Mean–variance portfolio selection with random parameters

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Abstract

This paper concerns the continuous-time, mean–variance portfolio selection problem in a complete market with random interest rate, appreciation rates and volatility coefficients. The problem is tackled using the results of stochastic linear–quadratic (LQ) optimal control and backward stochastic differential equations (BSDEs), two theories that have been extensively studied and developed in recent years. Specifically, the mean–variance problem is formulated as a linearly constrained stochastic LQ control problem. Solvability of this LQ problem is reduced, in turn, to proving global solvability of a stochastic Riccati equation. The proof of existence and uniqueness of this Riccati equation, which is a fully nonlinear and singular BSDE with random coefficients, is interesting in its own right and relies heavily on the structural properties of the equation. Efficient investment strategies as well as the mean–variance efficient frontier are then analytically derived in terms of the solution of this equation. In particular, it is demonstrated that the efficient frontier in the mean–standard deviation diagram is still a straight line or, equivalently, risk-free investment is still possible, even when the interest rate is random. Finally, a version of the Mutual Fund Theorem is presented.

Key words– Dynamic mean–variance portfolio selection; stochastic linear–quadratic optimal control; backward stochastic differential equation; stochastic Riccati equation; efficient frontier.

1 Introduction

Mean–variance portfolio selection is concerned with the allocation of wealth amongst a basket of securities (consisting of one bond and several stocks) so as to achieve a satisfactory trade-off between the return of the investment and the associated risk. In this paper, we consider the dynamic mean–variance portfolio selection problem under the assumptions that the market is complete and security trading takes place in continuous time. An important feature of this problem is that the interest rate of the bond and the appreciation and volatility rates of the stocks are allowed to be random processes. This contrasts with the model that is usually adopted in which these quantities are taken as deterministic (and in many cases, time-invariant); see for example Duffie and Jackson (1990), Duffie and Richardson (1991) and Follmer and Sondermann (1986).

The mean–variance problem was pioneered by Markowitz in the single-period setting; see Markowitz (1952, 1959). In the seminal paper Markowitz (1952), the variance of the final wealth, Var $x(T)$, is used as a measure of the risk associated with a given portfolio and the problem of finding a satisfactory trade-off between risk and the expected return is posed as a quadratic programming problem; see also Merton (1972) where this single-period problem is solved analytically under the assumptions that the stock covariance matrix is positive definite and shorting is allowed.

On the other hand, work on the multi-period portfolio selection problem has taken, in general, a somewhat different, though related, tack to Markowitz's original formulation of the single-period problem. Rather than treating the $\operatorname{Var} x(T)$ and $Ex(T)$ of a portfolio as separate quantities and finding the relationship between them, as done in Markowitz (1952), Merton (1972), and Perold (1984) for the single-period problem, a single quantity, the expected utility of terminal wealth $EU(x(T))$, is considered instead. The conflicting 'profit-seeking yet risk-averse' nature of the investor is captured by the utility function U, which is commonly a power, log, exponential, or quadratic form. It should be noted that mean–variance analysis and expected utility approach are two different tools in dealing with portfolio selection, accounting for different degrees of risk-aversion. For example, the former enjoys a better performance when the market is less volatile, while the latter performs better when the outcome of the stock prices occurs at the tails of the distribution. (For this and a more detailed comparison between the two, see Zhao and Ziemba (2000)). As a consequence, optimal portfolios determined by utility functions are usually not mean–variance efficient (as understood in the Markowitz framework). One exception is the case of the quadratic utility function; see Duffie and Richardson (1991) where this relationship is shown in the setting of the related mean–variance hedging problem.

In a recent paper by Li and Ng (1999), the discrete-time multi-period mean–variance problem is studied in the framework of multi-objective optimization, where $\mathrm{Var}\, x(T)$ and $Ex(T)$ are viewed as competing objectives. They are combined to give the following single-objective cost: For every $\mu > 0$ (representing the weights imposed by the investor on the two objectives),

$$J(u(\cdot)) := -Ex(T) + \mu \mathrm{Var}\, x(T). \tag{1}$$

One feature of this paper is an embedding technique, introduced to deal with (1). This is required since dynamic programming can not be used when the cost functional contains the term $\mathrm{Var}\, x(T)$. By the above embedding, dynamic programming can be applied and the mean–variance efficient frontier is explicitly obtained.

In the paper by Zhou and Li (2000), the continuous-time mean–variance problem (with deterministic coefficients) is formulated, for the first time, as a stochastic linear–quadratic (LQ) optimal control problem. The solution of this problem is obtained using the embedding technique introduced in Li and Ng (1999) and results from stochastic LQ control. It is important to recognize, however, that the solution of this problem could *also* have been obtained (after embedding) via dynamic programming and the associated Hamilton-Jacobi-Bellman (HJB) partial differential equation because the coefficients are assumed to be deterministic. Nevertheless, there are many advantages of using the framework of stochastic LQ control to study dynamic mean–variance problems. First, powerful results from the theory of stochastic control can be used to gain further insights into this problem. For example, recent results on the optimal control of indefinite/singular stochastic LQ problems (see, e.g., Chen, Li and Zhou (1998), Yong and Zhou (1999)) reveals the crucial role of uncertainty in stochastic control problems. This, in turn, has important implications in finance applications. Second, a unified study of various mean–variance type problems in finance can be undertaken in the framework of stochastic LQ control. For example, in addition to the portfolio selection problem, a Black–Scholes model with mean–variance hedging is naturally formulated as a stochastic LQ problem and solved using the LQ framework in Kohlmann and Zhou (2000). Thirdly, constraints on the state and/or control variable and randomness in the parameters are part of the natural language of stochastic control; such considerations in turn have important interpretations in the setting of finance. Finally, we will show in this paper that the LQ approach, unlike the

HJB equation, can be applied to the mean-variance problem even when the parameters are random.

In this paper, we consider the continuous-time mean–variance portfolio selection problem with random coefficients. We emphasize, however, that this generalization to random coefficients is by no means routine and does give rise to difficulties not encountered in the deterministic coefficient case. Firstly, the embedding technique can not be used and dynamic programming is difficult to apply when the coefficients are not deterministic. In particular, the usual HJB equation is not valid when the coefficients are random. (To be specific, the HJB equation involves a terminal boundary condition and for this reason, will become a backward stochastic partial differential equation in the case when coefficients are random. Such equations are complicated and difficult to handle). If one is to employ the LQ approach as done in Zhou and Li (2000), then the key difficulty is the necessity of proving global solvability of the so-called *stochastic Riccati equation* (SRE) associated with this problem. This brings us to the second, and crucial, difference between the mean–variance problem with random coefficients, and that with deterministic coefficients. When the coefficients are deterministic, the SRE reduces to a *linear* deterministic *ordinary differential equation* (ODE); existence and uniqueness follows immediately from standard results. When the coefficients are random, however, the SRE is a fully *nonlinear*, singular *backward stochastic differential equation* (BSDE) for which the usual assumptions (such as the Lipschitz and linear growth conditions; see Pardoux and Peng (1990) or Yong and Zhou (1999)) are *not* satisfied. For these reasons, global solvability of this equation can not be established using standard techniques and must be proved by exploiting the special structure of the SRE arising in the setting of mean–variance portfolio problem. It is worth emphasizing that studying such a nonlinear BSDE is interesting in its own right. (In recent years, BSDE theory has been developed extensively and enjoys profound applications in many areas, especially in finance; see Ma and Yong (1999) or Yong and Zhou (1999) for the latest accounts of the theory and applications). Afterwards, the efficient frontier and the optimal investment policy of the original mean–variance problem are derived analytically in terms of the solution to the above SRE, by employing results from convex optimization and stochastic LQ control. A rather interesting implication of our result is that the efficient frontier in the mean–standard deviation diagram is still a straight line or, equivalently, risk-free investment is still possible, even when the interest rate is random. This, however, can be explained by the fact that the risk arising from the random interest rate can be *perfectly* hedged by composing an appropriate portfolio of the stocks, under the basic assumptions of this paper that the interest rate is adapted to the stocks and the market is complete. Finally, we show that the Mutual Fund theorem is still valid in the continuous-time, random parameter setting, namely, any efficient portfolio can be constructed through the risk-free portfolio and another pre-specified efficient portfolio.

