Continuous-Time Mean–Risk Portfolio Selection

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Abstract

This paper is concerned with continuous-time portfolio selection models in a complete market where the objective is to minimize the risk subject to a prescribed expected payoff at the terminal time. The risk is measured by the expectation of a certain function of the deviation of the terminal payoff from its mean. First of all, a model where the risk has different weights on the upside and downside variance is solved explicitly. The limit of this weighted mean–variance problem, as the weight on the upside variance goes to zero, is the mean–semivariance model which is shown to admit *no* optimal solution. This negative result is further generalized to a mean–downside-risk portfolio selection problem where the risk has non-zero value only when the terminal payoff is lower than its mean. Finally, a general model is investigated where the risk function is convex. Sufficient and necessary conditions for the existence of optimal portfolios are given. Moreover, optimal portfolios are obtained when they do exist. The solution is based on completely solving certain static, constrained optimization problems of random variables.

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1 Introduction

Risk is a central issue in financial investment, yet it is a subjective notion as opposed to return. Therefore, a fundamental problem is how risk should be measured. In the early 1950s, Markowitz [17] proposed the single-period mean–variance (M–V) portfolio selection model, where he used the variance to measure the risk. This seminal work has been widely recognized to have laid the foundation of modern portfolio theory. However, there has also been substantial amount of objection to the measurement of risk by variance. The main aspects of the M–V theory under criticism include the penalty on the upside return, and the equal weight on the upside and downside whereas the asset return distribution is generally assymmetric. Consequently, some alternative risk measures were proposed, notably the so-called downside risk, where only the return below its mean or a target level is counted as risk [6, 22, 19]. One of the downside risk measures is the semivariance. In [18] Markowitz himself agreed that "semivariance seems more plausible than variance as a measure of risk". On the other hand, in a single-period financial market, other risk measures have also been proposed and studied, including VaR [9], mean–absolute deviation [12], and minimax measure [2]. For a recent survey on the Markowitz model and models with various risk measures, refer to [23].

The M–V approach "has received comparably little attention in the context of long-term investment planning" ([23, p.32]), especially in continuous time setting, until very recently. In a series of papers [26, 14, 13, 25, 1] the continuous-time Markowitz models have been investigated thoroughly with closed-form solutions obtained in most cases. In this paper, we will study continuous-time portfolio selection models, in a complete market, with risk measures different from the variance. We will start with a weighted mean–variance problem where the risk has different weights on upside and downside returns. Explicit solution will be obtained for this model. While the weighted mean–variance model is important in its own right, it also converges to the mean–semivariance model when the weight on the upside variance goes to zero. Surprisingly and in sharp contrast to the single-period setting, based on this convergence approach we will show that the mean–semivariance model has *no* optimal solution, although asymptotically optimal solution can be obtained from the solution to the weighted mean–variance model. This "negative" result motivates us to study a general mean–downside-risk model where only the downside return is penalized, not necessarily in the fashion of variance. It turns out that this general downside-risk model provides no optimal solution either, under a very mild condition.

Finally, we will study a "most general" mean–risk model, where the risk is measured by the expectation of a convex function of the deviation of the terminal payoff from its mean.

For this model, we give a complete solution in terms of characterization of the existence of optimal portfolio and presentation of the solution when it exists.

The basic approach to solving the dynamic mean–risk portfolio selection is to reduce the problem into two subproblems: one is to solve a constrained static optimization problem on the terminal wealth, and the other is to replicate the optimal terminal wealth. This approach is rather standard; see [20, 7, 11, 1]. The second subproblem is straightforward to solve in view of the completeness of the market. The main contribution of this paper is that we solve the first subproblem thoroughly for very general functions that define the underlying risk. This subproblem is sufficiently interesting in its own right, from the viewpoints of both probability and optimization.

The rest of this paper is organized as follows. In section 2, we specify the continuous-time financial market under consideration, and introduce the equivalent static optimization problem for a dynamic portfolio selection problem. In section 3, we investigate the weighted mean–variance problem, and in section 4, we treat the mean–semivariance model based on the results in section 3 and a convergence approach. Section 5 is devoted to the study on the mean–downside-risk problem. In section 6, we turn to the general mean–risk model, and find the sufficient and necessary conditions for the problem to admit optimal solutions. Several examples are presented to illustrate the general results obtained. Finally, the paper is closed in section 7 with some concluding remarks.

2 Problem formulation

In this paper T is a fixed terminal time and $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ a fixed filtered complete probability space on which is defined a standard $\mathcal{F}_t$-adapted m-dimensional Brownian motion $W(t) \equiv (W^1(t), \cdots, W^m(t))'$ with $W(0) = 0$. It is assumed that $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$. For $q \geq 1$, we denote by $L^q_{\mathcal{F}}(0, T; \mathbf{R}^d)$ the set of all $\mathbf{R}^d$-valued, $\mathcal{F}_t$-adapted measurable stochastic processes $f(\cdot) = \{f(t) : 0 \leq t \leq T\}$ such that $E\int_0^T |f(t)|^q dt < +\infty$, and by $L^q(\mathcal{F}_T, \mathbf{R}^d)$ the set of all $\mathbf{R}^d$-valued, $\mathcal{F}_T$-measurable random variables X such that $E|X|^q < +\infty$. Throughout this paper, a.s. signifies that the corresponding statement holds true with probability 1 (with respect to P).

Suppose there is a market in which $m + 1$ assets (or securities) are traded continuously. One of the assets is the bank account whose price process $S_0(t)$ is subject to the following (stochastic) ordinary differential equation:

$$dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T]; \quad S_0(0) = s_0 > 0,$$

where the interest rate $r(t)$ is a uniformly bounded, $\mathcal{F}_t$-adapted, scalar-valued stochastic process. Note that normally one would assume that $r(t) \geq 0$; yet this assumption is not necessary in our subsequent analysis. The other m assets are stocks whose price processes $S_i(t)$, $i = 1, \cdots, m$, satisfy the following stochastic differential equation (SDE):

$$dS_i(t) = S_i(t)[b_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)], \quad t \in [0,T]; \quad S_i(0) = s_i > 0,$$

where $b_i(t)$ and $\sigma_{ij}(t)$, the appreciation and dispersion (or volatility) rates, respectively, are scalar-valued, $\mathcal{F}_t$-adapted, uniformly bounded stochastic processes.

Define the volatility matrix $\sigma(t) := (\sigma_{ij}(t))_{m \times m}$. A basic assumption throughout this paper is that the covariance matrix

$$\sigma(t)\sigma(t)' \geq \delta I_m, \quad \forall t \in [0,T], \quad \text{a.s.},$$

for some $\delta > 0$, where I_m is the $m \times m$ identity matrix. This assumption ensures that the market is complete.

Consider an agent whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Assume that the trading of shares takes place continuously in a self-financing fashion (i.e., there is no consumption or income) and there are no transaction costs. Then $x(\cdot)$ satisfies (see e.g. Karatzas and Shreve [10] and Elliott and Kopp [5])

$$dx(t) = \Big\{r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)]\pi_i(t)\Big\}dt + \sum_{j=1}^{m}\sum_{i=1}^{m} \sigma_{ij}(t)\pi_i(t)dW^j(t), \quad x(0) = x_0 \geq 0,$$

where $\pi_i(t)$, $\; i = 0, 1, 2 \cdots, m$, denotes the total market value of the agent's wealth in the i-th asset. We call $\pi(\cdot) \equiv (\pi_1(\cdot), \cdots, \pi_m(\cdot))'$ the *portfolio* of the agent.

Set $B(t) := (b_1(t) - r(t), \cdots, b_m(t) - r(t))$, and define the *risk premium process* $\theta(t) \equiv (\theta_1(t), \cdots, \theta_m(t)) := B(t)(\sigma(t)')^{-1}$ and the pricing kernel

$$\rho(t) := \exp\Big\{-\int_0^t [r(s) + \frac{1}{2}|\theta(s)|^2]ds - \int_0^t \theta(s)dW(s)\Big\}. \tag{1}$$

With this notation, wealth equation becomes

$$dx(t) = [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \quad x(0) = x_0. \tag{2}$$

Before we formulate our continuous-time portfolio selection model, we specify the "allowable" investment policies with

Definition 2.1 A portfolio $\pi(\cdot)$ is said to be *admissible* if $\pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^m)$.

The various portfolio selection models we are going to consider in this paper are all special cases of the following general problem

$$\begin{array}{ll} \text{Minimize} & Ef(x(T) - Ex(T)), \\ \text{subject to} & \begin{cases} \pi(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbf{R}^m), \\ (x(\cdot),\pi(\cdot)) \text{ satisfies equation (2) with initial wealth } x_0, \\ Ex(T) = z, \end{cases} \end{array} \tag{3}$$

where $x_0, z \in \mathbf{R}$ and the function $f : \mathbf{R} \to \mathbf{R}$ are given. In words, problem (3) is to minimize the risk, measured by certain function of the deviation of the terminal wealth from its mean, via continuous trading, subject to an initial budget constraint (specified by x_0) and a target expected terminal payoff (specified by z). The trade-off between return and risk is realized by achieving the minimum possible risk after one specifies the target return. A mean–risk efficient frontier will then be traced out as z varies over certain range. The Markowitz mean–variance problem is a special case of (3) with $f(x) = x^2$.

Applying [4, p. 22, Proposition 2.2] to equation (2) we have

$$x(t) = \rho(t)^{-1}E(\rho(T)x(T)|\mathcal{F}_t), \quad \forall t \in [0,T], \text{ a.s.}. \tag{4}$$

In particular,

$$x_0 = E[\rho(T)x(T)].$$

Hence, as in [1] the portfolio selection problem (3) can be decomposed into a static optimization problem and a wealth replication problem. The static optimization problem is

$$\begin{array}{ll} \text{Minimize} & Ef(X - z), \\ \text{subject to} & EX = z,\ E[\rho(T)X] = x_0,\ X \in L^2(\mathcal{F}_T, \mathbf{R}). \end{array} \tag{5}$$

Suppose X^* is an optimal solution to (5), then the replication problem is to find a portfolio such that its terminal wealth hits X^*; in other words, the problem is to find $(x(\cdot),\pi(\cdot))$ that solves the following equation

$$dx(t) = [r(t)x(t)dt + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t), \ \ x(T) = X^*. \tag{6}$$

Theorem 2.1 *If $(x^*(\cdot),\pi^*(\cdot))$ is optimal for problem (3), then $x^*(T)$ is optimal for problem (5) and $(x^*(\cdot),\pi^*(\cdot))$ satisfies (6). Conversely, if X^* is optimal for problem (5), then (6) must have a solution $(x^*(\cdot),\pi^*(\cdot))$ which is an optimal solution for (3).*

Proof: The proof is the same as that of [1, Theorem 2.1]. $\square$

Remark 2.1 The replication problem (6) is essentially a backward stochastic differential equation (BSDE); refer to [15, 16, 24] for various approaches in solving BSDEs. Indeed, the unique solution $(x^*(\cdot), \pi^*(t))$ of (6) is given by

$$\pi^*(t) = (\sigma(t)')^{-1} y^*(t), \tag{7}$$

whereas $(x^*(\cdot), y^*(\cdot))$ is the unique solution to the BSDE

$$dx(t) = [r(t)x(t) + \theta(t)y(t)]dt + y(t)'dW(t), \;\; x(T) = X^*. \tag{8}$$

Thus, according to Theorem 2.1 the key is to solve the static optimization problem (5). The remainder of this paper will be mainly devoted to solving problem (5) for various situations.

