Portfolio Choice via Quantiles*

Xue Dong He[†] and Xun Yu Zhou[‡]

March 4, 2010

Abstract

A portfolio choice model in continuous time is formulated for both complete and incomplete markets, where the quantile function of the terminal cash flow, instead of the cash flow itself, is taken as the decision variable. This formulation covers a wide body of existing and new models with law-invariant preference measures, including expected utility maximisation, mean-variance, goal reaching, Yaari's dual model, Lopes' SP/A model, behavioural model under prospect theory, as well as those explicitly involving VaR and CVaR in objectives and/or constraints. A solution scheme to this quantile model is proposed, and then demonstrated by solving analytically the goal-reaching model and Yaari's dual model. A general property derived for the quantile model is that the optimal terminal payment is anti-comonotonic with the pricing kernel (or with the minimal pricing kernel in the case of an incomplete market) if the investment opportunity set is deterministic. As a consequence, the mutual fund theorem still holds in a market where rational and irrational agents co-exist.

Key Words. Portfolio choice, continuous time, quantile function, law invariant measure, utility maximisation, Yaari's dual theory, goal-reaching, behavioural finance, probability distortion, mutual fund theorem.

---

*This work was presented at the 2008 Special Semester on Stochastics with Emphasis on Finance in Linz, Austria, the 2009 Midlands Probability Theory Seminars in Warwick, the 2009 Workshop on Optimal Stopping and Singular Stochastic Control Problems in Finance in Singapore, the 2010 Workshop on Foundations of Mathematical Finance at the Fields Institute, Toronto, and at seminars at Columbia, Chinese Academy of Sciences (CAS), Indian Institute of Science (IISc), Oxford, Oslo, Swedish Royal Institute of Technology (KTH), Vienna Institute of Finance, and Yale. We are grateful to the participants at these events, in particular to Nicole El Karoui, Paul Embrechts, Hanqing Jin, Dilip Madan, Jan Obloj, Walter Schachermayer, and Thaleia Zariphopoulou, for their comments. We thank Chris Rogers for making us aware of Dybvig's work in a discussion not directly related to this paper. Zhou acknowledges financial support from Nomura Centre for Mathematical Finance and a start-up fund of the University of Oxford, and both He and Zhou acknowledge research grants from the Oxford–Man Institute of Quantitative Finance.

†Mathematical Institute and Oxford–Man Institute of Quantitative Finance, The University of Oxford, 24–29 St Giles, Oxford OX1 3LB, UK. Email: <hex@maths.ox.ac.uk>.

‡Mathematical Institute and Nomura Centre for Mathematical Finance, and Oxford–Man Institute of Quantitative Finance, The University of Oxford, 24–29 St Giles, Oxford OX1 3LB, UK, and Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. Email: <zhouxy@maths.ox.ac.uk>.

1 Introduction

Study on continuous-time portfolio choice has predominantly centred around expected utility maximisation (including the mean–variance model - although it has its own subtly unique features) since the seminal papers of Samuelson (1969) and Merton (1969). Abundant research around, there have been essentially two approaches developed to solve the utility model. One is the stochastic control or dynamic programming approach, initially proposed by Merton (1969, 1971), which transforms the problem into solving a partial differential equation called the Hamilton-Jacobi-Bellman (HJB) equation. The other one is the martingale approach. This approach, developed by Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), and Pliska (1986), employs a martingale characterisation to turn the dynamic wealth equation into a static budget constraint and then identifies the optimal terminal wealth via solving a static optimisation problem. If the market is complete, an optimal strategy is derived by replicating the optimal terminal wealth in the same spirit of perfectly hedging a contingent claim. Karatzas and Shreve (1998) gives a systematic account on this approach. In an incomplete market with possible portfolio constraints, the martingale approach is further developed to include the so-called convex duality machinery; see, e.g., Cvitanić and Karatzas (1992), Kramkov and Schachermayer (1999), and Goll and Rüschendorf (2001).

However, it has been known for a long time that some of the basic tenets of the expected utility as a risk preference measure are systematically violated in practice. Hence, many alternative preference measures have been put forth, notably Yaari's "dual theory of choice" (Yaari 1987) which attempts to resolve a number of puzzles and paradoxes associated with the expected utility theory (although, as Yaari 1987 admits, the dual theory would lead to other paradoxes). In this theory, instead of applying a utility which is essentially a "distortion" in payment, one distorts the probability decumulative function of the payment. This probability distortion function, as Yaari shows, represents the risk preference in a different way. In particular, risk aversion is characterised by a convex - rather than concave - distortion. Other theories developed along this line of involving subjective probability distortions include Lopes' SP/A model (Lopes 1987 and Lopes and Oden 1999) and, most significantly, Kahneman and Tversky's prospect theory (Kahneman and Tversky 1979 and Tversky and Kahneman 1992), both in the modern behavioural decision-making paradigm.

It is a natural problem to formulate and solve a portfolio choice model involving probability distortions; yet a key technical challenge is that such a distortion renders a nonlinear expectation that destroys the time-consistency necessary for the dynamic programming approach as well as the convexity necessary for the convex duality approach.

Another large set of portfolio choice problems could involve *explicitly* probability and VaR/CVaR/quantile, instead of expectation, in their objectives and/or constraints. For instance, the goal-reaching problem, initiated by Kulldorff (1993) and Heath (1993), investigated extensively by Browne (1999, 2000), and later extended to hedging of contingent claims by Föllmer and Leukert (1999) and Spivak and Cvitanić (1999), is to maximise the probability of the terminal cash flow in excess of a given level or a given benchmark. Other models could include VaR/CVaR/quantile as risk measures.[1] It is well known that these

[1]Oddly enough, despite an extensive literature search we have not found any study on *continuous-time* diffusion models with VaR/CVaR/quantile appearing explicitly either in objectives or in constraints (note in particular that, although the title of Föllmer and Leukert 1999 includes the word "quantile", the paper deals with a problem of maximising the probability of successfully hedging a contingent claim). Kataoka (1963)

problems cannot be solved, at least directly, by conventional approaches. For example, although one could write a probability as an expectation of an indicator function, the latter is inherently non-convex. Dynamic programming, on the other hand, becomes inapplicable for problems explicitly involving VaR/quantile.

In summary, there could be many alternative, ad hoc, and economically sensible portfolio choice models where the nice properties (such as time-consistency and convexity) we have all along taken for granted would be missing.[2] Ad hoc approaches have been developed to solve a very limited number of these models. A question arises as to whether it is possible to establish/develop a unified, general framework/approach to cover/solve all the aforementioned models (and many others) once and for all. The answer is, as we will show subsequently, affirmative, and all it takes is a new perspective compared to the standard portfolio selection literature in mathematical finance.

To reach the answer to the preceding question, this paper explores and exploits two essential commonalities among all the seemingly different models mentioned above. One is that all the preference/performance measures involved are *law-invariant*. That is, agents care about only the probability distribution of the terminal cash flow, rather than the cash flow itself. The other commonality is that all the preferences can be written as a *distorted mean* where both the payment and its distribution function are alterted; see (15) below. An analysis shows that if we change the decision variable from the terminal cash flow X to $G(Z)$, where G is the quantile function of X and Z is *any* uniform random variable on $[0, 1]$, then the preference reduces to a linear expectation (under a possibly different probability measure)! This change of variable does not change the preference value because X and $G(Z)$ are *always* identical in law. There is, however, another issue to be addressed with this technique because the budget constraint is inherently law-*variant* so the preceding change of variable would in general violate the constraint. However, a dual argument originally due to Dybvig (1988), in the classical economic spirit that (loosely speaking) maximising a performance measure is equivalent to minimising the associated cost, reveals that X can also be replaced by $G(Z)$ in the budget constraint where Z is a *particular* uniform variable generated by the pricing kernel.

Based on these analyses, we are prompted to formulate a portfolio choice model, very general in the sense that it covers all the aforementioned models and many others, where the optimal quantile function of the terminal payment is to be chosen. Since we have recovered linear expectation in the quantile model, we are able to propose a general solution scheme based on the Lagrange approach and a weak/strong duality argument. Once the optimal quantile function is obtained, the corresponding optimal terminal cash flow can be recovered by a simple formula, which as a by-product indicates that it is anti-comonotonic with the pricing kernel. If the market is complete, then the optimal portfolio is the one replicating the obtained terminal payoff. If the market is incomplete, then we seek the

---

probably is the first to include a quantile criterion in a single-period portfolio choice model. Grigor'ev and Kan (2004) and Dhaene *et al.* (2005) both consider dynamic portfolio selection problems with explicitly present quantile-related criteria, but their settings are both very specific and restrictive (the former studies a discrete-time model with uniform distribution of returns, and the latter confines the portfolios to the class of "constant mix" ones). For more recent study/survey on quantile-related performance measures see Föllmer and Schied (2004), Dhaene *et al.* (2004), and Cherny and Madan (2008).

[2]Some models may lack both the time-consistency and the convexity. For examples, Lopes' SP/A model has both a probability distortion and a probability constraint, and Kahneman and Tversky's propsect model has probability distortions and an S-shaped utility function.

so-called minimal pricing kernel which exists in some cases such as when the investment opportunity set is deterministic.

We demonstrate our formulation and solution procedure by applying them to the goal-reaching model and Yaari's model. Analytical solutions are obtained for both models which turn out to be of the same binary, "win-or-lose-all" structure, although there are subtle - and indeed substantial - differences between the two in terms of the implied risk–return preferences. It should be noted that, while our approach gives an alternative way to that of Browne (1999) in solving the goal-reaching model, we actually extend the setting to include possibly stochastic opportunity sets for which Browne's HJB method would fail.[3] Moreover, the formulation and solution to the continuous-time Yaari's dual model are completely new to our best knowledge.

The quantile formulation also enables us to establish a mutual fund theorem at least in the case of a deterministic opportunity set (complete or incomplete market regardless). This has a potentially important consequence in developing a capital asset pricing model for a market where rational (utility maximising) and irrational (behavioural) agents co-exist.

We finally remark that it is not new at all in the economics (including mathematical economics) literature to express risk preferences in terms of quantiles or distribution functions; see, to name but a few, Machina (1982), Yaari (1987), and Dybvig (1988). However, to our best knowledge the quantile formulation and its general solution procedure for possibly non-convex/concave utility functions and non-convex/concave probability distortions are new in the portfolio choice literature especially in the continuous-time setting[4, 5]. The idea was in fact around in Jin and Zhou (2008) for overcoming the difficulties arising from the nonconcavity and time-inconsistency in the continuous-time portfolio selection model under the prospect theory, but it was used there in an ad hoc nature. The present paper

[3]The statistical hypothesis testing argument of Föllmer and Leukert (1999) and the martingale approach of Spivak and Cvitanić (1999) could also solve the goal-reaching model with a stochastic opportunity set.

[4]Despite its title, Dybvig (1988) does not formulate or solve any specific class of portfolio choice problems per se. Instead, it is concerned with the *dual problem* of portfolio choice, namely, to characterise the lowest cost of any given terminal distribution. As discussed above, the dual argument is indeed one of the main theoretical foundations of the quantile formulation here - although we were not aware of Dybvig' work when we were carrying out this research.

[5]After this paper was accepted, the papers by Schied (2004) and Carlier and Dana (2006) came to our attention. Schied (2004) introduces a quantile-based optimization technique to solve a specific class of convex, robust portfolio selection problems. In Carlier and Dana (2006), a more general class of quantile-based calculus of variations problems with law-invariant *concave* criteria are formulated, and the issues of existence of solutions, necessary conditions for optimality, and sufficient conditions for the regularity of solutions are addressed. These results are closely related to the ones in this paper, but there are important differences. On p.130, Carlier and Dana (2006), it is stated that "we shall also require v to be concave (in the *random variable* X) ...". This requirement is violated by our model (or indeed any model with nontrivial probability distortions). Of course, in Section 3.1 there (which contains results closest to ours), this assumption does not seem to be necessary. However, therein the criterion v is required to be strictly second-order stochastic dominance (SSD) preserving, which is necessary in proving the key Proposition 3.1. Notice that being SSD preserving is quite a strong assumption; it is strictly stronger than law invariance plus monotonicity (which are the only two essential assumptions imposed in our paper here) – see the bottom of p. 130 in Carlier and Dana (2006). Only when v is concave do the two coincide – see Proposition 2.4. In other words, certain concavity of the criteria is implicitly assumed and seems to be critical in the arguments of Carlier and Dana (2006). In contrast, one of the key points of our paper is to abandon the convexity/concavity (be it in the utilities or in the probability distortions) assumption altogether. Indeed, because of the S-shaped utility functions and the reversed S-shaped probability distortions involved, the criterion in a general prospect theory model is inherently nonconcave in either cash flows or in there quantiles.

attempts to systematically utilise and develop the quantile approach to solving (amongst others) possibly non-expected and non-convex/concave portfolio choice problems.