It is worth mentioning that Korn (1997) considers a continuous-time mean–variance portfolio selection problem with non-negativity constraints on the terminal wealth under the assumption that all coefficients are deterministic and time-invariant. Basically the Lagrange multiplier method is employed to tackle the problem. What is interesting is the way of dealing with the non-negativity constraint: the initial wealth is decomposed into two parts; one part is invested in a European put option to ensure the non-negativity of the terminal wealth, while the other is invested in the market without constraints. Certainly the problem studied in Korn (1997) is different from the one in this paper; also it is not clear how to extend the idea there to the case where all the market coefficients, including the interest rate, are random. Finally, duality methods play an important role in the problem of utility maximization as studied in Cox and Huang (1989, 1991) and Karatzas, Lehoczky and Shreve (1987). (For a summary of these results as well as generalizations to problems with portfolio constraints (e.g. Cvitanic and Karatzas (1992)), we direct the reader to the book Karatzas and Shreve (1998)). In these papers, convex duality is used to obtain the random terminal condition that must be satisfied by the wealth process under the optimal investment strategy. Through this, a linear partial differential equation for the optimal wealth and an expression for the optimal investment policy are obtained. On the other hand, duality methods are used in this paper to reduce the mean variance problem into an unconstrained stochastic LQ control problem that can be studied via BSDEs.

The outline of this paper is as follows. In Section 2, we introduce the mean–variance problem and formulate it as a stochastic optimization problem with linear equality constraints. In Section 3, we treat the mean–variance problem as a linearly constrained stochastic LQ problem. In particular, we show that global solvability of a singular Riccati BSDE is sufficient for the solvability of the LQ problem. In Section 4, we prove existence and uniqueness of solutions to the singular Riccati BSDE. In Section 5, we use these results to derive the mean–variance efficient frontier and optimal policy. In Section 6, we end with some summarizing comments.

2 Formulation of mean–variance portfolio selection

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space such that $\mathcal{F}_0$ is augmented by all the P-null sets of $\mathcal{F}$ (and hence $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous on $[0,T)$). Let $W(t) = (W^1(t), \cdots, W^m(t))'$ be an $\mathbb{R}^m$-valued standard Brownian motion (with $W(0) = 0$) on this space, and assume that $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by $W(\cdot)$. Throughout this paper, for any $1 \leq q < \infty$ we denote the set of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes f on $[0, T]$ such that $E\int_0^T |f(t)|^q\, dt < \infty$ by $L^q_{\mathcal{F}}(0, T; \mathbb{R}^m)$, the set of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes which are uniformly bounded by $L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^m)$, the set of continuous $\{\mathcal{F}_t\}_{t\geq 0}$ adapted processes such that $E\{\sup_{t\in[0,T]} |f(t)|^2\} <$

∞ by $L^2_{\mathcal{F}}(\Omega;\, C(0,T;\mathbb{R}^m))$, and the set of all $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, uniformly bounded continuous processes by $L^\infty_{\mathcal{F}}(\Omega;\, C(0,T;\mathbb{R}^m))$. Finally, the set of bounded, $\mathbb{R}^m$-valued, $\mathcal{F}_T$-adapted random variables will be denoted by $L^\infty_{\mathcal{F}_T}(\Omega,\, \mathbb{R}^m)$.

Consider a market with $m+1$ securities, consisting of a bond and m stocks. The bond price $P_0(t)$ satisfies the (stochastic) ordinary differential equation:

$$\begin{cases} dP_0(t) = r(t)\, P_0(t)\, dt, \quad t \in [0,\, T], \\ P_0(0) = p_0 > 0, \end{cases} \tag{2}$$

where the interest rate $r(t) > 0$ is a uniformly bounded, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, scalar-valued stochastic process. The price of each of the stocks, $P_1(t), \cdots, P_m(t)$, satisfies the stochastic differential equation (SDE):

$$\begin{cases} dP_i(t) = P_i(t)\Big\{b_i(t)\, dt + \sum_{j=1}^m \sigma_{ij}(t)\, dW^j(t)\Big\}, \quad t \in [0,\, T], \\ P_i(0) = p_i > 0, \end{cases} \tag{3}$$

where $b_i(t) > 0$ and $\sigma_i(t) = [\sigma_{i1}, \cdots, \sigma_{im}(t)]$ are the appreciation rate and dispersion (or volatility) rate of the i^{th} stock, respectively. Once again, we assume that $b_i(t)$ and $\sigma_{ij}(t)$ are scalar-valued, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, uniformly bounded stochastic processes. Note that the assumption that $r(t) > 0$ and $b_i(t) > 0$ is made simply because this matches what happens in practice. However, the main results in this paper do not rely on this assumption. Denoting

$$\sigma(t) = \begin{bmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{bmatrix} \in \mathbb{R}^{m\times m}, \tag{4}$$

we assume throughout that $\sigma(t)$ is uniformly non-degenerate: that is, there exists $\delta > 0$ such that

$$\sigma(t)\, \sigma(t)' \geq \delta\, I, \quad \forall\, t \in [0,\, T],\ P - a.s.. \tag{5}$$

In particular, $\sigma(t)$ must be non-singular a.e. $t \in [0,\, T]$, P-a.s..

Suppose that the total wealth of an agent at time $t \geq 0$ is denoted by $x(t)$. If transaction costs and consumption are ignored, and share trading takes place in continuous time, then we obtain:

$$\begin{cases} dx(t) = \Big\{r(t)\, x(t) + \sum_{i=1}^m [b_i(t) - r(t)]\, u_i(t)\Big\}\, dt \\ \qquad\qquad + \sum_{j=1}^m \sum_{i=1}^m \sigma_{ij}(t)\, u_i(t)\, dW^j(t), \quad t \in [0,\, T], \\ x(0) = x_0 > 0, \end{cases} \tag{6}$$

where $u_i(t)$ is the total market value of the agent's wealth in the i^{th} asset. If $u_i(t) < 0$ $(i = 1, \cdots, m)$ then the agent is short selling the i^{th} stock. If $u_0(t) < 0$, then the agent is borrowing the amount $|u_0(t)|$ at rate $r(t)$. We refer to $u(t) = (u_1(t), \cdots, u_m(t))'$ as the *portfolio* of the agent. Note that $u_0(t)$ has been excluded from a portfolio since it is completely determined by the allocation of stocks and the total wealth $x(t)$.

The agent's objective is to find a portfolio $u(\cdot)$ such that the expected terminal wealth satisfies $Ex(T) = d$, for some $d \in \mathbb{R}$, while the risk measured by the variance of the terminal wealth

$$\text{Var } x(T) := E(x(T) - Ex(T))^2 = E(x(T) - d)^2 \tag{7}$$

is minimized. The problem of finding such a portfolio $u(\cdot)$ is referred to as the *mean–variance portfolio selection problem.*

Definition 2.1 *A portfolio $u(\cdot)$ is said to be admissible if it is $\mathbb{R}^m$-valued, integrable (i.e., $E\int_0^T |u(t)|dt < +\infty$), $\{\mathcal{F}_t\}_{t\geq 0}$-adapted and the SDE (6) has a unique solution $x(\cdot)$ corresponding to $u(\cdot)$. In this case, we refer to $(x(\cdot), u(\cdot))$ as an admissible pair.*

Therefore, the mean–variance problem can be formulated as a linearly constrained stochastic optimization problem:

$$\begin{cases} \min J_{\text{MV}}(u(\cdot)) := E\,\frac{1}{2}\,(x(T) - d)^2, \\ \text{subject to:} \\ \quad Ex(T) = d, \\ \quad u(\cdot) \text{ is admissible.} \end{cases} \tag{8}$$

Finally, an optimal portfolio to the above problem is called an *efficient portfolio* corresponding to d, the corresponding $(\text{Var } x(T), d)$ is called an *efficient point*, whereas the set of all the efficient points is called an *efficient frontier.*

3 A stochastic LQ framework

It is clear from (8) that the mean–variance problem may be formulated as a linearly constrained LQ optimal control problem. In this section, we derive a *sufficient conditions* for the solvability of (8). In the first step of this derivation, the constrained problem (8) is reduced to an unconstrained problem through the introduction of a Lagrange multiplier. Next, basic results from convex optimization are used to show that solvability of this unconstrained problem is sufficient for solvability of the constrained problem. In Section 3.1, a sufficient condition for solvability of this unconstrained problem is derived using ideas from stochastic LQ control. In particular, it is shown that the mean–variance problem boils down to proving existence and uniqueness of solutions of an equation known in the literature as the stochastic Riccati equation (SRE). In Section 3.2, expressions for the optimal solution of the constrained problem (8) are obtained in terms of the SRE. The issue of solvability of the SRE is addressed in Section 4.

3.1 An unconstrained mean–variance problem

Rewrite the dynamics (6) as

$$\begin{cases} dx(t) &= [r(t)x(t) + B(t)\,u(t)\,]\,dt + u(t)'\sigma(t)dW(t), \quad t \in [0,\,T], \\ x(0) &= x_0, \end{cases} \tag{9}$$

where $B(t) = [b_1(t) - r(t), \cdots, b_m(t) - r(t)]$. Consider the cost functional:

$$J(u(\cdot)) = E\frac{1}{2}\Big\{Hx(T)^2 + 2cx(T)\Big\}, \tag{10}$$

where the parameters H and c are $\mathcal{F}_T$-measurable random variables. Clearly $J(u(\cdot))$ and $J_{\mathrm{MV}}(u(\cdot))$ (see (8)) are equivalent, in terms of the minimization problem, with $H = 1$ and $c = -d$. However, we choose to parameterize $J(u(\cdot))$ since in the sequel we need to handle different problems with different parameter values. Moreover, our analysis applies to a class of problems more general than the mean–variance problem (for instance, H and c may be random variables). Since this is of independent interest, we shall use the notation as stated in (9) and (10) and specialize to the mean–variance case later in the paper.

