Markowitz Strategies Revised*

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Abstract

Continuous-time Markowitz's mean-variance efficient strategies are modified by parameterising a critical quantity. It is shown that these parameterised Markowitz strategies could reach the original mean target with arbitrarily high probabilities. This, in turn, motivates the introduction of certain stopped strategies where stock holdings are liquidated whenever the parameterised Markowitz strategies reach the present value of the mean target. The risk aspect of the revised Markowitz strategies are examined via expected discounted shortfall from the initial budget. A new portfolio selection model is suggested based on the results of the paper.

Key Words. Continuous-time portfolio selection, Markowitz efficient strategies, goal-reaching probability, stopping time, expected shortfall

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1 Introduction

Markowitz's Nobel-prize-winning mean-variance portfolio selection model (Markowitz 1952) is to minimise the variance of the terminal wealth subject to archiving a prescribed mean target in a single-period investment. The continuous-time extension of the model has been studied extensively in the literaute; see, e.g., Richardson (1989), Bajeux-Besnainous and Portait (1998), Zhou and Li (2000), and Lim (2004). Li and Zhou (2006) shows that with a continuous-time mean-variance efficient strategy the discounted mean target could be reached at or before the terminal time with a probability of at least 80% for any complete market with a deterministic investment opportunity set. This somewhat surprising result suggests that, while they are derived from the mean-variance criterion, the Markowitz strategies are quite good in terms of possessing high goal-reaching probabilities.

This paper is motivated by the preceding 80% rule. It starts with a very simply idea: now that a Markowitz strategy reaches a given goal with an at least 80% chance, if we parameterise certain quantity in such a strategy and then optimise over this parameter we should be able to further increase the goal-reaching probability. The idea indeed works, and we will show that the goal-reaching probability can be made as close to 100% as one wants. Therefore, we may revise a Markowitz strategy as follows: we follow the parameterised strategy until the goal is reached when we would liquidate the stock holdings. This stopped strategy would achieve the given target with a very high probability. There is, however, another issue. Goal-archiving alone cannot serve as the sole performance measure because the wealth process of a revised Markowitz strategy may go very low (indeed negatively low) before it eventually hits the target. Hence, we need to consider the risk aspect of such strategies at the same time. A natural and reasonable risk measure is the expected shortfall from the initial endowment. We will derive the expected shortfall analytically, and show that it is bounded even though the corresponding goal-reaching probability may be very close to 1. Our results further motivate us to propose a new portfolio selection model.

The remainder of the paper is organised as follows. In section 2 we set up the market and introduce the parameterised Markowitz strategies. Section 3 is devoted to the goalachieving probabilities of these parameterised Markowitz strategies. In section 4 stopped Markowitz strategies are proposed and their various properties are investigated including the expected shortfall. Finally, section 5 concludes.

2 Parameterised Markowitz Portfolios

Consider a capital market in which there are m + 1 assets. One is a risk-free asset (bank account) whose value process $S_0(t)$ is give by

$$S_0(t) = s_0 e^{\int_0^t r(s)ds}, \ t \ge 0,$$

where r(t) > 0 is the interest rate. The others are *m* risky assets (stocks) whose price processes $S_1(t), \dots, S_m(t)$ satisfy the following SDEs:

$$dS_i(t) = S_i(t) \left(\mu_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW^j(t) \right), \quad S_i(0) = s_i > 0, \quad t \ge 0,$$
(2.1)

where $\mu_i(t)$ is the appreciation rate, $\sigma_{ij}(t)$ is the volatility rate of the stocks, and $W(t) = (W^1(t), \dots, W^m(t))^{\tau}$ is a standard *m*-dimensional Brownian motion living in a standard probability space (Ω, \mathcal{F}, P) . Here and henceforth, τ denotes the matrix transpose. We assume that all the given parameters $r(t), \mu_i(t)$ and $\sigma_{ij}(t)$ are bounded deterministic functions on [0, T], where T > 0 is given. Furthermore, we assume that for each $t \in [0, T]$ the matrix $\sigma(t) = (\sigma_{ij}(t))$ is uniformly non-degenerate. Thus the market is complete.

Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the Brownian motion $W(\cdot)$. If an (\mathcal{F}_t) -adapted process $\pi(t) = (\pi_1(t), \cdots, \pi_m(t))^{\tau}$, where $\pi_i(t)$ is the total market value at time t of an agent's wealth in the *i*-th stock, satisfies $E \int_0^T |\pi(s)|^2 ds < \infty$, we call it an admissible portfolio (strategy). The wealth process under an admissible portfolio $\pi(\cdot)$ satisfies the following SDE (see, e.g., Karatzas and Shreve 1999):

$$dx(t) = \left[r(t)x(t) + \pi(t)^{\tau}b(t) \right] dt + \pi(t)^{\tau}\sigma(t)dW(t), \quad x(0) = x_0, \tag{2.2}$$

where $x_0 > 0$ is the initial endowment of an agent, and $b(t) = (\mu_1(t) - r(t), \dots, \mu_m(t) - r(t))^{\tau}$. Consequently, under a portfolio $\pi(\cdot)$, the agent's wealth invested in the bank account at time t equals $x(t) - \pi(t)^{\tau} \mathbf{1}$, where **1** denotes the m-dimensional column vector of 1's.