The remainder of this paper is organised as follows. Section 2 proposes the general quantile model motivated by five concrete models. In Section 3, a solution scheme is described for the general model, followed by its application to two specific models – the goal-reaching and Yaari's models – with explicit solutions. Economic interpretations of the solutions obtained are discussed. Section 4 is devoted to the incomplete market, and Section 5 to the mutual fund theorem. Finally, Section 6 concludes.

2 A New Portfolio Choice Formulation

In this section we set up the continuous-time market, and explain the background and motivation of a new portfolio choice formulation via five concrete models.

2.1 A continuous-time market

Let $T > 0$ be given and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ be a filtered probability space on which is defined a standard $\mathcal{F}_t$-adapted n-dimensional Brownian motion $W(t) \equiv (W^1(t), \cdots, W^n(t))^\top$ with $W(0) = 0$. It is assumed that $\mathcal{F}_t = \sigma\{W(s) : 0 \le s \le t\}$, augmented by all the P-null sets. Here and henceforth $A^\top$ denotes the transpose of a matrix A, and $a_+ := \max(a, 0)$, $a_- := \max(-a, 0)$ for $a \in \mathbb{R}$.

We define a continuous-time financial market following Karatzas and Shreve (1998). In the market there are $m + 1$ assets being traded continuously. One of the assets is a bank account whose price process $S_0(t)$ is subject to the following equation:

$$dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T]; \quad S_0(0) = s_0 > 0, \tag{1}$$

where the interest rate $r(\cdot)$ is an $\mathcal{F}_t$-progressively measurable, scalar-valued stochastic process with $\int_0^T |r(s)|ds < +\infty$ a.s.. The other m assets are stocks whose price processes $S_i(t)$, $i = 1, \cdots, m$, satisfy the following stochastic differential equation (SDE):

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW^j(t)\right], \quad t \in [0, T]; \quad S_i(0) = s_i > 0, \tag{2}$$

where $b_i(\cdot)$ and $\sigma_{ij}(\cdot)$, the appreciation and volatility rates respectively, are scalar-valued, $\mathcal{F}_t$-progressively measurable stochastic processes with

$$\int_0^T \left[\sum_{i=1}^{m} |b_i(t)| + \sum_{i=1}^{m}\sum_{j=1}^{n} |\sigma_{ij}(t)|^2\right] dt < +\infty, \text{ a.s..}$$

Set the excess rate of return process

$$B(t) := (b_1(t) - r(t), \cdots, b_m(t) - r(t))^\top,$$

and define the volatility matrix process $\sigma(t) := (\sigma_{ij}(t))_{m\times n}$. Basic assumptions imposed on the market parameters throughout this paper are summarised as follows:

Assumption 1 *There exists an $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^n$-valued process $\theta_0(\cdot)$ with $Ee^{\frac{1}{2}\int_0^T |\theta_0(t)|^2 dt} < +\infty$ such that*

$$\sigma(t)\theta_0(t) = B(t), \quad \textit{a.s., a.e. } t \in [0, T].$$

Assumption 2 *There exist $s_2 \geq s_1 > 0$ such that $s_1 \leq S_0(T) \leq s_2$.*

Assumption 1 is only slightly stronger than the standard no-arbitrage assumption due to the additional Novikov condition; see Karatzas and Shreve (1998) for details. Assumption 2 holds when the risk-free rate is bounded.

Consider an agent, with an initial endowment $x_0 > 0$ and an investment horizon $[0, T]$, whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Assume that the trading of shares takes place continuously in a self-financing fashion and there are no transaction costs. Then $x(\cdot)$ satisfies (see, e.g., Karatzas and Shreve 1998)

$$dx(t) = \left[r(t)x(t) + B^\top(t)\pi(t)\right]dt + \pi(t)^\top\sigma(t)dW(t), \quad t \in [0, T]; \quad x(0) = x_0, \tag{3}$$

where $\pi_i(t), \quad i = 1, 2 \cdots, m,$ denotes the total market value of the agent's wealth in the i-th asset at time t. The process $\pi(\cdot) \equiv (\pi_1(\cdot), \cdots, \pi_m(\cdot))^\top$ is called a *portfolio* if it is $\mathcal{F}_t$-progressively measurable with

$$\int_0^T |\sigma(t)^\top\pi(t)|^2 dt < +\infty \ \text{ and } \int_0^T |B(t)^\top\pi(t)|dt < +\infty, \ \text{ a.s..}$$

and it is tame (i.e., the corresponding discounted wealth process, $S_0(t)^{-1}x(t)$, is almost surely bounded from below – although the bound may depend on $\pi(\cdot)$). It is standard in the continuous-time literature that a portfolio be required to be tame so as to, among other things, exclude the notorious doubling strategy.

There may be other constraints on the portfolios specific to a given problem, such as prohibition of shorting or bankruptcy. A portfolio is called *admissible* if it satisfies all the given constraints. Let Π be the set of all admissible portfolios. It is important to note that Π does not depend on the initial position x_0. The agent evaluates each admissible portfolio $\pi(\cdot)$ via a certain performance (or preference) measure, denoted by $J(x_0, \pi(\cdot))$. The precise forms of $J(x_0, \pi(\cdot))$ are dictated by individual problems, and will be discussed fully in the sequel. The objective of a portfolio selection problem is, for a given initial endowment x_0, to choose an optimal portfolio whose performance value achieves the supremum of $J(x_0, \pi(\cdot))$ over Π. Denote by $v(x_0)$ this supremum value.

This paper aims to introduce a very general portfolio choice formulation which in particular covers both the neoclassical (utility maximisation) and behavioural models. To do so we need the following "minimal" assumption on the models we are able to include.

Assumption 3 *For an initial position x_0 and an admissible portfolio $\pi(\cdot)$, if $\hat{x}_0 > x_0$ then there is an admissible portfolio $\hat{\pi}(\cdot)$ such that $J(\hat{x}_0, \hat{\pi}(\cdot)) > J(x_0, \pi(\cdot))$.*

The economic sensibility of this assumption is clear: with more initial budget the agent will be able to do strictly better. One may appreciate that this is a very weak assumption[6], and any portfolio model violating this would be abnormal. Indeed, all the five concrete models to be presented in the next subsection satisfy this assumption; see discussions at the end of Section 2.2.

[6]Indeed, it is easy to show that Assumption 3 is even weaker than the following very reasonable assumption: $v(\hat{x}_0) > v(x_0)\ \forall \hat{x}_0 > x_0$.

2.2 Five motivating models

In this subsection we motivate our new portfolio choice formulation via five concrete models. These models appear quite different in terms of their economical interpretations and mathematical formulations; yet the commonalities among them will be explored, leading to a universal framework and approach covering all of them (and more). In the remainder of this section we assume that the underlying continuous-time market is complete or equivalently the process $\theta_0(\cdot)$ in Assumption 1 is unique. The study of an incomplete market will be deferred to Section 4. Define

$$\rho(t) := \exp\left\{-\int_0^t \left[r(s) + \frac{1}{2}|\theta_0(s)|^2\right] ds - \int_0^t \theta_0(s)^\top dW(s)\right\}, \tag{4}$$

the pricing kernel or state density price process. Denote $\rho := \rho(T)$. It is clear that under Assumptions 1 and 2, $0 < \rho < +\infty$ a.s. and $0 < E\rho < +\infty$. Let

$$\begin{aligned} \bar{\rho} &\equiv \text{esssup}\ \rho := \sup\{a \in \mathbb{R} : P\{\rho > a\} > 0\}, \\ \underline{\rho} &\equiv \text{essinf}\ \rho := \inf\{a \in \mathbb{R} : P\{\rho < a\} > 0\}. \end{aligned} \tag{5}$$

In view of the martingale approach a portfolio choice problem in this economy boils down to determining the optimal terminal wealth.

Let $\mathbb{F}$ be the set of cumulative distribution functions (CDFs hereafter) of all the lower bounded random variables taking values on $\mathbb{R}$, i.e.

$$\mathbb{F} = \{F(\cdot) : \mathbb{R} \to [0,1],\ \text{nondecreasing, } \textit{càdlàg},\ F(a-) = 0 \text{ for some } a \in \mathbb{R} \text{ and } F(+\infty) = 1\}.$$

The lower boundedness above corresponds to the required tameness of portfolios. For any $F(\cdot) \in \mathbb{F}$, denote by $F^{-1}(\cdot)$ its left-inverse, i.e.,

$$F^{-1}(t) = \inf\{x \in \mathbb{R} :\ F(x) \geq t\} = \sup\{x \in \mathbb{R} :\ F(x) < t\}, \quad t \in [0,1].$$

Let $\mathbb{G} := \{F^{-1}(\cdot) :\ F(\cdot) \in \mathbb{F}\}$ be the corresponding set of quantile functions, or

$$\mathbb{G} = \left\{G(\cdot) : [0,1] \to \mathbb{R}^+,\ \text{nondecreasing, left continuous, } G(0) = -\infty, G(0+) > -\infty\right\},$$

where $G(1) := G(1-)$.

Model 1: Expected Utility Maximisation

$$\begin{array}{ll} \underset{X}{\text{Max}} & Eu(X) \\ \text{subject to} & E[\rho X] = x_0,\ X \geq 0,\ X \text{ is } \mathcal{F}_T \text{ measurable,} \end{array} \tag{6}$$

where $u(\cdot)$ is a utility function, $E[\rho X] = x_0$ is the budget constraint, and $X \geq 0$ is the no-bankruptcy constraint (which may be absent in some variants of the model). This is the classical utility model initiated by Samuelson (1969) and Merton (1969) with extensive research thereafter. Under the concavity assumption on the utility function (representing the agent risk-aversion) it is a simple exercise via a Lagrange technique to solve the above optimisation problem. As explained earlier the solution X^* to this static optimisation problem is the optimal terminal cash flow that ought to be achieved. The optimal portfolio will then be the one replicating X^*.

It is interesting to look more closely at the preference measure, $Eu(X)$, in this model. Recall for a random payoff $X \geq 0$ its mean is

$$E[X] = \int_0^{+\infty} x dF_X(x) \tag{7}$$

where $F_X(\cdot)$ is the CDF of X, while

$$Eu(X) = \int_0^{+\infty} u(x) dF_X(x). \tag{8}$$

Hence, compared with the mean evaluation (7), the expected utility (8) essentially applies a utility function to *distort* the payment outcomes when evaluating a random payment. The way the distortion takes place reflects the agent attitude towards risk, which is captured mathematically by the convexity and/or concavity of the utility function.

Model 2: Goal Reaching

$$\begin{array}{ll} \underset{X}{\text{Max}} & P(X \geq b) \\ \text{subject to} & E[\rho X] = x_0,\ X \geq 0,\ X \text{ is } \mathcal{F}_T \text{ measurable,} \end{array} \tag{9}$$

where $b > 0$ is the goal (level of wealth) intended to be reached by time T. This is called the goal-reaching problem, which was proposed by Kulldorff (1993), Heath (1993), and studied extensively (including various extensions) by Browne (1999, 2000).

Economically, the goal-reaching model is different from the expected utility model; see a detailed discussion in Browne (1999). Technically, it is not covered by the standard utility model either, since

$$P(X \geq b) = \int_0^{\infty} \mathbf{1}_{(x \geq b)} dF_X(x), \tag{10}$$

and the indicator function $\mathbf{1}_{(x \geq b)}$ is not concave. Browne (1999, 2000) primarily employs the dynamic programming and HJB equation to solve the problem.

Model 3: Yaari's Dual Theory

$$\begin{array}{ll} \underset{X}{\text{Max}} & \int_0^{\infty} w(P(X > x))dx \\ \text{subject to} & E[\rho X] = x_0,\ X \geq 0,\ X \text{ is } \mathcal{F}_T \text{ measurable,} \end{array} \tag{11}$$

where $w : [0,1] \to [0,1]$ is a function called a *probability distortion* or *weighting function* representing a subjective inflation/deflation of the true probability. It is a generally non-linear, non-decreasing (so the distortion at least preserves the order of the probabilities) function with $w(0) = 0$ and $w(1) = 1$ (so there is no distortion on sure events).

