\right)\right) u(a) + w\left(\frac{s-a}{b-a}\right) u(b) \right], \tag{4.1}$$

then

$$\tau_{(a^*, b^*)} := \begin{cases} \inf\{t \geqslant 0 : S_t \notin (a^*, b^*)\}, & \text{if } a^* < b^* \\ 0, & \text{if } a^* = b^* \end{cases} \tag{4.2}$$

is an optimal stopping to problem (2.5).

PROOF. Due to Lemma 4.1, we need only to find the optimal distribution function in $\mathcal{D}_2$ to maximize (3.1). For any $F \in \mathcal{D}_2$ with $F(x) = c\mathbf{1}_{[a,b)}(x) + \mathbf{1}_{[b,+\infty)}(x)$, $\quad x \in [0, +\infty)$, we have

$$J_D(F) = \int_0^\infty w(1 - F(x)) u'(x)\, \mathrm{d}x = (1 - w(1-c))u(a) + w(1-c)u(b),$$

and

$$\int_0^\infty (1 - F(x))\,\mathrm{d}x = ac + b(1-c).$$

Thus our problem boils down to

$$\text{(4.3)} \qquad \begin{array}{ll} \text{Maximize} & J(a,b,c) := (1 - w(1-c))u(a) + w(1-c)u(b) \\ \text{subject to} & ac + b(1-c) \leqslant s, \quad 0 < a \leqslant b, \quad 0 \leqslant c \leqslant 1. \end{array}$$

Clearly $a \leqslant s$, otherwise the first constraint of (4.3) would be violated. On the other hand, in maximizing $J(a,b,c)$ one should choose b as large as possible when a and c are fixed. So we need only to consider the range $0 < a \leqslant s \leqslant b$ when solving (4.3). Moreover, $J(a,b,c)$ is non-increasing in c when a and b are fixed; hence $c = \frac{b-s}{b-a}$ when $a < b$, while $c \in [0,1]$ can be arbitrarily chosen when $a = b$. Therefore,

$$\sup_{\tau \in \mathcal{T}} J(\tau) = \sup_{F \in \mathcal{D}_2} J_D(F) = \sup_{0 < a \leqslant s \leqslant b} \left[\left(1 - w\left(\frac{s-a}{b-a}\right)\right) u(a) + w\left(\frac{s-a}{b-a}\right) u(b) \right].$$

Now if (a^*, b^*) with $0 < a^* < b^*$ is determined by (4.1), then clearly $S_{\tau_{(a^*,b^*)}}$, where $\tau_{(a^*,b^*)}$ is defined by (4.2), has a two-point distribution $P(S_{\tau_{(a^*,b^*)}} = a^*) = c^*$ and $P(S_{\tau_{(a^*,b^*)}} = b^*) = 1 - c^*$. Moreover, $c^* = \frac{b^*-s}{b^*-a^*}$ by virtue of the optional sampling theorem. The CDF of $S_{\tau_{(a^*,b^*)}}$ is $F^*(x) = c^* \mathbf{1}_{[a^*,b^*)}(x) + \mathbf{1}_{[b^*,+\infty)}(x)$. Hence $\tau_{(a^*,b^*)}$ is an optimal solution. If $a^* = b^*$, then it must hold that $a^* = b^* = s$. Hence we have $\sup_{\tau \in \mathcal{T}} J(\tau) = J(a^*, b^*, c) = u(s) = J(\tau_{(a^*,b^*)})$. So $\tau_{(a^*,b^*)}$ defined by (4.2) is an optimal solution to problem (2.5). □

According to [35], a convex probability distortion overweighs "bad" outcomes and underweighs "good" ones in maximizing the underlying criterion; hence it captures the risk-aversion of an investor. The preceding theorem suggests that a risk averse agent's optimal strategy is to stop at one of the two thresholds, a^* and b^*. In the context of stock liquidation, this corresponds to the widely adopted "take-profit-or-cut-loss" strategy, namely one should sell a stock either when it has reached a pre-determined target b^* or sunk to a prescribed loss level a^* (note that the initial price s is in between a^* and b^*), for a stock that is not worth "buy and hold perpetually".

In particular, when there is no probability distortion, i.e., $w(x) \equiv x$, which is trivially convex, Theorem 4.2 recovers the results of [14] where an optimal stopping problem is studied with a specific S-shaped utility function $u(\cdot)$ without probability distortion. Indeed, Theorem 4.2 leads to a very general result in the absence of probability distortion: the optimality of the take-profit-or-cut-loss strategy is inherent regardless of the shape of $u(\cdot)$ – be it concave, convex or S-shaped – so long as it is non-decreasing.

It should be noted that in the current case no Skorokhod embedding technique is needed to recover the optimal stopping time τ^* from S_{τ^*}. This is because the explicit form of CDF of S_{τ^*} obtained reveals that S_{τ^*} is a two-point distribution; hence τ^* must be the exit time of an interval.

COROLLARY 4.3. *If $u(\cdot)$ is concave and $w(\cdot)$ is convex, then $\sup_{\tau\in\mathcal{T}} J(\tau) = u(s)$. Moreover, $\tau \equiv 0$ is an optimal stopping.*

PROOF. The convexity of $w(\cdot)$ along with $w(0) = 0$ and $w(1) = 1$ implies that $w(x) \leqslant x$, for all $x \in [0, 1]$; so we have

$$\begin{aligned}
\sup_{\tau\in\mathcal{T}} J(\tau) &= \sup_{0<a\leqslant s\leqslant b} \left[\left(1 - w\left(\frac{s-a}{b-a}\right)\right) u(a) + w\left(\frac{s-a}{b-a}\right) u(b)\right] \\
&= \sup_{0<a\leqslant s\leqslant b} \left[u(a) + w\left(\frac{s-a}{b-a}\right)(u(b) - u(a))\right] \\
&\leqslant \sup_{0<a\leqslant s\leqslant b} \left[u(a) + \frac{s-a}{b-a}(u(b) - u(a))\right] \\
&= \sup_{0<a\leqslant s\leqslant b} \left[\frac{b-s}{b-a}u(a) + \frac{s-a}{b-a}u(b)\right] \leqslant u(s) = J(0),
\end{aligned}$$

where we used the concavity property of $u(\cdot)$ to obtain the last inequality. □

This result stipulates that when $w(\cdot)$ is convex and $u(\cdot)$ is concave, the two thresholds degenerate into one which is the initial state s. From some of the examples in Section 2 (e.g. when the original payoff function is power or logarithmic), $u(\cdot)$ being concave corresponds to a "bad" asset. So if the agent is risk averse (reflected by the convexity of $w(\cdot)$) or risk neutral (no distortion) and the asset is unfavorable, then the optimal stopping strategy is to stop immediately.

4.2. *Convex $u(\cdot)$.* Next we consider the case when $u(\cdot)$ is convex while $w(\cdot)$ has an arbitrary shape. In this case the quantile formulation (3.2) is more convenient to deal with. The following result is an analog of Lemma 4.1, whose proof is however much simpler.

LEMMA 4.4. *If $u(\cdot)$ is convex, then*

$$\sup_{G\in\mathcal{Q}} J_Q(G) = \sup_{G\in\mathcal{Q}_2} J_Q(G),$$

where $\mathcal{Q}_2$ is defined as

$$\mathcal{Q}_2 := \{G \in \mathcal{Q} : G = a\mathbf{1}_{(0,c]} + b\mathbf{1}_{(c,1)},\ 0 < a \leqslant b,\ 0 < c \leqslant 1\}.$$

See Appendix D for a proof of the above lemma.

THEOREM 4.5. *If $u(\cdot)$ is convex, then*

$$\sup_{\tau\in\mathcal{T}} J(\tau) = \sup_{0<a\leqslant s\leqslant b}\left[\left(1-w\left(\tfrac{s-a}{b-a}\right)\right)u(a) + w\left(\tfrac{s-a}{b-a}\right)u(b)\right] = \sup_{x\in(0,1]}\left[w(x)u\left(\tfrac{s}{x}\right)\right]. \tag{4.4}$$

Moreover, if (a^*, b^*) is determined by (4.1) then $\tau_{(a^*,b^*)}$ defined by (4.2) is an optimal stopping to problem (2.5).