Let γ be a real number. Consider the following portfolio in a feedback form which would generate a portfolio-wealth pair $(\pi_{\gamma}(\cdot), x_{\gamma}(\cdot))$:

$$\pi_{\gamma}(t) = -\left[x_{\gamma}(t) - \gamma e^{\int_0^t r(s)ds}\right] (\sigma(t)\sigma(t)^{\tau})^{-1}b(t), \quad t \ge 0.$$

$$(2.3)$$

This class of portfolios is inspired by the form of a continuous-time Markowitz efficient portfolio where γ has a specific value related to the given mean target (see Remark 3 below). Substituting (2.3) into (2.2) we get the following SDE

$$dx_{\gamma}(t) = \left([r(t) - |\theta(t)|^2] x_{\gamma}(t) + \gamma e^{\int_0^t r(s)ds} |\theta(t)|^2 \right) dt + \left(\gamma e^{\int_0^t r(s)ds} - x_{\gamma}(t) \right) \theta(t)^{\tau} dW(t),$$
(2.4)

where $\theta(t) = \sigma(t)^{-1}b(t)$. Clearly this equation has a unique solution which is the wealth process generated by the portfolio $\pi_{\gamma}(\cdot)$. We call $\pi_{\gamma}(\cdot)$ a parameterised Markowitz portfolio with parameter γ .

Set
$$y_{\gamma}(t) = x_{\gamma}(t) - \gamma e^{\int_0^t r(s)ds}$$
. Then by (2.4) we have

$$dy_{\gamma}(t) = \left(r(t) - |\theta(t)|^2\right) y_{\gamma}(t)dt - y_{\gamma}(t)\theta(t)^{\tau}dW(t), \quad y_{\gamma}(0) = x_0 - \gamma.$$
(2.5)

Thus,

$$y_{\gamma}(t) = (x_0 - \gamma) \exp\left(\int_0^t r(s)ds - \varphi(t)\right), \qquad (2.6)$$

where

$$\varphi(t) = \frac{3}{2}\beta(t) + \int_0^t \theta(s)dW(s), \quad \beta(t) = \int_0^t |\theta(s)|^2 ds.$$
(2.7)

Remark 1 In view of (2.6) it is easy to see that $x_{\gamma}(t) < \gamma e^{\int_0^t r(s)ds}$ if $\gamma > x_0$, $x_{\gamma}(t) > \gamma e^{\int_0^t r(s)ds}$ if $\gamma < x_0$, and $x_{\gamma}(t) = \gamma e^{\int_0^t r(s)ds}$ if $\gamma = x_0$. So γ represents some ultimate (discounted) upper or lower bound of the wealth trajectory of a parameterised Markowitz portfolio.

Remark 2 If we consider $|y_{\gamma}(\cdot)|$ as the price process of a fictitious asset, then the parametrised Markowitz portfolio with parameter γ is nothing else than a *static* portfolio putting amount γ into the bank account and holding or shorting one share of the fictitious asset.

Remark 3 If $z > x_0$ and $\gamma = \frac{z - x_0 e^{-\beta(T)}}{1 - e^{-\beta(T)}}$ (it is easy to see that $\gamma > z > x_0$), then $\pi_{\gamma}(\cdot)$ is the mean-variance efficient portfolio (in the Markowitz sense) corresponding to the time-*T* mean target $ze^{\int_0^T r(s)ds}$; see Zhou and Li (2000).

In this paper we will use the following formulae extensively; see Borodin and Saminen (2002), pp. 251 and 252:

$$P\left(\sup_{0\le s\le t}(\mu s + W(s)) \ge y, \mu t + W(t) \in dz\right) = \frac{1}{\sqrt{2\pi t}}e^{\mu z - \mu^2 t/2 - (|z-y|+y)^2/2t}dz, \ y \ge 0,$$
(2.8)

$$P\left(\inf_{0\le s\le t}(\mu s + W(s))\le y, \mu t + W(t) \in dz\right) = \frac{1}{\sqrt{2\pi t}}e^{\mu z - \mu^2 t/2 - (|z-y|-y|^2/2t}dz, \ y\le 0.$$
(2.9)

3 Goal-Reaching Probabilities

In Li and Zhou (2005), it is shown that the Markowitz efficient strategy with a mean target $ze^{\int_0^T r(s)ds}$ indeed hits the target at or before the terminal time with a probability no less than 80%. Since the Markowitz efficient strategy is a special one of the parameterised strategies (2.3), we expect that this "goal-reaching" probability could be higher by choosing other values of γ . This section shows that, in fact, the goal-reaching probability can be made as high as one wants.