The preference measure in (11) was first put forward by Yaari (1987) as a "dual theory of choice under risk" to the expected utility theory. If we write (via Fubini's theorem)

$$\int_0^{\infty} w(P(X > x))dx = \int_0^{\infty} x d\left[-w(1 - F_X(x))\right], \tag{12}$$

then we see that, in contrast to the expected utility (8) that distorts the payment, Yaari's measure (12) distorts the CDF of the payment instead. Yaari (1987), Theorem 2, further shows that the risk preference can also be captured by this distortion; specifically the agent is risk-averse if and only if w is convex.

Yaari's dual measure is one of the so-called *non-expected utilities* which, as Yaari (1987) argues, can explain a number of paradoxes associated with the expected utility theory, although it leads to new "dual" paradoxes at the same time. It would be interesting to explore what solutions this dual measure would generate in the context of portfolio choice. There is a rather preliminary study, Hamada, Sherris and van der Hoek (2001), on a discrete-time portfolio choice model featuring Yaari's measure, whereas the continuous-time model (11) is new to our best knowledge. The technical difficulties in solving (11) include the non-concavity of (12) in X due to the distortion w, and the time-inconsistency of the measure because of (12) being essentially a nonlinear expectation (also known as the Choquet expectation) under the capacity $w \circ P$. In particular, time-consistency is the foundation of the dynamic programming, the latter being the primary approach in treating dynamic portfolio choice problems.

Model 4: Lopes' SP/A Theory

$$
\begin{array}{ll}
\underset{X}{\text{Max}} & \int_0^\infty w(P(X > x))dx \\
\text{subject to} & P(X \geq A) \geq \alpha, \\
& E[\rho X] = x_0,\ X \geq 0,\ X \text{ is } \mathcal{F}_T \text{ measurable,}
\end{array}
\tag{13}
$$

where w is now called (in Lopes' terminology) the *decumulative weighting function* in the SP/A theory, A the aspiration level, and α the confidence level of the final payment exceeding the aspiration. The SP/A theory, developed by Lopes (1987), is widely regarded as an instantiation of the psychological/behavioural decision-making model, where SP stands for a *security–potential* criterion and A for an *aspiration* criterion. Model (13) looks similar to the Yaari model (11) except for the additional aspiration constraint; nevertheless w in (13) actually has a more specific economical interpretation. Lopes (1987) specifies w as a weighted combination of a convex function and a concave one, where the convexity represents the security (risk-aversion) and the concavity captures the potential (risk-seeking).

Lopes and Oden (1999) apply the SP/A theory to formulate and solve a single-period portfolio selection model. However, it appears that the continuous-time counterpart (13) has not been studied in the literature at all.

Model 5: Kahneman and Tversky's Prospect Theory

$$
\begin{array}{ll}
\underset{X}{\text{Max}} & \int_0^\infty w_+\left(P\left(u_+\left((X-B)_+\right) > x\right)\right)dx \\
& \quad - \int_0^\infty w_-\left(P\left(u_-\left((X-B)_-\right) > x\right)\right)dx \\
\text{subject to} & E[\rho X] = x_0,\ X \text{ is } \mathcal{F}_T \text{ measurable and a.s. bounded from below,}
\end{array}
\tag{14}
$$

where B, an $\mathcal{F}_T$ measurable random variable, is a *reference point* in wealth, $u_+(\cdot)$ and $u_-(\cdot)$ are the utility and disutility functions of *gains* (excesses of wealth over B) and *losses* (shortfalls from B) respectively, and w_+ and w_- are probability distortions on gains and losses respectively.

The preference measure in the above model was proposed and developed by Kahneman and Tversky, which is the most important component in the Nobel-prize-winning propsect theory[7]. There has been active research in incorporating propsect theory into portfolio choice, albeit mainly restricted hitherto to the single-period case. Study on continuous-time models such as (14) has been started only recently; see Berkelaar, Kouwenberg and Post (2004), and Jin and Zhou (2008).

It remains to show that the standing Assumption 3 holds naturally for the above models under reasonable conditions. For the utility model (6), the performance measure is $J(x_0, \pi(\cdot)) = Eu(x(T))$ where $x(T) := X$ is the terminal wealth under portfolio $\pi(\cdot)$ starting from the initial endowment $x(0) = x_0$. If $\hat{x}_0 > x_0$, we then define

$$\hat{X} := \frac{\hat{x}_0 - x_0}{E\rho} + X.$$

Clearly, $\hat{X}$ is $\mathcal{F}_T$ measurable, $\hat{X} > X$ a.s. and $E[\rho\hat{X}] = \hat{x}_0$. Now, assuming that $u(\cdot)$ is strictly increasing and letting $\hat{\pi}(\cdot)$ be the replicating portfolio of $\hat{X}$, we have $J(\hat{x}_0, \hat{\pi}(\cdot)) = Eu(\hat{X}) > Eu(X) = J(x_0, \pi(\cdot))$.

The same argument applying to Yaari's model (11) and Lopes' model (13) yields that Assumption 3 is valid for the two if w is strictly increasing. For the same reason, the prospect model (14) satisfies the assumption if all $w_\pm$ and $u_\pm$ are strictly increasing.

The above argument, however, does not apply to the goal-reaching model (9) because in general it could hold that $P(\hat{X} \geq b) = P(X \geq b)$ even though $\hat{X} > X$ a.s. We use a different technique instead. Let $J(x_0, \pi(\cdot)) = P(X \geq b)$ where X is the terminal wealth under portfolio $\pi(\cdot)$ starting from the initial endowment $x(0) = x_0$. Consider $\hat{x}_0$ with $bE[\rho] > \hat{x}_0 > x_0$. (If $bE[\rho] \leq \hat{x}_0$ then the corresponding optimal value, $v(\hat{x}_0)$, is 1 which is a trivial case.) Then $P(X < b) > 0$. Find an $\mathcal{F}_T$ measurable set $A \subset \{X < b\}$ such that $P(A) > 0$ and $bE[\rho 1_A] \leq \hat{x}_0 - x_0$. Define $\hat{X} := a1_A + X1_{A^c}$ where

$$a := \frac{\hat{x}_0 - E[\rho X 1_{A^c}]}{E[\rho 1_A]} \geq \frac{x_0 + bE[\rho 1_A] - E[\rho X 1_{A^c}]}{E[\rho 1_A]} \geq b.$$

In fact the above a was chosen so that $E[\rho\hat{X}] = \hat{x}_0$. Clearly $\hat{X} \geq 0$ and

$$P(\hat{X} \geq b) = P(\hat{X} \geq b|A)P(A) + P(\hat{X} \geq b|A^c)P(A^c) = P(A) + P(X \geq b) > P(X \geq b).$$

Therefore, Assumption 3 holds *unconditionally* for (9).

2.3 Formulation via quantiles

Among the preceding five models, the last three involve non-expected utilities due to the probability distortions; hence the standard approaches such as convex duality and dynamic programming fail to apply. The questions we are going to address are whether we can

[7]The *prospect theory* was first introduced in Kahneman and Tversky (1979), and later modified to the so-called *cumulative prospect theory* in Tversky and Kahneman (1992) so as to be consistent with the first-order stochastic dominance. On the other hand, in the works of Kahneman and Tversky the behavioural measure is defined on prospects with discrete outcomes, while the one in (14) is a natural generalisation that covers both continuous and discrete outcomes.

solve the last three models, and whether in addition we can establish/develop a unified framework/approach to cover/solve all the five models (and many others) at the same time. We will show that the answers are positive if we take a different perspective compared with the one taken for granted in expected utility maximisation. A first and key step to reach the answers is to find the commonalities among the models above. Notice that all the preference measures in those models can be written in the following general form:

$$C(X) := \int_{-\infty}^{\infty} u(x) d\left[w(F_X(x))\right] \tag{15}$$

where $u(\cdot)$ and $w : [0,1] \to [0,1]$ are both nonlinear[8]. In essence, (15) is a *modified* mean of the cash flow X where *both* the cash flow and its probability distribution are distorted[9]. While $C(X)$ appears to be a non-expected measure mainly due to the presence of w, by letting $z = F_X(x)$ in (15) we have (assuming that w is differentiable)

$$C(X) = \int_0^1 u\left(F_X^{-1}(z)\right) d\left(w(z)\right) = \int_0^1 u\left(F_X^{-1}(z)\right) w'(z) dz = E\left[u(G(Z))w'(Z)\right] \tag{16}$$

where Z is *any* uniform random variable on $(0,1)$ (we write $Z \sim U(0,1)$) and $G = F_X^{-1}$, the *quantile function* of X. Hence, by regarding G (a quantile) as the decision variable, instead of X (a random variable), we recover linear expectation.[10]

Note that the law-invariant nature of the performance measure $C(X)$ is essential in the above treatment. Next, to have a complete quantile formulation, it remains to express the budget constraint $E[\rho X] = x_0$, a constraint inherent to any continuous-time portfolio choice model, in terms of also the quantile of X. An obstacle for doing this is that ρ and X are two possibly correlated random variables, and the correlation is generally unknown. In other words, $E[\rho X]$ is law-variant (in X). To get around we need to exploit some "dual" property of the underlying optimisation problem, subtly based upon the minimal assumption introduced earlier, namely Assumption 3, along with the following additional assumption on the pricing kernel ρ:

Assumption 4 *ρ admits no atom.*

This assumption will be in force hereafter. It is satisfied, in particular, when $r(\cdot)$ and $\theta(\cdot)$ are deterministic with $\int_0^T |\theta(t)|^2 dt \neq 0$ (in which case ρ is a nondegenerate lognormal random variable).

Denote by $F_\rho(\cdot)$ the CDF of ρ and $Z_\rho := 1 - F_\rho(\rho)$. Because ρ is atom-less, $Z_\rho \sim U(0,1)$ and we can express ρ in terms of Z_ρ: $\rho = F_\rho^{-1}(1 - Z_\rho)$ a.s.. The following lemma, derived[11] in Jin and Zhou (2008) Theorem B.1, is crucial.

---

[8]Strictly speaking, the preference measure of the propsect model, Model 5, is the difference between the two terms of the form (15). As discussed below, our approach here applies to Model 5 as well.

[9]General preference measures involving both utility functions and non-additive probabilities have been proposed in, e.g., Quiggin (1982) and Schmeidler (1989), albeit for discrete probability spaces.

[10]Alternatively (but equivalently), we may take the random variable $Y := G(Z)$ as the new decision variable, and (16) reduces to the classical expected utility criterion under a new probability measure that has a Radon–Nikodym density $w'(Z)$ with respect to P. Note that $G(Z)$ has the same probability law as X, but in general $G(Z) \neq X$ as random variables.

[11]The essential ideas contained in Lemma 1 were first put forth in Dybvig (1988), Theorems 2 and 3. The exact form of the lemma needed for the present paper was proved, with a different proof than Dybvig (1988), in Jin and Zhou (2008).

Lemma 1 *Suppose Assumption 4 holds. Then $E[\rho G(Z_\rho)] \leq E[\rho X]$ for any lower bounded random variable X whose quantile is G. Furthermore, if $E[\rho G(Z_\rho)] < \infty$, then the inequality becomes equality if and only if $X = G(Z_\rho)$, a.s..*

Recall that $E[\rho X]$ is the $t = 0$ price of a future ($t = T$) random cash flow X. The economic interpretation of this lemma is that one can always replace a random payment X by $Y := G(Z_\rho)$, which has the *same probability law* as X, yet with no greater (and possibly smaller) cost. Notice this replacement would not change the preference measure (16), including of course those of the aforementioned five models, due to the law-invariance. Hence a dual argument yields that at an optimal solution X^* it must hold that $E[\rho G^*(Z_\rho)] = E[\rho X^*]$ where G^* is the quantile of X^*. (Indeed, if $y_0 := E[\rho G^*(Z_\rho)] < E[\rho X^*] =: x_0$, then $G^*(Z_\rho)$ would achieve the same performance value with a strictly smaller budget. By Assumption 3, the agent could *strictly* increase the performance value with the original budget $x_0 > y_0$, hence contracting the optimality of X^*.) This, in turn, leads to $X^* = G^*(Z_\rho)$, a.s. in view of Lemma 1.