The class of *admissible controls* is the set

$$\mathcal{U} = \Big\{u(\cdot) \in L^1_{\mathcal{F}}(0,\,T;\,\mathbb{R}^m)\,\Big|\,(9) \text{ has a unique solution under } u(\cdot)\Big\}.$$

Clearly $\mathcal{U}$ is a convex set. If $u(\cdot) \in \mathcal{U}$ and $x(\cdot)$ is the associated solution of (9), then we refer to $(x(\cdot),\,u(\cdot))$ as an *admissible pair.*

Throughout this section, we shall assume the following:

Assumption (A1): $H,\, c \in L^\infty_{\mathcal{F}_T}(\Omega;\,\mathbb{R}),\ H \geq \delta$ for some $\delta > 0$.

Note in particular that H and c may be random. The (unconstrained) *stochastic LQ problem* associated with (9)-(10) is as follows:

$$\begin{cases} \min J(u(\cdot)), \\ \text{subject to:} \\ \quad (x(\cdot),\,u(\cdot)) \text{ admissible for (9).} \end{cases} \tag{11}$$

The problem (11) is said to be *finite* if there exists some finite constant $K \in \mathbb{R}$ such that

$$J(u(\cdot)) \geq K, \quad \forall\, u(\cdot) \in \mathcal{U},$$

and *solvable* if there exists a control $u^*(\cdot) \in \mathcal{U}$ such that

$$J(u^*(\cdot)) \leq J(u(\cdot)), \quad \forall\, u(\cdot) \in \mathcal{U}.$$

In this case, the control $u^*(\cdot)$ is referred to as the *optimal control.* We say that (11) is *uniquely solvable* if it is solvable and the optimal control is unique. Note that a finite LQ problem is not necessarily solvable.

Before stating the main result in this subsection, we introduce the following BSDEs (the argument t is suppressed):

$$\begin{cases} dp = -\Big\{\Big[2r - B(\sigma\,\sigma')^{-1}B'\Big]p - 2B(\sigma^{-1})'\,\Lambda - \frac{1}{p}\Lambda'\Lambda\Big\}\,dt \\ \qquad +\Lambda'\,dW, \quad t \in [0,\,T], \\ p(T) = H, \\ p(t) > 0, \quad \forall\, t \in [0,\,T], \end{cases} \tag{12}$$

$$\begin{cases} dh = \Big\{r\,h + B(\sigma^{-1})'\,\eta\Big\}\,dt + \eta'\,dW \\ h(T) = \frac{c}{H}, \end{cases} \tag{13}$$

and the forward SDE:

$$\begin{cases} dx = \Big\{\Big[r - B(\sigma\sigma')^{-1}\Big(B' + \sigma\frac{\Lambda}{p}\Big)\Big]\,x - B(\sigma\sigma')^{-1}\Big[\Big(B' + \sigma\frac{\Lambda}{p}\Big)h + \sigma\eta\Big]\Big\}\,dt \\ -\Big\{\Big[\sigma^{-1}B' + \frac{\Lambda}{p}\Big]\,(x+h) + \eta\Big\}'\,dW, \\ x(0) = x_0. \end{cases} \tag{14}$$

The BSDE (12) is a special case of the SRE. A solution of the SRE is a pair of square–integrable, adapted processes $(p,\,\Lambda) \in L^\infty_{\mathcal{F}}(\Omega,\,C(0,\,T)) \times L^2_{\mathcal{F}}(0,\,T;\,\mathbb{R}^m)$ which satisfies the system of equations (12). Note that the inequality $p > 0$ is part of this system of equations and needs to be verified when proving solvability. Similarly, a solution of the BSDE (13) is a pair of square–integrable, adapted processes $(h,\,\eta) \in L^2_{\mathcal{F}}(\Omega,\,C(0,\,T)) \times L^2_{\mathcal{F}}(0,\,T;\,\mathbb{R}^m)$ that satisfies the SDE and terminal condition in (13). The SRE (12) and BSDE (13) play a fundamental role in the solution of the LQ problem (11) (and hence, the mean–variance problem (8)).

In Proposition 3.2, it will be shown that existence of solutions of the equation (12) is sufficient for solvability of the LQ problem (11). When the coefficients r, B, σ, H and c, are all deterministic, we have $\Lambda = 0$ and $\eta = 0$. Hence, (12)-(13) become linear (backward) ODEs and (14) is a linear SDE in $x(\cdot)$ with bounded coefficients. In this situation, the existence and uniqueness of solutions of (12)-(14) follows immediately from the standard theory; see Yong and

Zhou (1999) or Zhou and Li (2000) for example. In the case of random parameters the equation (13) is a linear BSDE with a bounded terminal condition. Standard theory, once again, applies and guarantees the existence and uniqueness of a solution $(h, \eta) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^n)$; see Yong and Zhou (2000). On the other hand, existence and uniqueness for (12) and (14) is by no means trivial when coefficients are random. In this case, the SRE (12) is a complicated nonlinear BSDE. In particular, the right hand side of this equation is not globally Lipschitz continuous nor linearly growing (due to the term $\Lambda'\Lambda/p$) and hence, the standard results on existence and uniqueness of solutions can not be applied. In the case of the SDE (14), the term $\Lambda \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ associated with the Riccati equation (12) appears in the coefficient of the state x in both drift and diffusion. Although linear, the question of existence of solutions of (14) no longer lies in the domain of standard theory, which requires the coefficients of x to be uniformly bounded. In this section, we prove that existence and uniqueness of solutions of (14) is implied by the existence and uniqueness of solutions of the SRE (12). The proof of the existence and uniqueness for (12) will be deferred to the next section.

Proposition 3.1 *Suppose that* **(A1)** *holds. If (12) has a unique solution* $(p, \Lambda) \in L^\infty_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$*, then (14) has a unique strong solution* x*. Moreover, for every* $1 \le q < \infty$*,* $x \in L^q_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}))$.

Proof: Consider the following linear SDE:

$$\begin{cases} dX = -r\,X dt - XB(\sigma^{-1})'\,dW \\ X(0) = p(0)\,x_0 + p(0)h(0). \end{cases} \tag{15}$$

By standard theory, there exists a unique strong solution $X \in L^q_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R}))$ $(1 \le q < \infty)$ of the SDE (15); see Yong and Zhou (1999). Define

$$x = \frac{X}{p} - h \in L^q_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R})), \quad 1 \le q < \infty. \tag{16}$$

(In particular, the right hand side of (16) is well defined since solvability of (12) implies $p > 0$). Using Ito's formula, it can be shown that x is a solution of the SDE (14). Uniqueness follows immediately from the relation (16) as well as the uniqueness for the equations (12), (13) and (15). ∎

The following result shows that solvability of the SRE (12) is a sufficient condition for solvability of the LQ problem (11).

Proposition 3.2 *Assume that* **(A1)** *holds and the BSDE (12) has a unique solution* $(p, \Lambda) \in L^\infty_{\mathcal{F}}(\Omega, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$*. Then the associated stochastic LQ problem (11) is solvable. The unique optimal feedback control is given by*

$$u(t) = -(\sigma(t)\sigma(t)')^{-1}\Big[\Big(B(t)' + \sigma(t)\,\frac{\Lambda(t)}{p(t)}\Big)(x(t) + h(t)) + \sigma(t)\,\eta(t)\Big] \tag{17}$$

and the associated optimal cost is

$$J^* = \frac{1}{2}\Big\{p(0)(x_0 + h(0))^2 - E\Big(\frac{c^2}{H}\Big)\Big\}. \tag{18}$$

Proof: It will be convenient to use the following equivalent expression for the BSDE (12):

$$\begin{cases} dp &= -\Big[2r - \Big(B' + \sigma\frac{\Lambda}{p}\Big)'(\sigma\sigma')^{-1}\Big(B' + \sigma\frac{\Lambda}{p}\Big)\Big]pdt + \Lambda' dW, \quad t \in [0,\, T], \\ p(T) &= H, \\ p &> 0. \end{cases} \tag{19}$$