In the following we fix $z > x_0$ (if $z \le x_0$ then the problem is trivial). For any T > 0, we put

$$\tau_{\gamma}(T) = \inf \left\{ 0 < t \le T : x_{\gamma}(t) = z e^{\int_0^t r(s) ds} \right\}.$$

Here and henceforth we convent that $\inf \emptyset := \infty$. By (2.6) we obtain

$$\tau_{\gamma}(T) = \inf \left\{ 0 < t \le T : \exp(\varphi(t)) = \frac{\gamma - x_0}{\gamma - z} \right\}.$$
(3.1)

If $z > \gamma \ge x_0$, then $\frac{\gamma - x_0}{\gamma - z} \le 0$, and we have $\tau_{\gamma}(T) = \infty$. If $\gamma > z$ then $\frac{\gamma - x_0}{\gamma - z} > 1$; and if $\gamma < x_0$, then $0 < \frac{\gamma - x_0}{\gamma - z} < 1$. In the last two cases, for notational simplicity we denote $L(\gamma) = \log \frac{\gamma - x_0}{\gamma - z}$.

Theorem 3.1 We have

$$\lim_{\gamma \to \infty} P(\tau_{\gamma}(T) \le T) = 1, \quad \lim_{\gamma \downarrow z} P(\tau_{\gamma}(T) \le T) = 0, \tag{3.2}$$

and

$$\lim_{\gamma \uparrow x_0} P(\tau_{\gamma}(T) \le T) = 0, \quad \lim_{\gamma \to -\infty} P(\tau_{\gamma}(T) \le T) = 1.$$
(3.3)

Proof. First of all, notice that by a well-known time change technique

$$\varphi(t) = \frac{3}{2}\beta(t) + \widehat{W}(\beta(t)), \quad t \ge 0$$

where $\widehat{W}(\cdot)$ is a standard Brownian motion.

Now consider $\gamma > z$. Since $\varphi(0) = 0$ and $L(\gamma) > 0$, by (3.1) and (2.8) we have

$$P(\tau_{\gamma}(T) \leq T) = P\left(\sup_{0 \leq t \leq T} \varphi(t) \geq L(\gamma)\right)$$

= $P\left(\sup_{0 \leq t \leq \beta(T)} \left(\frac{3}{2}t + \widehat{W}(t)\right) \geq L(\gamma)\right)$
= $N\left(\frac{3}{2}\sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}}\right) + e^{3L(\gamma)}N\left(-\frac{3}{2}\sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}}\right),$ (3.4)

where $N(\cdot)$ is the cumulative distribution function of a standard normal random variable. It then follows immediately that (3.2) holds.

Next we assume $\gamma < x_0$. Notice in this case $L(\gamma) < 0$. Then

$$P(\tau_{\gamma}(T) \leq T) = P\left(\inf_{0 \leq t \leq T} \varphi(t) \leq L(\gamma)\right)$$

= $P\left(\inf_{0 \leq t \leq \beta(T)} \left(\frac{3}{2}t + \widehat{W}(t)\right) \leq L(\gamma)\right)$
= $N\left(-\frac{3}{2}\sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}}\right) + e^{3L(\gamma)}N\left(\frac{3}{2}\sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}}\right),$ (3.5)

leading to (3.3). The proof is complete.

So, the goal-reaching probability can be made as high as possible with the parameterised Markowitz strategies by sending the parameter γ to either positive or negative infinity. On the other hand, when γ is close to z from above or close to x_0 from below the goal-reaching probability is very small.

These results indicate that goal-reaching probability alone is not a good performance measure, because the risk part is not properly taken care of. They also show that the very popular risk measure VaR (Value at Risk) may not be a proper measure in guiding investment practice, because it gives only the probability of certain losses to occur, but not the magnitude of potential losses. As a result, VaR often induces agents to gamble on an event with a small probability. Browne (1999) studies the goal-reaching model with a non-negativity constraint on the wealth trajectory, and derives that the optimal terminal wealth profile is a digital option, which corresponds to exactly a gambling strategy; see also He and Zhou (2008). In the next section, we will address the risk-control issue of the parameterised Markowitz strategies.