The above argument shows that an optimal solution X^* of all the models in this section, or indeed any continuous-time model satisfying Assumptions 3 and 4, must be in the form $G^*(Z_\rho)$ where G^* is a quantile and Z_ρ is a *particular* uniform random variable $Z_\rho = 1 - F_\rho(\rho)$. In other words, to find an optimal solution we need only to search among the random variables of the form $G(Z_\rho)$ where $G \in \mathbb{G}$. Since $\rho = F_\rho^{-1}(1 - Z_\rho)$ a.s., we can replace the budget constraint $E[\rho X] = x_0$ by

$$E[F_\rho^{-1}(1 - Z_\rho)G(Z_\rho)] = x_0. \tag{17}$$

Now we are ready to formulate our general portfolio choice model via quantiles:

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U(G(\cdot)) = E\left[u(G(Z_\rho))w'(Z_\rho)\right] \\ \text{subject to} & E[F_\rho^{-1}(1 - Z_\rho)G(Z_\rho)] = x_0, \\ & G(\cdot) \in \mathbb{G} \cap \mathbb{M}, \end{array} \tag{18}$$

where $Z_\rho = 1 - F_\rho(\rho)$, $\mathbb{G}$ is the set of quantile functions of lower bounded random variables and $\mathbb{M}$ specifies some other constraints. For instance, the no-bankruptcy constraint $X \geq 0$ can be translated into $\mathbb{M} = \{G(\cdot) : G(0+) \geq 0\}$.

Sometimes it is more convenient to consider the following integral version of (18):

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U(G(\cdot)) = \int_0^1 u\left(G(z)\right) w'(z) dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(1 - z)G(z)dz = x_0. \\ & G(\cdot) \in \mathbb{G} \cap \mathbb{M}. \end{array} \tag{19}$$

We have demonstrated that the above formulation generalises the five concrete models presented in the previous subsection[12]. In fact, it is general enough to cover many other models such as the continuous-time Markowitz model[13], models explicitly involving VaR, CVaR or quantile functions in performance measures and/or constraints.

[12] To be precise, Model 5 is not *directly* covered by the above formulation because its objective is the difference of two terms. However, a key step in solving Model 5, as carried out in Jin and Zhou (2008), is to decompose the problem into two subproblems each of which is of the form (18).

[13] Assumption 3 holds for the Markowitz model if one is interested in only the nonsatiation portion of the Markowitz efficient frontier.

Finally, we reiterate that the above formulation depends on the market completeness since it involves explicitly a pricing kernel via Z_ρ. The incomplete market case will be dealt with in Section 4.

3 Solutions

In this section we first outline the general solution scheme to solving (18) or (19) in a complete market, and then illustrate the scheme by solving explicitly the goal-reaching model and Yaari's dual model.[14] Notice that the results on Yaari's model are completely new to our best knowledge.

The general scheme starts with removing the budget constraint in (19) via a Lagrange multiplier $\lambda \in \mathbb{R}$ and considering the following problem

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U_\lambda(G(\cdot)) := \int_0^1 u\left(G(z)\right) w'(z)dz - \lambda\left(\int_0^1 F_\rho^{-1}(1-z)G(z)dz - x_0\right) \\ \text{subject to} & G(\cdot) \in \mathbb{G} \cap \mathbb{M}. \end{array} \tag{20}$$

In solving the above problem one usually ignores the constraint, $G(\cdot) \in \mathbb{G} \cap \mathbb{M}$, in the first instance, since in many cases the optimal solution of the resulting unconstrained problem could be modified (without affecting the objective value) to satisfy this constraint under some reasonable assumptions (see concrete examples below). For some cases such a modification could be technically challenging; see for example the SP/A model tackled in He and Zhou (2008). In other cases the constraint may need to be dealt with separately, via techniques specific to each problem.

Once (20) is solved with an optimal solution $G^*_\lambda(\cdot)$, one then finds $\lambda^* \in \mathbb{R}$ that binds the original budget constraint, namely,

$$\int_0^1 F_\rho^{-1}(1-z)G^*_{\lambda^*}(z)dz = x_0.$$

The existence of such λ^* can usually be obtained by examining the monotonicity and continuity of $f(\lambda) := \int_0^1 F_\rho^{-1}(1-z)G^*_\lambda(z)dz$ in λ. Moreover, if the strict monotonicity can be established, then λ^* is unique. See the examples below.

Finally, $G^*(\cdot) := G^*_{\lambda^*}(\cdot)$ can be proved to be the optimal solution to (18) or (19). This is shown in the following way. Let $v(x_0)$ and $v_\lambda(x_0)$ be respectively the optimal value of (19) and (20). By their very definitions we have the following *weak duality*

$$v(x_0) \le \inf_{\lambda \in \mathbb{R}} v_\lambda(x_0) \quad \forall x_0 \in \mathbb{R}.$$

However,

$$v(x_0) \le \inf_{\lambda \in \mathbb{R}} v_\lambda(x_0) \le v_{\lambda^*}(x_0) = U_{\lambda^*}(G^*(\cdot)) = U(G^*(\cdot)) \le v(x_0).$$

[14]Among the five models presented earlier, the expected utility model has been well studied, and the prospect model has been solved quite completely in Jin and Zhou (2008). The SP/A model can be solved by the scheme suggested here, which however would involve substantial technicalities among other subtle issues unique to this model. We hence decide, in order not to distract the main focus of this paper, to investigate the SP/A model in a separate paper He and Zhou (2008).

This implies that $G^*(\cdot)$ is optimal to (19) (and the *strong duality* $v(x_0) = \inf_{\lambda\in\mathbb{R}} v_\lambda(x_0)$ holds).

The uniqueness of the optimal solution can also be derived from that of (20). Indeed, suppose we have established the uniqueness of optimal solution to (20) for $\lambda = \lambda^*$, and λ^* is such that $G^*_{\lambda^*}(\cdot)$ binds the budget constraint. Then $G^*_{\lambda^*}(\cdot)$ is the unique optimal solution to (18). To see this, assume there exists another optimal solution $\tilde{G}^*(\cdot)$ to (18). Then

$$U_{\lambda^*}(\tilde{G}^*(\cdot)) \le U_{\lambda^*}(G^*_{\lambda^*}(\cdot)) = v(x_0) = U(\tilde{G}^*(\cdot)) = U_{\lambda^*}(\tilde{G}^*(\cdot)).$$

Hence, by the uniqueness of optimal solution to (20), we conclude $\tilde{G}^*(\cdot) = G^*_{\lambda^*}(\cdot)$.

Finally, once (18) or (19) has been solved with the optimal solution $G^*(\cdot)$, the corresponding optimal terminal cash flow can be recovered by

$$X^* = G^*(Z_\rho) \equiv G^*(1 - F_\rho(\rho)). \tag{21}$$

The above expression shows that the optimal terminal wealth of the model (18) or (19) is *anti-comonotonic* with the pricing kernel ρ in a complete market. This underlines one of the most important, common properties of the quantile model (which covers a wide range of portfolio selection problems from neoclassical to behavioural). It will also play a significant role in treating incomplete markets and in establishing the mutual fund theorem; see the next two sections.

We remark that while the above solution scheme is outlined under $\lambda \in \mathbb{R}$, it extends readily to the situation where λ is restricted to a smaller subset, typically the positive axis $\mathbb{R}^+ \setminus \{0\}$[15]; see examples below.

Now we apply this general scheme to two concrete models presented earlier. Recall we are dealing with a complete market for now. Consequently, we assume the following on the market in Section 2.1 throughout the remainder of this section:

Assumption 5 *$m = n$ and $\sigma(t)$ is invertible a.s., a.e. $t \in [0, T]$.*

3.1 Goal-reaching model

Consider the goal-reaching problem (9). Browne (1999) has solved this problem, assuming that the investment opportunity set is deterministic, using rather ad hoc method based on the HJB equation and the associated verification theorem. Here, without assuming a deterministic investment opportunity set, we demonstrate that our quantile formulation will lead to a rather simple approach.

First, it is easy to see that if $x_0 \ge bE[\rho]$, then a trivial optimal solution is $X^* = b$ and the optimal value is 1. Therefore we confine us to the only interesting case $0 < x_0 < bE[\rho]$, which means that the goal is at least more ambitious than the risk-free payoff. Notice

$$P(X \ge b) = \int_{\mathbb{R}} \mathbf{1}_{\{x\ge b\}} dF_X(x) = \int_0^1 \mathbf{1}_{\{G(z)\ge b\}} dz,$$

[15]This is because, due to Assumption 3, the equality constraint in (19) could be revised to the less-or-equal inequality constraint without essentially changing the model.

and $X \geq 0$ is equivalent to $G(0+) \geq 0$. Hence problem (9) can be formulated in the following quantile version:

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U(G(\cdot)) = \int_0^1 \mathbf{1}_{\{G(z)\geq b\}} dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(1-z)G(z)dz = x_0, \\ & G(\cdot) \in \mathbb{G}, \quad G(0+) \geq 0. \end{array} \tag{22}$$

This, certainly, specialises the general model (19) with a non-convex/concave "utility" function.

Introducing the Lagrange multiplier $\lambda > 0$ (as discussed earlier as well as evident from below in this case we need only to consider positive multipliers), we have the following family of problems

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U_\lambda(G(\cdot)) := \int_0^1 \left[\mathbf{1}_{\{G(z)\geq b\}} - \lambda F_\rho^{-1}(1-z)G(z)\right] dz + \lambda x_0 \\ \text{Subject to} & G(\cdot) \in \mathbb{G}, \quad G(0+) \geq 0. \end{array} \tag{23}$$

Ignore the constraints for now, and consider the pointwise maximisation of the integrand above in the argument $x = G(z)$: $\max_{x\geq 0}[\mathbf{1}_{\{x\geq b\}} - \lambda F_\rho^{-1}(1-z)x]$. It is an easy exercise to show that its optimal value is $\max\{1 - \lambda F_\rho^{-1}(1-z)b, 0\}$ attained at $x^* = b\mathbf{1}_{\{1-\lambda F_\rho^{-1}(1-z)b\geq 0\}}$. Moreover, such an optimal solution is unique whenever $1 - \lambda F_\rho^{-1}(1-z)b > 0$. Thus, we define

$$G_\lambda^*(z) := b\mathbf{1}_{\{1-\lambda F_\rho^{-1}(1-z)b\geq 0\}}, \quad 0 < z < 1,$$

which is nondecreasing in z. It may not be left continuous; however, the value of $U_\lambda(G(\cdot))$ is unchanged if $G(\cdot)$ is altered only at countable points on $[0,1]$. Hence we can take the left-continuous modification of $G_\lambda^*(\cdot)$ to be the optimal solution of (23), and the optimal solution is unique up to a null Lebesgue measure. On the other hand, the modification above would generate the same (in the sense of a.s.) random payment (21) since ρ has no atom. So the above $G_\lambda^*(\cdot)$ can be regarded as the optimal solution to (23).

Now we are to find $\lambda^* > 0$ binding the budget constraint so as to conclude that $G_{\lambda^*}^*(\cdot)$ is the optimal solution to (22). To this end, let

$$\begin{aligned} f(\lambda) :&= \int_0^1 F_\rho^{-1}(1-z)G_\lambda^*(z)dz \\ &= b\int_0^1 F_\rho^{-1}(1-z)\mathbf{1}_{\{F_\rho^{-1}(1-z)\leq 1/(\lambda b)\}}dz \\ &= b\int_0^{+\infty} x\mathbf{1}_{\{x\leq 1/(\lambda b)\}}dF_\rho(x) \\ &= bE\left[\rho\mathbf{1}_{\{\rho\leq 1/(\lambda b)\}}\right], \quad \lambda > 0. \end{aligned}$$

It is easy to see that $f(\cdot)$ is nonincreasing, continuous on $(0,+\infty)$, with $\lim_{\lambda\downarrow 0} f(\lambda) = bE[\rho]$ and $\lim_{\lambda\uparrow +\infty} f(\lambda) = 0$. Therefore, for any $0 < x_0 < bE[\rho]$, there exists $\lambda^* > 0$ such that $f(\lambda^*) = x_0$ or the budget constraint holds. As per discussed in the general solution scheme the corresponding $G_{\lambda^*}^*(\cdot)$ solves (22) and the terminal payment $X^* = G_{\lambda^*}^*(1 - F_\rho(\rho)) = b\mathbf{1}_{\{\rho\leq c^*\}}$, where $c^* \equiv (\lambda^* b)^{-1}$ is such that the initial budget constraint binds, solves the original problem (9). Finally, the optimal solution is unique and the optimal value is $P(X^* \geq b) = P(\rho \leq c^*) = F_\rho(c^*)$.