PROOF. Due to Lemma 4.4, we need only to find the optimal quantile function in $\mathcal{Q}_2$ to maximize (3.2). For any $G \in \mathcal{Q}_2$ with $G(x) = a\mathbf{1}_{(0,c]}(x) + b\mathbf{1}_{(c,1)}(x)$, $x \in [0,1)$, we have

$$J_Q(G) = \int_0^1 u(G(x))w'(1-x)\,\mathrm{d}x = (1-w(1-c))u(a) + w(1-c)u(b),$$

and

$$\int_0^1 G(x)\,\mathrm{d}x = ac + b(1-c).$$

This leads to exactly the same optimization problem (4.3), and one follows exactly the same lines of proof of Theorem 4.2 to conclude that the first equality of (4.4) is valid and $\tau_{(a^*,b^*)}$ defined by (4.2) is an optimal solution.

It remains to prove the second equality of (4.4). Since both $u(\cdot)$ and $w(\cdot)$ are continuous, we have

$$\begin{aligned}&\sup_{0<a\leqslant s\leqslant b}\left[\left(1-w\left(\frac{s-a}{b-a}\right)\right)u(a) + w\left(\frac{s-a}{b-a}\right)u(b)\right]\\ \geqslant &\sup_{a=0,s\leqslant b}\left[\left(1-w\left(\frac{s-a}{b-a}\right)\right)u(a) + w\left(\frac{s-a}{b-a}\right)u(b)\right] = \sup_{x\in(0,1]}\left[w(x)u\left(\frac{s}{x}\right)\right].\end{aligned}$$

Fix $0 < a \leqslant s \leqslant b$ with $a < b$, and let $G(x) = a\mathbf{1}_{(0,c]}(x) + b\mathbf{1}_{(c,1)}(x)$, $x \in [0,1)$, where $c = \frac{b-s}{b-a}$. Rewriting

$$G(x) = \frac{a}{s}s + \frac{(b-a)(1-c)}{s}\frac{s}{1-c}\mathbf{1}_{(c,1)}(x), \quad x \in [0,1),$$

we deduce by the convexity of $u(\cdot)$ that

$$u(G(x)) \leqslant \frac{a}{s}u(s) + \frac{(b-a)(1-c)}{s}u\left(\frac{s}{1-c}\mathbf{1}_{(c,1)}(x)\right), \quad x \in [0,1);$$

hence

$$\int_0^1 u(G(x))w'(1-x)\,\mathrm{d}x \leqslant \tfrac{a}{s}u(s) + \tfrac{(b-a)(1-c)}{s}u\left(\tfrac{s}{1-c}\right)w(1-c) \leqslant \sup_{x\in(0,1]}\left[w(x)u\left(\tfrac{s}{x}\right)\right]. \tag{4.5}$$

In other words,

$$\left(1-w\left(\frac{s-a}{b-a}\right)\right)u(a) + w\left(\frac{s-a}{b-a}\right)u(b) = \int_0^1 u(G(x))w'(1-x)\,\mathrm{d}x \leqslant \sup_{x\in(0,1]}\left[w(x)u\left(\frac{s}{x}\right)\right].$$

This completes the proof. □

COROLLARY 4.6. *If $u(\cdot)$ is convex, then $\tau^* \equiv 0$ is an optimal solution to problem (2.5) if and only if $u(s) = \sup_{x\in(0,1]}\left[w(x)u\left(\frac{s}{x}\right)\right]$. Moreover, if $u(s) < \sup_{x\in(0,1]}\left[w(x)u\left(\frac{s}{x}\right)\right]$, then the maximum in (4.1) is not achievable.*

PROOF. Clearly $\tau^* \equiv 0$ if and only if $\sup_{\tau\in\mathcal{T}} J(\tau) = u(s)$, which is equivalent to $u(s) = \sup_{x\in(0,1]}\left[w(x)u\left(\frac{s}{x}\right)\right]$ by virtue of Theorem 4.5.

If $u(s) < \sup_{x\in(0,1]}\left[w(x)u\left(\frac{s}{x}\right)\right]$, then the last inequality in (4.5) is strict unless $a = 0$. This implies that the maximum in (4.1) is not achievable. □

In the context of asset selling, $u(\cdot)$ is convex when the underlying asset ranges from "intermediate" to "bad" depending on the form of the original payoff function $U(\cdot)$; see the examples in Section 2. Theorem 4.5 shows that in this case an optimal strategy is *in general* still of a "take-profit-or-cut-loss" form. However, one must note that it is also possible that the maximum in (4.1) is not achievable (as indicated in Corollary 4.6). In that case suppose $x^* = \operatorname{argmax}_{x\in(0,1]}\left[w(x)u\left(\frac{s}{x}\right)\right]$ exists. Let $b^* = s/x^*$, and

$$\tau_{(0,b^*)} := \inf\{t \geqslant 0 : S_t \notin (0,b^*)\}.$$

Then we have

$$\sup_{\tau\in\mathcal{T}} J(\tau) = \sup_{x\in(0,1]} \left[w(x)u\left(\frac{s}{x}\right)\right] = J(\tau_{(0,b^*)}).$$

However, when $b^* > s$, $\tau_{(0,b^*)}$ is not a finite stopping time (i.e. $P(\tau_{(0,b^*)} = +\infty) > 0$). The interpretation of this fact is that only a stop-gain threshold b^* is set if one applies $\tau_{(0,b^*)}$; but as S_t will never reach 0, with a positive probability the process never exits the interval $(0, b^*)$.

5. Concave $u(\cdot)$. In this section, we study the case when $u(\cdot)$ is concave. Again, we employ the quantile formulation (3.2) where the objective functional $J_Q(\cdot)$ becomes concave. In sharp contrast to the case when $u(\cdot)$ is convex, in general the maxima of (3.2) are now in the interior of the constraint set, which can be obtained using the classical Lagrange method. Let us first describe the general solution procedure.

Consider a family of relaxed problems

$$\begin{aligned} J_Q^\lambda(G) &:= \int_0^1 u(G(x))w'(1-x)\,\mathrm{d}x - \lambda\left(\int_0^1 G(x)\,\mathrm{d}x - s\right) \\ &= \int_0^1 \Big(u(G(x))w'(1-x) - \lambda G(x)\Big)\,\mathrm{d}x + \lambda s = \int_0^1 f^\lambda(x, G(x))\,\mathrm{d}x + \lambda s, \end{aligned} \tag{5.1}$$

where $\lambda \geqslant 0$ and $f^\lambda(x,y) := u(y)w'(1-x) - \lambda y$. To maximize $J_Q^\lambda(\cdot)$ it suffices to maximize $f^\lambda(x,\cdot)$ for each x. Recall that we do not assume $u(\cdot)$ to be smooth (which is the case when, e.g., the original payoff function $U(\cdot)$ is that of a call option). Define

$$\begin{aligned} u'(x) &:= \limsup_{h\to 0+} \frac{u(x+h)-u(x)}{h}, \\ (u')_u^{-1}(x) &:= \inf\left\{y \geqslant 0 : u'(y) < x\right\}, \\ (u')_l^{-1}(x) &:= \inf\left\{y \geqslant 0 : u'(y) \leqslant x\right\}. \end{aligned}$$

It is easy to see that both $(u')_l^{-1}$ and u' are right-continuous, while $(u')_u^{-1}$ is left-continuous. Fix x. As $f^\lambda(x,\cdot)$ is concave on $\mathbb{R}^+$, y maximizes $f^\lambda(x,\cdot)$ on $\mathbb{R}^+$ if and only if

$$y \in \left[(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right), (u')_u^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right].$$

To proceed, we need to further specify the shape of the probability distortion function $w(\cdot)$. The case when $w(\cdot)$ is convex has been solved in Section 4 where $\tau^* = 0$ is an optimal stopping time. The other cases will be studied in the next three subsections respectively.