3 The Weighted Mean–Variance Model

The classical mean–variance portfolio selection problem uses the variance as the measure for risk, which puts the same weight on the downside and upside (in relation to the mean) of the return. In this section, we study the "weighted" mean–variance portfolio selection model where the weights on the downside and upside may be different. Specifically, for given $\alpha > 0, \beta > 0, z \in \mathbf{R}, x_0 \in \mathbf{R}$, we consider problem (3) with $f(x) = \alpha x_+^2 + \beta x_-^2$, where $x_+ \geq 0$ and $x_- \geq 0$ denote the positive and negative parts of x respectively. It reduces to the classical mean–variance model when $\alpha = \beta$.

As discussed at the end of Section 2, to solve the above problem it suffices to solve a static optimization problem (5) in terms of X. Define $Y := X - z$, then (5) specializes to

$$\begin{aligned} &\text{Minimize} \quad E(\alpha Y_+^2 + \beta Y_-^2), \\ &\text{subject to} \quad EY = 0, \; E[\rho Y] = y_0, \; Y \in L^2(\mathcal{F}_T, \mathbf{R}), \end{aligned} \tag{9}$$

where $\rho := \rho(T)$ and $y_0 := x_0 - zE\rho$.

Since the above is a static convex optimization problem with a nonnegative infimum, using the Lagrange multiplier approach (see [1, Proposition 4.1]), we conclude that Y^* is an optimal solution of (9) if and only if Y^* is a feasible solution of (9) and there exists a pair (λ, μ) such that Y^* is an optimal solution of the following problem

$$\min_{Y \in L^2(\mathcal{F}_T, \mathbf{R})} E[\alpha Y_+^2 + \beta Y_-^2 - 2(\lambda - \mu\rho)Y]. \tag{10}$$

Lemma 3.1 *Problem (10) admits a unique optimal solution* $Y^* = \frac{(\lambda-\mu\rho)_+}{\alpha} - \frac{(\lambda-\mu\rho)_-}{\beta}$.

Proof: For any $Y \in L^2(\mathcal{F}_T, \mathbf{R})$, we have, sample-wisely,

$$\begin{aligned}
& \alpha Y_+^2 + \beta Y_-^2 - 2(\lambda - \mu\rho)Y \\
= \; & \alpha(Y_+^2 - 2\frac{\lambda - \mu\rho}{\alpha}Y_+) + \beta(Y_-^2 + 2\frac{\lambda - \mu\rho}{\beta}Y_-) \\
= \; & \alpha(Y_+ - \frac{\lambda - \mu\rho}{\alpha})^2 - \frac{(\lambda - \mu\rho)^2}{\alpha} + \beta(Y_- + \frac{\lambda - \mu\rho}{\beta})^2 - \frac{(\lambda - \mu\rho)^2}{\beta} \\
\geq \; & -\frac{(\lambda - \mu\rho)_+^2}{\alpha} - \frac{(\lambda - \mu\rho)_-^2}{\beta} \\
= \; & \alpha(Y_+^*)^2 + \beta(Y_-^*)^2 - 2(\lambda - \mu\rho)Y^*.
\end{aligned}$$

This shows that Y^* is an optimal solution. The uniqueness of the optimal solution follows from the strict convexity of the problem (10). □

Proposition 3.1 *For any* y_0*, there exists a unique pair* (λ, μ) *such that the optimal solution* Y^* *in Lemma 3.1 satisfies* $EY^* = 0, E[\rho Y^*] = y_0$*. Moreover,* $\lambda < 0, \mu < 0$ *if* $y_0 > 0$, $\lambda > 0, \mu > 0$ *if* $y_0 < 0$*, and* $\lambda = \mu = 0$ *if* $y_0 = 0$.

Proof: If $y_0 = 0$, then we simply take $\lambda = \mu = 0$ (in which case $Y^* = 0$).

If $y_0 < 0$, then it is easy to see, using the mean-value theorem of a continuous function, that the following equation admits a unique solution $\zeta > 0$:

$$E(\zeta - \rho)_+/\alpha = E(\zeta - \rho)_-/\beta. \tag{11}$$

Set

$$a := E[\rho(\zeta - \rho)_+]/\alpha - E[\rho(\zeta - \rho)_-]/\beta. \tag{12}$$

Since $EY^* = 0, E[\rho Y^*] = y_0$, ρ cannot be a constant, by (11) we have

$$a < \zeta E[(\zeta - \rho)_+]/\alpha - \zeta E(\zeta - \rho)_-/\beta = 0.$$

Take $\mu := y_0/a > 0, \lambda := \zeta\mu > 0$. Then it is straightforward that (λ, μ) is the desired pair.

Finally, if $y_0 > 0$, then let $\xi > 0$ be the unique solution of equation

$$E(\xi - \rho)_-/\alpha = E(\xi - \rho)_+/\beta. \tag{13}$$

Set

$$b := E[\rho(\xi - \rho)_-]/\alpha - E[\rho(\xi - \rho)_+]/\beta. \tag{14}$$

An argument similar to above yields $b > 0$. Take $\mu := -y_0/b < 0, \lambda := \xi\mu < 0$. Then (λ, μ) is the desired pair.

For the uniqueness, it is not difficult to prove by discussing for the cases $\mu < 0$ and $\mu > 0$ respectively. □

Theorem 3.1 *The unique optimal solution for problem (9) is*

$$Y^* = \frac{(\lambda - \mu\rho)_+}{\alpha} - \frac{(\lambda - \mu\rho)_-}{\beta}$$

where (λ, μ) is the unique solution of the system of equations:

$$\begin{cases} \frac{E(\lambda-\mu\rho)_+}{\alpha} - \frac{E(\lambda-\mu\rho)_-}{\beta} = 0, \\ \frac{E[\rho(\lambda-\mu\rho)_+]}{\alpha} - \frac{E[\rho(\lambda-\mu\rho)_-]}{\beta} = y_0. \end{cases} \tag{15}$$

Moreover, for the case $y_0 < 0, y_0 = 0$, and $y_0 > 0$, the minimum value $E[\alpha(Y_+^*)^2 + \beta(Y_-^*)^2]$ of the problem (9) is equal to $-y_0^2/a, 0, y_0^2/b$, respectively, where a is given by (12) and b is given by (14).

Proof: The first part of the theorem is immediate from Lemma 3.1 and Proposition 3.1. To prove the second part, note that the case when $y_0 = 0$ is trivial; so we consider $y_0 \neq 0$. One has

$$\begin{aligned} -\frac{1}{\beta}E(\lambda - \mu\rho)_-^2 &= \frac{1}{\beta}E[(\lambda - \mu\rho)_-(\lambda - \mu\rho)] \\ &= \lambda\frac{E(\lambda - \mu\rho)_-}{\beta} - \mu\frac{E[\rho(\lambda - \mu\rho)_-]}{\beta} \\ &= \lambda\frac{E(\lambda - \mu\rho)_+}{\alpha} - \mu\left\{\frac{E[\rho(\lambda - \mu\rho)_+]}{\alpha} - y_0\right\} \\ &= \frac{1}{\alpha}E[(\lambda - \mu\rho)_+(\lambda - \mu\rho)] + \mu y_0 \\ &= \frac{1}{\alpha}E(\lambda - \mu\rho)_+^2 + \mu y_0, \end{aligned}$$

where we have utilized the equations (15). Consequently,

$$E[\alpha(Y^*)_+^2 + \beta(Y^*)_-^2] = \frac{1}{\alpha}E(\lambda - \mu\rho)_+^2 + \frac{1}{\beta}E(\lambda - \mu\rho)_-^2 = -\mu y_0.$$

By the proof of Propositioin 3.1, we obtain immediately the desired result. □

Translating back to the weighted mean–variance portfolio selection problem (3), in view of Theorem 2.1, the unique optimal portfolio corresponding to $z > 0$ is the replicating portfolio for the terminal contingent claim $x^*(T) = \frac{(\lambda-\mu\rho)_+}{\alpha} - \frac{(\lambda-\mu\rho)_-}{\beta} + z$. Details are left to the reader. We note that if $z = \frac{x_0}{E\rho}$, then $\lambda = \mu = 0$ implying that $x^*(T) = z$ a.s. under the optimal portfolio. Hence in this case the optimal portfolio is a risk-free portfolio or a zero-coupon bound. As a by-product, we have proved that a risk-free portfolio is available (which involves exposure to the stocks) even though the interest rate is random.

4 The Mean–Semivariance Model

In this section we consider the mean–semivariance problem, where only the downside return is penalized. This is a case of (3) with $f(x) = x_-^2$.

As before we denote $\rho := \rho(T)$ where $\rho(\cdot)$ is defined by (1). Define

$$\rho_0 := \inf\{\eta \in \mathbf{R} : P(\rho < \eta) > 0\}, \quad \rho_1 := \sup\{\eta \in R : P(\rho > \eta) > 0\}. \tag{16}$$

Lemma 4.1 *Let $\zeta(\alpha)$, $\alpha \in (0,1)$, be the solution to (11) with $\beta = 1-\alpha$, then* $\lim_{\alpha\downarrow 0}\zeta(\alpha) = \rho_0$. *Similarly, let $\xi(\alpha)$, $\alpha \in (0,1)$, be the solution to (13) with $\beta = 1-\alpha$, then* $\lim_{\alpha\downarrow 0}\xi(\alpha) = \rho_1$.

Proof: Define $f(\zeta) := \frac{E(\zeta-\rho)_+}{E(\zeta-\rho)_-}$, $\zeta \in (\rho_0, \rho_1)$. Then equation (11) is equivalent to $f(\zeta) = \frac{\alpha}{1-\alpha}$. Obviously, $f(\zeta)$ is a strictly positive and strictly increasing function on $\zeta \in (\rho_0, \rho_1)$; hence $\zeta(\alpha)$ is strictly increasing on $\alpha \in (0,1)$, and in this interval, $\rho_0 < \zeta(\alpha) < \rho_1$.

Denote $\lim_{\alpha\downarrow 0}\zeta(\alpha) = \zeta_0$, then $\zeta_0 \geq \rho_0$. If $\zeta_0 > \rho_0$, then take $\zeta \in (\rho_0, \zeta_0)$. Since $\zeta < \zeta_0 = \lim_{\alpha\downarrow 0}\zeta(\alpha)$, we have $\frac{E(\zeta-\rho)_+}{E(\zeta-\rho)_-} = 0$, implying $E(\zeta-\rho)_+ = 0$. However, $\zeta > \rho_0$, so $P(\rho < \zeta) > 0$ leading to a contradiction. Therefore, $\zeta_0 = \rho_0$.

Similarly, we can prove the other part of the lemma in terms of $\xi(\alpha)$. □

We are now in a position to prove the following *negative* result.

Theorem 4.1 *The mean–semivariance problem (3) with $f(x) = x_-^2$ does not admit any optimal solution so long as $z \neq \frac{x_0}{E\rho}$.*

Proof: In view of Theorem 2.1, it suffices to prove that the static optimization problem

$$\begin{aligned} &\text{Minimize} \quad E(Y_-^2), \\ &\text{subject to} \quad EY = 0,\ E[\rho Y] = y_0 \equiv x_0 - zE\rho,\ Y \in L^2(\mathcal{F}_T, \mathbf{R}) \end{aligned} \tag{17}$$

has no optimal solution. Consider problem (9) with $\beta = 1-\alpha$ and $\alpha \in (0,1)$. It has been proved in the proof of Proposition 3.1 that there exists a pair $(\lambda(\alpha), \mu(\alpha))$ such that

$Y(\alpha) = \frac{(\lambda(\alpha)-\mu(\alpha)\rho)_+}{\alpha} - \frac{(\lambda(\alpha)-\mu(\alpha)\rho)_-}{\beta}$ satisfies $EY(\alpha) = 0, E[\rho Y(\alpha)] = y_0$. This implies that each $Y(\alpha)$ is feasible for problem (17).