To summarise, we have

Theorem 1 *Assume that* $0 < x_0 < bE[\rho]$. *Then the unique solution to the goal-reaching problem (9) is* $X^* = b\mathbf{1}_{\{\rho \le c^*\}}$ *where* $c^* > 0$ *is the one such that* $E[\rho X^*] = x_0$. *The optimal value is* $F_\rho(c^*)$.

The solution above certainly reduces to that of Browne (1999) when the investment opportunity set is deterministic. It is, however, important to highlight the advantages of our approach. First, the approach in Browne (1999) is rather ad hoc, in that a value function of the problem is *conjectured* and then verified to be the solution of the HJB equation, without an explanation as to how the function was come up with in the first place. Here we *derive* the solution (without having to know its form a priori) based on the quantile approach. Thus our method could be easily adapted to more general settings. Second, the HJB equation fails to work with a stochastic investment opportunity set,[16] which however can be treated by our approach here. Finally, our result can even be extended to an incomplete market with a deterministic opportunity set; see the next section for details.

Föllmer and Leukert (1999) and Spivak and Cvitanić (1999) extend the goal-achieving problem to the context of hedging contingent claims, allowing more general settings involving random goals, stochastic opportunity sets, and/or continuous semimartingales as asset prices. The approaches they develop (Neyman–Pearson lemma and martingale respectively) are again somewhat specific to the probability maximisation problems. In contrast, the quantile approach of this paper is general enough to cover many models beyond probability maximisation.[17]

We end this subsection by noting an interesting feature of the solution derived. The optimal terminal wealth profile for the goal-reaching is a digital option.[18] Optimally the agent either obtains a fixed payment upon a "winning event" or else loses all the money on a "losing event" at the end of the investment horizon. Whether the world ends up with a winning event is completely dictated by the pricing kernel not exceeding a critical level c^*. Moreover, since $X^* = b\mathbf{1}_{\{\rho \le c^*\}}$, the payoff b in case of a winning is fixed while the winning probability, $P(\rho \le c^*)$, monotonically decreases with b/x_0, a quantity that measures the aspiration (or indeed the greed) of the agent. This is seen by

$$E[\rho\mathbf{1}_{\{\rho \le c^*\}}] = (b/x_0)^{-1}.$$

From a different perspective, given the initial wealth x_0, there is a tradeoff between the winning amount b and the winning chance represetned by c^*, since

$$bE[\rho\mathbf{1}_{\{\rho \le c^*\}}] = x_0.$$

So the higher goal the agent sets the less chance the goal will be reached, and vice versa. This in turn suggests that, although the notion of risk preference is not explicitly presented in the goal-reaching model, it is *implied* in the following sense: the more risk-averse the

[16]In this case one would have to involve the so-called backward stochastic partial differential equation in formulating the corresponding HJB equation, which is in general very complicated and extremely hard to deal with.

[17]In this paper the goal b in (9) is assumed to be deterministic, althouth it is not essential. If b is random, then by considering a new decision variable $Y = X/b$ one could mathematically recover (9) with $b = 1$. Of course, some subtle technical consideration is required if b is not almost surely strictly positive; we leave the details to the interested readers.

[18]It is an easy problem to replicate a digital option to obtain the optimal trading strategy; see, e.g., Appendix E in Jin and Zhou (2008).

agent is, the more weight should be put on the winning *chance* and the less on the winning *amount*. More on this in the next subsection.

3.2 Yaari's dual model

In this subsection we turn to the portfolio choice model (11) under Yaari's dual theory. We assume $x_0 > 0$ to exclude a trivial case. In view of (12), (15), and (16) the problem has the following quantile formulation:

$$\begin{array}{ll} \underset{G(\cdot)}{\text{Max}} & U(G(\cdot)) = \int_0^1 G(z)w'(1-z)dz \\ \text{subject to} & \int_0^1 F_\rho^{-1}(1-z)G(z)dz = x_0. \\ & G(\cdot) \in \mathbb{G}, \quad G(0+) \geq 0. \end{array} \tag{24}$$

We first impose the following assumption on the distortion function $w(\cdot)$.

Assumption 6 $w(\cdot) : [0,1] \to [0,1]$ *is continuous and strictly increasing with* $w(0) = 0$, $w(1) = 1$. *Furthermore,* $w(\cdot)$ *is continuously differentiable on* $(0,1)$.

Other than the differentiability which is purely technical, the economical sensibility of the assumption is evident.

Before we attempt to solve (11) or (24), notice the preference measure is linear in the payment; see (12). Hence its value can be possibly made as large as one wants. Define by $v(x_0)$ the optimal value of (24). We say the model is *ill-posed* if $v(x_0) = +\infty$; otherwise it is *well-posed*. An ill-posed model is one where the incentives implied by the model are wrong, and in the context of portfolio choice an ill-posed model usually leads to trading strategies that take the greatest possible leverages (hence the agent is most aggressive); see Jin and Zhou (2008) for a detailed discussion and treatment of the ill-posedness. A well-posed Yaari's model requires some consistency between the probability distortion and the market. This is made precise in the following theorem.

Theorem 2 *Under Assumption 6, model (24) is ill-posed if* $\liminf_{z\downarrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} = +\infty$, *and well-posed if* $\limsup_{z\downarrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} < +\infty$.

Proof If $\liminf_{z\downarrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} = +\infty$, then for any $n > 0$, there exists $z_1 \in (0,1)$ such that $w'(z) \geq \frac{n}{x_0} F_\rho^{-1}(z)$ for any $z \in (0, z_1]$. Construct $G(\cdot) \in \mathbb{G}$ in the following way: it is 0 on $[0, 1-z_1]$ and is a constant b on $(1-z_1, 1]$. Because $\rho > 0$, we have $F_\rho^{-1}(z) > 0\ \forall z > 0$. Hence we can select b such that $\int_0^1 G(z)F_\rho^{-1}(1-z)dz = x_0 > 0$. Consequently, we have

$$\int_0^1 G(z)w'(1-z)dz \geq \frac{n}{x_0}\int_0^1 G(z)F_\rho^{-1}(1-z)dz = n.$$

This indicates that $v(x_0) = +\infty$ or the underlying model is ill-posed.

If $\limsup_{z\downarrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} < +\infty$, then there exists $K_1 > 0$ and $0 < z_1 < 1$ such that $w'(z) \leq K_1 F_\rho^{-1}(z)$ for any $z \in (0, z_1]$. Now for any feasible $G(\cdot)$ to problem (24), we have

$$G(1-z_1) \leq \frac{\int_{1-z_1}^1 G(z)F_\rho^{-1}(1-z)dz}{\int_{1-z_1}^1 F_\rho^{-1}(1-z)dz} \leq \frac{x_0}{\int_{1-z_1}^1 F_\rho^{-1}(1-z)dz} =: K_2 < +\infty.$$

Thus

$$\begin{aligned}
U(G(\cdot)) &= \int_0^{1-z_1} G(z)w'(1-z)dz + \int_{1-z_1}^1 G(z)w'(1-z)dz \\
&\leq G(1-z_1)\int_0^{1-z_1} w'(1-z)dz + K_1\int_{1-z_1}^1 G(z)F_\rho^{-1}(1-z)dz \\
&\leq K_2 + K_1x_0.
\end{aligned}$$

This shows that $v(x_0) \leq K_2 + K_1x_0 < +\infty$ and the model is well-posed. ∎

If w is concave and differentiable, and $\underline{\rho} = 0$, then

$$\liminf_{z\downarrow 0} \frac{w'(z)}{F_\rho^{-1}(z)} \geq \liminf_{z\downarrow 0} \frac{w'(1/2)}{F_\rho^{-1}(z)} = +\infty.$$

So a concave distortion leads to an ill-posed problem or the agent is most aggressive in taking the risk. However, this is perfectly consistent with Yaari's theory that a concave distortion is equivalent to the risk-seeking preference.

Next, we apply the Lagrange method to solve (24), for which we introduce an additional assumption in terms of a function $M(z) := \frac{w'(1-z)}{F_\rho^{-1}(1-z)},\ 0 < z < 1$.

Assumption 7 *$M(\cdot)$ is continuous on $(0,1)$, and there exists $z_0 \in (0,1)$ such that $M(\cdot)$ is strictly increasing on $(0, z_0)$ and strictly decreasing on $(z_0, 1)$.*

Assumption 7 can be weakened to the one where $M(\cdot)$ may have a finite number of monotonic pieces. However, such a generalisation only incurs notational complexity in the approach below rather than any essential difference. On the other hand, we will show later in this subsection that Assumption 7 holds naturally for some common and interesting cases. Finally note that, in view of Theorem 2, problem (24) is well-posed under this assumption.

Consider the following family of problems with the parameter $\lambda > 0$ being the Lagrange multiplier:

$$\begin{array}{ll}
\underset{G(\cdot)}{\text{Max}} & U_\lambda(G(\cdot)) := \int_0^1 G(z)\left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right]dz + \lambda x_0 \\
\text{subject to} & G(\cdot) \in \mathbb{G},\quad G(0+) \geq 0.
\end{array} \tag{25}$$

Denote by $v_\lambda(x_0)$ the optimal value of (25). The following proposition solves (25) completely.

Proposition 1 *Let Assumptions 6-7 hold. Then there is the unique root $\lambda^* > 0$ of the following function on $(0, +\infty)$:*

$$h(\lambda) := \int_0^1 \left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right]_+ dz - \int_{z_0}^1 \left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right]_- dz.$$

Moreover,

(i) *If $0 < \lambda < \lambda^*$, then $v_\lambda(x_0) = +\infty$.*

(ii) *If $\lambda > \lambda^*$, then $v_\lambda(x_0) = \lambda x_0$ and the unique optimal solution to* (25) *is $G^*_\lambda(\cdot) \equiv 0$.*

(iii) *If* $\lambda = \lambda^*$*, then* $v_\lambda(x_0) = \lambda x_0$*, and the set of optimal solutions to* (25) *is* $\{G(\cdot) \in \mathbb{G} : \ G(z) = b\mathbf{1}_{\{z(\lambda^*)<z\le 1\}}, \ b \ge 0\}$ *where* $0 < z(\lambda^*) \le z_0$ *is the one satisfying* $M(z(\lambda^*)) = \lambda^*$.

Proof Rewrite, for each $\lambda > 0$,

$$U_\lambda(G(\cdot)) = \int_0^1 G(z)F_\rho^{-1}(1-z)f_\lambda(z)dz + \lambda x_0$$

where $f_\lambda(z) := M(z) - \lambda$. If $f_\lambda(z) < 0 \ \forall z \in (0,1)$ then the obvious unique optimal $G_\lambda^*(\cdot) \equiv 0$ which leads to (ii). Hence we assume that $f_\lambda(z) \ge 0$ for at least one $z \in (0,1)$. Let $z(\lambda) := \inf\{z \in (0,z_0] : \ f_\lambda(z) \ge 0\}$ with the convention that $\inf \emptyset := z_0$. A crucial step in what follows is to show that the optimal solution to (25) must be attained in a subclass of $\mathbb{G}$, consisting of certain step functions, defined as

$$\mathbb{G}_\lambda := \{G(\cdot) \in \mathbb{G} : \ G(z) = b\mathbf{1}_{\{z(\lambda)<z\le 1\}}, \ \ b \ge 0\}.$$

It is clear that $v_\lambda(x_0) \ge \sup_{G(\cdot)\in\mathbb{G}_\lambda} U_\lambda(G(\cdot))$. To show the opposite inequality, consider $\bar{z}(\lambda) := \inf\{z \in [z_0,1) : \ f_\lambda(z) < 0\}$ with $\inf\emptyset := 1$. By virtue of Assumption 7, $f_\lambda(\cdot)$ is positive on $(z(\lambda), \bar{z}(\lambda))$ and negative on $(0, z(\lambda)) \cup (\bar{z}(\lambda), 1)$. Now, for any feasible $G(\cdot)$ to (25), we have

$$\begin{aligned}
U_\lambda(G(\cdot)) &= \int_0^1 G(z)F_\rho^{-1}(1-z)f_\lambda(z)dz + \lambda x_0 \\
&= \int_0^{z(\lambda)} G(z)F_\rho^{-1}(1-z)f_\lambda(z)dz \\
&\quad + \int_{z(\lambda)}^{\bar{z}(\lambda)} G(z)F_\rho^{-1}(1-z)f_\lambda(z)dz + \int_{\bar{z}(\lambda)}^1 G(z)F_\rho^{-1}(1-z)f_\lambda(z)dz + \lambda x_0 \\
&\le \alpha \int_{z(\lambda)}^{\bar{z}(\lambda)} F_\rho^{-1}(1-z)f_\lambda(z)dz + \alpha \int_{\bar{z}(\lambda)}^1 F_\rho^{-1}(1-z)f_\lambda(z)dz + \lambda x_0 \\
&\le \sup_{g(\cdot)\in\mathbb{G}_\lambda} U_\lambda(g(\cdot)),
\end{aligned}$$

where $\alpha := \lim_{z\downarrow\bar{z}(\lambda)} G(z)$. The first inequality above becomes equality if and only if $G(\cdot) = \alpha\mathbf{1}_{\{z(\lambda)<z\le 1\}}$. Therefore, if $G^*(\cdot)$ is optimal to (25), then $G^*(\cdot) \in \mathbb{G}_\lambda$.