Applying Ito's formula to the expression $(x(t) + h(t))^2$, it can be shown that:

$$\begin{aligned} & d(x(t) + h(t)) \\ = & \Big\{u'\sigma\sigma' u + 2u'\Big(B'(x+h) + \sigma\eta\Big) + 2r(x+h)^2 + 2(x+h)B'(\sigma^{-1})'\eta + \eta'\eta\Big\}dt \\ & +2(x+h)(\eta'\sigma + \eta')dW. \end{aligned}$$

Applying Ito's formula to $p(t)(x(t) + h(t))^2$, using the BSDE (19), we obtain:

$$\begin{aligned} & p(T)(x(T) + h(T))^2 = p(0)(x(0) + h(0))^2 \\ & +E\int_0^T p\Big\{u + (\sigma(t)\sigma(t)')^{-1}\Big[\Big(B(t)' + \sigma(t)\,\frac{\Lambda(t)}{p(t)}\Big)(x(t) + h(t)) + \sigma(t)\,\eta(t)\Big]\Big\}' \\ & \times\sigma(t)\sigma(t)'\Big\{u + (\sigma(t)\sigma(t)')^{-1}\Big[\Big(B(t)' + \sigma(t)\,\frac{\Lambda(t)}{p(t)}\Big)(x(t) + h(t)) + \sigma(t)\,\eta(t)\Big]\Big\}dt. \end{aligned}$$

Since

$$\frac{1}{2}p(T)(x(T) + h(T))^2 = \frac{1}{2}\Big\{Hx(T)^2 + 2cx(T)\Big\} + \frac{1}{2}\frac{c^2}{H}$$

it follows that:

$$\begin{aligned} & J(u(\cdot)) = \frac{1}{2}p(0)(x(0) + h(0))^2 - E\Big(\frac{c^2}{H}\Big) \\ & +E\,\frac{1}{2}\int_0^T p\Big\{u + (\sigma(t)\sigma(t)')^{-1}\Big[\Big(B(t)' + \sigma(t)\,\frac{\Lambda(t)}{p(t)}\Big)(x(t) + h(t)) + \sigma(t)\,\eta(t)\Big]\Big\}' \\ & \times\sigma(t)\sigma(t)'\Big\{u + (\sigma(t)\sigma(t)')^{-1}\Big[\Big(B(t)' + \sigma(t)\,\frac{\Lambda(t)}{p(t)}\Big)(x(t) + h(t)) + \sigma(t)\,\eta(t)\Big]\Big\}dt. \end{aligned}$$

It follows immediately that (17) is the unique optimal control and (18) is the optimal cost. ∎

Before concluding this subsection, we present a result on the convexity properties of the cost functional $J(u(\cdot))$, defined by (9)-(10), which will be required later.

Proposition 3.3 *Suppose that* **(A1)** *holds. Then the functional $J(u(\cdot))$ defined by (9)-(10) is strictly convex.*

Proof: This is an immediate consequence of the linearity of the dynamics (9) and the fact that $H \geq \delta > 0$ for some constant $\delta \in \mathbb{R}$. ∎

3.2 Constrained mean–variance problem

We are now in the position to study the original mean–variance portfolio selection problem (8), based on the results of the previous subsection on the unconstrained version of the problem. We will keep the same setup, notation and assumptions of the previous subsection, with an additional constraint represented by the following functional:

$$J_1(u(\cdot)) = E\Big\{c_1\, x(T)\Big\}, \quad u(\cdot) \in \mathcal{U}. \tag{20}$$

We shall make the following assumptions:

Assumption (A2): H and c satisfy **(A1)** and $c_1 \in L^{\infty}_{\mathcal{F}_T}(\Omega,\, \mathbb{R})$.

Let $d \in \mathbb{R}$ be given and fixed. Consider the following linearly constrained LQ problem:

$$\begin{cases} J^* = \inf J(u(\cdot)), \\ \text{subject to:} \\ \quad J_1(u(\cdot)) = d, \\ \quad (x(\cdot),\, u(\cdot)) \text{ admissible for (9).} \end{cases} \tag{21}$$

The problem is equivalent to the original mean–variance problem (8) with $B(t) = [b_1(t) - r(t), \cdots, b_m(t) - r(t)]$, $H = 1$, $c = -d$ and $c_1 = 1$. However, as before, we keep the general notation to ease the exposition as well as to see more general properties of the solution.

An admissible control $u(\cdot) \in \mathcal{U}$ is said to be a *feasible control* for (21) if it satisfies the constraint in (21). If there exists a feasible control, then the problem (21) is said to be *feasible*. Note that by convention, if (21) is not feasible, then $J^* = \infty$. We refer to (21) as being *finite* if it is feasible and $J^* > -\infty$. Finally, if (21) is finite and the optimal cost is achieved by a feasible control $u^*(\cdot)$, then (21) is said to be *solvable* and $u^*(\cdot)$ is an *optimal control.*

As in the previous subsection, we shall assume throughout this subsection that the SRE (12) has a unique solution $(p,\, \Lambda) \in L^{\infty}_{\mathcal{F}}(\Omega,\, C(0,\, T;\, \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$. It follows from the linearity of (13) and the boundedness of the coefficients in this equation that the BSDEs (13) and

$$\begin{cases} d\bar{h} &= \Big\{r\,\bar{h} + B(\sigma^{-1})'\,\bar{\eta}\Big\}\,dt + \bar{\eta}'\,dW, \quad t \in [0,\, T], \\ \bar{h}(T) &= \frac{c_1}{H}. \end{cases} \tag{22}$$

have unique solutions $(h,\, \eta)$, $(\bar{h},\, \bar{\eta}) \in L^{\infty}_{\mathcal{F}}(\Omega;\, C(0,\, T;\, \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$, respectively, and from Proposition 3.1 that the SDEs (14) and

$$\begin{cases} d\bar{x} &= \Big\{\Big[r - B(\sigma\sigma')^{-1}\Big(B' + \sigma\frac{\Lambda}{p}\Big)\Big]\bar{x} \\ & \quad -B(\sigma\sigma')^{-1}\Big[(B' + \sigma\frac{\Lambda}{p})\bar{h} + \sigma\bar{\eta}\Big]\Big\}\, dt \\ & \quad -\Big\{\Big[\sigma^{-1}B' + \frac{\Lambda}{p}\Big](\bar{x} + \bar{h}) + \bar{\eta}\Big\}'\, dW, \quad t \in [0,\, T], \\ \bar{x}(0) &= 0. \end{cases} \tag{23}$$

have unique solutions $x,\ \bar{x}$ such that $x,\ \bar{x} \in L^q_{\mathcal{F}}(\Omega;\ C(0,\, T;\ \mathbb{R}))$ for every $1 \le q < \infty$, respectively. We have the following result on the feasibility of (21).

Proposition 3.4 *Suppose that* **(A2)** *holds. Let* $(\psi,\, \xi) \in L^2_{\mathcal{F}}(\Omega;\ C(0,\, T;\ \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T;\ \mathbb{R}^m)$ *denote the unique solution of the BSDE*

$$\begin{cases} d\psi(t) &= -r(t)\,\psi(t)\,dt + \xi(t)'\,dW(t), \quad t \in [0,\, T], \\ \psi(T) &= c_1. \end{cases} \tag{24}$$

Then the LQ problem (21) is feasible for every $d \in \mathbb{R}$ if and only if

$$E\int_0^T \Big|\psi(t)\,B(t) + \xi(t)'\,\sigma(t)'\Big|^2\, dt > 0. \tag{25}$$

Proof: To prove sufficiency, let $u(t) = B(t)'\psi(t) + \sigma(t)\xi(t)$ and $\hat{u}(t) = \lambda\, u(t)$, where $\lambda \in \mathbb{R}$. Denote by $\hat{x}(t)$ the corresponding state under $\hat{u}(\cdot)$ for the system (9). Then it follows that $\hat{x}(t) = y(t) + \lambda\, z(t)$, where $y(\cdot),\ z(\cdot) \in L^q_{\mathcal{F}}(\Omega;\ C(0,\, T;\ \mathbb{R}))$ $(1 \le q < \infty)$ are the unique solutions of the SDEs

$$\begin{cases} dy(t) &= r(t)\,y(t)\,dt, \quad t \in [0,\, T], \\ y(0) &= x_0, \end{cases}$$

$$\begin{cases} dz(t) &= [r(t)\,z(t) + B(t)u(t)]\,dt + u(t)'\sigma(t)dW(t), \quad t \in [0,\, T], \\ z(0) &= 0, \end{cases}$$

respectively. Therefore, $E[c_1\hat{x}(T)] = E[c_1 y(T)] + \lambda E[c_1 z(T)]$ and (21) is feasible if for any $d \in \mathbb{R}$ there exists a $\lambda \in \mathbb{R}$ such that $E[c_1\hat{x}(T)] = d$. Equivalently, (21) is feasible if $E[c_1 z(T)] \neq 0$. Our result follows from the observation that

$$E[c_1 z(T)] = E\int_0^T \Big[\psi(t)\,B(t) + \xi(t)'\,\sigma(t)'\Big]\,\hat{u}(t)\,dt \tag{26}$$

(see p. 353 of Yong and Zhou (1999)) and the definition of $\hat{u}(t)$.