Since $z \neq \frac{x_0}{E\rho}$, we have $y_0 \neq 0$. First consider the case when $y_0 < 0$. It was proved in the proof of Proposition 3.1 that $\lambda(\alpha) > 0, \mu(\alpha) > 0$. Let $\zeta(\alpha) = \lambda(\alpha)/\mu(\alpha)$. Then $\zeta(\alpha)$ is the solution to (11) with $\beta = 1-\alpha$. Lemma 4.1 along with its proof yields $\zeta(\alpha) > \rho_0$, and $\zeta(\alpha) \to \rho_0$ as $\alpha \downarrow 0$. Consequently,

$$\begin{aligned} 0 \leq E[(\rho-\rho_0)(\zeta(\alpha)-\rho)_+]/\alpha &\leq (\zeta(\alpha)-\rho_0)E(\zeta(\alpha)-\rho)_+/\alpha \\ &= (\zeta(\alpha)-\rho_0)E(\zeta(\alpha)-\rho)_-/(1-\alpha) \\ &\leq (\zeta(\alpha)-\rho_0)E\rho/(1-\alpha) \to 0, \quad \text{as } \alpha \downarrow 0, \end{aligned}$$

and

$$E[(\rho-\rho_0)(\zeta(\alpha)-\rho)_-]/(1-\alpha) \to E(\rho-\rho_0)^2, \quad \text{as } \alpha \downarrow 0.$$

However, by (11) we have

$$\mu(\alpha) = \frac{y_0}{a(\alpha)} = \frac{y_0}{E[(\rho-\rho_0)(\zeta(\alpha)-\rho)_+]/\alpha - E[(\rho-\rho_0)(\zeta(\alpha)-\rho)_-]/(1-\alpha)},$$

so that $\mu(\alpha) \to -y_0/E(\rho-\rho_0)^2$, as $\alpha \downarrow 0$. Therefore,

$$E[Y(\alpha)_-^2] = \frac{\mu(\alpha)^2 E(\zeta(\alpha)-\rho)_-^2}{(1-\alpha)^2} \to y_0^2/E(\rho-\rho_0)^2, \quad \text{as } \alpha \downarrow 0. \tag{18}$$

On the other hand, for any feasible solution Y of problem (17), Cauchy–Schwartz's inequality yields $\{E[(\rho-\rho_0)Y_-]\}^2 \leq E[Y_-]^2 E[(\rho-\rho_0)^2 \mathbf{1}_{Y<0}]$. Note that $E[(\rho-\rho_0)^2\mathbf{1}_{Y<0}] \neq 0$, for otherwise $P(Y \geq 0) = 1$ which together with $EY = 0$ would imply $P(Y=0) = 1$ and hence $y_0 = 0$. As a result,

$$E[Y_-]^2 \geq \frac{\{E[(\rho-\rho_0)Y_-]\}^2}{E[(\rho-\rho_0)^2\mathbf{1}_{Y<0}]} = \frac{\{E[(\rho-\rho_0)Y_+]-y_0\}^2}{E[(\rho-\rho_0)^2\mathbf{1}_{Y<0}]} > \frac{y_0^2}{E(\rho-\rho_0)^2}, \tag{19}$$

where the last *strict* inequality is due to the facts that $y_0 < 0$ and $EY = 0$. Comparing (18) and (19) we conclude that there is no optimal solution for (17) in this case.

For the case $y_0 > 0$, we have proved that $\lambda(\alpha) < 0, \mu(\alpha) < 0$ and $\xi(\alpha) := \lambda(\alpha)/\mu(\alpha) > 0$, where $\xi(\alpha)$ is the solution to (13) with $\beta = 1-\alpha$. According to Lemma 4.1, $\xi(\alpha) \to \rho_1$ as $\alpha \downarrow 0$. First assume that $\rho_1 < +\infty$. Then an argument completely analogous to the above yields

$$E[Y(\alpha)_-^2] \to y_0^2/E(\rho_1-\rho)^2, \quad \text{as } \alpha \downarrow 0, \tag{20}$$

whereas $E[Y_-]^2 > y_0^2/E(\rho_1-\rho)^2$ for any feasible solution Y of problem (17). Thus there is no optimal solution for (17).

Note that

$$\begin{aligned} b(\alpha) \geq\ & \xi(\alpha)E(\xi(\alpha)-\rho)_-/\alpha - E[\rho(\xi(\alpha)-\rho)_+]/(1-\alpha) \\ =\ & \xi(\alpha)E(\xi(\alpha)-\rho)_+/(1-\alpha) - E[\rho(\xi(\alpha)-\rho)_+]/(1-\alpha) \\ =\ & E(\xi(\alpha)-\rho)_+^2/(1-\alpha). \end{aligned} \tag{21}$$

Consequently, by (21) and the fact that $\mu(\alpha) = -y_0/b(\alpha)$ we have

$$E[Y(\alpha)_-^2] = \frac{\mu(\alpha)^2 E(\xi(\alpha)-\rho)_+^2}{(1-\alpha)^2} \leq \frac{\mu(\alpha)^2}{1-\alpha} b(\alpha) = \frac{y_0^2}{(1-\alpha)b(\alpha)}. \tag{22}$$

Thus, if $\xi(\alpha) \to \rho_1 = +\infty$, as $\alpha \downarrow 0$, then $E[Y(\alpha)_-^2] \to 0$, as $\alpha \downarrow 0$. On the other hand, for any feasible solution Y , if $EY_-^2 = 0$, then $Y = 0$ implying $y_0 = 0$. This, once again, proves that (17) has no optimal solution. □

Remark that if $z = \frac{x_0}{E\rho}$, then there is a risk-free portfolio under which the terminal wealth is exactly z. This portfolio is therefore an optimal portfolio for (17). Also, although the mean–semivariance problem in general does not admit optimal solutions, the infimum of the problem has been obtained explicitly in the proof of Theorem 4.1. Specifically, the infimum is $\frac{y_0^2}{E(\rho-\rho_0)^2}$ if $y_0 < 0$, and is $\frac{y_0^2}{E(\rho_1-\rho)^2}$ if $y_0 > 0$. Moreover, asymptotically optimal portfolios can be obtained by replicating $Y(\alpha)$ as $\alpha \to 0$.

Theorem 4.1 shows that, quite contrary to the single-period case, the mean–semivariance portfolio selection problem in a complete continuous-time financial market does not admit a solution (save for the trivial case when $z = \frac{x_0}{E\rho}$). In the next section, we shall extend this "negative" result to a general model that concerns only the downside risk.

5 The Mean–Downside-risk Model

Some alternative measures for risk have been proposed in lieu of the variance, and one of such measures is the downside risk which concerns only the downside deviation of the return from the mean. The semivariance studied in the previous section is a typical type of downside risk measure. In this section, we will generalize the result obtained in Section 4 to a general portfolio selection model with downside risk.

Before we formulate the underlying portfolio selection problem, let us investigate an abstract static optimization problem, which is interesting in its own right. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $q \geq 1$, we denote by $L^q(\mathcal{F}, \mathbf{R})$ the set of all $\mathcal{F}$-measurable real random variables X such that $|X|^q$ is integrable under P. Let ξ be a strictly positive real random variable, with the property that

$$P\{\xi \in (M_1, M_2)\} > 0, \text{ and } P\{\xi = M_1\} = P\{\xi = M_2\} = 0, \ \forall 0 \leq M_1 < M_2 \leq +\infty. \quad (23)$$

Consider the following optimization problem, with a given $y_0 \in \mathbf{R}$:

$$\begin{array}{ll} \text{Minimize} & Ef(Y), \\ \text{subject to} & EY = 0, \ E[\xi Y] = y_0, \ Y \in L^q(\mathcal{F}, \mathbf{R}), \end{array} \quad (24)$$

where $f : \mathbf{R} \to \mathbf{R}$ is a given function. Throughout this section we impose the following assumption on f:

Assumption 5.1 $f \geq 0$, left continuous at 0, strictly decreasing on $\mathbf{R}^-$, and $f(x) = 0$ $\forall x \in \mathbf{R}^+$.

An example of such a function is $f(x) = (x_-)^p$ for some $p \geq 0$. By virtue of the assumed properties of f, problem (24) has a finite (nonnegative, in fact) infimum.

Theorem 5.1 *Problem (24) admits no optimal solution for any $y_0 \neq 0$.*

This theorem will be proved via several intermediate results. Denote $L^q(\mathcal{F}, \mathbf{R}^-) := \{X \in L^q(\mathcal{F}, \mathbf{R}) : X \leq 0\}$. For any $a \leq 0$, define

$$h(a) := \inf_{Z \in L^q(\mathcal{F}, \mathbf{R}^-),\ E[\xi Z] = a} Ef(Z).$$

Lemma 5.1 *$h(a)$ is decreasing on $\mathbf{R}^-$. Moreover, if for a given $a_1 < 0$, there exists $\bar{Z} \in L^q(\mathcal{F}_T, \mathbf{R}^-)$ such that $E[\xi \bar{Z}] = a_1, Ef(\bar{Z}) = h(a_1)$, then $h(a_1) > h(a_2)$ $\forall a_2 \in (a_1, 0)$.*

Proof: For any $a_1 < a_2 < 0$, we have

$$h(a_2) \leq \inf_{Z \in L^q(\mathcal{F}_T, \mathbf{R}^-),\ E[\xi Z] = a_1} Ef(\frac{a_2}{a_1} Z) \leq \inf_{Z \in L^q(\mathcal{F}_T, \mathbf{R}^-),\ E[\xi Z] = a_1} Ef(Z) = h(a_1).$$

If there exists a $\bar{Z} \in L^q(\mathcal{F}_T, \mathbf{R}^-)$ with $E[\xi \bar{Z}] = a_1, Ef(\bar{Z}) = h(a_1)$, then

$$h(a_2) \leq Ef(\frac{a_2}{a_1} \bar{Z}) < Ef(\bar{Z}) = h(a_1).$$

This completes the proof. □

Lemma 5.2 *For any $\alpha > 0$, $\delta > 0$, and $0 < \beta < \alpha\delta$, there exists a uniformly bounded random variable $\bar{Y} \geq 0$ such that $E\bar{Y} = \alpha, E[\xi\bar{Y}] = \beta$, and $\bar{Y} = 0$ on the set $\{\omega \in \Omega : \xi \geq \delta\}$.*

Proof: Take $\delta_1 < \delta_2 < \delta$ so that $E(\xi|\delta_1 \leq \xi < \delta_2) = \beta/\alpha$. The property of the distribution of ξ and the fact that $\beta/\alpha < \delta$ ensure the existence of such δ_1, δ_2. Define $\bar{Y} = \frac{\alpha}{P(\delta_1 \leq \xi < \delta_2)}\mathbf{1}_{\delta_1 \leq \xi < \delta_2}$. Then $\bar{Y}$ satisfies all the desired requirements. □

Lemma 5.3 *For any $y_0 < 0$ and $\epsilon > 0$, there exists a feasible solution Y for problem (24) such that $Ef(Y) < h(y_0) + \epsilon$.*

Proof: For any $\epsilon > 0$, there exists $Z \in L^q(\mathcal{F}_T, \mathbf{R}^-)$ such that $E[\xi Z] = y_0$, and $h(y_0) \leq Ef(Z) < h(y_0) + \epsilon$. Since $\frac{a}{y_0}E[\xi Z] = a \; \forall a < y_0$, we have $h(a) \leq Ef(\frac{a}{y_0}Z)$. Fix $a < y_0$. Since the distribution of ξ has no atom by the assumption, there exists $\delta_0(a) > 0$ such that

$$\frac{a}{y_0}E[Z\xi\mathbf{1}_{\xi \geq \delta_0(a)}] = y_0.$$

As a result, one can take $\delta_1(a) > 0$ with $\delta_1(a) < \delta_0(a)$ and

$$\frac{-E[\frac{a}{y_0}Z\mathbf{1}_{\xi \geq \delta_1(a)}]}{y_0 - \frac{a}{y_0}E[Z\xi\mathbf{1}_{\xi \geq \delta_1(a)}]} > \frac{1}{\delta_1(a)}.$$

It is easy to see that $\lim_{a \uparrow y_0} \delta_0(a) = 0$; hence $\lim_{a \uparrow y_0} \delta_1(a) = 0$.

