Erratum to "A corrected proof of the stochastic verification theorem within the framework of viscosity solutions"

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Abstract

We correct the proof of Theorem 4.1 in Gozzi, Święch and Zhou [SIAM J. Control Optim., 43 (2005), pp. 2009–2019] by imposing additional conditions on the viscosity subsolution U.

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The proof of the stochastic verification theorem, Theorem 4.1 of [3], is not correct without additional assumptions on the function U there, which is a viscosity subsolution of the underlying HJB equation. This theorem was originally stated in [6], Theorem 3.1 (see also [5, Theorem 5.3, Chapter 5, Section 5.2]) with a gap in its proof, and [3] was an attempt to provide a complete proof. Unfortunately in [3] the inequality [3, (19)] and the inequality following it are not justified. We do not yet know how to fix the proof in the general case. Our proof as well as that in [5] relied on Lemma 5.2 of [5, Chapter 5, Section 5.2] which, as pointed to us by S. Federico (see [1]), is incorrect. However the proof still works under certain additional regularity assumptions on the subsolution U. We do not intend to develop the most general condition here; instead we will provide the additional conditions that will enable us to fill the gaps in the proof of Theorem 4.1. These conditions are: 1) the subsolution function U satisfies

$$U(t+h,x) - U(t,x) \le C(1+|x|^m)h, \quad m \ge 0, \text{ for all } x \in \mathbb{R}^n, 0 < t < t+h \le T; \tag{1}$$

and 2) U is semiconcave, uniformly in t, i.e. there exists $C_0 \ge 0$ such that for every $t \in (0,T]$

$$U(t,\cdot) - C_0|\cdot|^2 \quad \text{is concave on } \mathbb{R}^n. \tag{2}$$

We remark that these conditions are satisfied when U is taken as the value function V under reasonable assumptions on the data of the original stochastic control problem, while in applying a verification theorem one indeed mostly applies the value function. See the end of this note for details.

To justify the application of Fatou's Lemma in [3, (19)] (where lim sup can be taken along any subsequence) we need to know that the convergence is dominated from above. Therefore we need to know that

$$\mathbb{E}_{\omega_0} \frac{1}{h}[U(t_0+h, x^*(t_0+h)) - U(t_0, x^*(t_0))] \le \rho_1(\omega_0) \tag{3}$$

for $h \le h_0$, for some $h_0 > 0$ and $\rho_1 \in L^1(\Omega;\mathbb{R})$. Moreover it is easy to see that the conclusion of Lemma 5.2 of [5] is true if one replaces (5.6) there by a stronger assumption that $\frac{1}{h}[g(t+h) - g(t)] \le \rho_2(t)$ for $h \le h_0$,

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for some $h_0 > 0$ and $\rho_2 \in L^1(t_0, T; \mathbb{R})$. Therefore, since we apply it to the function $g(t) = \mathbb{E}U(t, x^*(t))$, we need

$$\mathbb{E}\frac{1}{h}[U(t+h, x^*(t+h)) - U(t, x^*(t))] \le \rho_2(t) \tag{4}$$

for $h \le h_0$, for some $h_0 > 0$ and $\rho_2 \in L^1(t_0, T; \mathbb{R})$. With this condition the inequality following [3, (19)] is true.

The proofs that (3) and (4) hold if (1) and (2) are satisfied are similar. We will only show (4).

By (2) we have that if $(p, q, Q) \in D^{1,2,+}_{t^+,x} U(t, x)$ then

$$U(t+h, y) \le U(t, x) + C(1 + |x|^m)h + \langle q, y - x\rangle + C_0|y - x|^2 \tag{5}$$

for all $y \in \mathbb{R}^n, 0 < h \le T - t$. Therefore we have

$$U(t+h, x^*(t+h)) - U(t, x^*(t)) \le C(1+|x^*(t)|^m)h + \langle q^*(t), x^*(t+h) - x^*(t)\rangle + C_0|x^*(t+h) - x^*(t)|^2. \tag{6}$$

We notice that by [3, (6)] and (2) there exists $m_1 \ge 1$ such that

$$|q^*(t)| \le C_1(1 + |x^*(t)|^{m_1}). \tag{7}$$

Moreover by the assumptions of [3] we have that for every $l \ge 1$

$$\sup_{t \le s \le T} \mathbb{E}|x^*(s)|^l \le K_l(1 + |x|^l). \tag{8}$$

Using (7), (8), [3, Hypothesis 2.1], and the fact that $q^*(t)$ is $\mathcal{F}_t$ measurable, we have

$$\mathbb{E}C(1 + |x^*(t)|^m)h \le C_2(1 + |x|^m)h,$$

$$\mathbb{E}\langle q^*(t), x^*(t+h) - x^*(t)\rangle = \mathbb{E}\langle q^*(t), \int_t^{t+h} b(s, x^*(s), u^*(s))ds + \int_t^{t+h} \sigma(s, x^*(s), u^*(s))dW(s)\rangle$$

$$\le C_1(\mathbb{E}[(1 + |x^*(t)|^{m_1})^2])^{1/2}(\mathbb{E}[\int_t^{t+h} b(s, x^*(s), u^*(s))ds]^2)^{1/2} \le C_3(1 + |x|^{m_2})h$$

and

$$\mathbb{E}C_0|x^*(t+h) - x^*(t)|^2 \le 2C_0(\mathbb{E}[\int_t^{t+h} b(s, x^*(s), u^*(s))ds]^2) + 2C_0(\mathbb{E}[\int_t^{t+h} \sigma(s, x^*(s), u^*(s))dW(s)]^2)$$

$$\le C_4(1 + |x|^2)(h^2 + h)$$

for some absolute constants C_2, C_3, C_4 and some positive number m_2. Therefore, if $h_0 = 1$, (4) is satisfied with a constant function $\rho_2(t) = C_5(1 + |x|^{m_3})$ for some C_5 and $m_3 = \max(m, m_2, 2)$.

It is difficult to expect that a viscosity subsolution U satisfies (1) and (2) in general, yet these conditions may be satisfied when U is the value function V, i.e. is the viscosity solution of the HJB equation [3, (2)], which is the most interesting case[1]. It is well known that V satisfies (2) if, in addition to [3, Hypothesis 2.1], h is semiconcave, $f(t, \cdot, u)$ is semiconcave, uniformly in t and u, and $b_x(t, \cdot, u)$ and $\sigma_x(t, \cdot, u)$ are Lipschitz continuous, uniformly in t and u, see [5, Chapter 4, Proposition 4.5]. Conditions guaranteeing the Lipschitz continuity of V in the spatial variable x (which is stronger than (1)) are also well known. We refer to [2, Section IV.8], [4], and [5, Chapter 4, Proposition 3.1].

We conclude this note by reiterating that, while it is mathematically interesting to state and prove the most general verification theorem, the additional conditions we introduce here are reasonable and adequate in *applying* the theorem.

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[1] Note that in the original statements of the stochastic verification theorem, [6, Theorem 3.1] and [5, Chapter 5, Theorem 5.3], it is indeed the viscosity solution that is used.

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