Conversely, suppose that (21) is feasible for every $d \in \mathbb{R}$. Then, for each $d \in \mathbb{R}$, there is a $u(t) = \hat{u}(t)$ such that $E[c_1 x(T)] = E[c_1 y(T)] + E[c_1 z(T)] = d$. Since $E[c_1 y(T)]$ is independent of $u(\cdot)$, this implies that there exists $\hat{u}(\cdot)$ such that $E[c_1 z(T)] \neq 0$. Hence, it follows from (26) that (25) is true. ■

From Proposition 3.4 and the condition (25), we obtain certain minimum requirements for the LQ problem (21) to be feasible for every $d \in \mathbb{R}$. In particular, it is necessary that $c_1 \neq 0$ (since $(\psi, \xi) = (0, 0)$ is the unique solution of (24) when $c_1 = 0$) and $B \neq 0$ or $\sigma \neq 0$. In view of (21), these requirements for feasibility are as expected. On the other hand, if $(\psi, \xi) \neq (0, 0)$ (which is the case when $c_1 \neq 0$), then (25) is easily satisfied since (ψ, ξ) is independent of B and σ. In particular, if $(\psi, \xi) \neq (0, 0)$ but (25) does not hold, then there are 'arbitrarily small' perturbations of B and σ that will give rise to a new problem that is feasible for every d. Hence, assuming feasibility of (21) for every $d \in \mathbb{R}$ (or equivalently, that (25) holds) is very mild.

We have the following result.

Proposition 3.5 *Assume that* **(A2)** *and (25) are satisfied. Let $d \in \mathbb{R}$ be given and fixed. Then (21) is finite for every $d \in \mathbb{R}$. Moreover, if the BSDE (12) has a unique solution $(p, \Lambda) \in L^\infty_{\mathcal{F}}(\Omega, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$, then the inequality*

$$p(0)\,\bar{h}(0)^2 < E\Big(\frac{c_1^2}{H}\Big) \tag{27}$$

is satisfied, and the optimal cost of (21) is

$$\begin{aligned} J^* &= E\frac{1}{2}\Big\{p(0)\Big(x_0 + h(0)\Big)^2 - \frac{c_0^2}{H}\Big\} \\ &\quad + \frac{1}{2}\frac{\Big(p(0)\,\bar{h}(0)\,[x_0 + h(0)] - E\Big(\frac{c_0 c_1}{H}\Big) - d\Big)^2}{E\Big(\frac{c_1^2}{H}\Big) - p(0)\bar{h}(0)^2}. \end{aligned} \tag{28}$$

Furthermore, if (21) is solvable, then its optimal feedback control is

$$\begin{aligned} u^*(t) &= -(\sigma(t)\sigma(t)')^{-1}\Big[\Big(B(t)' + \sigma(t)\frac{\Lambda(t)}{p(t)}\Big)(x^*(t) + h(t)) + \sigma(t)\,\eta(t) \\ &\quad + \lambda^*\Big(\Big(B(t)' + \sigma(t)\frac{\Lambda(t)}{p(t)}\Big)\bar{h}(t) + \sigma(t)\bar{\eta}(t)\Big)\Big], \end{aligned} \tag{29}$$

and the corresponding optimal state trajectory is

$$x^*(t) = x(t) + \lambda^* \bar{x}(t), \tag{30}$$

where $x(\cdot)$ and $\bar{x}(\cdot)$ are the solutions to (14) and (23), respectively, and

$$\lambda^* = \frac{p(0)\,\bar{h}(0)\,[x_0 + h(0)] - E\Big(\frac{cc_1}{H}\Big) - d}{E\Big(\frac{c_1^2}{H}\Big) - p(0)\,\bar{h}(0)^2}. \tag{31}$$

Proof: Let

$$\mathcal{X} = \Big\{ (x,\, u) \in L^1_{\mathcal{F}}(0,\, T;\, \mathbb{R}) \times L^1_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)) \,\Big|\, (x,\, u) \text{ is admissible} \Big\}.$$

Clearly, $\mathcal{X}$ is a convex subset of $L^1_{\mathcal{F}}(0,\, T;\, \mathbb{R}) \times L^1_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$. Hence, we may treat $J(\cdot)$ either as a function of $u(\cdot) \in \mathcal{U}$, or equivalently, as a function of $(x(\cdot),\, u(\cdot)) \in \mathcal{X}$. (In this case, we shall often write $J(x(\cdot),\, u(\cdot)))$. Similarly, treating $J_1(\cdot)$ as a function of $(x(\cdot),\, u(\cdot))$, it follows that $J_1(x(\cdot),\, u(\cdot))$ is linear on $L^1_{\mathcal{F}}(0,\, T;\, \mathbb{R}) \times L^1_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$. Moreover, it follows from Proposition 3.3 that $J(x(\cdot),\, u(\cdot))$ is convex over $\mathcal{X}$. Therefore, (21) is equivalent to the following optimization problem with convex cost and linear constraints:

$$\begin{cases} J^* := \inf J(x(\cdot),\, u(\cdot)) \\ \text{Subject to:} \\ \quad J_1(x(\cdot),\, u(\cdot)) = d, \\ \quad (x(\cdot),\, u(\cdot)) \in \mathcal{X}. \end{cases} \tag{32}$$

Now, by assumption and Proposition 3.4 the constrained LQ problem (21) is feasible. Moreover, it follows from Proposition 3.2 that existence for (12) implies that the unconstrained LQ problem (11), with cost $J(u(\cdot)) = E(1/2)(Hx(T)^2 + 2cx(T))$, is solvable with a finite optimal cost. Therefore, (21) is finite. Hence it follows from Luenberger (1968) that

$$J^* = \max_{\lambda \in \mathbb{R}} \inf_{u(\cdot) \in \mathcal{U}} J(u(\cdot);\, \lambda), \tag{33}$$

where

$$J(u(\cdot);\, \lambda) := J(u(\cdot)) + \lambda(J_1(u(\cdot)) - d).$$

By Proposition 3.2, for every fixed $\lambda \in \mathbb{R}$, the unconstrained LQ problem

$$J^*(\lambda) := \inf_{u(\cdot) \in \mathcal{U}} J(u(\cdot);\, \lambda)$$

is solvable with optimal cost

$$\begin{aligned} &J^*(\lambda) \\ &= \frac{1}{2}\Big\{ p(0)\Big[x_0 + h(0) + \lambda \bar{h}(0)\Big]^2 - E\Big(\frac{[c + \lambda c_1]^2}{H}\Big)\Big\} - \lambda\, d \\ &= \frac{1}{2}\Big\{ p(0)\Big(x_0 + h(0)\Big)^2 - E\Big(\frac{c^2}{H}\Big)\Big\} \\ &\quad + \frac{1}{2}\Big\{ 2\lambda\Big[p(0)\,\bar{h}(0)[x_0 + h(0)] - E\Big(\frac{c\,c_1}{H}\Big) - d\Big] \\ &\quad - \lambda^2\Big[E\Big(\frac{c_1^2}{H}\Big) - p(0)\bar{h}(0)^2\Big]\Big\}, \end{aligned} \tag{34}$$

and optimal control

$$u \;=\; -(\sigma\sigma')^{-1}\Big[\Big(B' + \sigma\frac{\Lambda}{p}\Big)(x + h) + \sigma\,\eta + \lambda\Big(\Big(B' + \sigma\frac{\Lambda}{p}\Big)\bar{h} + \sigma\bar{\eta}\Big)\Big]. \tag{35}$$

Since (21) is feasible for every $d \in \mathbb{R}$, the right hand side of (33) is bounded and equal to J^*, and this maximum is achieved by some λ^*. Hence it follows from (34) that $E(c_1^2/H) - p(0)\bar{h}(0)^2 \geq 0$. However, since the right hand side of (33) is bounded, it follows that if $E(c_1^2/H) - p(0)\bar{h}(0)^2 = 0$, we must have

$$p(0)\,\bar{h}(0)[x_0 + h(0)] - E\Big(\frac{c\,c_1}{H}\Big) - d = 0$$

for every $d \in \mathbb{R}$, which is a contradiction. Therefore, (27) must hold and λ^* is given by (31) and J^* by (28). Furthermore, if (21) is solvable, then it follows from (35) that the optimal control is given by (29). Substituting (29) into (9), it follows that the optimal state trajectory is given by (30). ■