Define $Y_a = \frac{a}{y_0}Z\mathbf{1}_{\xi \geq \delta_1(a)} + \bar{Y}_a\mathbf{1}_{\xi < \delta_1(a)}$, where $\bar{Y}_a \geq 0$ is such that $\bar{Y}_a = 0$ on the set $\{\omega \in \Omega : \xi \geq \delta_1(a)\}$, and

$$\begin{aligned} E\bar{Y}_a &= -E[\tfrac{a}{y_0}Z\mathbf{1}_{\xi \geq \delta_1(a)}], \\ E[\xi\bar{Y}_a] &= y_0 - E[\tfrac{a}{y_0}\xi Z\mathbf{1}_{\xi \geq \delta_1(a)}]. \end{aligned}$$

The existence of such $\bar{Y}_a$ is implied by Lemma 5.2. Consequently, $EY_a = 0, E[\xi Y_a] = y_0$, meaning that Y_a is feasible for problem (24).

Now $Ef(Y_a) = E[f(\frac{a}{y_0}Z)\mathbf{1}_{\xi \geq \delta_1(a)}] + E[f(\bar{Y}_a)\mathbf{1}_{\xi < \delta_1(a)}] = E[f(\frac{a}{y_0}Z)\mathbf{1}_{\xi \geq \delta_1(a)}]$. Thus, we have

$$Ef(\frac{a}{y_0}Z) \geq Ef(Y_a) \geq E[f(Z)\mathbf{1}_{\xi \geq \delta_1(a)}]$$

which implies $\lim_{a \uparrow \eta} Ef(Y_a) = Ef(Z) < h(y_0) + \epsilon$. Thus, we can take $a < y_0$ such that $Ef(Y_a) < h(y_0) + \epsilon$. □

Proposition 5.1 *Problem (24) admits no optimal solution for any $y_0 < 0$.*

Proof: In view of Lemma 5.3 it suffices to show that $Ef(Y) > h(y_0)$ for any feasible solution Y of (24). To this end, first note that $E[\xi Y^+] > 0$, for otherwise $Y^+ = 0$ which along with $EY = 0$ yields $Y = 0$ and hence $y_0 = 0$. Therefore, $a := E[-\xi Y^-] < y_0$, suggesting $h(a) \geq h(y_0)$ by virtue of Lemma 5.1. If $h(a) = h(y_0)$, then the contrapositive of Lemma 5.1 implies that $Ef(-Y^-) > h(a)$. Since $f(x) = f(-x^-)$, we have $Ef(Y) = Ef(-Y^-) > h(a) = h(y_0)$. Otherwise, if $h(a) > h(y_0)$, then $Ef(Y) \geq h(a) > h(y_0)$. □

Now let us turn to the case when $y_0 > 0$.

Proposition 5.2 *Problem (24) admits no optimal solution for any* $y_0 > 0$.

Proof: Since $y_0 > 0$, any feasible solution Y of problem (24) satisfies $Ef(Y) > 0$. Thus we only need to show that there exists a sequence $\{Y_n\}$ of feasible solutions for problem (24) with $\lim_{n\to+\infty} Ef(Y_n) = 0$. Indeed, for any $n > 0$, define $Y_n = -a_n \mathbf{1}_{\xi<n} + b_n \mathbf{1}_{\xi\geq n}$, where a_n, b_n are defined by

$$a_n = \frac{y_0}{[E(\xi|\xi \geq n) - E(\xi|\xi < n)]P(\xi < n)}, \quad b_n = \frac{y_0}{[E(\xi|\xi \geq n) - E(\xi|\xi < n)]P(\xi \geq n)}.$$

Then it is easy to verify that $a_n > 0, b_n > 0$, $\lim_{n\to+\infty} a_n = 0$, and $EY_n = 0, E[\xi Y_n] = y_0$. Thus, $\{Y_n\}$ are feasible solutions for (24), and

$$0 \leq Ef(Y_n) = E[f(-a_n)\mathbf{1}_{\xi<n}] \leq f(-a_n).$$

Since f is left continuous at 0, we conclude $\lim_{n\to+\infty} Ef(Y_n) = 0$. □

Remark 5.1 In the proof of Proposition 5.2, only the following properties of $f(\cdot)$ was utilized: $f(x) > 0$ if $x < 0$, $f(x) = 0$ if $x \geq 0$, and $\lim_{x\uparrow 0} f(x) = 0$. The strictly decreasing property of $f(\cdot)$ was not necessary.

Combining Proposition 5.1 and Proposition 5.2 yields the conclusion of Theorem 5.1.

Now we turn to the continuous-time portfolio selection problem (3) where f satisfies Assumption 5.1. The way the function f is given suggests that only the downside deviation of the terminal wealth from its mean is penalized; hence the model constitutes a (very general) mean–downside-risk portfolio selection problem.

Let $\rho(\cdot)$ be the price kernel defined by (1). We impose the following assumption:

Assumption 5.2 For any $0 \leq M_1 < M_2 \leq +\infty$, $P\{\rho(T) \in (M_1, M_2)\} > 0$ and $P\{\rho(T) = M_1\} = P\{\rho(T) = M_2\} = 0$.

This assumption is satisfied when, say, $r(\cdot)$ and $\theta(\cdot)$ are deterministic and $\int_0^T |\theta(t)|^2 dt > 0$.

The corresponding static optimization problem (5), after taking a transformation $Y := X - z$, is exactly the problem (24) with $q = 2$. Hence, by virtue of Theorems 5.1 and 2.1, we conclude the following result.

Theorem 5.2 *Under Assumptions 5.1 and 5.2, Problem (3) admits no optimal solution for any $z \neq \frac{x_0}{E\rho(T)}$. On the other hand, if $z = \frac{x_0}{E\rho(T)}$, then (3) has an optimal portfolio which is the risk-free portfolio.*

Theorem 5.2 claims that a mean–downside-risk portfolio selection problem does not generally attain an optimal solution in a complete continuous-time financial market. It is a very general result; however it does not completely cover Theorem 4.1 since the latter does not require Assumption 5.2.

6 The General Mean–Risk Model

We have shown in the last section that in the continuous-time setting, the mean–downside-risk model does not achieve optimality in general. In other words, problem (24) does not admit an optimal solution if the function f has the property that it vanishes on the nonnegative half real axis. Notice that for this negative result to hold the function f is *not* required to be convex. In this section, we will study model (24) where a general *convex* function f is used to measure the risk. We will give a complete solution to the problem in terms of telling exactly when the problem possesses an optimal solution and, when it does, giving the explicit solution.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and ξ a strictly positive real random variable on it satisfying (23). Consider a convex (hence continuous) function $f : \mathbf{R} \to \mathbf{R}$, not necessarily differentiable. For any $x \in \mathbf{R}$, its subdifferential $\partial f(x)$ in the sense of convex analysis (see, e.g., [21]), is defined as the set

$$\partial f(x) := \{x^* \in \mathbf{R} : \ f(y) - f(x) \geq x^*(y - x), \ \ \forall y \in \mathbf{R}\} \equiv [f'_-(x), f'_+(x)], \qquad (25)$$

where $f'_-(x)$ and $f'_+(x)$ are the left and right derivatives of f at x respectively. The set $\partial f(x)$ is a non-empty bounded set for every $x \in \mathbf{R}$ ([21, Theorem 23.4]). Moreover, the convexity of f implies that the subdifferential is non-decreasing in the sense that

$$f'_+(x_1) \leq f'_-(x_2), \ \ \forall x_1 \leq x_2. \qquad (26)$$

We call a convex function f to be strictly convex at $x_0 \in \mathbf{R}$ if

$$f(x_0) < \kappa f(x_1) + (1-\kappa) f(x_2)$$

for any $x_1 < x_0 < x_2$ and $\kappa \in (0,1)$ with $\kappa x_1 + (1-\kappa)x_2 = x_0$. A convex function is called strictly convex if it is strictly convex at every $x \in \mathbf{R}$. Some properties of a convex function that are useful in this paper are presented in an appendix.

Throughout this section we assume that f satisfies

Assumption 6.1 f is convex, and strictly convex at 0.

Note that the strict convexity at 0 is a very mild condition, which is valid in many meaningful cases (see the examples at the end of this section).

In view of Jensen's inequality, one has $Ef(Y) \geq f(EY) = f(0)$ for any feasible solution Y of (24). Hence problem (24) has a finite infimum if its feasible region is non-empty. Also we see that if $y_0 = 0$, then (24) has (trivially) an optimal solution $Y^* = 0$ a.s.. On the other hand, due to the convexity of f, we can apply [1, Proposition 4.1] to conclude that (24) admits an optimal solution Y^* if and only if Y^* is feasible for (24) and there exists a pair (λ, μ) such that Y^* solves the following problem

$$\min_{Y \in L^q(\mathcal{F}, \mathbf{R})} E[f(Y) - (\lambda - \mu\xi)Y]. \tag{27}$$

Lemma 6.1 *$Y^* \in L^q(\mathcal{F}, \mathbf{R})$ is an optimal solution to (27) if and only if*

$$f(Y^*) - (\lambda - \mu\xi)Y^* = \min_{y \in \mathbf{R}} [f(y) - (\lambda - \mu\xi)y], \quad \text{a.s..}$$

Proof: The "if" part is obvious. We now prove the "only if" part. Suppose $Y^* \in L^q(\mathcal{F}, \mathbf{R})$ is an optimal solution to (27). Define $h(y) := f(y) - (\lambda - \mu\xi)y, \quad y \in \mathbf{R}$, and $c := \inf_{y \in \mathbf{R}} h(y)$. Let $Z := \cup_{n \in \mathbf{N}} \{(z_1, \cdots, z_n) : z_i \in \mathbf{Q}\}$, where $\mathbf{Q}$ is the set of rational numbers, and $\bar{h}(z) := \inf_{1 \leq i \leq n} h(z_i, \omega)$ for $z = (z_1, \cdots, z_n) \in Z$. Since $h(y)$ is continuous in y, we have $c = \inf_{z \in Z} \bar{h}(z)$. Now, if Y^* is not almost surely a minimum point of $h(\cdot)$, namely, $P\{c < h(Y^*)\} > 0$, then there exists $z = (z_1, \cdots, z_n) \in Z$ such that $P\{\bar{h}(z) < h(Y^*)\} > 0$. It is easy to see then that there is $y^* \in \mathbf{Q}$ with $P\{h(y^*) < h(Y^*)\} > 0$. Put $A := \{\omega : h(y^*, \omega) < h(Y^*(\omega), \omega)\}$, and $Y' := y^* \mathbf{1}_A + Y^* \mathbf{1}_{A^c}$. Then $Y' \in L^q(\mathcal{F}, \mathbf{R})$, and $Eh(Y') < Eh(Y^*)$, leading to a contradiction. $\square$

Define a set-valued function G: $\cup_{x \in \mathbf{R}} \partial f(x) \to 2^{\mathbf{R}}$

$$G(y) := \{x \in \mathbf{R} : y \in \partial f(x)\}, \quad \forall y \in \cup_{x \in \mathbf{R}} \partial f(x),$$

and define g: $\cup_{x\in\mathbf{R}}\partial f(x)\to\mathbf{R}$ as the "inverse function" of ∂f as follows

$$g(y):=\mathrm{argmin}_{x\in G(y)}|x|,\quad \forall y\in\cup_{x\in\mathbf{R}}\partial f(x).$$

In Appendix we prove that g is a well-defined function (on its domain), and the set of y's where $G(y)$ is not a singleton is countable. In other words, denoting

$$\Gamma:=\{y\in\cup_{x\in\mathbf{R}}\partial f(x):G(y)\text{ is a singleton}\},$$

then the set $[\cup_{x\in\mathbf{R}}\partial f(x)]\setminus\Gamma$ is countable. Moreover, $g(\cdot)$ is increasing on $\cup_{x\in\mathbf{R}}\partial f(x)$ and continuous at points in Γ (Proposition A.5).