On the other hand, a simple exercise shows that

$$h(\lambda) \equiv \int_{z(\lambda)}^1 \left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right] dz.$$

Clearly, $h(\cdot)$ is continuous and strictly decreasing on $(0,+\infty)$ with $\lim_{\lambda\downarrow 0} h(\lambda) = 1$, $\lim_{\lambda\uparrow\infty} h(\lambda) = -\infty$. So $h(\lambda)$ admits a unique root $\lambda^* > 0$. Now,

$$v_\lambda(x_0) = \sup_{G(\cdot)\in\mathbb{G}_\lambda} U_\lambda(G(\cdot)) = \lambda x_0 + \sup_{b\ge 0}[bh(\lambda)] = \lambda x_0 + h(\lambda)\sup_{b\ge 0} b.$$

Since $h(\lambda)$ is positive when $\lambda < \lambda^*$, negative when $\lambda > \lambda^*$, and identical to 0 when $\lambda = \lambda^*$, the desired results (i)-(iii) follow immediately. ∎

Now we are ready to give the complete solution to (11) or equivalently (24).

Theorem 3 *Suppose Assumptions 6-7 hold, and let λ^* be the one in Proposition 1. Then*

(i) *The strong duality holds, i.e.,* $v(x_0) = \inf_{\lambda>0} v_\lambda(x_0) = \lambda^* x_0$.

(ii) $X^* = b^* \mathbf{1}_{\{\rho \le c\}}$ *is the unique optimal solution to* (11) *where* c *is the unique root of the following function*

$$\varphi(x) := x w(F_\rho(x)) - w'(F_\rho(x)) \int_0^x s dF_\rho(s) \tag{26}$$

on $(F_\rho^{-1}(1-z_0), \overline{\rho})$ *and* $b^* > 0$ *is the one binding the initial budget constraint, i.e.,* $E[\rho X^*] = x_0$.

Proof We clearly have the weak duality: $v(x_0) \le \inf_{\lambda>0} v_\lambda(x_0) = \lambda^* x_0$ where the equality is due to Proposition 1. Now, take $G^*(z) = b^* \mathbf{1}_{\{z(\lambda^*) < z \le 1\}}, \;\; 0 \le z \le 1$, such that $\int_0^1 G^*(z) F_\rho^{-1}(1-z)dz = x_0$. Then $v(x_0) \ge U(G^*(\cdot)) = \lambda^* x_0$ where the equality is again by Proposition 1. This proves the strong duality and, moreover, $G^*(\cdot)$ is optimal to (24).

If there is another optimal solution $\tilde{G}^*(\cdot)$ to (24), then $\int_0^1 \tilde{G}^*(z) F_\rho^{-1}(1-z)dz = x_0$ and $\tilde{G}^*(\cdot)$ is optimal to (25) with the multiplier λ^*. Proposition 1-(iii) then implies that $\tilde{G}^*(\cdot) \in \mathbb{G}_{\lambda^*}$ and consequently $\tilde{G}^*(\cdot) = G^*(\cdot)$. This proves the uniqueness of the optimal solution to (24).

Next we recover the optimal terminal payoff via (21):

$$X^* = b^* \mathbf{1}_{\{z(\lambda^*) < 1 - F_\rho(\rho) \le 1\}} = b^* \mathbf{1}_{\{\rho \le c\}}$$

where $c := F_\rho^{-1}(1 - z(\lambda^*))$.

It remains to show that the above c is the unique root of $\varphi(\cdot)$ defined in (26). Recalling $h(\lambda) = \int_{z(\lambda)}^1 \left[w'(1-z) - \lambda F_\rho^{-1}(1-z)\right] dz$ and $\lambda = M(z(\lambda))$ when $0 < z(\lambda) \le z_0$, we conclude that $z(\lambda^*)$ is the unique root of the following function (in y)

$$\int_y^1 w'(1-z)dz - M(y) \int_y^1 F_\rho^{-1}(1-z)dz \equiv w(1-y) - M(y) \int_y^1 F_\rho^{-1}(1-z)dz.$$

A change of integrand variable $s = F_\rho^{-1}(1-z)$ in evaluating the integral above reveals immediately that $c = F_\rho^{-1}(1 - z(\lambda^*))$ is the unique root of φ defined on the interval $(F_\rho^{-1}(1-z_0), \overline{\rho})$. The proof is completed. ∎

The following example first shows the validity of Assumption 7 for a broad and interesting class of distortion functions and pricing kernels, and then gives the corresponding optimal solution to Yaari's model.

Example 1 Let ρ follow lognormal distribution[19], i.e., $F_\rho(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$ for some $\mu \in \mathbb{R}$, $\sigma > 0$ where $\Phi(\cdot)$ is the CDF of the standard Normal distribution. Take $w(z) = z^\gamma$ for some $\gamma > 1$; so $w(\cdot)$ is convex and reflects the risk aversion of the investor according to Yaari's theory. We now verify Assumption 7 or, equivalently, the monotonicity of

$$f(x) := \frac{w'(F_\rho(x))}{x} = \frac{\gamma \left[\Phi\left(\frac{\ln x - \mu}{\sigma}\right)\right]^{\gamma-1}}{x}, \;\; x > 0.$$

[19]This covers, e.g., the case with a deterministic investment opportunity set.

A calculation shows

$$f'(x) = \frac{\gamma\left[\Phi\left(\frac{\ln x-\mu}{\sigma}\right)\right]^{\gamma-2}}{x^2} f_1\left(\frac{\ln x-\mu}{\sigma}\right)$$

where

$$f_1(y) := \frac{\gamma-1}{\sigma}\phi\left(y\right) - \Phi\left(y\right)$$

and $\phi(\cdot)$ is the density function of the standard Normal distribution. Again taking derivative on $f_1(y)$, we have

$$f_1'(y) = -\phi(y)\left(1 + \frac{\gamma-1}{\sigma}y\right).$$

Therefore f_1 takes its maximum at $y_1 = -\frac{\sigma}{\gamma-1} < 0$, strictly increases on $(-\infty, y_1)$, and strictly decreases on $(y_1, +\infty)$. Moreover,

$$f_1(y_1) = -\left[\Phi\left(y_1\right) + \frac{1}{y_1}\phi\left(y_1\right)\right] = -\left[1 - \Phi\left(-y_1\right) - \frac{1}{-y_1}\phi\left(-y_1\right)\right] > 0.$$

On the other hand, we have

$$f_1(-\infty) = 0, \qquad f_1(+\infty) = -1 < 0.$$

Hence $f_1(\cdot)$ has a unique root y_2 such that $f_1(y) > 0$ on $(-\infty, y_2)$ and $f_1(y) < 0$ on $(y_2, +\infty)$. Furthermore $y_2 > y_1 = -\frac{\sigma}{\gamma-1}$. Let $\rho_0 := \exp(\sigma y_2 + \mu) > \exp\left(\mu - \frac{\sigma^2}{\gamma-1}\right)$ and $z_0 := 1 - F_\rho(\rho_0) = 1 - \Phi(y_2)$. Then Assumption 7 holds.

The function $\varphi(\cdot)$ defined in Theorem 3 is

$$\varphi(x) = \left[\Phi\left(\frac{\ln x-\mu}{\sigma}\right)\right]^{\gamma-1}\left[x\Phi\left(\frac{\ln x-\mu}{\sigma}\right) - \gamma e^{\mu+\frac{\sigma^2}{2}}\Phi\left(\frac{\ln x-\mu}{\sigma} - \sigma\right)\right].$$

This function has a unique root c on $\left(\rho_0, \gamma\exp\left(\mu + \frac{\sigma^2}{2}\right)\right)$. Therefore, the optimal solution is given as $X^* = b^*\mathbf{1}_{\{\rho\le c\}}$ with $E[\rho X^*] = x_0$.

As with the goal-reaching model, it turns out that the optimal solution to the Yaari's model has the same digital or "win-or-lose-all" structure[20]. However, there are subtle differences between the two models. In the goal-reaching model, the winning payoff b is

[20] As discussed in Madan and Zhou (2008) the structure of digital claims being optimal is arguably not well formed and appears to be an artificial consequence of the problem formulations – in particular the linearity in both preferences and constraints – than a real structural property. Here both the goal-reaching model and the Yaari model have linear payment in their preference measures (Yaari's criterion only distorts the probability and not the payment), which is the essential economic reason behind the digital solutions. A more plausible optimal claim is what we call a *gambling strategy*, that is, it is a known claim X in certain states of the world and another claim Y otherwise, and X and Y usually have *distinct* economic interpretations. So people gamble on the occurrence of the former states (the good states). The propsect model Jin and Zhou (2008) has exactly such an optimal structure where X is the gain and Y the loss. See also He and Zhou (2008) for the SP/A model where a utility function is applied to distort the payment in addition to the probability distortion. A digital option is certainly a special case and, more importantly, an approximation of the general form. We stress that the general quantile model formulated in this paper has rich optimal structures; it just so happens that the two models demonstrated here have the digital structure.

exogenously *chosen* by the agent while the winning chance is endogenously *implied* by the model; the latter being affected by the initial wealth x_0 (or more precisely, by the aspiration level b/x_0). In Yaari's model, $X^* = b^* \mathbf{1}_{\{\rho \leq c\}}$, so both the winning chance and the winning amount are endogenous. In particular, c is completely determined by the market and the agent risk preference; see (26). In other words there is a cap on the winning chance *regardless* of the initial wealth whereas the winning payoff b^* depends linearly on the initial wealth.

As Yaari (1987) argues, risk preference is explicitly present and reflected by the probability distortion w in his dual criterion. Now let us examine how w would affect the winning chance represented by c, and whether Yaari's model is consistent with the notion (as discussed earlier) that a more risk-averse agent would put more weight on the winning chance and less weight on the winning payoff.