5.1. *Concave $w(\cdot)$.*

THEOREM 5.1. *If both $u(\cdot)$ and $w(\cdot)$ are concave, and there exists $\lambda^* \geqslant 0$ such that $(u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right) > 0$, $\forall\, x \in (0,1)$ and*

$$\int_0^1 (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right) \mathrm{d}x = s, \tag{5.2}$$

then $G^*(x) := (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)$ is an optimal solution to problem (3.2).

PROOF. Clearly $G^*(x)$ maximizes $f^{\lambda^*}(x,\cdot)$ on $\mathbb{R}^+$, for each $x \in (0,1)$. Since $w'(1-x)$ is non-decreasing in x, G^* is non-decreasing and left continuous. By defining $G^*(0) = 0$ we see G^* is indeed a quantile function given that $G^*(x) > 0\ \forall\, x \in (0,1)$. Moreover, $G^* \in \mathcal{Q}$ by virtue of (5.2). On the other hand, for any $G \in \mathcal{Q}$,

$$\begin{aligned} J_Q(G) \leqslant J_Q^{\lambda^*}(G) &= \int_0^1 f^{\lambda^*}(x, G(x))\,\mathrm{d}x + \lambda^* s \\ &\leqslant \int_0^1 f^{\lambda^*}(x, G^*(x))\,\mathrm{d}x + \lambda^* s = J_Q^{\lambda^*}(G^*) = J_Q(G^*). \end{aligned}$$

So G^* is an optimal solution to problem (3.2). □

The above general result involves an assumption that λ^* exists so that (5.2) holds. Conceptually, non-existence of the Lagrange multiplier is an indication of the ill-posedness or the non-attainability of the underlying optimization problem; see Section 3 of [17] for a detailed study in the context of utility maximization. Mathematically, when $u(\cdot)$ and $w(\cdot)$ are given in specific forms (see e.g. Example 1 below) it is straightforward to check the validity of the assumption. In more general cases, one constructs the function $\varphi(\lambda) := \int_0^1 (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right) dx$, and then checks the validity of (5.2) by examining the continuity of φ and its values at $\lambda = 0$ and $\lambda \uparrow \infty$.

In general, the quantile function, G^*, of the optimally stopped state does no longer correspond to a two-point distribution; or there is no threshold level that would *directly* trigger a stopping. We have discussed in the previous section that $u(\cdot)$ being concave corresponds to, at least in some cases of interest, an "unfavorable" underlying stochastic process. On the other hand, a concave $w(\cdot)$ suggests that the agent is risk-seeking in that she exaggerates the probability of the underlying process reaching a very high state. In the context of stock selling, our result indicates that a speculative agent, when holding a "bad" stock, will not set any specific cut-loss or stop-gain prices.

Moreover, since $w(\cdot)$ is concave, we have

$$\underline{b} := (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-)}\right) \leqslant G_{\lambda^*}(x) \leqslant (u')_l^{-1}\left(\frac{\lambda^*}{w'(0+)}\right) =: \bar{b}, \quad \forall\, x \in (0,1).$$

In particular, if $w'(0+) < \infty$, then $\bar{b} < +\infty$; hence the optimally stopped state will never exceed $\bar{b}$, or one will have already stopped before the process ever reaches $\bar{b}$. Similarly, if $w'(1-) > 0$, then the optimally stopped state will never fall below $\underline{b}$. If the range of w' is a singleton which must be $\{1\}$, then the range of the possible stopped states is also a singleton, which is necessarily $\{s\}$. This shows that, in the case of stock liquidation, if there is no probability distortion then a bad stock will be sold immediately, which is also consistent with Corollary 4.3. In other words, if an agent is still holding an unfavorable stock then it is an indication that the agent is distorting probability scale hoping for extraordinarily return.

We now provide the following example to illustrate the general result of Theorem 5.1.

Example 1. Consider a model of asset selling with a concave function $u(x) = \frac{1}{\gamma}x^\gamma$, $0 < \gamma < 1$ and a concave distortion function $w(x) = x^\alpha$, $0 < \alpha \leqslant 1$. We have $(u')^{-1}(x) = x^{\frac{1}{\gamma-1}}$, $w'(1-x) = \alpha(1-x)^{\alpha-1}$.

First we assume that $1 > \alpha > \gamma$, namely, the agent is only moderately risk-seeking (relative to the original payoff function and the quality of the asset). The equation (5.2) for λ^* is $\frac{\alpha-\gamma}{1-\gamma}\left(\frac{\lambda^*}{\alpha}\right)^{\frac{1}{\gamma-1}} = s$, which clearly has a unique solution. Then the optimal quantile function is

$$G^*(x) = s\frac{\alpha-\gamma}{1-\gamma}\left(\frac{1}{1-x}\right)^{\frac{1-\alpha}{1-\gamma}}, \quad x \in (0,1). \tag{5.3}$$

The corresponding CDF of the optimally stopped price is

$$F^*(x) = \begin{cases} 1 - \left(s\frac{\alpha-\gamma}{1-\gamma}\right)^{\frac{1-\gamma}{1-\alpha}} x^{-\frac{1-\gamma}{1-\alpha}}, & x \geqslant s\frac{\alpha-\gamma}{1-\gamma}; \\ 0, & x < s\frac{\alpha-\gamma}{1-\gamma}. \end{cases} \tag{5.4}$$

This is a Pareto distribution[8] with the Pareto index $\frac{1-\gamma}{1-\alpha} > 1$. In particular, one should never stop when the asset price is below $s\frac{\alpha-\gamma}{1-\gamma}$, a true fraction of the initial price s. Pareto index is a measure of the "fatness" of the tail of the stopped price. The larger the Pareto index (i.e. the lower γ or the higher α),

[8]Pareto distribution was put forth by Italian economist Vilfredo Pareto ([23]) to describe the allocation of wealth among individuals in a society.

the lighter tailed the distribution (and hence the smaller the proportion of very high stopped prices). This makes perfect sense since a higher α implies a less exaggeration of the probability of the asset achieving very high prices, hence more likely the agent stops at a moderate price.

There are infinitely many stopping times generating the same distribution F^* in this case. However, a convenient one is the so-called Azéma–Yor stopping time (see [1])

$$\tau_{AY} = \inf\left\{ t \geqslant 0 \;:\; S_t \leqslant \frac{\alpha-\gamma}{1-\gamma} \max_{0\leqslant s\leqslant t} S_s \right\}, \tag{5.5}$$

which is an optimal solution to problem (2.5). Azéma–Yor theorem is applicable in our case since $\int_0^\infty x\,\mathrm{d}F^*(x) \equiv \int_0^1 G^*(x)\,\mathrm{d}x = s$. Such a stopping strategy is to stop at the first time when the boundary of the *drawdown constraint* $S_t \geqslant \frac{\alpha-\gamma}{1-\gamma} \max\limits_{0\leqslant s\leqslant t} S_s$ is touched upon. This implies that one sells as soon as the current stock price falls below a true fraction of the historical high price.

If $\alpha = 1$ (i.e. there is no distortion), then $\tau_{AY} = 0$. Hence an agent who distorts the probability scale will hold an asset which would be otherwise sold immediately by one who does not. This shows that probability distortion does change the optimal stopping behavior.

If $\alpha < \gamma$ so that the agent is sufficiently risk-taking, then choose *any* $0 < \eta < 1$ satisfying $\alpha < (1-\eta)\gamma$. Take $G^*(x) = \eta s(1-x)^{\eta-1}$. It is easy to check that $G^* \in \mathcal{Q}$ while

$$J_Q(G^*) = \int_0^1 \frac{1}{\gamma}(\eta s(1-x)^{\eta-1})^\gamma \alpha(1-x)^{\alpha-1}\,\mathrm{d}x = +\infty.$$

So the optimal value of (2.5) in this case is $+\infty$ and G^* is an optimal solution. Since G^* also follows a Parato distribution, the corresponding Azéma-Yor stopping time is given by

$$\tau_{AY} = \inf\left\{ t \geqslant 0 \;:\; S_t \leqslant \eta \max_{0\leqslant s\leqslant t} S_s \right\}.$$

Finally, when $\alpha = \gamma$ we construct $G_n(x) = \frac{1}{n}s(1-x)^{\frac{1}{n}-1},\;\; n > 0$. Then $G_n \in \mathcal{Q}$ with

$$J_Q(G_n) = \int_0^1 \frac{1}{\gamma}(\frac{1}{n}s(1-x)^{\frac{1}{n}-1})^\gamma \alpha(1-x)^{\alpha-1}\,\mathrm{d}x = \frac{1}{\gamma}s^\gamma n^{1-\gamma}.$$

Hence the optimal value of the stopping problem is $+\infty$. The corresponding Azéma-Yor stopping time is

$$\tau_{AY,n} = \inf\left\{ t \geqslant 0 \;:\; S_t \leqslant \frac{1}{n} \max_{0\leqslant s\leqslant t} S_s \right\}.$$

It is not hard to show that there is no optimal solution in this case.