4 Stopped Markowitz Strategies

The previous section shows that the revised, parameterised Markowitz portfolios could realize a given target before the terminal time with very high probabilities. This in turn suggests the following further revision of the strategies: whenever the target is realized half way through one stops and withdraws funds from the stocks. To be precise, let $z > x_0$ be the t = 0 value of the given target. We construct the following *stopped Markowitz portfolio* with parameter γ where $\gamma > z$ or $\gamma < x_0$: If $\tau_{\gamma}(T) \leq T$ then we liquidate all the stocks holdings at time $\tau_{\gamma}(T)$ and put the proceeds into the bank account to achieve the final total wealth $ze \int_0^T r(s) ds$ at time T; otherwise (i.e., $\tau_{\gamma}(T) = \infty$), we continue to use the (parameterised) Markowitz portfolio until T and get the final wealth $x_{\gamma}(T)$ (which will be *strictly* less than $ze \int_0^T r(s) ds$). Consequently, the wealth at time T of such a stopped Markowitz portfolio with parameter γ is

$$\xi_{\gamma}(T) = z e^{\int_0^T r(s)ds} I_{[\tau_{\gamma}(T) \le T]} + x_{\gamma}(T) I_{[\tau_{\gamma}(T) = \infty]}.$$
(4.1)

Define the expected discounted wealth at time T of a stopped Markowitz portfolio with parameter γ

$$W_{\gamma}(T) = E[e^{-\int_0^T r(s)ds} \xi_{\gamma}(T)].$$
(4.2)

Proposition 4.1 If $\gamma > z$ of $\gamma < x_0$ then $W_{\gamma}(T) < z$.

Proof. Since $e^{-\int_0^T r(s)ds} x_{\gamma}(T) < z$ on $[\tau_{\gamma}(T) = \infty]$ and $P(\tau_{\gamma}(T) = \infty) > 0$ by virtue of (3.4) or (3.5), we obtain the desired inequality immediately from (4.1).

So the stopped Markowitz portfolio underachieves the original *mean* target presented in the mean-variance model. However, this is compensated by the increased goal-reaching probability when $|\gamma|$ is sufficiently large. The following gives more precise estimates of the mean losses with stopped Markowitz strategies when γ converges to the ends of its ranges.

Proposition 4.2 We have

$$\lim_{\gamma \to \infty} W_{\gamma}(T) = z + (z - x_0) \left(e^{-\beta(T)} N(-\frac{1}{2}\sqrt{\beta(T)}) - 3N(-\frac{3}{2}\sqrt{\beta(T)}) \right),$$

$$\lim_{\gamma \downarrow z} W_{\gamma}(T) = z - (z - x_0) e^{-\beta(T)},$$
(4.3)

 $\lim_{\gamma \to -\infty} W_{\gamma}(T) = z + (z - x_0) \left(e^{-\beta(T)} N(\frac{1}{2}\sqrt{\beta(T)}) - 3N(\frac{3}{2}\sqrt{\beta(T)}) \right),$ $\lim_{\gamma \uparrow x_0} W_{\gamma}(T) = x_0.$ (4.4)

Proof. Since

$$E\left[e^{-\varphi(T)}, \ \tau_{\gamma}(T) = \infty\right] = E\left[e^{-\varphi(T)}\right] - E\left[e^{-\varphi(T)}, \tau_{\gamma}(T) \le T\right]$$
$$= e^{-\beta(T)} - E\left[e^{-\varphi(T)}, \tau_{\gamma}(T) \le T\right],$$

we deduce

$$W_{\gamma}(T) = E[e^{-\int_{0}^{T} r(s)ds}\xi_{\gamma}(T)]$$

$$= zP(\tau_{\gamma}(T) \leq T) + E\left[\gamma + (x_{0} - \gamma)e^{-\varphi(T)}, \ \tau_{\gamma}(T) = \infty\right]$$

$$= (z - \gamma)P(\tau_{\gamma}(T) \leq T) + \gamma$$

$$+ (x_{0} - \gamma)\left(e^{-\beta(T)} - E\left[e^{-\varphi(T)}, \tau_{\gamma}(T) \leq T\right]\right).$$
(4.5)

We first consider the $\gamma > z$ case where $L(\gamma) > 0$. By (3.1) and (2.8) we get

$$\begin{split} E\left[e^{-\varphi(T)}, \tau_{\gamma}(T) \leq T\right] &= E\left[e^{-\varphi(T)}, \sup_{0 \leq t \leq T} \varphi(t) \geq L(\gamma)\right] \\ &= E\left[e^{-\left(\frac{3}{2}\beta(T) + \widehat{W}(\beta(T))\right)}, \sup_{0 \leq t \leq \beta(T)} \left(\frac{3}{2}t + \widehat{W}(t)\right) \geq L(\gamma)\right] \\ &= \frac{e^{-9\beta(T)/8 - 2L(\gamma)^2/\beta(T)}}{\sqrt{2\pi\beta(T)}} \int_{-\infty}^{L(\gamma)} \exp\left(-\frac{z^2 - [\beta(T) + 4L(\gamma)]z}{2\beta(T)}\right) dz \\ &+ \frac{e^{-9\beta(T)/8}}{\sqrt{2\pi\beta(T)}} \int_{L(\gamma)}^{\infty} \exp\left(-\frac{z^2 - \beta(T)z}{2\beta(T)}\right) dz \\ &= e^{L(\gamma) - \beta(T)} N\left(-\frac{1}{2}\sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}}\right) + e^{-\beta(T)} N\left(\frac{1}{2}\sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}}\right). \end{split}$$