Recall that in the proof of Theorem 3, we have proved that $z := 1 - F_\rho(c)$ is the unique root of the following function

$$g(y) := w(1-y) - M(y) \int_y^1 F_\rho^{-1}(1-s)ds$$

on $(0, z_0)$. Noting $g'(y) = -M'(y) \int_y^1 F_\rho^{-1}(1-s)ds$ along with Assumption 7, $g(\cdot)$ strictly decreases on $(0, z_0)$ and strictly increases on $(z_0, 1)$. Consequently, $g(y) > 0$ on $(0, z)$ and $g(y) < 0$ on $(z, 1)$. Rewrite $g(\cdot)$ as

$$g(y) = w'(1-y) \left[\frac{w(1-y)}{w'(1-y)} - \frac{\int_y^1 F_\rho^{-1}(1-s)ds}{F_\rho^{-1}(1-y)} \right].$$

Suppose now we have two distortion functions, $w_1(\cdot)$ and $w_2(\cdot)$, both satisfying Assumptions 6 and 7, such that

$$\frac{w_1'(1-y)}{w_1(1-y)} \leq \frac{w_2'(1-y)}{w_2(1-y)}. \tag{27}$$

The corresponding functions $g(\cdot)$, $\varphi(\cdot)$ and the quantities z and c are now affixed with a subscript $i = 1, 2$ to indicate the correspondence to the two distortions w_i, $i = 1, 2$. Then we have

$$0 = g_1(z_1) \geq w_1'(1-z_1) \left[\frac{w_2(1-z_1)}{w_2'(1-z_1)} - \frac{\int_{z_1}^1 F_\rho^{-1}(1-s)ds}{F_\rho^{-1}(z_1)} \right].$$

This implies $g_2(z_1) \leq 0$ and, consequently, $z_1 \geq z_2$ or equivalently, $c_1 \leq c_2$. If the inequality in (27) is strict, so are all the subsequent inequalities above. If we accept that in the tradeoff between the winning chance and the winning amount, a risk-averse agent favours the former, then the above analysis shows that the index $\frac{w'(z)}{w(z)}$ can be used to measure the degree of "risk-aversion" in Yaari's model, i.e., the greater the index the higher winning chance the agent wishes to achieve. We remark that in Machina (1982), Theorem 4-(ii), based on the notion of risk aversion introduced by Rothschild and Stiglitz (1970), it is shown that the risk aversion levels of different preference functionals can be ranked according to a criterion. It is easy to show that the criterion in the current Yaari setting is exactly the Arrow–Pratt index $\frac{w''(z)}{w'(z)}$. Here, we have suggested a different risk aversion index as implied by the Yaari portfolio choice model.[21]

4 Incomplete Market

In this section we discuss the incomplete market case. A crucial advantage with a complete market is that there is a unique pricing kernel ρ; so one can turn a dynamic portfolio choice problem into a static one in terms of the terminal cash flow. Once the optimal terminal wealth is derived the corresponding portfolio is nothing else than the one to replicate it. With an incomplete market, on the other hand, one needs to specify the set of terminal cash flows that are replicable (attainable) before finding an optimal terminal wealth position.[22]

We take the continuous-time market formulated in Section 2.1. Now the dimension of the Brownian motion, n, is not necessarily the same as the number of stocks, m. Moreover, there is an explicit constraint on portfolios

$$\pi(t) \in K, \text{ a.s., a.e. } t \in [0,T], \tag{28}$$

where $K \subset \mathbb{R}^m$ is a given closed convex cone.[23] In addition to achieving more generality, including this constraint will also be useful in proving the mutual fund theorem in the next section.

An $\mathcal{F}_T$ measurable contingent claim (random variable) ξ is called *attainable* or *replicable* if there exists an initial endowment $x \in \mathbb{R}$ and an admissible portfolio satisfying (28) whose terminal wealth is $x(T) = \xi$. In the following, we identify a set of attainable contingent claims. To this end, we introduce the following.

Let K^* be the dual cone of K, i.e.,

$$K^* := \{x \in \mathbb{R}^m : \ x \cdot y \geq 0 \text{ for all } y \in K\}.$$

Define the set of $\mathcal{F}_t$-progressively measurable, $\mathbb{R}^n$-valued processes:

$$\Theta := \{\theta(\cdot) : \ \sigma(t)\theta(t) - B(t) \in K^*, \text{ a.s., a.e., } t \in [0,T] \text{ with } Ee^{\frac{1}{2}\int_0^T |\theta(t)|^2 dt} < +\infty\}.$$

Assumption 1 implies that Θ is nonempty, and it is easy to see that Θ is convex. For any $\theta(\cdot) \in \Theta$, define a pricing kernel process

$$\rho_\theta(t) := \exp\left\{-\int_0^t \left[r(s) + \frac{1}{2}|\theta(s)|^2\right] ds - \int_0^t \theta(s)^\top dW(s)\right\}, \tag{29}$$

and call $\rho_\theta := \rho_\theta(T)$ a pricing kernel. Notice in an incomplete market there could be many pricing kernels.

Introduce the notation $A(K) := \{Ax : x \in K\}$ for any $n \times m$ matrix A. Clearly, $A(K)$ is a convex cone. We assume that

[21] In the case of a power distortion function, i.e., $w(z) = z^\gamma, \gamma > 1$, these two indices are indeed consistent in ordering the risk aversion level.

[22] The discussion in this section follows standard lines in dealing with incomplete markets in the continuous-time portfolio selection literature. The main finding is that this approach turns out to work well within the quantile framework too.

[23] A typical example is the so-called no-shorting constraint. Indeed, even though $m = n$ and $\sigma(t)$ is uniformly non-degenerate, the presence of such a portfolio constraint renders an incomplete market.

Assumption 8 *$\sigma(t)^\top(K)$ is closed, a.s., a.e. $t \in [0, T]$.*

Generally speaking, $A(K)$ may not be closed even if K is a closed convex cone. However, if rank$(A) = n$, then $A(K)$ is closed for any closed convex cone K. If $K = \{y : By \geq 0\}$ for some matrix B, then $A(K)$ is closed for any A. Therefore, Assumption 8 covers some interesting cases such as the no-shorting constraint.

The following is a classical result.

Proposition 2 *Suppose Assumptions 1 and 8 hold. Let ξ be an $\mathcal{F}_T$ measurable random variable such that $S_0(T)^{-1}\xi$ is bounded from below. If there exists $\hat{\theta}(\cdot) \in \Theta$ such that*

$$x := E[\rho_{\hat{\theta}}\xi] = \sup_{\theta \in \Theta} E[\rho_\theta \xi] < +\infty,$$

then ξ is attainable with the initial wealth x.

If there is no cone constraint, i.e., $K = \mathbb{R}^m$, then Proposition 2 is exactly the classical result for incomplete markets; see, e.g., Theorem 8 in Jacka (1992). In the presence of constraints, Karatzas and Shreve (1998) deal with the case of general closed convex constraints in complete market. Föllmer and Kramkov (1997) consider a general market via optional decompositions. Proposition 2 is a special case of the results in Föllmer and Kramkov (1997). Meanwhile, Assumption 8 is related to Assumption 3.1 in Föllmer and Kramkov (1997).

Proposition 2 characterises the attainability of a contingent claim ξ by an optimisation problem

$$\max_{\theta \in \Theta} E[\rho_\theta \xi]. \tag{30}$$

If this problem is solved by some $\hat{\theta} \in \Theta$, then ξ is attainable. However, in general $\hat{\theta}$ depends on ξ and there is no *common* $\hat{\theta}$ for all the tame claims. This would cause a major problem to the continuous-time portfolio model and our quantile approach, because in this case the constraint $x_0 = E[\rho_{\hat{\theta}}\xi] = \sup_{\theta \in \Theta} E[\rho_\theta \xi]$ has essentially infinitely many constraints and it remains an open problem as to how to formulate the quantile model accordingly. However, if the investment opportunity set is deterministic, then it is possible to select a common $\hat{\theta}$ amongst a certain set (to be specified below) of contingent claims, in which case the quantile model can be formulated as in the complete market case.

We now introduce

Assumption 9 *$r(\cdot), B(\cdot), \sigma(\cdot)$ are deterministic.*

Denote by $\Theta(t) := \{\theta : \ \sigma(t)\theta - B(t) \in K^*\}, \quad 0 \leq t \leq T$. Under Assumption 9, define

$$\hat{\theta}(t) := \text{argmin}_{\theta \in \Theta(t)} |\theta|^2. \tag{31}$$

It is clear that $\hat{\theta}(\cdot)$ is uniquely defined and deterministic. Furthermore, via the measurable selection theorem, $\hat{\theta}(\cdot)$ is measurable. Also, by Assumption 1 and the definition of $\hat{\theta}(\cdot)$, we have $\int_0^T |\hat{\theta}(t)|^2 dt < +\infty$. Therefore, $\hat{\theta}(\cdot) \in \Theta$. We call $\hat{\theta}(\cdot)$ the *minimal price of risk process* and call $\rho_{\hat{\theta}}$ the *minimal pricing kernel*.

Let $\mathbb{H}$ be the set of all non-increasing functions $g : \mathbb{R}_+ \backslash \{0\} \to \mathbb{R}$ which is bounded from below. The following result, which essentially follows from Theorem 6.6.4 in Karatzas and Shreve (1998), indicates that any contingent claim in the form $g(\rho_{\hat{\theta}})$ where $g \in \mathbb{H}$ can be replicated.

Theorem 4 *Suppose Assumptions 1, 8 and 9 hold. Let $\hat{\theta}(\cdot)$ be as in* (31)*, and $g(\cdot) \in \mathbb{H}$ be such that $x := E[\rho_{\hat{\theta}} g(\rho_{\hat{\theta}})] < \infty$. Then $g(\rho_{\hat{\theta}})$ is attainable with the initial wealth x.*

Now, take $\rho := \rho_{\hat{\theta}}$, and assume (as before) that ρ has no atom. Noticing that Lemma 1 holds for any atomless, positive random variable ρ (see Jin and Zhou 2008, Theorem B.1]), we may go through exactly the same argument as that in Section 2.3 and formulate the portfolio choice model (19). If (19) is solved with an optimal solution G^*, then the optimal terminal wealth is $X^* = g(\rho)$ where $g(x) = G^*(1 - F_\rho(x))$, according to (21). However, it is indeed true that $g \in \mathbb{H}$; hence X^* is replicable by the initial wealth x_0, while the replicating portfolio is the optimal strategy to the portfolio selection model.

In the case of a stochastic investment opportunity set, we do not yet have a general quantile formulation. However, we may at least include the case of a *weak complete market*, a notion proposed by Schachermayer, Sirbu and Taflin (2009). Let $\hat{\theta}(\cdot)$ be defined as in (31) (even without Assumption 9). The market is weak complete if $g(\rho_{\hat{\theta}})$ is replicable for any bounded non-increasing function g. (So Theorem 4 says that a market with a deterministic investment opportunity set and with cone constraints is weak complete.) Clearly, for a weak complete market the quantile formulation is valid so long as $\rho_{\hat{\theta}}$ admits no atom.

Therefore, the solutions to the goal-reaching problem and Yaari's model, obtained in Section 3, as well as that to the behavioural model (see Theorem 4.1 in Jin and Zhou 2008), can be extended readily to incomplete markets with deterministic investment opportunity sets and with conic constraints (or even to weak complete markets), where the unique pricing kernel ρ is replaced by the minimal pricing kernel $\rho_{\hat{\theta}}$.

5 Mutual Fund Theorem

The *Mutual Fund Theorem*, also called the *two-fund theorem* or *separation theorem*, states that under some assumptions, agents achieve optimality by simply allocating money between the bank account and a risky portfolio called the *mutual fund*. The key feature is that the mutual fund is same for all agents. The mutual fund theorem dates back to the Markowitz mean–variance portfolio analysis in single period where it can be shown that if all the investors are mean–variance efficiency seeker, then the mutual funds theorem holds even though different investors may have difference risk-return preferences. This becomes the foundation of the capital asset pricing model (CAPM). Merton (1971) shows that for the continuous-time Black–Scholes model (where the opportunity set is deterministic) the mutual fund theorem holds if all the agents are power utility maximisers. Such a result has been generalised to the case of general concave utility functions; see Karatzas and Shreve (1998). Recently, Schachermayer, Sirbu and Taflin (2009) discuss in a general setting when a mutual fund theorem holds true, assuming all the agents are expected utility maximisers.

Now, thanks to the quantile formulation developed in this paper especially the general expression of an optimal terminal wealth (21), we are able to prove that the mutual fund theorem holds in any market (complete or incomplete, with possible conic constraints on portfolios) having a deterministic opportunity set so long as all the agents follow the general model (19). Note that such a model covers a very diversified risk–return preferences including those of the classical utility maximisation, mean-variance and various behavioural models.

Consider the (possibly incomplete) market presented in Section 4 where portfolios are constrained in a given closed convex cone K. Under Assumption 9, let $\hat{\theta}(\cdot)$ be the minimal price of risk process and define $N(t) := \rho_{\hat{\theta}}(t)^{-1},\ 0 \le t \le T$. It is well known that $N(\cdot)$ is the wealth process of the optimal portfolio under log utility maximisation with initial wealth 1. We call this portfolio the *numéraire portfolio*; see, e.g., Schachermayer, Sirbu and Taflin (2009).