The objective of this subsection is to identify the ranges of y_0 where problem (24) admits optimal solution(s) and, when it does, to obtain an optimal solution in various situations of f. It follows from Lemma 6.1 that problem (24) admits an optimal solution if and only if there exists a pair (λ,μ) satisfying the following condition:

$$\begin{aligned}&\lambda-\mu\xi\in\cup_{x\in\mathbf{R}}\partial f(x)\text{ a.s., and there is }Y^*\in L^q(\mathcal{F},\mathbf{R})\text{ with }Y^*\in G(\lambda-\mu\xi),\text{ a.s., }EY^*=0,\\&\quad\text{and }E[\xi Y^*]=y_0.\end{aligned}\tag{28}$$

Moreover, when there exists a pair (λ,μ) satisfying the above condition, Y^* is one of the optimal solutions for (24). Remark that if $\mu\neq 0$, then, since the set $[\cup_{x\in\mathbf{R}}\partial f(x)]\setminus\Gamma$ is countable and the distribution of ξ has no atom, we have $P\{\lambda-\mu\xi\in\Gamma\}=1$. In this case $G(\lambda-\mu\xi)$ is almost surely a singleton; hence problem (24) has a unique optimal solution $Y^*=g(\lambda-\mu\xi)$.

We will solve problem (24) in each of the following four (mutually exclusive) cases:

Case 1: The set $\cup_{x\in\mathbf{R}}\partial f(x)$ is upper bounded but not lower bounded;

Case 2: The set $\cup_{x\in\mathbf{R}}\partial f(x)$ is lower bounded but not upper bounded;

Case 3: $\cup_{x\in\mathbf{R}}\partial f(x)=\mathbf{R}$;

Case 4: The set $\cup_{x\in\mathbf{R}}\partial f(x)$ is both upper and lower bounded.

Let us first focus on Case 1. In this case, it follows from Proposition A.1 that $\cup_{x\in\mathbf{R}}\partial f(x)$ is either a closed interval $(-\infty,\bar{k}]$ or an open one $(-\infty,\bar{k})$ where

$$\bar{k}:=\lim_{x\to+\infty}f'_+(x)\in\mathbf{R}.\tag{29}$$

It is also clear that $\lim_{y\to-\infty} g(y) = -\infty$. Moreover, in this case one only needs to consider $\mu \geq 0$ in searching for (λ, μ) satisfying condition (28), for otherwise $\cup_{x\in\mathbf{R}}\partial f(x)$ would be unbounded from above.

The following technical lemma plays an important role in the subsequent analysis.

Lemma 6.2 *In Case 1, assume that there are $\lambda_0 > f'_-(0), \mu_0 > 0$ such that $g(\lambda_0 - \mu_0\xi) \in L^q(\mathcal{F}, \mathbf{R})$. Then for any $\mu_1 \in (0, \mu_0), \lambda_1 \in (f'_-(0), \lambda_0)$, there exists $\gamma \in L^q(\mathcal{F}, \mathbf{R}^+)$, such that $|g(\lambda - \mu\xi)| \leq \gamma$ for any $\mu \in [0, \mu_1]$ and $\lambda \in [f'_-(0), \lambda_1]$. If in addition $\xi g(\lambda_0 - \mu_0\xi) \in L^q(\mathcal{F}, \mathbf{R})$, then γ satisfies $\xi\gamma \in L^q(\mathcal{F}, \mathbf{R})$.*

Proof: Since $g(\cdot)$ is increasing (Proposition A.5), for any $\mu \in [0, \mu_1], \lambda \in [f'_-(0), \lambda_1]$, we have

$$g(f'_-(0) - \mu_1\xi) \leq g(\lambda - \mu\xi) \leq g(\lambda_1).$$

On the other hand, on the set $\{\omega : \xi(\omega) \leq \frac{\lambda_0 - f'_-(0)}{\mu_0 - \mu_1}\}$, we have

$$g(f'_-(0) - \mu_1\xi) \geq g\left(\frac{\mu_0 f'_-(0) - \lambda_0\mu_1}{\mu_0 - \mu_1}\right);$$

and on the set $\{\omega : \xi(\omega) > \frac{\lambda_0 - f'_-(0)}{\mu_0 - \mu_1}\}$ we have

$$g(f'_-(0) - \mu_1\xi) \geq g(\lambda_0 - \mu_0\xi).$$

Thus, if we put

$$\gamma := g(\lambda_1) + \left|g\left(\frac{\mu_0 f'_-(0) - \lambda_0\mu_1}{\mu_0 - \mu_1}\right)\right| + |g(\lambda_0 - \mu_0\xi)|,$$

then γ meets the requirement. □

Lemma 6.3 *In Case 1, for any given $\lambda \in (-\infty, \bar{k})$, $g_\lambda(\mu) := Eg(\lambda - \mu\xi)$ is strictly decreasing in $\mu \in \mathbf{R}^+$.*

Proof: Since $g(\cdot)$ is increasing, $g_\lambda(\cdot)$ is decreasing. Moreover, for any $\mu > 0$, $g_\lambda(\mu) < g_\lambda(0)$. Indeed, if $g_\lambda(\mu) = g_\lambda(0)$, then $Eg(\lambda - \mu\xi) = Eg(\lambda)$ leading to $g(\lambda - \mu\xi) = g(\lambda)$. This, in turn, implies that $\lambda - \mu\xi \in \partial f(g(\lambda))$ which contradicts to the boundedness of $\partial f(g(\lambda))$.

Next, for any $0 < \mu_1 < \mu_2$, if $g_\lambda(\mu_1) = g_\lambda(\mu_2)$, then $g(\lambda - \mu_1\xi) = g(\lambda - \mu_2\xi)$ a.s.. We are to show that in this case $g(\cdot)$ must be constant on $(-\infty, \lambda - 1]$. In fact, if $g(\cdot)$ is not constant on $(-\infty, \lambda - 1]$, then for any $\epsilon > 0$, there exists $y_1 \leq \lambda - 1$ such that $g(y_1) < g(y_1 + \epsilon)$. Take $\epsilon = (\mu_2 - \mu_1)/(2\mu_2)$. Then it is straightforward to verify that $\frac{\lambda - y_1}{\mu_2} < \frac{\lambda - (y_1 + \epsilon)}{\mu_1}$. Now, if $\xi \in [\frac{\lambda - y_1}{\mu_2}, \frac{\lambda - (y_1 + \epsilon)}{\mu_1}]$, then the monotonicity of $g(\cdot)$ yields $g(\lambda - \mu_1\xi) \geq$

$g(y_1+\epsilon)$ and $g(\lambda-\mu_2\xi)\le g(y_1)$. It then follows from the inequality $g(y_1)<g(y_1+\epsilon)$ that $P\{g(\lambda-\mu_2\xi)<g(\lambda-\mu_1\xi)\}\ge P\{\xi\in[\frac{\lambda-y_1}{\mu_2},\frac{\lambda-(y_1+\epsilon)}{\mu_1}]\}>0$, which contradicts the assumption that $g(\lambda-\mu_1\xi)=g(\lambda-\mu_2\xi)$ a.s..

We have shown that $g(\cdot)$ is constant on $(-\infty,\lambda-1]$; nevertheless this is impossible because $\lim_{y\to-\infty}g(y)=-\infty$. The proof is complete. □

Theorem 6.1 *In Case 1, assume that there are $\lambda_0>f'_-(0),\mu_0>0$ such that $g(\lambda_0-\mu_0\xi)\in L^q(\mathcal{F},\mathbf{R})$ and $Eg(\lambda_0-\mu_0\xi)=0$. Then for any $\lambda\in[f'_-(0),\lambda_0]$, there exists a unique $0\le\mu(\lambda)\le\mu_0$ such that $g(\lambda-\mu(\lambda)\xi)\in L^q(\mathcal{F},\mathbf{R})$ and $Eg(\lambda-\mu(\lambda)\xi)=0$. Moreover, $\mu(\lambda)=0$ for $\lambda\in[f'_-(0),f'_+(0)]\equiv\partial f(0)$, and $\mu(\cdot)$ is continuous and strictly increasing on $[f'_+(0),\lambda_0]$.*

Proof: For any fixed $\lambda\in(f'_-(0),\lambda_0)$, define $g_\lambda(\mu):=Eg(\lambda-\mu\xi)$ for $\mu\in[0,\mu_0)$. It follows from Lemma 6.2 that for any $\mu_1\in(0,\mu_0)$, the family of random variables $\{g(\lambda-\mu\xi):\mu\in[0,\mu_1]\}$ are uniformly integrable. Hence by the dominated convergence theorem $g_\lambda(\cdot)$ is continuous on $[0,\mu_0)$. On the other hand, $g(\lambda-\mu\xi)$ is decreasing when $\mu\uparrow\mu_0$, and when $\mu_0>\mu>\mu_0/2$, $g(\lambda-\mu\xi)\le g(\lambda-\xi\mu_0/2)\in L^q(\mathcal{F},\mathbf{R})$. Hence, the monotonic convergence theorem yields

$$\lim_{\mu\uparrow\mu_0}Eg(\lambda-\mu\xi)=E\lim_{\mu\uparrow\mu_0}g(\lambda-\mu\xi)=Eg(\lambda-\mu_0\xi).$$

Note that the above equality may take the value of $-\infty$. If $Eg(\lambda-\mu_0\xi)>-\infty$, then the strict monotonicity of g leads to $Eg(\lambda-\mu_0\xi)<Eg(\lambda_0-\mu_0\xi)=0$. Thus it always holds that $\lim_{\mu\uparrow\mu_0}g_\lambda(\mu)<0$. But $g_\lambda(0)\equiv Eg(\lambda)\ge Eg(f'_-(0))=0$; so it follows from the facts that $g_\lambda(\cdot)$ is *strictly* decreasing (Lemma 6.3) and continuous on $[0,\mu_0)$ that there exists a unique $\mu(\lambda)\in[0,\mu_0)$ with $g_\lambda(\mu(\lambda))\equiv Eg(\lambda-\mu(\lambda)\xi)=0$. Moreover, Lemma 6.2 ensures that $g(\lambda-\mu(\lambda)\xi)\in L^q(\mathcal{F},\mathbf{R})$.

To prove the second part of the theorem, first notice that $\lambda_0>f'_+(0)$. Indeed, if it is not true, then $\lambda_0\in\partial f(0)$ and hence $g(\lambda_0)=0$. However, appealing to Lemma 6.3 we have $Eg(\lambda_0-\mu_0\xi)>g(\lambda_0)=0$ which is a contradiction. Now, whenever $\lambda\in[f'_-(0),f'_+(0)]\equiv\partial f(0)$, we have $Eg(\lambda)=g(\lambda)=0$; thus the uniqueness of $\mu(\lambda)$ yields $\mu(\lambda)=0$. Next, consider $\lambda_0\ge\lambda_1>\lambda_2\ge f'_+(0)$. Since $\mu(\lambda)>0$ whenever $\lambda>f'_+(0)$, and $Eg(\lambda_2-\mu\xi)<Eg(\lambda_1-\mu\xi)$ whenever $\mu>0$, we have $g_{\lambda_2}(\mu(\lambda_1))\equiv Eg(\lambda_2-\mu(\lambda_1)\xi)<Eg(\lambda_1-\mu(\lambda_1)\xi)\equiv 0\equiv g_{\lambda_2}(\mu(\lambda_2))$. Since $g_{\lambda_2}(\cdot)$ is strictly decreasing, we conclude $\mu(\lambda_1)>\mu(\lambda_2)$, proving that $\mu(\cdot)$ is strictly increasing on $[f'_+(0),\lambda_0]$.