5.2. *Reverse S-shaped $w(\cdot)$.*

THEOREM 5.2. *Assume that $u(\cdot)$ is concave, and $w(\cdot)$ is reverse S-shaped, i.e. it is concave on $[0,1-q]$ and convex on $[1-q,1]$ for some $q\in(0,1)$. If (a^*,λ^*) with $a^*>0$ is a solution to the following mathematical program*

(5.6)
$$\begin{array}{ll}\text{Maximize} & (1-w(1-q))u(a)+\int_q^1 u\left(a\vee (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right)w'(1-x)\,\mathrm{d}x\\ \text{subject to} & \lambda\geqslant 0,\ a\geqslant 0,\ aq+\int_q^1 a\vee (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\mathrm{d}x=s,\end{array}$$

then

(5.7)
$$G^*(x)=a^*\mathbf{1}_{(0,q]}(x)+\left(a^*\vee (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)\right)\mathbf{1}_{(q,1)}(x),\quad x\in[0,1)$$

is an optimal solution to problem (3.2).

The proof of this theorem is rather technical, and it is delayed to Appendix E.

Reverse S-shaped probability distortion has been used and studied by many authors; see, e.g., [31, 24, 16] and in particular by Kahneman and Tversky in the celebrated CPT ([32]). For a reverse S-shaped distortion $w(\cdot)$, $w'(x)>1$ around both $x=0$ and $x=1$. This implies, as seen from (3.2), that such distortion puts higher weights on *both* very good and very bad outcomes. In other words, the agent exaggerates the small probabilities of both very good and very bad scenarios. In [12], exaggeration of small probabilities for extremely good and bad outcomes is used to model the emotion of *hope* and *fear* respectively. In the current context of optimal stopping, the expression (5.7) shows, qualitatively, that the agent sets a cut-loss level a (because she has fear) and does not set any stop-gain level (because she has hope). This is widely known as the "cut loss and let profit run" strategy in stock trading.

EXAMPLE 2. Consider a concave $u(x)=\frac{1}{\gamma}x^\gamma$, $0<\gamma<1$, and a reverse S-shaped distortion function

$$w(x)=\begin{cases}2x-2x^2, & 0\leqslant x\leqslant \frac{1}{2};\\ 2x^2-2x+1, & \frac{1}{2}<x\leqslant 1.\end{cases}$$

Then constraints in (5.6) become

$$\lambda \geqslant 0, \quad a \geqslant 0, \quad \frac{1}{2}a + \int_{\frac{1}{2}}^{1} a \vee \left(\frac{\lambda}{4x-2}\right)^{\frac{1}{\gamma-1}} \mathrm{d}x = s,$$

and the objective function in (5.6) is

$$J(a,\lambda) = \frac{1}{2\gamma}a^{\gamma} + \frac{1}{\gamma}\int_{\frac{1}{2}}^{1} a^{\gamma} \vee \left(\frac{\lambda}{4x-2}\right)^{\frac{\gamma}{\gamma-1}} (4x-2)\,\mathrm{d}x.$$

Define

$$\bar{c} = \inf\left\{x \geqslant \frac{1}{2} : a \leqslant \left(\frac{\lambda}{4x-2}\right)^{\frac{1}{\gamma-1}}\right\} \wedge 1 \in [0.5, 1].$$

If $\bar{c} = 1$, then the problem reduces to maximize $J(a,\lambda) = \frac{1}{\gamma}s^{\gamma}$ subject to $\lambda \geqslant 0$, $a = s$, which is trivial. If $\bar{c} \in [0.5, 1)$, then the above constraints are equivalent to

(5.8)
$$a = \frac{s}{\bar{c} + \frac{1-\gamma}{2(2-\gamma)}\left((2\bar{c}-1)^{\frac{1}{\gamma-1}} - (2\bar{c}-1)\right)}, \quad \lambda = (4\bar{c}-2)a^{\gamma-1}, \quad 0.5 \leqslant \bar{c} < 1,$$

and thus our objective is to maximize $\frac{1}{\gamma}s^{\gamma}g(\bar{c})$ in $\bar{c} \in [0.5, 1)$, where

$$g(\bar{c}) := \left(\frac{1}{\bar{c} + \frac{1-\gamma}{2(2-\gamma)}\left((2\bar{c}-1)^{\frac{1}{\gamma-1}} - (2\bar{c}-1)\right)}\right)^{\gamma} \times \left(1 - 2\bar{c} + 2\bar{c}^2 + \frac{1-\gamma}{2-\gamma}\left((2\bar{c}-1)^{\frac{\gamma}{\gamma-1}} - (2\bar{c}-1)^2\right)\right).$$

Now, $g(0.5) = \left(\frac{1-\gamma}{2-\gamma}\right)^{1-\gamma} 2^{\gamma} = \left(\frac{1-\gamma}{4(2-\gamma)}2^{\frac{2-\gamma}{1-\gamma}}\right)^{1-\gamma}$ and $g(1-) = g(1) = 1$. Noting $2^t > 4t \quad \forall t \in (4, \infty)$, we conclude $\frac{1-\gamma}{4(2-\gamma)}2^{\frac{2-\gamma}{1-\gamma}} < 1$, if $\frac{2-\gamma}{1-\gamma} > 4$ or $0 < \gamma < \frac{2}{3}$. Thus $g(0.5) < g(1-)$, if $0 < \gamma < \frac{2}{3}$; in other words the maximum value of the objective function is achieved at some point $\bar{c}^* \in (0.5, 1]$.

We have now deduced the optimal quantile function

$$G^*(x) = a\mathbf{1}_{(0,\bar{c}]}(x) + a\left(\frac{4\bar{c}-2}{4x-2}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{(\bar{c},1)}(x), \quad x \in [0,1),$$

where $\bar{c} \equiv \bar{c}^* \in (0.5, 1]$, and $a > 0$ is determined via (5.8). The corresponding optimal CDF is

$$F^*(x) = \begin{cases} 0, & x < a; \\ (\bar{c} - 0.5)(x/a)^{1-\gamma} + 0.5, & a \leqslant x < (2\bar{c} - 1)^{\frac{1}{\gamma-1}} a; \\ 1, & x \geqslant (2\bar{c} - 1)^{\frac{1}{\gamma-1}} a. \end{cases}$$

The barycenter function (also called the Hardy-Littlewood maximal function) defined by Azéma and Yor (1979) is then given by

$$\Psi(x) = \frac{\int_{[x,\infty)} y \, \mathrm{d}F^*(y)}{1 - F^*(x-)} = \begin{cases} s, & x \leqslant a; \\ \frac{1-\gamma}{2-\gamma} \frac{(x/a)^{2-\gamma} - (2\bar{c}-1)^{\frac{2-\gamma}{1-\gamma}}}{(x/a)^{1-\gamma} - (2\bar{c}-1)} a, & a \leqslant x < (2\bar{c} - 1)^{\frac{1}{\gamma-1}} a; \\ (2\bar{c} - 1)^{\frac{1}{\gamma-1}} a, & x \geqslant (2\bar{c} - 1)^{\frac{1}{\gamma-1}} a, \end{cases}$$

whereas the corresponding Azéma-Yor stopping time is

$$\tau_{AY} = \inf\left\{ t \geqslant 0 \; : \; \Psi(S_t) \leqslant \max_{0 \leqslant s \leqslant t} S_s \right\}.$$

Suppose now that $\gamma = 0.3$. Then the optimal $\bar{c}^* \approx 0.70$, and $a \approx 0.72s$, $\lambda \approx s^{-0.7}$ are determined via (5.8). Therefore the optimal quantile function presented in Theorem 5.2 is

$$G^*(x) \approx 0.72s\mathbf{1}_{(0,0.7]}(x) + (4x - 2)^{1.43} s\mathbf{1}_{(0.7,1)}(x).$$

Since the barycenter function Ψ is increasing, the Azéma-Yor stopping time is the first time when S_t hits $\Psi^{-1}(\max_{0 \leqslant s \leqslant t} S_s)$, a moving level that is related to the running maximum (rather than a proportion of the running maximum as in Example 1).