From (4.5) we obtain an expression of the expected discounted wealth at time T as follows:

$$W_{\gamma}(T) = (z - \gamma)P(\tau_{\gamma}(T) \leq T) + \gamma + (x_0 - \gamma)e^{-\beta(T)} \left[N\left(-\frac{1}{2}\sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}} \right) - e^{L(\gamma)}N\left(-\frac{1}{2}\sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}} \right) \right].$$

$$(4.6)$$

and

By l'Hôpital's rule, it is easy to show that

$$\lim_{\gamma \to \infty} \left[\gamma (1 - P(\tau_{\gamma} \le T)) \right] = (z - x_0) \left(\frac{2}{\sqrt{2\pi\beta(T)}} e^{-\frac{9}{8}\beta(T)} - 3N(-\frac{3}{2}\sqrt{\beta(T)}) \right),$$

and

$$\lim_{\gamma \to \infty} \left\{ (x_0 - \gamma) e^{-\beta(T)} \left[N \left(-\frac{1}{2} \sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}} \right) - e^{L(\gamma)} N \left(-\frac{1}{2} \sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}} \right) \right] \right\}$$
$$= (z - x_0) \left(e^{-\beta(T)} N \left(-\frac{1}{2} \sqrt{\beta(T)} \right) - \frac{2}{\sqrt{2\pi\beta(T)}} e^{-\frac{9}{8}\beta(T)} \right).$$

The first equation of (4.3) then follows immediately. The second one is straightforward. Now consider $\gamma < x_0$, where $L(\gamma) < 0$. Then

$$\begin{split} E\left[e^{-\varphi(T)}, \tau_{\gamma}(T) \leq T\right] &= E\left[e^{-\varphi(T)}, \inf_{0 \leq t \leq T} \varphi(t) \leq L(\gamma)\right] \\ &= E\left[e^{-\left(\frac{3}{2}\beta(T) + \widehat{W}(\beta(T))\right)}, \inf_{0 \leq t \leq \beta(T)} \left(\frac{3}{2}t + \widehat{W}(t)\right) \leq L(\gamma)\right] \\ &= \frac{e^{-9\beta(T)/8 - 2L(\gamma)^2/\beta(T)}}{\sqrt{2\pi\beta(T)}} \int_{L(\gamma)}^{\infty} \exp\left(-\frac{z^2 - [\beta(T) + 4L(\gamma)]z}{2\beta(T)}\right) dz \\ &+ \frac{e^{-9\beta(T)/8}}{\sqrt{2\pi\beta(T)}} \int_{-\infty}^{L(\gamma)} \exp\left(-\frac{z^2 - \beta(T)z}{2\beta(T)}\right) dz \\ &= e^{L(\gamma) - \beta(T)} N\left(\frac{1}{2}\sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}}\right) + e^{-\beta(T)} N\left(-\frac{1}{2}\sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}}\right). \end{split}$$

Thus, from (4.5) we obtain an expression of the expected discounted wealth at time T as follows:

$$W_{\gamma}(T) = (z - \gamma)P(\tau_{\gamma}(T) \leq T) + \gamma$$

+ $(x_0 - \gamma)e^{-\beta(T)} \left[N\left(\frac{1}{2}\sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}}\right) - e^{L(\gamma)}N\left(\frac{1}{2}\sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}}\right) \right].$ (4.7)

It can be shown that

$$\lim_{\gamma \to -\infty} [\gamma (1 - P(\tau_{\gamma} \le T))] = (z - x_0) \left(-\frac{2}{\sqrt{2\pi\beta(T)}} e^{-\frac{9}{8}\beta(T)} - 3N(\frac{3}{2}\sqrt{\beta(T)}) \right),$$

$$\lim_{\gamma \to -\infty} \left\{ (x_0 - \gamma) e^{-\beta(T)} \left[N \left(\frac{1}{2} \sqrt{\beta(T)} - \frac{L(\gamma)}{\sqrt{\beta(T)}} \right) - e^{L(\gamma)} N \left(\frac{1}{2} \sqrt{\beta(T)} + \frac{L(\gamma)}{\sqrt{\beta(T)}} \right) \right] \right\}$$
$$= (z - x_0) \left(e^{-\beta(T)} N \left(\frac{1}{2} \sqrt{\beta(T)} \right) + \frac{2}{\sqrt{2\pi\beta(T)}} e^{-\frac{9}{8}\beta(T)} \right).$$
(4.8)

This leads to the first equation of (4.4). The second one is straightforward.