Next we show by contradiction the right continuity of $\mu(\cdot)$ on $[f'_+(0),\lambda_0)$. Assume that there exists $\lambda\in[0,\lambda_0)$, and $\epsilon>0$ such that for any $\lambda'>\lambda$, $\mu(\lambda')>\mu(\lambda)+\epsilon$. Without loss of generality, suppose $\mu(\lambda)+\epsilon<\mu(\lambda_0)$. Then

$$0=\lim_{\lambda'\downarrow\lambda} Eg(\lambda'-\mu(\lambda')\xi)\le \lim_{\lambda'\downarrow\lambda} Eg(\lambda'-(\mu(\lambda)+\epsilon)\xi).$$

On the other hand, it follows from Lemma 6.2 that the family of random variables $\{g(\lambda'-(\mu(\lambda)+\epsilon)\xi):\lambda'\in[\lambda,\lambda_1]\}$, for any fixed $\lambda_1\in(\lambda,\lambda_0)$, is uniformly integrable. Therefore we have

$$\lim_{\lambda'\downarrow\lambda} Eg(\lambda'-(\mu(\lambda)+\epsilon)\xi)=Eg(\lambda-(\mu(\lambda)+\epsilon)\xi)<Eg(\lambda-\mu(\lambda)\xi)=0,$$

leading to a contradiction.

It finally remains to prove the left continuity of $\mu(\cdot)$ on $(f'_+(0),\lambda_0]$. Assume that there exists $\lambda\in(f'_+(0),\lambda_0]$ and $\epsilon>0$ such that for any $\lambda'<\lambda$, $\mu(\lambda')<\mu(\lambda)-\epsilon$. Without loss of generality, suppose $\mu(\lambda)-\epsilon>0$. Then

$$0=\lim_{\lambda'\uparrow\lambda} Eg(\lambda'-\mu(\lambda')\xi)\ge \lim_{\lambda'\uparrow\lambda} Eg(\lambda'-(\mu(\lambda)-\epsilon)\xi).$$

Obviously, $g(\lambda'-(\mu(\lambda)-\epsilon)\xi)$ is increasing when $\lambda'\uparrow\lambda$, and when $\lambda'>\lambda/2$, $g(\lambda'-(\mu(\lambda)-\epsilon)\xi)\ge g(\lambda/2-(\mu(\lambda)-\epsilon)\xi)\in L^q(\mathcal{F},\mathbf{R})$ by virtue of Lemma 6.2. Hence by the monotonic convergence theorem,

$$\lim_{\lambda'\uparrow\lambda} Eg(\lambda'-(\mu(\lambda)-\epsilon)\xi)=Eg(\lambda-(\mu(\lambda)-\epsilon)\xi)>Eg(\lambda-\mu(\lambda)\xi)=0.$$

Again, this is a contradiction. □

Define

$$\begin{cases}\bar\Lambda:=\{\lambda\in[f'_-(0),\bar k]:\text{There exists }\mu=\mu(\lambda)\text{ so that }g(\lambda-\mu(\lambda)\xi)\in L^q(\mathcal{F},\mathbf{R}),\\ \qquad\qquad Eg(\lambda-\mu(\lambda)\xi)=0,\ \ \xi g(\lambda-\mu(\lambda)\xi)\in L^1(\mathcal{F},\mathbf{R})\},\\ \bar\lambda:=\sup_{\lambda\in\bar\Lambda}\lambda,\\ \tilde g(\lambda):=E[\xi g(\lambda-\mu(\lambda)\xi)],\ \ \lambda\in[f'_-(0),\bar\lambda).\end{cases}\tag{30}$$

Notice that $\bar\Lambda\neq\emptyset$, since $\partial f(0)\subseteq\bar\Lambda$. As a result $f'_+(0)\le\bar\lambda\le\bar k$. Also, by virtue of Lemma 6.2 and Theorem 6.1, $[f'_-(0),\bar\lambda)\subseteq\bar\Lambda$.

Theorem 6.2 *In Case 1, $\tilde g(\lambda)=0$ for $\lambda\in[f'_-(0),f'_+(0)]\equiv\partial f(0)$, and $\tilde g(\cdot)$ is continuous and strictly decreasing on $[f'_+(0),\bar\lambda)$. Moreover, if $\bar\lambda\in\bar\Lambda$ and $\bar\lambda<\bar k$, then $\tilde g(\cdot)$ is also left continuous at $\bar\lambda$.*

Proof: Theorem 6.1 provides that $\mu(\lambda) = 0$ for any $\lambda \in \partial f(0)$; hence $\tilde{g}(\lambda) = E[\xi g(\lambda)] = 0$. Furthermore, for $\bar{\lambda} > \lambda_1 > \lambda_2 \geq f'_+(0)$ (if $\bar{\lambda} \in \bar{\Lambda}$, then λ_1 may take the value of $\bar{\lambda}$), it follows from Theorem 6.1 that $\mu(\lambda_1) > \mu(\lambda_2) \geq 0$. Denote $\xi_0 := \frac{\lambda_1 - \lambda_2}{\mu(\lambda_1) - \mu(\lambda_2)} > 0$. If $\xi \geq \xi_0$, then $\lambda_1 - \mu(\lambda_1)\xi \leq \lambda_2 - \mu(\lambda_2)\xi$ resulting in $g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi) \leq 0$. Similarly, if $\xi < \xi_0$, then $g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi) \geq 0$. As a consequence,

$$
\begin{aligned}
& \tilde{g}(\lambda_1) - \tilde{g}(\lambda_2) \\
= \; & E\left\{\xi[g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi)]\right\} \\
= \; & E\left\{\xi[g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi)]\mathbf{1}_{\xi \geq \xi_0}\right\} + E\left\{\xi[g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi)]\mathbf{1}_{\xi < \xi_0}\right\} \\
\leq \; & \xi_0 E\left\{[g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi)]\mathbf{1}_{\xi \geq \xi_0}\right\} + \xi_0 E\left\{[g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi)]\mathbf{1}_{\xi < \xi_0}\right\} \\
= \; & \xi_0 E[g(\lambda_1 - \mu(\lambda_1)\xi) - g(\lambda_2 - \mu(\lambda_2)\xi)] \\
= \; & 0.
\end{aligned}
$$

Moreover, if $\tilde{g}(\lambda_1) - \tilde{g}(\lambda_2) = 0$, then $g(\lambda_1 - \mu(\lambda_1)\xi) = g(\lambda_2 - \mu(\lambda_2)\xi)$ a.s.. By a reasoning similar to that in the proof of Lemma 6.3, we can prove that this is impossible. So $\tilde{g}(\cdot)$ is strictly decreasing on $[f'_+(0), \bar{\lambda})$.

Fix $\lambda \in [f'_+(0), \bar{\lambda})$. There is $\lambda_0 \in \bar{\Lambda}$ with $\lambda < \lambda_0$. By Lemma 6.2, the family $\{\xi g(\lambda' - \mu(\lambda')\xi) : \lambda' \in [0, (\lambda + \lambda_0)/2]\}$ is uniformly integrable. Thus by the continuity of $\mu(\cdot)$, we have

$$
\begin{aligned}
\lim_{\lambda' \to \lambda} \tilde{g}(\lambda') &= \lim_{\lambda' \to \lambda} E[\xi g(\lambda' - \mu(\lambda')\xi)] \\
&= E[\lim_{\lambda' \to \lambda} \xi g(\lambda' - \mu(\lambda')\xi)] \\
&= E[\xi g(\lambda - \mu(\lambda)\xi)] \\
&= \tilde{g}(\lambda).
\end{aligned}
$$

This proves the continuity of $\tilde{g}(\cdot)$ on $[f'_+(0), \bar{\lambda})$.

Finally, in the case when $\bar{\lambda} \in \bar{\Lambda}$ and $\bar{\lambda} < \bar{k}$, one has

$$
\tilde{g}(\bar{\lambda}) \leq \lim_{\lambda' \uparrow \bar{\lambda}} \tilde{g}(\lambda') = \lim_{\lambda' \uparrow \bar{\lambda}} E[\xi g(\lambda' - \mu(\lambda')\xi)] \leq \lim_{\lambda' \uparrow \bar{\lambda}} E[\xi g(\bar{\lambda} - \mu(\lambda')\xi)].
$$

On the other hand, since $g(\cdot)$ is increasing, we have $|g(\bar{\lambda} - \mu(\lambda')\xi)| \leq |g(\bar{\lambda})| + |g(\bar{\lambda} - \mu(\bar{\lambda})\xi)|$. Thus the dominated convergence theorem yields

$$
\lim_{\lambda' \uparrow \bar{\lambda}} E[g(\bar{\lambda} - \mu(\lambda')\xi)\xi] = E[g(\bar{\lambda} - \mu(\bar{\lambda})\xi)\xi] = \tilde{g}(\bar{\lambda}).
$$

Therefore, $\tilde{g}(\cdot)$ is left continuous at $\bar{\lambda}$. □

The following result gives the complete solution to problem (24) for Case 1.

Theorem 6.3 *Consider Case 1.*

(i) *If $\bar{\lambda} \notin \bar{\Lambda}$, then (24) admits an optimal solution if and only if $y_0 \in (\underline{y}, 0]$, where $\underline{y} = \lim_{\lambda \uparrow \bar{\lambda}} \tilde{g}(\lambda)$. If $\bar{\lambda} \in \bar{\Lambda}$, then (24) admits an optimal solution if and only if $y_0 \in \{\tilde{g}(\bar{\lambda})\} \cup (\underline{y}, 0]$. If in addition $\bar{\lambda} < \bar{k}$, then $\tilde{g}(\bar{\lambda}) = \underline{y}$.*

(ii) *When $y_0 = 0$, $Y^* := 0$ is the unique optimal solution to (24).*

(iii) *When $y_0 < 0$ and the existence of optimal solution is assured, $Y^* := g(\lambda - \mu(\lambda)\xi)$ is the unique optimal solution to (24), where λ is the unique solution to $\tilde{g}(\lambda) = y_0$.*

Proof: (i) The "if" part follows immediately from Theorem 6.2. To prove the "only if" part, suppose that (24) admits an optimal solution Y^*, then there exists a pair (λ, μ) satisfying condition (28). If $\lambda < f'_-(0)$, then $\mu = 0$ (for otherwise $Eg(\lambda - \mu\xi) < Eg(\lambda) \leq g(f'_-(0)) = 0$). Hence it follows from (28) that $EY^* = 0$ and $Y^* \in G(\lambda)$, a.s. or $\lambda \in \partial f(Y^*)$, a.s.. If $P(Y^* = 0) < 1$, then $P(Y^* > 0) > 0, P(Y^* < 0) > 0$. Therefore $\lambda \in [\cup_{x>0} \partial f(x)] \cap [\cup_{x<0} \partial f(x)]$, which is impossible by Proposition A.2 and the fact that f is strictly convex at 0. Thus $P(Y^* = 0) = 1$ and, consequently, $y_0 = E[\xi Y^*] = 0$. On the other hand, if $\lambda \geq f'_-(0)$, then the conclusion follows from Theorem 6.2.

(ii) If $y_0 = 0$, it follows from Jensen's inequality that, for any feasible solution Y of (24), $Ef(Y) \geq f(EY) = f(0) \equiv Ef(0)$. Hence $Y^* := 0$ is an optimal solution. To prove that Y^* is the only solution, let Y be any feasible solution of (24) with $P(Y \neq 0) > 0$. Since f is strictly convex at 0, there exists an affine function $g(x) = ax + b$ so that $f(0) = g(0)$ and $f(x) > g(x) \ \ \forall x \neq 0$. Therefore $P(f(Y) > g(Y)) > 0$, resulting in $Ef(Y) > Eg(Y) = g(EY) = g(0) = f(0) = Ef(0)$. This shows that Y is not optimal.