5.3. *S-shaped $w(\cdot)$.*

THEOREM 5.3. *Assume that $u(\cdot)$ is concave, and $w(\cdot)$ is S-shaped, i.e. it is convex on $[0, 1-q]$ and concave on $[1-q, 1]$ for some $q \in (0, 1)$. If (a^*, λ^*) is a solution to the following mathematical program*

$$\begin{array}{ll} \text{Maximize} & \int_0^q u\left(a \wedge (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right) w'(1-x)\, \mathrm{d}x + w(1-q)u(a) \\ \text{subject to} & \lambda \geqslant 0, \; a \geqslant 0, \; \int_0^q a \wedge (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right) \mathrm{d}x + a(1-q) = s, \end{array}$$

and $(u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right) > 0 \; \forall \, x \in (0, q]$, then

$$G^*(x) = \left(a^* \wedge (u')_l^{-1}\left(\frac{\lambda^*}{w'(1-x)}\right)\right) \mathbf{1}_{(0,q]}(x) + a^*\mathbf{1}_{(q,1)}(x), \quad x \in [0, 1)$$

is an optimal solution to problem (3.2).

PROOF. The proof is similar to that of Theorem 5.2; hence omitted. □

The economic interpretation of the result for this case is just opposite to the reverse S-shaped counterpart. An S-shaped probability distortion underlines an agent who under-weighs probabilities of extreme events (both good and bad). So she sets an upper target level simply because she is not hopeful for a dramatically high price, while she does not prescribe a cut-loss level since she believes the asset will not go catastrophically wrong.

5.4. *Discussion.* We have obtained the quantile functions of the optimally stopped states for the three cases discussed in this section. In order to finally solve the original distorted optimal stopping problem (2.5), we need to recover optimal stopping times from the quantile functions. Unlike the cases investigated in Section 4 where the optimal distribution/quantile functions are those of two-point or one-point distributions and optimal stopping times can be uniquely determined, there could be infinitely many stopping times corresponding to the same distribution of the stopped state.[9] As demonstrated in Examples 1 and 2, the Azéma-Yor stopping time would provide a convenient solution that is related to the running maximum of the underlying process, which is also commonly incorporated in practice. On the other hand, for many applications how optimally stopped states are probabilistically distributed already reveals important qualitative information. For instance, we have shown in this section when the agent would put a cut-loss floor or a target state or simply set none, depending on her risk preferences. An optimally stopped state distribution is also adequate in calculating the optimal payoff value function, which is relevant in the context of, say, option pricing or irreversible investment.

The results in this section have also demonstrated how probability distortion affects optimal stopping strategies. In the previous section we have proved that if there is no probability distortion, optimal stopping strategies are always of the threshold-type with at most two thresholds. Thus one stops only at (at most) two states. Strategies are qualitatively changed when there is probability distortion, where one sets only one-sided threshold or simply none. Moreover there could be infinitely many stopped states.

6. S-shaped $u(\cdot)$. We now consider the case when $u(\cdot)$ is S-shaped. If the distortion $w(\cdot)$ is convex, then the result has already be derived in Section 4. If $w(\cdot)$ is concave, then we can utilize the same idea as in Subsection 5.3 to get similar results, thanks to the duality between $u(\cdot)$ and $w(\cdot)$. If $w(\cdot)$ is

[9] In the Skorokhod embedding literature, one usually introduces additional criteria in order to uniquely determine the stopping time; see, e.g. [21].

S-shaped, we can also apply similar techniques. We leave the details to the interested readers. In this section, we will focus on the most interesting case when $w(\cdot)$ is a reverse S-shaped distortion function.[10]

Henceforth in this section u is convex on $[0,\theta]$ and concave on $[\theta,\infty)$ for some $\theta>0$, and w is concave on $[0,1-q]$ and convex on $[1-q,1]$ for some $q\in(0,1)$.

Fix $G_0\in\mathcal{G}$. Let $x_0=\sup\{x\in[0,1)\mid G_0(x)\leqslant\theta\}\wedge 1$. Then $G_0(x_0)\leqslant\theta$ since G_0 is left-continuous, and

$$J_Q(G_0)=\int_0^{x_0}u\left(G_0(x)\right)w'(1-x)\,\mathrm{d}x+\int_{x_0}^1 u\left(G_0(x)\right)w'(1-x)\,\mathrm{d}x.$$

Consider two subproblems:

$$\max_{G\in\mathcal{G}^-}\int_0^{x_0}u(G(x))w'(1-x)\,\mathrm{d}x, \tag{6.1}$$

$$\max_{G\in\mathcal{G}^+}\int_{x_0}^{1}u(G(x))w'(1-x)\,\mathrm{d}x, \tag{6.2}$$

where

$$\mathcal{G}^-=\left\{G\in\mathcal{G}\;\middle|\;G(x_0)\leqslant G_0(x_0),\int_0^{x_0}G(x)\,\mathrm{d}x\leqslant\int_0^{x_0}G_0(x)\,\mathrm{d}x\right\},$$
$$\mathcal{G}^+=\left\{G\in\mathcal{G}\;\middle|\;G(x_0+)\geqslant G_0(x_0),\int_{x_0}^1G(x)\,\mathrm{d}x\leqslant\int_{x_0}^1G_0(x)\,\mathrm{d}x\right\}.$$

Subproblem (6.1) is a convex maximization problem. Using the idea of the proof of Lemma C.1, we can show that the optimal solution to the subproblem (6.1) is of the form

$$G(x)=a_1\mathbf{1}_{(0,c_1]}(x)+a_2\mathbf{1}_{(c_1,c_2]}(x)+G_0(x_0)\mathbf{1}_{(c_2,x_0]}(x),\quad\forall\,x\in(0,x_0].$$

For subproblem (6.2), we can use the idea of proof of Theorem 5.2 to show that the optimal solution must be of the form

$$G(x)=G_0(x_0)\mathbf{1}_{(x_0,q]}(x)+\left(G_0(x_0)\vee(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right)\mathbf{1}_{(q,1)}(x),\ \forall\,x\in(x_0,1).$$

Now we conclude that the optimal solution is of the form

$$\begin{aligned}G^*(x)=&\;a_1\mathbf{1}_{(0,c_1]}(x)+a_2\mathbf{1}_{(c_1,c_2]}(x)+a_3\mathbf{1}_{(c_2,q]}(x)\\&+\left(a_3\vee(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right)\mathbf{1}_{(q,1)}(x),\ \forall\,x\in(0,1),\end{aligned}$$

[10] If $u(\cdot)$ is interpreted as a utility function, then the case when $u(\cdot)$ is S-shaped while $w(\cdot)$ is reverse S-shaped is consistent with the CPT of [32].

where parameters a_1, a_2, a_3, c_1, c_2 and λ are subject to

$$\lambda \geqslant 0, \quad 0 < a_1 \leqslant a_2 \leqslant a_3 \leqslant \theta, \quad 0 \leqslant c_1 \leqslant c_2 \leqslant q,$$
$$a_1 c_1 + a_2(c_2 - c_1) + a_3(q - c_2) + \int_q^1 a_3 \vee (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right) \mathrm{d}x \leqslant s.$$

The objective is

$$\begin{aligned} J_Q(G^*) = (1 - w(1-c_1))u(a_1) &+ (w(1-c_1) - w(1-c_2))u(a_2) \\ &+ (w(1-c_2) - w(1-q))u(a_3) \\ &+ \int_q^1 u\left(a_3 \vee (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right) w'(1-x)\,\mathrm{d}x. \end{aligned}$$

Hence, the original problem reduces to the above mathematical program, which can be solved readily.