To conclude this section, we remark that the results of Section 3.1, in particular Proposition 3.2, can be extended to a more general stochastic LQ problem where the dynamics is

$$\left\{\begin{array}{rcl} dx(t) & = & [A(t)\,x(t) + B(t)\,u(t)\,]\,dt \\ & & + \sum_{j=1}^{k}[C_j(t)\,x(t) + D_j(t)\,u(t)]\,dW^j(t), \quad t \in [0,\,T], \\ x(0) & = & x_0, \end{array}\right. \tag{36}$$

and the cost functional is

$$J(u(\cdot)) = E\frac{1}{2}\Big\{\int_0^T \Big(x(t)'Q(t)x(t) + u(t)'R(t)u(t)\Big)\,dt + x(T)'Hx(T) + 2c'x(T)\Big\}. \tag{37}$$

Here A, B, C_j, D_j, Q and R are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted, matrix-valued processes, $x_0 \in \mathbb{R}^n$ is non-random, and H and c are $\mathcal{F}_T$-random variables. Assuming that the following SRE

$$\left\{\begin{array}{rcl} dP & = & -\Big\{P\,A + A'\,P + \sum_{j=1}^{k}(\Lambda_j\,C_j + C_j'\,\Lambda_j + C_j'\,P\,C_j) + Q \\ & & -\Big[P\,B + \sum_{j=1}^{k}(C_j'P + \Lambda_j)\,D_j\Big]\,K^{-1}\,\Big[B'P + \sum_{j=1}^{k} D_j'\,\,(PC_j + \Lambda_j)\Big]\Big\}\,dt \\ & & + \sum_{j=1}^{k}\Lambda_j\,dW^j, \quad t \in [0,\,T], \\ P(T) & = & H, \\ K & = & R + \sum_{j=1}^{k} D_j'\,P\,D_j \; > \; 0 \end{array}\right. \tag{38}$$

has a solution $(P,\,\Lambda)$, the optimal feedback control and optimal cost can be obtained explicitly based on a completion-of-square argument similar to the one employed in the proof of Proposition 3.2. Details are left to the interested readers.

4 Solvability of stochastic Riccati equation

An immediate consequence of Proposition 3.5 is that the mean–variance problem (8) is solvable if the SRE (12) has a unique solution. The one remaining gap in our solution of the mean–variance problem is a proof of global solvability of the SRE (12). The SRE (12) is a special case (corresponding to the mean–variance problem (8)) of the multi-dimensional SRE (38) associated with the general stochastic LQ optimal control problem (36)–(37). This general SRE (38) is a highly nonlinear, matrix-valued BSDE, and very little is known about the solvability properties of the equation. Chen and Yong (2000) proved some results on local solvability of (38). For global solvability of (38), partial results have been obtained in certain special cases (e.g. Bismut (1976), Peng (1992) and Kohlmann and Tang (2000)) though none of these results cover the situation (8) that we are interested in. In particular, in Kohlmann and Tang (2000) the coefficients are random processes adapted to only certain components of the underlying Brownian motion, while the coefficients D_j's corresponding to these components are assumed to be zero. This is essentially equivalent to the case when the random coefficients are *independent* of the Brownian motion. As a consequence, the corresponding SRE in the context of the mean–variance problem in Kohlmann and Tang (2000) is a *linear* BSDE. On the other hand, we are interested in the situation where the coefficients are *adapted* to the Brownian motion and, in the mean–variance problem (8), the SRE remains nonlinear (12).

Fortunately, the SRE (12) studied in this paper has a special structure due, in particular, to the fact that the state variable $x(t)$ is scalar-valued and the cost $J(u(\cdot))$ depends only on the terminal wealth $x(T)$. In this section, we shall prove existence and uniqueness of solutions of (12) by fully exploiting these special features. Note that the proof is by no means trivial even though it is a scalar BSDE; in fact the approach to the SRE (12) is interesting in its own right.

To begin, consider the following linear scalar BSDE:

$$\left\{\begin{array}{rcl} dY & = & -\{FY + G'Z\}\,dt + Z'dW, \quad t \in [0,\, T], \\ Y(T) & = & \xi. \end{array}\right. \tag{39}$$

A solution $(Y,\, Z)$ of (39), where $Z \in L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$, is said to be a *bounded solution* if $(Y,\, Z) \in L^\infty_{\mathcal{F}}(\Omega,\, C(0,\, T;\, \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$. Also, if $Y_1,\, Y_2 \in L^\infty_{\mathcal{F}}(\Omega;\, C(0,\, T; \mathbb{R})))$ are processes satisfying $Y_1(t) \geq Y_2(t)$ for all $t \in [0,\, T]$, P-a.s., then we write $Y_1 \geq Y_2$.

Proposition 4.1 *Let $F \in L^\infty_{\mathcal{F}}(0,\, T;\, \mathbb{R})$, $G \in L^\infty_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$ and $\xi \in L^\infty_{\mathcal{F}_T}(\Omega;\, \mathbb{R})$ such that $\xi \geq \delta$ P-a.s. for some constant $\delta > 0$. Then the BSDE (39) has a unique solution $(Y,\, Z) \in L^\infty_{\mathcal{F}}(\Omega;\, C(0,\, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$. Moreover, there is a constant $k > 0$ such that $Y \geq k$.*

Proof: Since (39) is a linear BSDE with bounded coefficients, it has a unique *square integrable* solution $(Y,\, Z) \in L^2_{\mathcal{F}}(\Omega;\, C(0,\, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^m)$; see Theorem 2.2, p. 349 of Yong and Zhou (1999). To show that $Y \in L^\infty_{\mathcal{F}}(\Omega;\, C(0,\, T; \mathbb{R}))$, note that Y has the representation:

$$Y(t) = \tilde{E}\Big\{\xi e^{-\int_t^T F(s)\,ds} \,\Big|\, \mathcal{F}_t\Big\} \tag{40}$$

where $\tilde{E}$ is the expectation with respect to another probability measure $\tilde{P}$ with Radon-Nikodym derivative:

$$\frac{d\tilde{P}}{dP} = e^{-\frac{1}{2}\int_0^T G(t)'G(t)\,dt + G(t)'dW(t)};$$

see Proposition 2.2 of El Karoui, Peng and Quenez (1987). (Note that Novikov's condition is satisfied since G is uniformly bounded; see Theorem 5.1, p. 191 and Corollary 5.13, p. 199 of Karatzas and Shreve (1988)). It follows now from the assumptions on ξ and F that $Y \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$ and $Y \geq k$ for some $k > 0$. ∎

Theorem 4.1 *Suppose all the parameters involved satisfy* **(A1)**. *Then there exists a unique solution* $(p, \Lambda) \in L^\infty_{\mathcal{F}}(\Omega, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ *of the stochastic Riccati equation (12).*

Proof: Clearly, the existence and uniqueness of solutions of the BSDE

$$\left\{\begin{array}{rcl} dp &=& -\{Fp + G'\Lambda - \frac{1}{p}\Lambda'\Lambda\}\,dt + \Lambda'dW, \quad t \in [0, T], \\ p(T) &=& M, \\ p(t) &>& 0, \quad \forall\, t \in [0, T], \end{array}\right. \tag{41}$$

for arbitrary $F \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R})$, $G \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^m)$ and $M \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R})$ such that $M > 0$ and $M^{-1} \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R})$ implies existence and uniqueness for (12). To prove existence for (41), consider the following BSDE:

$$\left\{\begin{array}{rcl} dY &=& -\{-FY + G'Z\}\,dt + Z'dW, \quad t \in [0, T], \\ Y(T) &=& \frac{1}{M}. \end{array}\right. \tag{42}$$

Since $M \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R})$ and $M > 0$, there is a finite constant $\delta > 0$ such that $\frac{1}{M} \geq \delta$, P-a.s.. By Proposition 4.1, (42) has a unique solution $(Y, Z) \in L^\infty_{\mathcal{F}}(\Omega, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ and $k \leq Y \leq K$, P-a.s., for some constants $0 < k < K < \infty$. Therefore, $(p, \Lambda) = (\frac{1}{Y}, -\frac{Z}{Y^2})$ is well defined and $(p, \Lambda) \in L^\infty_{\mathcal{F}}(\Omega, C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ with $0 < \frac{1}{K} \leq p \leq \frac{1}{k}$. It can be directly verified, using Ito's formula, that (p, Λ) is a solution of the BSDE (41). To prove uniqueness, observe that if (p, Λ) is a solution of (41), then $(Y, Z) = (\frac{1}{p}, -\frac{\Lambda}{p^2})$ is a solution of (42). Uniqueness follows from the fact that (42) has a unique solution. ∎

5 Efficient portfolios and efficient frontier

Finally we are at the stage of solving the original mean–variance problem (8). The problem is a special case of the constrained LQ problem (21) with system parameters

$$B(t) = [b_1(t) - r(t), \cdots, b_m(t) - r(t)], \quad H = 1,\ c = -d,\ c_1 = 1, \tag{43}$$