By Theorem 3.1, the goal-reaching probabilities of the stopped Markowitz strategies are arbitrarily close to 1 when $|\gamma|$ is sufficiently large. Proposition 4.2 gives the precise prices, in terms of the resulting losses in terminal means, of the gains in goal-reaching probabilities. To elaborate, define

$$f(x) = e^{-x^2} N\left(-\frac{1}{2}x\right) - 3N\left(-\frac{3}{2}x\right), \ x \ge 0,$$

and

$$g(x) = e^{-x^2} N\left(\frac{1}{2}x\right) - 3N\left(\frac{3}{2}x\right), \ x \ge 0.$$

It is an easy exercise to show that -1 < f(x) < 0 and $-3 < g(x) < -1 \quad \forall x \ge 0$. See also Figures 1 and 2 for the graphs of the two functions. However, f(x) converges to 0 rather quickly when x becomes large. This, together with (4.3), suggests that the mean loss of the stopped Markowitz strategy with a large $\gamma > 0$ is small if $\beta(T)$ is large. Recall $\beta(T) = \int_0^T |\theta(s)|^2 ds$; so $\beta(T)$ being large is equivalent to T being large and/or the Sharpe ratio of the stock market being large. In other words, the stopped Markowitz strategies perform well when the market is good and the agent has a long investment horizon. In contrast, the stopped Markowitz strategy when $\gamma < 0$ is *not* preferred because, in view of the fact that g(x) < -1, there is a minimum loss in terminal mean (or $\lim_{\gamma\to -\infty} W_{\gamma}(T) \leq x_0$).

Now we are to study the problem of measuring the risk of this stopped Markowitz portfolio. From the last section we see that goal-reaching is not a proper performance measure without introducing a tradeoff in terms of risk control. Here we use the expected (discounted) shortfall, defined by

$$ES(\gamma) := E\left[\left(\xi_{\gamma}(T)e^{-\int_{0}^{T}r(s)ds} - x_{0}\right)_{-}\right],\tag{4.9}$$

as a reasonable risk measure, where $a_{-} = \max(-a, 0)$ for any real number a.

First of all, we have the following asymptotic properties of this risk measure for stopped Markowitz strategies.



Figure 1: $f(x) = e^{-x^2}N(-x/2) - 3N(-3x/2), x \ge 0.$



Figure 2: $g(x) = e^{-x^2} N(x/2) - 3N(3x/2), x \ge 0.$

Theorem 4.1 We have

$$\lim_{\gamma \to \infty} ES(\gamma) = (z - x_0) \left(-e^{-\beta(T)} N(-\frac{1}{2}\sqrt{\beta(T)}) + 3N(-\frac{3}{2}\sqrt{\beta(T)}) \right).$$

$$\lim_{\gamma \downarrow z} ES(\gamma) = (z - x_0) \left(e^{-\beta(T)} N\left(-\frac{1}{2}\sqrt{\beta(T)}\right) - N\left(-\frac{3}{2}\sqrt{\beta(T)}\right) \right).$$

$$(4.10)$$

and

$$\lim_{\gamma \to -\infty} ES(\gamma) = (z - x_0) \left(-e^{-\beta(T)} N(\frac{1}{2}\sqrt{\beta(T)}) + 3N(\frac{3}{2}\sqrt{\beta(T)}) \right),$$

$$\lim_{\gamma \uparrow x_0} ES(\gamma) = 0.$$
(4.11)

Proof. Since $[\xi_{\gamma}(T) \leq x_0 e^{\int_0^T r(s)ds}] \subset [\tau_{\gamma}(T) = \infty]$, we have

$$ES(\gamma) = E\left[\left(x_{\gamma}(T)e^{-\int_{0}^{T}r(s)ds} - x_{0}\right)_{-}, \ \tau_{\gamma}(T) = \infty\right], \tag{4.12}$$

where

$$x_{\gamma}(T)e^{-\int_{0}^{T}r(s)ds} - x_{0} = y_{\gamma}(T)e^{-\int_{0}^{T}r(s)ds} + (\gamma - x_{0})$$