7. Concluding Remarks. In this paper we have formulated an optimal stopping problem under distorted probabilities and developed an approach, primarily based on the distribution/quantile formulation and the Skorokhod embedding, to solving this new problem. Note that the optimal stopping strategies derived are *pre-committed*, instead of dynamically consistent. Precisely, while our solutions are optimal at $t = 0$, they are no longer optimal at $t = \varepsilon$ for any $\varepsilon > 0$. This is due to the inherent time-inconsistency arising from the distortion. There are recently studies on time-inconsistent optimal control, which use a time-consistent game equilibrium to replace the notion of "optimality"; see, e.g., [9] and [3]. It is, however, not clear how to extend this equilibrium idea to the optimal stopping setting. On the other hand, a pre-committed strategy is still important. It will determine the *value* of the problem at any given time. Moreover, in reality people often uphold strategies for a certain time period before changing them, even though they are dealing with time-inconsistent problems which may call for continuously changing strategies. For example, Barberis ([2]) analyzed in details, in the setting of casino gambling, the behavior of a "sophisticated" gambler who can commit to his initial exit strategy.

An important point to note is that, while our stopping strategies are time-inconsistent in terms of quantitative values, they are indeed time-consistent in terms of qualitative types. For example, Theorem 4.2 stipulates that if the current $S_t = s$, then the optimal stopping strategy is to stop either at a^* or b^*, whose values depend on s via (10). At the next moment the underlying process value becomes $S_{t+\varepsilon} = s'$, then one needs to re-calculate (10) to obtain the new thresholds a' and b'. Although the strategy has changed, its type (that of two-threshold) has not, which depends only on the risk preference of the agent and the property of the process.

We assume in this paper the underlying stochastic process to be a GBM for two reasons: 1) it is widely used in many applications especially in finance; and 2) we would like to concentrate on the new approach developed (which is already very complex) without being carried away by the complexity of a more general underlying process. The advantage of a GBM is that it can be turned into an exponential martingale via a simple transformation; thus the Skorokhod embedding applies. A more general process governed by a nonlinear SDE may still be transformed into a martingale, but more technicalities need to be taken care of, especially in terms of the range of the martingale. This will be studied in a forthcoming paper.

APPENDIX A: PROOF OF THEOREM 2.1

Because $u(\cdot)$ is non-increasing, we have

$$\sup_{\tau\in\mathcal{T}} J(\tau) = \sup_{\tau\in\mathcal{T}} \int_0^{u(0+)} w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x \leqslant \sup_{\tau\in\mathcal{T}} \int_0^{u(0+)} 1\,\mathrm{d}x = u(0+).$$

On the other hand,

$$\begin{aligned}\sup_{\tau\in\mathcal{T}} J(\tau) &\geqslant \limsup_{T\to+\infty} J(T) \geqslant \liminf_{T\to+\infty} J(T) \geqslant \liminf_{T\to+\infty} \int_0^\infty w(\mathbf{P}(u(S_T) > x))\,\mathrm{d}x \\ &\geqslant \int_0^\infty \liminf_{T\to+\infty} w(\mathbf{P}(u(S_T) > x))\,\mathrm{d}x \geqslant \int_0^\infty w(\liminf_{T\to+\infty} \mathbf{P}(u(S_T) > x))\,\mathrm{d}x \\ &= \int_0^\infty w(\mathbf{P}(u(0+) \geqslant x))\,\mathrm{d}x = u(0+) \geqslant \sup_{\tau\in\mathcal{T}} J(\tau),\end{aligned}$$

where we used the fact that $\lim_{t\to\infty} S_t = 0$ almost surely since S_t is an exponential martingale. This implies that $u(0+)$ is the optimal value of problem (2.5), and (2.6) holds.

Next, the fact that $\lim_{t\to\infty} S_t = 0$ implies $\tau_\ell \in \mathcal{T}$ for any $\ell > 0$. Now, if there is $\ell > 0$ such that $u(\ell) = u(0+)$, then $u(x) = u(0+)$ for each $x \in (0, \ell)$ since $u(\cdot)$ is non-increasing and therefore $\sup_{\tau\in\mathcal{T}} J(\tau) \leqslant u(0+) = J(\tau_\ell)$, proving that τ_ℓ solves problem (2.5).

If there is no $\ell > 0$ such that $u(\ell) = u(0+)$, then for every fixed $\tau \in \mathcal{T}$, we have $u(S_\tau) < u(0+)$ almost surely. Consequently,

$$J(\tau) = \int_0^{u(0+)} w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x < \int_0^{u(0+)} w(\mathbf{P}(u(0+) > x))\,\mathrm{d}x = u(0+),$$

which shows that there is no optimal solution to problem (2.5).

APPENDIX B: PROOF OF LEMMA 3.1

Let F and G be the CDF and the quantile function of S_τ, respectively, for a fixed $\tau \in \mathcal{T}$.

First we assume that $u(\cdot)$ is a strictly increasing, C^∞ function with $u(0) = 0$. Then

$$
\begin{aligned}
J(\tau) &= \int_0^\infty w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x = \int_0^\infty w(\mathbf{P}(u(S_\tau) > u(y)))\,\mathrm{d}u(y) \\
&= \int_0^\infty w(\mathbf{P}(S_\tau > x))\,\mathrm{d}u(x) = \int_0^\infty w(1 - F(x))\,\mathrm{d}u(x) \\
&= \int_0^\infty u(x)\,\mathrm{d}\left[-w(1 - F(x))\right] = \int_0^\infty u(x)w'(1 - F(x))\,\mathrm{d}F(x) \\
&= \int_0^1 u\left(G(x)\right) w'(1 - x)\,\mathrm{d}x,
\end{aligned}
$$

which proves (3.2), where the fifth equality is due to Fubini's theorem.

Next, given an absolutely continuous, non-decreasing function $u(\cdot)$ with $u(0) = 0$, for each $\varepsilon > 0$, we can find a strictly increasing, C^∞ function $u_\varepsilon(\cdot)$ such that $|u_\varepsilon(x) - u(x)| < \varepsilon$, for all $x \in \mathbb{R}^+$. It is easy to check that

$$
\left| \int_0^\infty w(\mathbf{P}(u_\varepsilon(S_\tau) > x))\,\mathrm{d}x - \int_0^\infty w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x \right| \leqslant \varepsilon,
$$

$$
\left| \int_0^1 u_\varepsilon\left(G(x)\right) w'(1 - x)\,\mathrm{d}x - \int_0^1 u\left(G(x)\right) w'(1 - x)\,\mathrm{d}x \right| \leqslant \varepsilon.
$$

We have proved that

$$
\int_0^\infty w(\mathbf{P}(u_\varepsilon(S_\tau) > x))\,\mathrm{d}x = \int_0^1 u_\varepsilon\left(G(x)\right) w'(1 - x)\,\mathrm{d}x.
$$

Therefore

$$
\left| \int_0^1 u\left(G(x)\right) w'(1 - x)\,\mathrm{d}x - \int_0^\infty w(\mathbf{P}(u(S_\tau) > x))\,\mathrm{d}x \right| \leqslant 2\varepsilon.
$$

Since ε is arbitrary, (3.2) follows.

To show (3.1), we note

$$
J(\tau) = \int_0^1 u\left(G(x)\right) w'(1 - x)\,\mathrm{d}x = \int_0^1 u\left(G(x)\right) \mathrm{d}\left[-w(1 - x)\right]
$$

$$
= \int_0^\infty u(x)\,\mathrm{d}\left[-w(1-F(x))\right] = \int_0^\infty w(1-F(x))\,\mathrm{d}u(x) = \int_0^\infty w\left(1 - F(x)\right) u'(x)\,\mathrm{d}x,
$$

where the forth equality is due to Fubini's theorem.

Finally, (3.3) is evident.

APPENDIX C: PROOF OF LEMMA 4.1

To prove this lemma we need some technical preliminaries.