(iii) This is evident from Theorem 6.2. □

Note that the "if" part of Theorem 6.3-(i) does not require the strict convexity of f at 0. However, this assumption can*not* be dropped for the "only if" part; see the following example.

Example 6.1 Take $f(x) = (x^2 - 1)\mathbf{1}_{x<-1}$, which is *not* strictly convex at 0. It is easy to see that $\cup_{x \in \mathbf{R}} \partial f(x) = (-\infty, 0]$. Pick $a \in \mathbf{R}$ such that $P(\xi > a) > \frac{1}{2} > P(\xi \leq a) > 0$, and take $Y^* := \frac{P(\xi \leq a)}{P(\xi > a)} \mathbf{1}_{\xi > a} - \mathbf{1}_{\xi \leq a}$. Then, $EY^* = 0$, and $y_0^* := E[\xi Y^*] = P(\xi \leq a)[E(\xi | \xi > a) - E(\xi | \xi \leq a)] > 0$. On the other hand, $Y^* \geq -1$ a.s., hence $Ef(Y^*) = 0$. This shows that problem (24) does admit an optimal solution Y^* even though $y_0 = y_0^* > 0$.

We have now completed the study on Case 1. As for Case 2, it can be turned into Case 1 by considering $\tilde{f}(x) = f(-x)$. Hence we only state the result.

Set

$$\underline{k} := \lim_{x \to -\infty} f'_-(x) \in \mathbf{R}, \tag{31}$$

and define

$$\begin{cases} \underline{\Lambda} := \{\lambda \in [\underline{k}, f'_+(0)] : \text{There exists } \mu = \mu(\lambda) \text{ so that } g(\lambda - \mu(\lambda)\xi) \in L^q(\mathcal{F}, \mathbf{R}), \\ \qquad\qquad Eg(\lambda - \mu(\lambda)\xi) = 0, \ \ \xi g(\lambda - \mu(\lambda)\xi) \in L^1(\mathcal{F}, \mathbf{R})\}, \\ \underline{\lambda} := \inf_{\lambda \in \underline{\Lambda}} \lambda, \\ \tilde{g}(\lambda) := E[\xi g(\lambda - \mu(\lambda)\xi)], \ \ \lambda \in (\underline{\lambda}, f'_+(0)]. \end{cases} \tag{32}$$

Theorem 6.4 *Consider Case 2.*

(i) *If $\underline{\lambda} \notin \underline{\Lambda}$, then (24) admits an optimal solution if and only if $y_0 \in [0, \bar{y})$, where $\bar{y} = \lim_{\lambda \downarrow \underline{\lambda}} \tilde{g}(\lambda)$. If $\underline{\lambda} \in \underline{\Lambda}$, then (24) admits an optimal solution if and only if $y_0 \in \{\tilde{g}(\lambda)\} \cup [0, \bar{y})$. If in addition $\underline{\lambda} > \underline{k}$, then $\tilde{g}(\bar{\lambda}) = \bar{y}$.*

(ii) *When $y_0 = 0$, $Y^* := 0$ is the unique optimal solution to (24).*

(iii) *When $y_0 > 0$ and the existence of optimal solution is assured, $Y^* := g(\lambda - \mu(\lambda)\xi)$ is the unique optimal solution to (24), where λ is the unique solution to $\tilde{g}(\lambda) = y_0$.*

Let us now turn to Case 3. It can be dealt with similarly combining the analyses for the previous two cases. Define

$$\begin{cases} \Lambda := \{\lambda \in \mathbf{R} : \text{There exists } \mu = \mu(\lambda) \text{ so that } g(\lambda - \mu(\lambda)\xi) \in L^q(\mathcal{F}, \mathbf{R}), \\ \qquad\qquad Eg(\lambda - \mu(\lambda)\xi) = 0, \ \ \xi g(\lambda - \mu(\lambda)\xi) \in L^1(\mathcal{F}, \mathbf{R})\}, \\ \bar{\lambda} := \sup_{\lambda \in \Lambda} \lambda, \ \ \underline{\lambda} := \inf_{\lambda \in \Lambda} \lambda, \\ \tilde{g}(\lambda) := E[\xi g(\lambda - \mu(\lambda)\xi)], \ \ \lambda \in (\underline{\lambda}, \bar{\lambda}), \\ \bar{y} := \lim_{\lambda \downarrow \underline{\lambda}} \tilde{g}(\lambda), \ \ \ \underline{y} := \lim_{\lambda \uparrow \bar{\lambda}} \tilde{g}(\lambda). \end{cases} \tag{33}$$

Theorem 6.5 *Consider Case 3. Problem (24) admits an optimal solution if and only if* $y_0 \in A \cup B$*, where*

$$A = \begin{cases} [\underline{y}, 0], & \text{if } \bar{\lambda} \in \Lambda \\ (\underline{y}, 0], & \text{if } \bar{\lambda} \notin \Lambda \end{cases}, \qquad B = \begin{cases} [0, \overline{y}], & \text{if } \underline{\lambda} \in \Lambda \\ [0, \overline{y}), & \text{if } \underline{\lambda} \notin \Lambda. \end{cases}$$

Moreover, when $y_0 = 0$, $Y^* := 0$ *is the unique optimal solution to (24), and when* $y_0 \neq 0$ *and the existence of optimal solution is assured,* $Y^* := g(\lambda - \mu(\lambda)\xi)$ *is the unique optimal solution to (24), where* λ *is the unique solution to* $\tilde{g}(\lambda) = y_0$.

The final case, Case 4, only has a trivial solution, as shown in the following theorem.

Theorem 6.6 *Consider Case 4. Problem (24) admits an optimal solution if and only if* $y_0 = 0$*, in which case the unique optimal solution is* $Y^* = 0$.

Proof: Suppose that Y^* is optimal to (24). Then there exists (λ, μ) so that $\lambda - \mu\xi \in \partial f(Y^*)$, a.s.. It follows from the uniform boundedness of $\partial f(x)$ that $\mu = 0$. Employing the same argument as in the proof of Theorem 6.3-(i) we conclude that $Y^* = 0$, a.s.. □

Once the static optimization problem (24) is completely solved, as before we can then immediately obtain the solution for the continuous-time portfolio selection problem (3) by applying Theorem 2.1. We omit the detailed statement of the results here; instead we give several examples to demonstrate the results.

Example 6.2 Let $f(x) = \alpha x_+^2 + \beta x_-^2$ with $\alpha, \beta > 0$. This corresponds to the weighted mean–variance model that has been studied in Section 3. f is strictly convex, $\cup_{x \in \mathbf{R}} \partial f(x) = \mathbf{R}$, and $g(y) = \frac{1}{2\alpha} y_+ - \frac{1}{2\beta} y_-$. For any $\lambda > 0$, it is straightforward to see that the equation $Eg(\lambda - \mu\rho) = 0$ has a unique solution $\mu(\lambda) = \lambda/\zeta$ where $\zeta > 0$ uniquely solves (11). Hence $\bar{\lambda} = +\infty$, and

$$\tilde{g}(\lambda) = \frac{E[\rho(\lambda - \mu(\lambda)\rho)_+]}{2\alpha} - \frac{E[\rho(\lambda - \mu(\lambda)\rho)_-]}{2\beta} = \lambda \tilde{g}(1).$$

As a result, $\lim_{\lambda \to +\infty} \tilde{g}(\lambda) = -\infty$ (recall that $\tilde{g}(1) < \tilde{g}(0) = 0$). Similarly, we can prove that $\underline{\lambda} = -\infty$ and $\lim_{\lambda \to -\infty} \tilde{g}(\lambda) = +\infty$. We can then apply Theorem 6.5 to conclude that the weighted mean-variance model admits a unique optimal solution for any $z \in \mathbf{R}$. Finally, the optimal portfolio obtained in Section 3 can be easily recovered. (It should be noted, however, the result in Section 3 cannot be superseded as Assumption 5.2 is not imposed there.)

Example 6.3 Let $f(x) = x_-^2$. This is the mean–semivariance model investigated in Section 4. Clearly, f is convex, strictly convex at 0, and $\cup_{x\in\mathbf{R}}\partial f(x) = (-\infty, 0]$. The inverse function $g(y) = \frac{1}{2}y, \quad y \leq 0$. It is easily seen that $\bar{\Lambda} = \{0\}$ and $\bar{\lambda} = 0 \in \bar{\Lambda}$. Now, $\tilde{g}(\lambda) = E[\rho g(\lambda - \mu\rho)] = \frac{1}{2}(E\rho - \frac{E\rho^2}{E\rho})\lambda$. Thus $\underline{y} = \lim_{\lambda\uparrow 0}\tilde{g}(\lambda) = 0$. It then follows from Theorem 6.3 that the mean–semivariance model admits an optimal solution if and only if $z = x_0/E\rho$. (Again, this does not recover Theorem 4.1 completely due to Assumption 5.2.)

Example 6.4 Let $f(x) = |x|$. The corresponding portfolio selection problem is called the mean–absolute-deviation model. Single-period mean–absolute-deviation model is studied in [12]. Now, f is strictly convex at 0, and $\cup_{x\in\mathbf{R}}\partial f(x) = [-1, 1]$. Thus in view of Theorem 6.6 the continuous-time mean–absolute-deviation model admits an optimal solution if and only if $z = x_0/E\rho$, in which case the optimal portfolio is simply the risk-free one.

Example 6.5 Let $f(x) = e^{-x}$. This function captures the situation where lager deviation of the terminal wealth from its mean is heavily penalized. Again, f is strictly convex, $\cup_{x\in\mathbf{R}}\partial f(x) = (-\infty, 0)$ (hence $\bar{k} = 0$), and $g(y) = -\ln(-y),\ y < 0$. Now, the equation $Eg(0-\mu\rho) = 0$ has a solution $\mu \equiv \mu(0) = e^{-E\ln\rho} > 0$, Moreover, $g(0-\mu(0)\rho) = \int_0^T[r(s) + \frac{|\theta(s)|^2}{2}]ds + \int_0^T\theta(s)dW(s) + E\ln\rho \in L^2(\mathcal{F}, \mathbf{R})$. It follows then from Theorem 6.1 that $\bar{\Lambda} = [-1, 0]$ and, consequently, $\bar{\lambda} = 0 = \bar{k}$. Furthermore, $\tilde{g}(0) = E[g(0 - \mu(0)\rho)\rho] = (E\rho)(E\ln\rho) - E(\rho\ln\rho)$. On the other other hand, when $-1 < \lambda \uparrow 0$, $g(\lambda - \mu(\lambda)\rho) = -\ln(-\lambda + \mu(\lambda)\rho) \geq -\ln(1 + \mu(0)\rho) \geq -\mu(0)\rho$, and $g(\lambda - \mu(\lambda)\rho) = -\ln(-\lambda + \mu(\lambda)\rho) \leq -\ln(\mu(\lambda)\rho) \leq -\ln(\mu(-1/2)) - \ln\rho$. Thus the dominated convergence theorem ensures that $\underline{y} \equiv \lim_{\lambda\uparrow 0}\tilde{g}(\lambda) = \tilde{g}(0)$. By Theorem 6.3, the mean–risk portfolio selection problem admits an optimal solution if and only if $x_0 - zE\rho \in [(E\rho)(E\ln\rho) - E(\rho\ln\rho), 0]$ or, equivalently, $z \in [\frac{x_0}{E\rho}, \frac{x_0-(E\rho)(E\ln\rho)+E(\rho\ln\rho)}{E\rho}]$. Finally, when the problem does admit an optimal solution, the optimal portfolio is the one replicating the claim $z - \ln(-\lambda + \mu\rho)$ where (λ, μ) is the unique solution pair to the following algebraic equation (which must admit a solution):

$$\begin{cases} E\ln(-\lambda + \mu\rho) = 0, \\ \\ E[\rho\ln(-\lambda + \mu\rho)] = zE\rho - x_0. \end{cases}$$