= $(\gamma - x_{0})\left(1 - e^{-\varphi(T)}\right).$

First we assume $\gamma > z$. In this case, we have

$$ES(\gamma) = (\gamma - x_0) E\left[\left(1 - e^{-\varphi(T)}\right)_{-}, \ \tau_{\gamma}(T) = \infty\right].$$

$$(4.13)$$

By (3.1) and (2.8) we get

$$\begin{split} &E\Big[\Big(1-e^{-\varphi(T)}\Big)_{-}, \ \tau_{\gamma}(T) \leq T\Big]\\ &= E\Big[\Big(1-e^{-(\frac{3}{2}\beta(T)+\widehat{W}(\beta(T))}\Big)_{-}, \ \sup_{0 < t \leq \beta(T)} \left(\frac{3}{2}t+\widehat{W}(t)\right) \geq L(\gamma)\Big]\\ &= \frac{e^{-9\beta(T)/8-2L(\gamma)^{2}/\beta(T)}}{\sqrt{2\pi\beta(T)}} \int_{-\infty}^{0} \exp\left(-\frac{z^{2}-[3\beta(T)+4L(\gamma)]z}{2\beta(T)}\right) \left(e^{-z}-1\right)dz\\ &= e^{L(\gamma)-\beta(T)}N\Big(-\Big(\frac{1}{2}\sqrt{\beta(T)}+\frac{2L(\gamma)}{\sqrt{\beta(T)}}\Big)\Big) - e^{3L(\gamma)}N\Big(-\Big(\frac{3}{2}\sqrt{\beta(T)}+\frac{2L(\gamma)}{\sqrt{\beta(T)}}\Big)\Big). \end{split}$$

On the other hand, we have

$$E[(1 - e^{-\varphi(T)})_{-}] = E[(1 - e^{-(\frac{3}{2}\beta(T) + \widehat{W}(\beta(T))})_{-}]$$

= $\frac{1}{\sqrt{2\pi\beta(T)}} \int_{-\infty}^{-\frac{3}{2}\beta(T)} (e^{-(\frac{3}{2}\beta(T) + z)} - 1)e^{-\frac{z^2}{2\beta(T)}} dz$
= $e^{-\beta(T)}N(-\frac{1}{2}\sqrt{\beta(T)}) - N(-\frac{3}{2}\sqrt{\beta(T)}).$

Thus, we obtain an expression for the expected (discounted) shortfall $ES(\gamma)$:

$$ES(\gamma) = (\gamma - x_0)k(\gamma), \quad \gamma > z, \tag{4.14}$$

where

$$k(\gamma) = e^{-\beta(T)} N\left(-\frac{1}{2}\sqrt{\beta(T)}\right) - N\left(-\frac{3}{2}\sqrt{\beta(T)}\right)$$
$$-e^{L(\gamma)-\beta(T)} N\left(-\left(\frac{1}{2}\sqrt{\beta(T)} + \frac{2L(\gamma)}{\sqrt{\beta(T)}}\right)\right)$$
$$+e^{3L(\gamma)} N\left(-\left(\frac{3}{2}\sqrt{\beta(T)} + \frac{2L(\gamma)}{\sqrt{\beta(T)}}\right)\right), \quad \gamma > z.$$
(4.15)

It is easy to see that

$$\lim_{\gamma \downarrow z} ES(\gamma) = (z - x_0) \left(e^{-\beta(T)} N \left(-\frac{1}{2} \sqrt{\beta(T)} \right) - N \left(-\frac{3}{2} \sqrt{\beta(T)} \right) \right).$$

Now we are going to investigate $\lim_{\gamma\to\infty} ES(\gamma)$. First of all, we have

$$\lim_{\gamma \to \infty} ES(\gamma) = \lim_{\gamma \to \infty} (\gamma - x_0) k(\gamma) = \lim_{\gamma \to \infty} \frac{k'(\gamma)}{-\frac{1}{(\gamma - x_0)^2}}$$
$$= -\lim_{\gamma \to \infty} (\gamma - x_0)^2 k'(\gamma).$$

Next we have

$$\begin{aligned} k'(\gamma) &= -L'(\gamma)e^{L(\gamma)-\beta(T)}N\Big(-\Big(\frac{1}{2}\sqrt{\beta(T)} + \frac{2L(\gamma)}{\sqrt{\beta(T)}}\Big)\Big) \\ &+ \frac{2L'(\gamma)}{\sqrt{2\pi\beta(T)}}e^{L(\gamma)-\beta(T)}e^{-\frac{1}{2}\left(\frac{1}{2}\sqrt{\beta(T)} + \frac{2L(\gamma)}{\sqrt{\beta(T)}}\right)^2} \\ &+ 3L'(\gamma)e^{3L(\gamma)}N\Big(-\Big(\frac{3}{2}\sqrt{\beta(T)} + \frac{2L(\gamma)}{\sqrt{\beta(T)}}\Big)\Big) \\ &- \frac{2L'(\gamma)}{\sqrt{2\pi\beta(T)}}e^{3L(\gamma)}e^{-\frac{1}{2}\left(\frac{3}{2}\sqrt{\beta(T)} + \frac{2L(\gamma)}{\sqrt{\beta(T)}}\right)^2}. \end{aligned}$$

Since $\lim_{\gamma\to\infty} (\gamma - x_0)^2 L'(\gamma) = x_0 - z$ and $\lim_{\gamma\to\infty} L(\gamma) = 0$, we obtain the first equation of (4.10).