LEMMA C.1. *For any $F^* \in \mathcal{S}_{n+1}$, $n = 2, 3, \cdots$, there exist F_1, $F_2 \in \mathcal{S}_n$ and $\theta \in [0, 1]$ such that*

$$F^* = \theta F_1 + (1-\theta)F_2,$$
$$\int_0^\infty (1 - F_1(x))\,\mathrm{d}x = \int_0^\infty (1 - F_2(x))\,\mathrm{d}x = \int_0^\infty (1 - F^*(x))\,\mathrm{d}x.$$

PROOF. We first prove the lemma for $n = 2$. Suppose $F^* \in \mathcal{S}_3$. Write

$$F^* = c_1\mathbf{1}_{[a_1,a_2)} + c_2\mathbf{1}_{[a_2,a_3)} + \mathbf{1}_{[a_3,\infty)}, \quad s_0 := \int_0^\infty (1 - F^*(x))\,\mathrm{d}x,$$

where $a_1 < a_2 < a_3$ (otherwise the desired result holds trivially). Note that

$$a_1 = \int_0^{a_1} (1 - F^*(x))\,\mathrm{d}x \leqslant \int_0^\infty (1 - F^*(x))\,\mathrm{d}x = \int_0^{a_3} (1 - F^*(x))\,\mathrm{d}x \leqslant a_3;$$

that is $a_1 \leqslant s_0 \leqslant a_3$. If $s_0 = a_1$, or $s_0 = a_3$, then $F^* \in \mathcal{S}_2$ and we are done by choosing $F_1 = F_2 = F^*$. Hence from now on, we assume $a_1 < s_0 < a_3$.

If $s_0 > a_2$, then let $F_1 := b_1\mathbf{1}_{[a_1,a_3)} + \mathbf{1}_{[a_3,\infty)}$ and $F_2 := b_2\mathbf{1}_{[a_2,a_3)} + \mathbf{1}_{[a_3,\infty)}$, where $b_1 = \frac{a_3-s_0}{a_3-a_1} \in (0, 1)$ and $b_2 = \frac{a_3-s_0}{a_3-a_2} \in (0, 1)$. It follows from

$$s_0 \equiv \int_0^\infty (1 - F^*(x))\,\mathrm{d}x = a_1 + (a_2 - a_1)(1 - c_1) + (a_3 - a_2)(1 - c_2)$$

that $\frac{c_1}{b_1} + \frac{c_2-c_1}{b_2} = 1$. It is now easy to see that F_1, F_2 and $\theta := c_1/b_1$ satisfy the desired requirements.

If $s_0 \leqslant a_2$, then let $F_1 := b_1\mathbf{1}_{[a_1,a_3)} + \mathbf{1}_{[a_3,\infty)}$ and $F_2 := b_2\mathbf{1}_{[a_1,a_2)} + \mathbf{1}_{[a_2,\infty)}$, where $b_1 = \frac{a_3-s_0}{a_3-a_1} \in (0, 1)$ and $b_2 = \frac{a_2-s_0}{a_2-a_1} \in [0, 1)$. Define $\theta_1 := \frac{c_1-c_2b_2}{b_1(1-b_2)}$, and $\theta_2 := \frac{c_2-c_1}{1-b_2} \geqslant 0$. Noting that

$$s_0 \equiv \int_0^\infty (1-F^*(x))\,\mathrm{d}x = a_1+(a_2-a_1)(1-c_1)+(a_3-a_2)(1-c_2) \geqslant a_1+(c_2-c_1)(a_2-a_1),$$

we deduce $\theta_2 \leqslant 1$. It is an easy exercise to verify

$$\theta_1(1 - F_1) + \theta_2(1 - F_2) - (\theta_1 + \theta_2)\mathbf{1}_{[0,a_3)} = 1 - F^* - \mathbf{1}_{[0,a_3)}.$$

Integrating both sides on $(0, \infty)$, we obtain $\theta_1 s_0+\theta_2 s_0-(\theta_1+\theta_2)a_3 = s_0-a_3$, which leads to $\theta_1 + \theta_2 = 1$ noting $s_0 < a_3$. Now we can easily verify that F_1, F_2 and $\theta := \theta_1$ satisfy the desired properties.

Now, let $F^* \in \mathcal{S}_{n+1}$ where $n = 3, 4, \cdots$. Write

$$F^* = c_1 \mathbf{1}_{[a_1,a_2)} + c_2 \mathbf{1}_{[a_2,a_3)} + c_3 \mathbf{1}_{[a_3,a_4)} + \sum_{i=4}^{n-1} c_i \mathbf{1}_{[a_i,a_{i+1})} + \mathbf{1}_{[a_n,\infty)}$$
$$= c_3 \mathbf{1}_{[a_1,a_4)} \left(\frac{c_1}{c_3} \mathbf{1}_{[a_1,a_2)} + \frac{c_2}{c_3} \mathbf{1}_{[a_2,a_3)} + \mathbf{1}_{[a_3,\infty)} \right) + \sum_{i=4}^{n-1} c_i \mathbf{1}_{[a_i,a_{i+1})} + \mathbf{1}_{[a_n,\infty)}.$$

Denote

$$\overline{F} = \frac{c_1}{c_3} \mathbf{1}_{[a_1,a_2)} + \frac{c_2}{c_3} \mathbf{1}_{[a_2,a_3)} + \mathbf{1}_{[a_3,\infty)} \in \mathcal{S}_3.$$

By what we have proved above, there exist $\overline{F}_1$, $\overline{F}_2 \in \mathcal{S}_2$ and $\theta \in [0, 1]$ such that

$$\overline{F} = \theta \overline{F}_1 + (1-\theta) \overline{F}_2,$$
$$\int_0^\infty (1 - \overline{F}_1(x))\,\mathrm{d}x = \int_0^\infty (1 - \overline{F}_2(x))\,\mathrm{d}x = \int_0^\infty (1 - \overline{F}(x))\,\mathrm{d}x.$$

Define

$$F_1 := c_3 \mathbf{1}_{[a_1,a_4)} \overline{F}_1 + \sum_{i=4}^{n-1} c_i \mathbf{1}_{[a_i,a_{i+1})} + \mathbf{1}_{[a_n,\infty)},$$
$$F_2 := c_3 \mathbf{1}_{[a_1,a_4)} \overline{F}_2 + \sum_{i=4}^{n-1} c_i \mathbf{1}_{[a_i,a_{i+1})} + \mathbf{1}_{[a_n,\infty)}.$$

Then F_1, F_2 and θ satisfy all the requirements. □

COROLLARY C.2. *For any $F^* \in \mathcal{D}_n$, $n = 2, 3, \cdots$, there exist $F_k \in \mathcal{D}_2$ and $\theta_k \in [0, 1]$, $k = 1, 2, \cdots, l$, such that*

$$F^* = \sum_{k=1}^{l} \theta_k F_k, \quad \sum_{k=1}^{l} \theta_k = 1.$$

PROOF. Since $F^* \in \mathcal{D}_n \subseteq \mathcal{S}_n$, it follows immediately from Lemma C.1 that there exist $F_k \in \mathcal{S}_2$ and $\theta_k \in [0, 1]$, $k = 1, 2, \cdots, l$, such that

$$F^* = \sum_{k=1}^{l} \theta_k F_k, \quad \sum_{k=1}^{l} \theta_k = 1, \quad \int_0^\infty (1 - F_k(x))\,\mathrm{d}x = \int_0^\infty (1 - F^*(x))\,\mathrm{d}x,$$

for $k = 1, 2, \cdots, l$. Because $F^* \in \mathcal{D}_n \subseteq \mathcal{D}$, we have

$$\int_0^\infty (1 - F_k(x))\,\mathrm{d}x = \int_0^\infty (1 - F^*(x))\,\mathrm{d}x \leqslant s,$$

which implies that $F_k \in \mathcal{D}$. Therefore, $F_k \in \mathcal{S}_2 \cap \mathcal{D} = \mathcal{D}_2$. □