Theorem 5 *Under Assumptions 8 and 9, any optimal cash flow of the model* (19) *can be attained by a (dynamic) portfolio of the risk-free asset* $S_0(\cdot)$ *and the numéraire portfolio* $N(\cdot)$*. Moreover, this portfolio never short sells* $N(\cdot)$*.*

Proof Itô's formula shows that

$$dN(t) = \left[r(t) + |\hat{\theta}(t)|^2\right] N(t)dt + N(t)\hat{\theta}(t)^\top dW(t), \quad N(0) = 1.$$

Define

$$\tilde{W}(t) := \int_0^t \left[\frac{1}{|\hat{\theta}(u)|\mathbf{1}_{\{\hat{\theta}(u)\neq 0\}}}\hat{\theta}(u)^\top + \frac{1}{\sqrt{n}}\mathbf{1}_{\{\hat{\theta}(u)=0\}}\mathbf{1}_n^\top\right] dW(u), \quad 0 \le t \le T.$$

By Lévy's characterisation, $\tilde{W}(\cdot)$ is a one-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. Then, $N(\cdot)$ satisfies

$$dN(t) = \left[r(t) + |\hat{\theta}(t)|^2\right] N(t)dt + |\hat{\theta}(t)|N(t)d\tilde{W}(t), \quad N(0) = 1.$$

Now the probability space $(\Omega, \mathcal{F}_T^{\tilde{W}}, (\mathcal{F}_t^{\tilde{W}})_{0\le t\le T}, P)$, where $\mathcal{F}_t^{\tilde{W}}$ is generated by $\tilde{W}(\cdot)$ and augmented by all the P-null sets, together with the risk-free asset $S_0(\cdot)$ and the risky asset $N(\cdot)$, constitutes a new, fictitious financial market. This market has a deterministic investment opportunity set. We further impose the no-shorting constraint in this market, i.e., the position of $N(\cdot)$ must be non-negative, which is a conic constraint. It is easy to show that the minimal price of risk process, as determined by (31) in general, is $|\hat{\theta}(\cdot)|$ in this new market; hence the corresponding minimal pricing kernel is $\rho_{\hat{\theta}}(T) := \rho$ itself. In other words, the minimal pricing kernels in the two markets are identical. However, it has been shown that with an optimal solution $G^*(\cdot)$ to (19) in the original market, the corresponding optimal terminal payoff is $X^* = g(\rho)$ where $g(x) = G^*(1 - F_\rho(x))$. Recall that $g \in \mathbb{H}$; hence Theorem 4 yields that X^* is replicable by an admissible portfolio in the *new* market. More precisely, there exists an $\mathcal{F}_t^{\tilde{W}}$-progressively measurable (and thus $\mathcal{F}_t$-progressively measurable) portfolio $\alpha(\cdot)$, with $\alpha(t) \ge 0$, a.s., a.e. $t \in [0, T]$, that replicates $X^* = g(\rho)$ from the initial wealth x_0. Here $\alpha(t)$ is the amount allocated to $N(\cdot)$ at time t; hence its non-negativity is due to the no-shorting constraint we have imposed. It follows that this replicating portfolio never short sells $N(\cdot)$. Finally, to see that this replicating portfolio does satisfy the given conic constraint, note that, again by Theorem 4, $N(\cdot) = \rho_{\hat{\theta}}(\cdot)^{-1}$ is replicable in the original market, and so is any non-negative position of $N(\cdot)$ since the portfolio constraint in the original market is conic. The proof is thus complete. ∎

The preceding theorem shows that $N(\cdot)$ is a mutual fund. This is probably the most general mutual fund theorem to date, at least to our best knowledge, due to the broad coverage of our quantile portfolio choice model (19). The result suggests that the mutual fund theorem is somewhat inherent in financial portfolio selection, at least in markets with deterministic opportunity sets. As a consequence, the same risky portfolio is being held across neoclassical (rational) and behavioural (irrational) agents in the market. This, in turn, will shed light on the market equilibrium and capital asset pricing in markets where rational and irrational agents co-exist.

There is an interesting application of our result to models featuring the so-called mental accounting. Mental accounting, a notion coined by Thaler (1980) and an important ingredient of the behavioural theory, argues that people group their assets into a number of non-fungible mental accounts. Das, Markowitz, Scheid and Statman (2009) consider a single-period portfolio optimisation with several separated mental accounts of different objectives. In particular, within each account the agent tries to maximise the expected return while lowering the risk which is identified as the probability that the terminal payoff is below some threshold. They show that if the returns of the assets are joint-normally distributed, then the optimal portfolios in those mental accounts are in the same mean–variance efficient frontier. Consequently, the agent will hold the same risky portfolio in each mental account, and so will she on aggregation. Now, consider an extension to the continuous-time market with a deterministic investment opportunity set. If within each mental account the agent follows an instance of our general quantile model (although across different accounts the preferences could be very different; say for Account A the agent is rational while for Account B she is behavioural), then she will hold the same risky portfolio $N(\cdot)$ in each mental account. As a result, she will hold the same risky portfolio in total.

6 Conclusions

Existing risk–return criteria (neoclassical and behavioural) in portfolio selection have introduced distortions in either payments or probabilities, or both, in evaluating uncertain payments. These distortions have various economical interpretations and significance. Yet they have given rise to difficulties, especially in the dynamic setting, for which traditional approaches fall apart. In this paper we propose to change the whole perspective of continuous-time portfolio choice: Instead of determining random terminal cash flows – specifications of values for all scenarios – one should consider quantiles – fractions of scenarios below given values, even if the underlying models may *not explicitly* involve quantiles in their objectives or constraints. The result is quite satisfying: it has sorted out the issues of nonlinear expectation and non-concavity simultaneously. We hope that the quantile formulation opens up a broad avenue to modelling and solving financial portfolio choice problems.

One should note that the quantile approach highly depends on the prerequisites that the preference measure is law-invariant and the pricing kernel is atomless. While we acknowledge a great wide variety of problems do satisfy these assumptions, it is a very challenging problem to explore beyond them.

References

[1] A. Berkelaar, R. Kouwenberg, and T. Post. Optimal portfolio choice under loss aversion. *The Review of Economics and Statistics*, 86(4):973–987, 2004.

[2] S. Browne. Reaching goals by a deadline: Digital options and continuous-time active portfolio management. *Advances in Applied Probability*, 31(2):551–577, 1999.

[3] S. Browne. Risk-constrained dynamic active portfolio management. *Management Science*, 46(9):1188–1199, Sept. 2000.

[4] A. Cherny and D.B. Madan. New measures for performance evaluation. Working paper, *http://ssrn.com/abstract=9554722008*, 2008.

[5] G. Carlier and R.-A. Dana. Law invariant concave utility functions and optimization problems with monotonicity and comonotonicity constraints. *Statistics & Decision*, 24: 127–152, 2006.

[6] J. Cvitanić and I. Karatzas. Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, 2(4):767–818, Nov. 1992.

[7] S.R. Das, H. Markowitz, J. Scheid, and M. Statman. Portfolio optimization with mental accounts. *Journal of Financial and Quantitative Analysis*, forthcoming, 2009.

[8] J. Dhaene, S. Vanduffel, Q. Tang, M. J. Goovaerts, R. Kaas, and D. Vyncke. Solvency capital, risk measure and comonotonicity: A survey. Working paper, *http://www.econ.kuleuven.ac.be/tew/academic/actuawet/pdfs/Riskmeasures.pdf*, 2004.

[9] J. Dhaene, S. Vanduffel, M. J. Goovaerts, R. Kaas, and D. Vyncke. Commonotonic approximations for optimal portfolio selection problems. *The Journal of Risk and Insurance*, 72(2):253–300, 2005.

[10] P. H. Dybvig. Distributional analysis of portfolio choice. *The Journal of Business*, 61(3):369–398, Jul. 1988.

[11] H. Föllmer and D. Kramkov. Optional decompositions under constraints. *Probability Theory and Related Fields*, 109(1):1–25, Sep. 1997.

[12] H. Föllmer and P. Leukert. Quantile hedging. *Finance & Stochastics*, 3, 251–273, 1999.

[13] H. Föllmer and A. Schied. *Stochastic Finance: An Introduction in Discrete Time.* 2nd Edition, Walter de Gruyter, 2004.

[14] T. Goll and L. Rüschendorf. Minimax and minimal distance martingale measures and their relationship to portfolio optimization. *Finance & Stochastics*, 5(4):557–581, Oct. 2001.

[15] P. V. Grigor'ev and Y. S. Kan. Optimal control of the investment portfolio with respect to the quantile criterion. *Automation and Remote Control*, 65(2):319–336, 2004.

[16] M. Hamada, M. Sherris, and J. van der Hoek. Martingale methods in dynamic portfolio allocation with distortion operators. Unpublished manuscript, *http://www.actuaries.org/AFIR/colloquia/Toronto/Hamada_Sherris_vanderHoek.pdf*, 2001.

[17] J. M. Harrison and D. M. Kreps. Martingales and arbitrage in multiperiod security markets. *Journal of Economic Theory*, 20(3):381–408, June 1979.

[18] J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11(3):215–260, Aug. 1981.

[19] J. M. Harrison and S. R. Pliska. A stochastic calculus model of continuous trading: Complete markets. *Stochastic Processes and their Applications*, 15(3):313–316, Aug. 1983.

[20] X. D. He and X. Y. Zhou. SP/A model in continuous time. Working Paper, University of Oxford, 2008.

[21] D. Heath. A continuous time version of Kulldorff's result. Unpublished manuscript, 1993.

[22] S. D. Jacka. A martingale representation result and an application to incomplete financial markets. *Mathematical Finance*, 2(4):239–250, Oct. 1992.

[23] H. Jin and X. Y. Zhou. Behavioral portfolio selection in continuous time. *Mathematical Finance*, 18:385–426, 2008. *Erratum*, to appear in *Mathematical Finance*, 2010.

[24] D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 47:263–291, 1979.

[25] I. Karatzas and S. E. Shreve. *Methods of Mathematical Finance*. Springer, 1998.

[26] S. Kataoka. On a stochastic programming model. *Econometrica*. 31: 181–196, 1963.

[27] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9(3):904–950, 1999.

[28] M. Kulldorff. Optimal control of favorable games with a time limit. *SIAM. Journal of Control and Optimization*, 31(1), Jan. 1993.

[29] L. L. Lopes. Between hope and fear: The psychology of risk. *Advances in Experimental Social Psychology*, 20:255–295, 1987.

[30] L. L. Lopes and G. C. Oden. The role of aspiration level in risk choice: A comparison of cumulative prospect theory and sp/a theory. *Journal of Mathematical Psychology*, 43(2):286–313, 1999.

[31] M. J. Machina. "Expected utility" analysis without the independence axiom. *Econometrica*, 50(2):277–322, 1982.

[32] D. B. Madan and X. Y. Zhou. Private communication. 2008.

[33] R. C. Merton. Lifetime portfolio selection under uncertainty: the continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, Aug. 1969.

[34] R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3:373–413, 1971.

[35] S. R. Pliska. A stochastic calculus model of continuous trading: Optimal portfolios. *Mathematics of Operations Research*, 11(2):371–382, May 1986.

[36] J. Quiggin. A theory of anticipated utility. *Journal of Economic Behavior and Organization*, 3:323–343, 1982.

[37] M. Rothschild and J. E. Stiglitz. Increasing risk: I. A definition. *Journal of Economic Theory*, 2(3):225–243, 1970.

[38] P. A. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51(3):239–246, Aug. 1969.

[39] W. Schachermayer, M. Sirbu, and E. Taflin. In which financial markets do mutual fund theorems hold true. *Finance & Stochastics*, 13(1):49–77, 2009.

[40] A. Schied. On the Neyman-Pearson problem for law-invariant risk measures and robust utility functionals. *Annals of Applied Probability*, 14:1398–1423, 2004.

[41] D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57(3):571–587, May 1989.

[42] G. Spivak and J. Cvitanić. Maximizing the probability of a perfect hedge. *The Annals of Applied Probability*, 9(4):1303–1328.

[43] R. Thaler. Towards a positive theory of consumer choice. *Journal of Economic Behavior and Organization*, 1:39–60, 1980.

[44] A. Tversky and D. Kahneman. Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, 5:297–323, 1992.

[45] M. E. Yaari. The dual theory of choice under risk. *Econometrica*, 55(1):95–115, Jan. 1987.