(which we shall assume throughout this section). Denoting

$$\rho(t) = B(t)(\sigma(t)\,\sigma(t)')^{-1}B(t)',$$

the SRE (12) can be rewritten as

$$\left\{\begin{array}{rcl} dp & = & -\Big\{(2r-\rho)\,p - 2B(\sigma^{-1})'\Lambda - \frac{1}{p}\Lambda'\Lambda\Big\}\,dt + \Lambda'\,dW, \quad t\in[0,\,T], \\ p(T) & = & 1. \end{array}\right. \tag{44}$$

Consider the following BSDE:

$$\left\{\begin{array}{rcl} dg & = & \Big\{rg + B(\sigma^{-1})'\zeta\Big\}\,dt + \zeta'\,dW, \quad t\in[0,\,T], \\ g(T) & = & 1. \end{array}\right. \tag{45}$$

Then the solutions to (13) and (22) (with parameters (43)) are

$$\begin{array}{rcl} (h(t),\,\eta(t)) & = & (-d\,g(t),\,-d\,\zeta(t)), \\ (\bar{h}(t),\,\bar{\eta}(t)) & = & (g(t),\,\zeta(t)). \end{array} \tag{46}$$

Finally, the SDEs (14) and (23) respectively become

$$\left\{\begin{array}{rcl} dx & = & \Big\{\Big[r - B(\sigma\sigma')^{-1}\Big(B' + \sigma\frac{\Lambda}{p}\Big)\Big]x \\ & & +d\,B(\sigma\,\sigma')^{-1}\Big[\Big(B' + \sigma\frac{\Lambda}{p}\Big)g + \sigma\zeta\Big]\Big\}\,dt \\ & & -\Big\{\Big[\sigma^{-1}B' + \frac{\Lambda}{p}\Big]\,(x - d\,g) - d\zeta\Big\}'\,dW, \quad t\in[0,\,T], \\ x(0) & = & x_0 \end{array}\right. \tag{47}$$

$$\left\{\begin{array}{rcl} d\bar{x} & = & \Big\{\Big[r - B(\sigma\sigma')^{-1}\Big(B' + \sigma\frac{\Lambda}{p}\Big)\Big]\bar{x} \\ & & -B(\sigma\,\sigma')^{-1}\Big[\Big(B' + \sigma\frac{\Lambda}{p}\Big)g + \sigma\zeta\Big]\Big\}\,dt \\ & & -\Big\{\Big[\sigma^{-1}B' + \frac{\Lambda}{p}\Big]\,(\bar{x} + g) + \zeta\Big\}'\,dW, \quad t\in[0,\,T], \\ \bar{x}(0) & = & 0. \end{array}\right. \tag{48}$$

We have the following result.

Theorem 5.1 (**Efficient portfolios and efficient frontier**) *Assume that (25) holds (with the parameters (43)). Then*

$$p(0)g(0)^2 < 1 \;\; \textit{and} \;\; g(0) > 0. \tag{49}$$

Moreover, the mean–variance problem (8) is finite for every $d \in \mathbb{R}$ *and the optimal value of* $\mathrm{Var}\, x(T)$*, amongst all the wealth processes* $x(\cdot)$ *satisfying* $Ex(T) = d$*, is*

$$\mathrm{Var}\, x(T) = \frac{p(0)\, g(0)^2}{1 - p(0)\, g(0)^2} \left[d - \frac{x_0}{g(0)} \right]^2. \tag{50}$$

Furthermore, if there exists a policy $u^*(\cdot)$ *that results in a wealth process* $x^*(\cdot)$ *that achieves* $Ex^*(T) = d$ *and (50), then*

$$x^*(t) = x(t) + \lambda^* \bar{x}(t) \tag{51}$$

and the optimal portfolio is

$$\begin{aligned} u^*(t) \;&=\; -(\sigma(t)\sigma(t)')^{-1} \Big[\Big(B(t)' + \sigma(t)\frac{\Lambda(t)}{p(t)} \Big) (x^*(t) - dg(t)) - d\sigma(t)\zeta(t) \\ &\quad + \lambda^* \Big(\Big(B(t)' + \sigma(t)\frac{\Lambda(t)}{p(t)} \Big) g(t) + \sigma(t)\zeta(t) \Big) \Big] \end{aligned} \tag{52}$$

where

$$\lambda^* = -\frac{p(0)\, g(0)^2}{1 - p(0)\, g(0)^2} \left[d - \frac{x_0}{g(0)} \right]. \tag{53}$$

Proof: Since the terminal condition of (45) satisfies $g(T) = 1$, it follows from Lemma 4.1 that there exists a finite constant $k > 0$ such that $g(t) \geq k$ for all $t \in [0, T]$, P-a.s.. This implies $g(0) > 0$. The remaining claims are an immediate consequence of Proposition 3.5. ■

In the case when $r(t)$ is deterministic, we have the following result.

Corollary 5.1 *Suppose that* $r(t)$ *is deterministic. Then the mean–variance problem (8) is finite for all* $d \in \mathbb{R}$ *if and only if* $B \neq 0$*. In this case*

$$(g(t),\, \zeta(t)) = \Big(e^{-\int_t^T r(s)\, ds},\, 0 \Big) \tag{54}$$

and the optimal variance associated with the return $Ex^*(T) = d$ *is*

$$\mathrm{Var}\, x^*(T) = \frac{p(0)\, e^{-2\int_0^T r(s)\, ds}}{1 - p(0) e^{-2\int_0^T r(s)\, ds}} \left[d - x_0\, e^{\int_0^T r(s)\, ds} \right]^2. \tag{55}$$

In addition, if $B(t)$ *and* $\sigma(t)$ *are also deterministic, then*

$$(p(t),\, \Lambda(t)) = \Big(e^{\int_t^T (2r(s) - \rho(s))\, ds},\, 0 \Big). \tag{56}$$

Proof: If $r(t)$ is deterministic, then (54) is the only solution of (45) due to the uniqueness of its solutions. Similarly, (24) becomes an ODE with a (unique) solution

$$(\psi(t),\, \xi(t)) = \Big(e^{\int_0^t r(s)\, ds},\, 0 \Big).$$

Since $r(t)$ is uniformly bounded, there are constants $0 < \delta_1 < \delta_2$ such that

$$\delta_1 E \int_0^T |B(t)|^2 \, dt \le E \int_0^T |\psi \, B + \xi' \sigma'|^2 \, dt \le \delta_2 E \int_0^T |B(t)|^2 \, dt.$$

Hence, it follows from Proposition 3.4 that (8) is finite for every $d \in \mathbb{R}$ if and only if $B \neq 0$. The remaining claims are immediate from Theorem 5.1. ■

We emphasize once again that the condition (25) for finiteness of the mean–variance problem is very mild, even in the case when all the parameters are random; see the comments after Proposition 3.4. In particular, we see from Corollary 5.1 that if $r(t)$ is deterministic *and* $B(t) \neq 0$ (which is a sensible assumption as the appreciation rates of stocks are supposed to be different from the bond rate), then the condition (25) is automatically satisfied and the associated mean–variance problem is finite. Hence, when $r(t)$ is deterministic and $B(t) \neq 0$, the efficient point associated with the mean–variance problem exists for *every* $d \in \mathbb{R}$. This resolves an open issue from Theorem 6.1 in Zhou and Li (1999) where the existence of an efficient point for any $d \in \mathbb{R}$ is assumed but not proven. On the other hand, if $r(t)$ is random and $B(t) = 0$, then the unique solution of (24) is $(\psi, \xi) \in L^2_{\mathcal{F}}(\Omega; C(0, T; \mathbb{R})) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ with $\xi \neq 0$ in general. Hence, in this situation, it is generally the case that

$$E \int_0^T |\psi B + \xi' \sigma'|^2 \, dt = E \int_0^T |\xi' \sigma'|^2 \, dt \ge \delta \, E \int_0^T |\xi|^2 \, dt > 0. \tag{57}$$

Therefore, if $r(t)$ is random, the mean–variance problem may still be feasible for every $d \in \mathbb{R}$, even in the pathological case when $B = 0$. In fact, when $r(t)$ is random, it is generally feasible. This is quite contrary to the case when $r(t)$ is deterministic and $B = 0$ for which we are generally unable to realize a given expected terminal wealth $Ex(T) = d$. This shows one fundamental difference that is brought about by the additional uncertainty.