Proof of Lemma 4.1. Suppose for some $F_0 \in \mathcal{D}$, we have $\sup_{F\in\mathcal{D}_2} J_D(F) < J_D(F_0) < \infty$. Construct a sequence of step CDFs, $F_m,\ m = 1, 2, \cdots$, satisfying $F_m \geqslant F_0$ and $\lim_{m\to\infty} F_m(x) = F_0(x)$ a.e. Clearly $F_m \in \mathcal{D}$, and it follows from the dominated convergence theorem that $\lim_{m\to\infty} J_D(F_m) = J_D(F_0)$. So there exists $F^* \in \mathcal{D}_n$ for some $n \geqslant 2$ such that $J_D(F^*) > \sup_{F\in\mathcal{D}_2} J_D(F)$. By Corollary C.2, there exist $\overline{F}_k \in \mathcal{D}_2$ and $\theta_k \in [0, 1]$, $k = 1, 2, \cdots, l$, such that

$$F^* = \sum_{k=1}^{l} \theta_k \overline{F}_k, \quad \sum_{k=1}^{l} \theta_k = 1.$$

However, recalling that w is convex, we have

$$J_D(F^*) = J_D(\sum_{k=1}^{l} \theta_k \overline{F}_k) \leqslant \sum_{k=1}^{l} \theta_k J_D(\overline{F}_k) \leqslant \sup_{F\in\mathcal{D}_2} J_D(F),$$

which leads to a contradiction.

APPENDIX D: PROOF OF LEMMA 4.4

Suppose

$$\sup_{G\in\mathcal{Q}} J_Q(G) > \sup_{G\in\mathcal{Q}_2} J_Q(G). \tag{D.1}$$

By the monotone convergence theorem, we can find a sequence of essentially bounded quantile functions $G_n \in \mathcal{Q}$, $n = 1, 2, \cdots$, so that $\lim_{n\to\infty} J_Q(G_n) = \sup_{G\in\mathcal{Q}} J_Q(G)$. For each fixed n, by the dominated convergence theorem, there is a sequence of step functions $G_{n,k} \in \mathcal{Q}$ with $\lim_{k\to\infty} J_Q(G_{n,k}) = J_Q(G_n)$. So we can find a step function $G_0 \in \mathcal{Q}$, written as

$$G_0(x) = a_0 + \sum_{i=1}^{n} b_i \mathbf{1}_{(c_i,1]}(x),\ a_0 > 0,\ b_i > 0,\ 0 < c_1 < \cdots < c_n < 1, \quad x \in (0, 1)$$

such that $J_Q(G_0) > \sup_{G\in\mathcal{Q}_2} J_Q(G)$. Since $G_0 \in \mathcal{Q}$, we have

$$\bar{s} := a_0 + \sum_{i=1}^{n} b_i(1 - c_i) \equiv \int_0^1 G_0(x)\,\mathrm{d}x \leqslant s.$$

Let $0 < \varepsilon < a_0$. Set

$$a_i = \varepsilon, \quad \alpha_i := \frac{b_i(1-c_i)}{\bar{s}-\varepsilon} > 0, \quad i = 1, \cdots, n, \quad \alpha_{n+1} := \frac{a_0-\varepsilon}{\bar{s}-\varepsilon} > 0,$$
$$G_i(x) := a_i + \frac{b_i}{\alpha_i}\mathbf{1}_{(c_i,1]}(x), \quad i = 1, \cdots, n, \quad G_{n+1}(x) := \bar{s}, \quad \forall\, x \in (0,1).$$

It is easy to check that $G_i \in \mathcal{Q}_2$, $i = 1, \cdots, n+1$, and

$$G_0(x) = \sum_{i=1}^{n+1} \alpha_i G_i(x), \quad \sum_{i=1}^{n+1} \alpha_i = 1, \quad \forall\, x \in (0,1).$$

Recalling that u is convex, we have

$$\sup_{G \in \mathcal{Q}_2} J_Q(G) < J_Q(G_0) = J_Q\left(\sum_{i=1}^{n+1} \alpha_i G_i\right) \leqslant \sum_{i=1}^{n+1} \alpha_i J_Q(G_i) \leqslant \sup_{G \in \mathcal{Q}_2} J_Q(G),$$

which is a contradiction. So (D.1) is false and the proof is complete.

APPENDIX E: PROOF OF THEOREM 5.2

The key idea of this proof is to show that one needs only to search among a special class of quantile functions in order to solve the relaxed problem (5.1). To this end, fix $G \in \mathcal{Q}$ and $\lambda \geqslant 0$, and let

$$x_0 := \sup\left\{x \in (0,q] \;\middle|\; G(x) \leqslant (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right\} \vee 0.$$

If $x_0 > 0$, we define

$$x_1 = \sup\left\{x \in (q,1) \;\middle|\; (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right) \leqslant G(x_0)\right\} \vee q,$$

and

$$\hat{G}_\lambda(x) = G(x_0)\mathbf{1}_{(0,x_1]}(x) + (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\mathbf{1}_{(x_1,1)}(x), \quad \forall\, x \in [0,1).$$

Then $\hat{G}_\lambda$ is also a quantile function. We now show that $J_Q^\lambda(G) \leqslant J_Q^\lambda(\hat{G}_\lambda)$. Noting $(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)$ is non-increasing in $x \in (0, x_0)$, we deduce

$$G(x) \leqslant G(x_0) = G(x_0-) \leqslant (u')_l^{-1}\left(\frac{\lambda}{w'((1-x_0)+)}\right) \leqslant (u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right), \quad \forall\, x \in (0,x_0).$$

Since $f^\lambda(x,\cdot)$ is non-decreasing on $\left[0,(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right]$ when $x\in(0,x_0)$, we have

$$f^\lambda(x,G(x))\leqslant f^\lambda\left(x,G(x_0)\right)=f^\lambda\left(x,\hat{G}_\lambda(x)\right),\quad \forall\, x\in(0,x_0).$$

Next, for any $x\in(x_0,x_1)$, $G(x)\geqslant G(x_0)\geqslant(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)$ and $f^\lambda(x,\cdot)$ is non-increasing on $\left[(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right),\infty\right)$. Hence

$$f^\lambda(x,(G(x))\leqslant f^\lambda\left(x,G(x_0)\right)=f^\lambda\left(x,\hat{G}_\lambda(x)\right),\quad \forall\, x\in(x_0,x_1).$$

Finally,

$$f^\lambda(x,G(x))\leqslant f^\lambda\left(x,(u')_u^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right)=f^\lambda\left(x,\hat{G}_\lambda(x)\right),\quad \forall\, x\in(x_1,1).$$

Therefore,

$$J_Q^\lambda(G)=\int_0^1 f^\lambda(x,G(x))\,\mathrm{d}x\leqslant\int_0^1 f^\lambda\left(x,\hat{G}_\lambda(x)\right)\mathrm{d}x=J_Q^\lambda(\hat{G}_\lambda).$$

If $x_0=0$, we define

$$x_1=\sup\left\{x\in(q,1)\;\middle|\;(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\leqslant G(0+)\right\}\vee q,$$

and

$$\hat{G}_\lambda(x)=G(0+)\mathbf{1}_{(0,x_1]}(x)+(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\mathbf{1}_{(x_1,1)}(x),\quad \forall\, x\in[0,1).$$

A similar argument as above shows that $J_Q^\lambda(G)\leqslant J_Q^\lambda(\hat{G}_\lambda)$.

We have now proved that in order to find an optimal quantile function one needs only to consider functions of the form $G(x)=a\mathbf{1}_{(0,q]}(x)+a\vee(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\mathbf{1}_{(q,1)}(x)$, where the parameters a and λ are subject to the constraints in (5.6). Note that the last equality constraint in (5.6) was due to the fact that the following payoff function is non-decreasing in a. The payoff under the above G is

$$J(a,\lambda):=J_Q(G)=(1-w(1-q))u(a)+\int_q^1 u\left(a\vee(u')_l^{-1}\left(\frac{\lambda}{w'(1-x)}\right)\right)w'(1-x)\,\mathrm{d}x,$$

which is exactly the objective function of (5.6). Since the optimal solution $a^*>0$, the corresponding G^* defined in (5.7) is a quantile. The proof is complete.

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