Example 6.6 Let $f(x) = (x-1)_-^2$. The corresponding portfolio selection model is a variant of the mean–semivariance model, except that the terminal wealth being less than its mean plus 1 is now considered as risk. In this case, f is not strictly convex everywhere; but it is indeed strictly convex at 0. It is easy to see that $\cup_{x\in\mathbf{R}}\partial f(x) = (-\infty, 0]$ (hence $\bar{k} = 0$), and $g(y) = y/2+1, \ y \leq 0$. Meanwhile the equation $Eg(0-\mu\rho)=0$ has a solution $\mu \equiv \mu(0) = 2/E\rho > 0$. By virtue of Theorem 6.1, $\bar{\Lambda} = [-2, 0]$ and, consequently, $\bar{\lambda} = 0 = \bar{k}$. Note that $g(0-\mu(0)\rho) = g(0-\mu\rho) = E\rho - E\rho^2/E\rho$, and $\underline{y} \equiv \lim_{\lambda \uparrow 0} \tilde{g}(\lambda) = \tilde{g}(0)$. By Theorem 6.3 the original portfolio selection problem admits an optimal solution if and only if $x_0 - zE\rho \in [E\rho - E\rho^2/E\rho, 0]$ or, equivalently, $z \in [\frac{x_0}{E\rho}, \frac{x_0}{E\rho} + \frac{E\rho^2}{(E\rho)^2} - 1]$. At last, when the problem does admit an optimal solution, the optimal portfolio is the one replicating the claim $z+1+\frac{\lambda-\mu\rho}{2}$ where (λ, μ) is the unique solution pair to the following *linear* algebraic equation:

$$\begin{cases} \lambda - \mu E\rho = -2, \\ \lambda E\rho - \mu E\rho^2 = 2x_0 - 2(1+z)E\rho. \end{cases}$$

Compared with Example 6.3 it is interesting to see that a shift of the mean makes the mean–semivariance model, which does not admit an optimal solution in any non-trivial case, possess non-trivial optimal solution.

7 Conclusion

In this paper we have first solved a weighted mean–variance portfolio selection model in a complete continuous-time financial market. Inspired by its result, we have proved that, other than a trivial case, the mean–semivariance problem in the same market is not well-posed in the sense that it does not have any optimal solution. This negative result has then been extended to a general mean–downside-risk mode. Furthermore, for the model with a general convex risk measure, delicate analysis has been carried out to obtain a complete solution. The results in this paper suggest that there are strikingly difference between the single-period and continuous-time markets.

There have been many researches on hedging and/or optimization problems; see [3, 4, 10, 7, 11] just to name a few. However, the constraint $Ex(T) = z$ is absent from these work (probably a sample-wise constraint such as $0 \leq x(T) \leq Y$ is present instead). We should emphasize that the constraint $Ex(T) = z$ is dictated by the very framework we are working within, i.e., a framework *à la* Markowitz, except that we went beyond Markowitz's original measure of risk – the variance – and considered risk measures determined by very general functions (as mentioned in introduction this mean–risk model has received little attention in the dynamic setting until very recently). On the other hand, it is this constraint that made our models different from those in the aforementioned papers as well as those utility based models, both in terms of economic interpretations and the mathematical techniques required to tackle them.

While the continuous-time portfolio selection models with the security price processes governed by diffusion processes are considered in this paper, our results readily extend to any semimartingale models, including the discrete-time case, as long as the completeness of the underlying market is assumed and some other technical assumptions are modified accordingly. On the other hand, the incomplete market case will be fundamentally different and more difficult to solve. In this case not every terminal contingent claim is replicable by admissible portfolios. In a recent paper [8], we have solved the mean–variance problem in an incomplete market by explicitly charactering the attainable terminal wealth set and solving the static optimization problem with the attainable set as an *additional* constraint. The general mean–risk problem in an incomplete market, however, is yet to be explored.

Appendix

A Some Properties of Convex Functions on R

In this appendix we present some properties of a convex function $f\colon \mathbf{R} \to \mathbf{R}$, which are useful in the main context. Let such a convex function f be fixed, and $\partial f(x)$ be its subdifferential at $x \in \mathbf{R}$.

Proposition A.1 *For any interval $A \subset \mathbf{R}$, $\cup_{x\in A}\partial f(x)$ is a convex set (and hence is an interval).*

Proof: Suppose $y_1 \in \partial f(x_1), y_2 \in \partial f(x_2)$ where $x_1, x_2 \in A$ with $x_1 < x_2$ and $y_1 < y_2$. It suffices to show that for any $y_0 \in (y_1, y_2)$, there is $x_0 \in [x_1, x_2]$ such that $y_0 \in \partial f(x_0)$.

It follows from the convexity that $x_1 \in \mathrm{argmin}_{x\in\mathbf{R}}\{f(x)-y_1x\}$ and $x_2 \in \mathrm{argmin}_{x\in\mathbf{R}}\{f(x)-y_2x\}$. On the other hand, the continuity of f ensures that there exists $x_0 \in [x_1, x_2]$ so that $f(x_0) - y_0x_0 = \min_{x\in[x_1,x_2]}\{f(x) - y_0x\}$. However, for any $x \le x_1$,

$$\begin{aligned} f(x) - y_0x &= f(x) - y_1x + (y_1 - y_0)x \\ &\ge f(x_1) - y_1x_1 + (y_1 - y_0)x \\ &= f(x_1) - y_0x_1 + (y_1 - y_0)(x - x_1) \\ &\ge f(x_1) - y_0x_1 \\ &\ge f(x_0) - y_0x_0. \end{aligned}$$

Similarly we can prove that $f(x) - y_0 x \geq f(x_0) - y_0 x_0$ for any $x \geq x_2$. Therefore $x_0 \in \text{argmin}_{x \in \mathbf{R}}\{f(x) - y_0 x\}$, which implies that $y_0 \in \partial f(x_0)$. $\square$

Proposition A.2 *If f is strictly convex at x_0, then*

$$(\cup_{x<x_0} \partial f(x)) \cap (\cup_{x>x_0} \partial f(x)) = \emptyset. \tag{34}$$

Proof: If the conclusion is not true, then there are $x_1 < x_0 < x_2$ so that $f'_-(x_2) \leq f'_+(x_1)$. Hence $f'_-(x_2) = f'_+(x_1)$ due to the non-decreasing property of the subdifferential of f. However, the convexity of f yields

$$f'_+(x_1) \leq f'_-(x_0) \leq f'_+(x_0) \leq f'_-(x_2).$$

Thus, all the above inequalities become equalities which, in turn, implies that f is not strictly convex at x_0. $\square$

Define a set-valued function G: $\cup_{x \in \mathbf{R}} \partial f(x) \to 2^{\mathbf{R}}$

$$G(y) := \{x \in \mathbf{R} : y \in \partial f(x)\}, \quad \forall y \in \cup_{x \in \mathbf{R}} \partial f(x).$$

If f is strictly convex, then $G(y)$ is a singleton for each y. In general, we have

Proposition A.3 *For any $y \in \cup_{x \in \mathbf{R}} \partial f(x)$, $G(y)$ is a closed interval in $\mathbf{R}$.*

Proof: First we prove that $G(y)$ is an interval. For any $x_1 \in G(y), x_2 \in G(y)$ with $x_1 \leq x_2$, and any $x \in (x_1, x_2)$, we have $f'_-(x) \leq f'_-(x_2) \leq y \leq f'_+(x_1) \leq f'_+(x)$. This implies $y \in \partial f(x)$, or $x \in G(y)$.

To show the closedness of $G(y)$, take $x_n \in G(y)$ with $x_n \to x \in \mathbf{R}$. Since $y \in \partial f(x_n)$, we have $f(x') - f(x_n) \geq y(x' - x_n)$ $\forall x' \in \mathbf{R}$. This yields $f(x') - f(x) \geq y(x' - x)$ $\forall x' \in \mathbf{R}$, implying that $y \in \partial f(x)$ or $x \in G(y)$. $\square$

Now, define the function g: $\cup_{x \in \mathbf{R}} \partial f(x) \to \mathbf{R}$ as

$$g(y) := \text{argmin}_{x \in G(y)} |x|, \quad \forall y \in \cup_{x \in \mathbf{R}} \partial f(x).$$

Thanks to Proposition A.3, g is well defined.

Proposition A.4 *The set $\{y \in \cup_{x \in \mathbf{R}} \partial f(x) : G(y)$ is not a singleton$\}$ is countable.*

Proof: Take any $y_1 < y_2$ such that $G(y_1)$ and $G(y_2)$ are not singletons. It follows from Proposition A.3 that both $\text{int}(G(y_1))$ and $\text{int}(G(y_2))$ are nonempty. Moreover, $\text{int}(G(y_1)) \cap \text{int}(G(y_2)) = \emptyset$. Indeed, if it is not true, then there exist $a < b$ such that $[a, b] \subset G(y_1) \cap G(y_2)$, leading to $f'_+(a) \geq y_2 > y_1 \geq f'_-(b)$ which is impossible. This proves the desired result. □

Denote $\Gamma := \{y \in \cup_{x\in\mathbf{R}} \partial f(x) : G(y) \text{ is a singleton}\}$.

Proposition A.5 *g is increasing on $\cup_{x\in\mathbf{R}}\partial f(x)$, and continuous at every $y \in \Gamma$.*

Proof: For any $y_1, y_2 \in \cup_{x\in\mathbf{R}}\partial f(x)$ with $y_1 < y_2$, if $x_1 := g(y_1) > g(y_2) =: x_2$, then $y_1 \geq f'_-(x_1) \geq f'_+(x_2) \geq y_2$, which is a contradiction. So $g(y_1) \leq g(y_2)$.

To prove the continuity at points in Γ, fix $y_0 \in \Gamma$ and let $x_0 := g(y_0)$. Since g is an increasing function, $\bar{x} := \lim_{y\downarrow y_0} g(y) \geq g(y_0) = x_0$. If $\bar{x} > x_0$, then for any $\epsilon > 0$ and $y > y_0$, one has $g(y) > \bar{x} - \epsilon$. Hence $y \geq f'_-(g(y)) \geq f'_+(\bar{x} - \epsilon)$, which implies

$$y_0 \geq f'_+(\bar{x} - \epsilon) \ \ \forall \epsilon > 0. \tag{35}$$

Now, for any $x \in (x_0, \bar{x})$ and $y \in \partial f(x)$, we have

$$y_0 \leq f'_+(x_0) \leq y \leq f'_+(x) \leq y_0$$

where the last inequality is due to (35). The above argument leads to $\cup_{x\in(x_0,\bar{x})}\partial f(x) = \{y_0\}$; so $G(y_0) \supseteq (x_0, \bar{x})$ is not a singleton, which contradicts the fact that $y_0 \in \Gamma$. This proves the right continuity of g. Similarly, one can show the left continuity of g. □

Corollary A.1 *If f is strictly convex, then g is increasing and continuous on $\cup_{x\in\mathbf{R}}\partial f(x)$.*

Proof: In view of Proposition A.5, it suffices to prove $\Gamma = \cup_{x\in\mathbf{R}}\partial f(x)$ or, equivalently, $G(y)$ is a singleton for any $y \in \cup_{x\in\mathbf{R}}\partial f(x)$.

Suppose $[x_1, x_2] \subset G(y)$, then $y \leq f'_+(x_1) \leq f'_-(x_2) \leq y$. Hence $f'_+(x_1) = f'_-(x_2) = y$ which implies that $\partial f(x) = \{y\}$ for all $x \in (x_1, x_2)$. Therefore $f(\cdot)$ is not strictly convex on (x_1, x_2). □

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