We have seen that the process $g(t)$ plays a central role in the solution to the mean–variance problem; see Theorem 5.1. It has a clear financial interpretation which we now elaborate. When $r(t)$ is deterministic, it follows from (54) that $g(t)$ is nothing but the discount rate between the current time t and the terminal time T. When $r(t)$ is random, then the equation (45) suggests that $g(t)$ is the price process of a contingent claim with a unit terminal payoff, while $(\sigma(t)^{-1})' \zeta(t)$ is the replicating portfolio of the claim; see Proposition 6.1, p. 394 of Yong and Zhou (1999). Thus, $g(t)$ is the *risk-adjusted discount rate* between t and T; in particular, $g(0)$ is the risk-adjusted discount rate for the entire horizon $[0, T]$. Quantitative properties of this discounting process follow immediately from the Comparison Theorem for nonlinear BSDEs (Theorem 5.1, p. 199 of El Karoui, Peng and Quenez (1997)).

Proposition 5.1 *If $r(t) \ge 0$ a.e. $t \in [0, T]$, P-a.s., then $0 < g(0) \le 1$. If $r(t) \ge \delta$ for some $\delta > 0$ a.e. $t \in [0, T]$, P-a.s., then $0 < g(0) < 1$. Moreover, if (h, η) and $(\tilde{h}, \tilde{\eta})$ are solutions of (45), corresponding to interest rates of $r(t)$ and $\tilde{r}(t)$, respectively, then $r(t) \le \tilde{r}(t)$ implies $h(t) \le \tilde{h}(t)$.*

We note that the optimal variance (50) involves a perfect square. In particular, choosing

$$d = d_0 \equiv \frac{x_0}{g(0)} \tag{58}$$

the associated optimal variance is $\mathrm{Var}\ x_0^*(T) = 0$. Substituting (58) into (53), it follows that the associated optimal investment policy corresponds to $\lambda_0^* = 0$. Substituting this into (52), we obtain the corresponding optimal control

$$u_0^* = -(\sigma\sigma')^{-1}\Big[\Big(B' + \sigma\frac{\Lambda}{p}\Big)(x_0^* - d_0 g) - d_0\sigma\zeta\Big]. \tag{59}$$

By (45), (46) and (47), we obtain

$$\begin{cases} d(x_0^* - d_0 g) &= \Big[A - B(\sigma\sigma')^{-1}\Big(B' + \sigma\frac{\Lambda}{p}\Big)\Big](x_0^* - d_0 g)\, dt \\ & \quad -(x_0^* - d_0 g)\Big[\sigma^{-1}B' + \frac{\Lambda}{p}\Big]'\, dW, \quad t \in [0, T], \\ x_0^*(0) - d_0 g(0) &= 0, \end{cases} \tag{60}$$

and hence,

$$x_0^*(t) - d_0 g(t) = 0, \quad \forall\, t \in [0, T], \ P - a.s.. \tag{61}$$

Therefore, (59) becomes

$$u_0^*(t) = \frac{x_0}{g(0)}\,(\sigma(t)^{-1})'\zeta(t). \tag{62}$$

If $r(t)$ is deterministic, then $\zeta(t) = 0$ and hence, $u_0^*(t) = 0$, meaning that the risk-free investment corresponds to putting all the wealth in the bond and nothing into stocks, which is precisely the result obtained in Zhou and Li (1999). On the other hand, if $r(t)$ is random, then $\zeta(t)$ is generally non-zero and hence $u_0^*(t) \neq 0$. Note that as discussed earlier, the part $(\sigma(t)^{-1})'\zeta(t)$ in (62) is nothing but the replicating portfolio for the contingent claim as specified in (45). In other words, if the interest rate is random, then a risk-free investment is still possible, and uses the above replicating portfolio to perfectly hedge the risk arising from the random interest rate. This leads to the following theorem.

Theorem 5.2 *(***Risk-free portfolio***) A risk-free investment is given by the portfolio (62) with the deterministic terminal wealth given by (58).*

Note that if $r(t) > \delta$ for some $\delta > 0$, then it follows from Proposition 5.1 that $g(0) < 1$ and hence in view of (58) an increase of initial wealth is therefore guaranteed. Moreover, it is seen that the agent can expect a terminal wealth of at least $d_0 \equiv \frac{x_0}{g(0)}$. Hence it is reasonable to restrict the parameter d in the problem (8) to be $d \geq d_0$. (This is why in many literatures the efficient frontier is limited to the portion where $d \geq d_0$.)

If we denote by $\sigma_{x(T)}$ the standard deviation of the terminal wealth, then (50) gives (taking into consideration the above discussion)

$$Ex(T) = \frac{x_0}{g(0)} + \sqrt{\frac{1-p(0)\,g(0)^2}{p(0)}}\frac{1}{g(0)}\sigma_{x(T)}. \tag{63}$$

Hence the efficient frontier in the mean–standard deviation diagram is still a straight line, which is also termed the *capital market line* (see, e.g., Luenberger (1998)). The slope of this line, $k = \sqrt{\frac{1-p(0)\,g(0)^2}{p(0)}}\frac{1}{g(0)}$, is called the *price of risk.*

The so-called *Mutual Fund Theorem*, originally due to Tobin (1958) for single-period investment, is a natural consequence of the mean–variance theory, and is the foundation of the CAPM (*Capital Asset Pricing Model*) (Sharpe (1964)). It turns out that a version of the Mutual Fund Theorem also holds in the continuous-time, random coefficient setting.

Theorem 5.3 (**Mutual Fund Theorem**) *Suppose an efficient portfolio $u_1^*(\cdot)$ is given by (52) corresponding to $d = d_1 > \frac{x_0}{g(0)}$. Then a portfolio $u^*(\cdot)$ is efficient if and only if there is $\alpha \geq 0$ such that*

$$u^*(t) = (1-\alpha)u_0^*(\cdot) + \alpha u_1^*(t), \quad t \in [0,T], \tag{64}$$

where $u_0^*(\cdot)$ is the risk-free portfolio defined in (62).

Proof: We first prove the "if" part. Since both $u_0^*(\cdot)$ and $u_1^*(\cdot)$ are efficient, by the explicit expression of any efficient portfolio given by (52), $u^*(t) = (1-\alpha)u_0^*(\cdot) + \alpha u_1^*(t)$ must be in the form of (52) corresponding to $d = (1-\alpha)d_0 + \alpha d_1$ (also noting that $x^*(\cdot)$ is linear in $u^*(\cdot)$). Hence $u^*(t)$ must be efficient.

Conversely, suppose $u^*(\cdot)$ is efficient corresponding to a certain $d > \frac{x_0}{g(0)} \equiv d_0$. Write $d = (1-\alpha)d_0 + \alpha d_1$. Multiplying

$$u_0^*(t) = -(\sigma(t)\sigma(t)')^{-1}\Big[-d_0\sigma\zeta\Big]$$

by $(1-\alpha)$, multiplying

$$\begin{aligned} u_1^*(t) = \ & -(\sigma(t)\sigma(t)')^{-1}\Big[\Big(B(t)' + \sigma(t)\tfrac{\Lambda(t)}{p(t)}\Big)(x_1^*(t) - d_1 g(t)) - d_1\sigma(t)\zeta(t) \\ & -\tfrac{p(0)\,g(0)^2}{1-p(0)\,g(0)^2}(d_1 - d_0)\Big(\Big(B(t)' + \sigma(t)\tfrac{\Lambda(t)}{p(t)}\Big)g(t) + \sigma(t)\zeta(t)\Big)\Big] \end{aligned}$$

by α, summing them up, and noticing that $\alpha(d_1 - d_0) = d - d_0$, we obtain that $(1-\alpha)u_0^*(t) + \alpha u_1^*(t)$ is represented by (52) with $x^*(t) = (1-\alpha)x_0^*(t) + \alpha x_1^*(t)$ and $d = (1-\alpha)d_0 + \alpha d_1$. This leads to (64). ∎

In the classical single-period setting, the mutual fund theorem asserts that any efficient portfolio is a combination of the risk-free asset and a fund consisting of only stocks (called the *tangent fund*). As a consequence, for any efficient portfolio the allocations among the stocks must have constant proportions independent of the total wealth of the agent. In the present case, however, the efficient portfolio obtained in (52) does not have the above feature. This suggests that the version of the mutual fund theorem in the single-period is no longer valid in the current setting. This is not surprising, though, for even the risk-free portfolio now has to include stocks (Theorem 5.2).

6 Conclusion

In this paper, we have studied the continuous-time mean–variance portfolio selection problem with random interest rate, appreciation rates and volatility coefficients. By treating this as a linearly constrained stochastic LQ problem and using results from the theory of convex optimization, the efficient frontier and associated optimal portfolio are explicitly derived. A key part of our analysis involves proving existence and uniqueness of solutions of a certain nonlinear BSDE. Although of independent interest, these existence results imply the solvability of the unconstrained LQ problems that arise with the introduction of Lagrange multipliers and plays a fundamental role in our derivation. The mean–variance efficient frontier for this problem is a perfect square, suggesting that risk-free investment is still possible when interest rates are random. This paper again demonstrates that the stochastic LQ control can serve as a powerful framework for dealing with certain financial application problems.

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