Now we consider the $\gamma < x_0$ case. In this case we have

$$ES(\gamma) = (x_0 - \gamma)E\Big[\Big(1 - e^{-\varphi(T)}\Big)_+, \ \tau_{\gamma}(T) = \infty\Big], \tag{4.16}$$

where $a_{+} = \max(a, 0)$ for any real number a. Noting that $L(\gamma) < 0$, by (3.1) and (2.9) we get

$$\begin{split} E\Big[\Big(1-e^{-\varphi(T)}\Big)_+, \ \tau_{\gamma}(T) \leq T\Big] &= E\Big[\Big(1-e^{-\varphi(T)}\Big)_+, \ \inf_{0\leq t\leq T}\varphi(t) \leq L(\gamma)\Big]\\ &= E\Big[\Big(1-e^{-(\frac{3}{2}\beta(T)+\widehat{W}(\beta(T))}\Big)_+, \ \inf_{0\leq t\leq \beta(T)}\left(\frac{3}{2}t+\widehat{W}(t)\right) \leq L(\gamma)\Big]\\ &= \frac{e^{-9\beta(T)/8-2L(\gamma)^2/\beta(T)}}{\sqrt{2\pi\beta(T)}} \int_0^\infty \exp\left(-\frac{z^2-[3\beta(T)+4L(\gamma)]z}{2\beta(T)}\right)\Big(1-e^{-z}\Big)dz\\ &= e^{3L(\gamma)}N\Big(\frac{2L(\gamma)}{\sqrt{\beta(T)}}+\frac{3}{2}\sqrt{\beta(T)}\Big) - e^{L(\gamma)-\beta(T)}N\Big(\frac{2L(\gamma)}{\sqrt{\beta(T)}}+\frac{1}{2}\sqrt{\beta(T)}\Big). \end{split}$$

On the other hand, we have

$$E[(1 - e^{-\varphi(T)})_{+}] = E[(1 - e^{-(\frac{3}{2}\beta(T) + \widehat{W}(\beta(T))})_{+}]$$

= $\frac{1}{\sqrt{2\pi\beta(T)}} \int_{-\frac{3}{2}\beta(T)}^{\infty} (1 - e^{-(\frac{3}{2}\beta(T) + z)}) e^{-\frac{z^{2}}{2\beta(T)}} dz$
= $N(\frac{3}{2}\sqrt{\beta(T)}) - e^{-\beta(T)}N(\frac{1}{2}\sqrt{\beta(T)}).$

Finally we obtain an expression for the expected (discounted) shortfall $ES(\gamma)$:

$$ES(\gamma) = (\gamma - x_0)l(\gamma), \quad \gamma < x_0, \tag{4.17}$$

where

$$l(\gamma) = N\left(\frac{3}{2}\sqrt{\beta(T)}\right) - e^{-\beta(T)}N\left(\frac{1}{2}\sqrt{\beta(T)}\right) - e^{3L(\gamma)}N\left(\frac{2L(\gamma)}{\sqrt{\beta(T)}} + \frac{3}{2}\sqrt{\beta(T)}\right)$$
(4.18)

$$+e^{L(\gamma)-\beta(T)}N\Big(\frac{2L(\gamma)}{\sqrt{\beta(T)}} + \frac{1}{2}\sqrt{\beta(T)}\Big), \quad \gamma < x_0.$$

$$(4.19)$$

Similarly as above we can prove (4.11).

The first equation of (4.10) implies that $|\lim_{\gamma\to\infty} ES(\gamma)| \leq z - x_0$. So the expected shortfall is capped when $\gamma > 0$ is sufficiently large. Moreover, as discussed earlier this cap could be reduced to be arbitrarily small when the stock market is sufficiently good and/or the investment horizon is sufficiently long. Recall that the goal-reaching probability is arbitrarily high when γ is sufficiently large. So the revised Markowitz strategies perform well judged by the goal-reaching probability and expected shortfall. On the other hand, the cap on the other limit, as seen by $|\lim_{\gamma\to-\infty} ES(\gamma)| \leq 3(z-x_0)$, is not as good.

We may define the expected (discounted) excess

$$EE(\gamma) := E\left[\left(\xi_{\gamma}(T)e^{-\int_0^T r(s)ds} - x_0\right)_+\right],\tag{4.20}$$

Corollary 4.1 We have

$$\lim_{\gamma \to \infty} EE(\gamma) = z - x_0.$$

$$\lim_{\gamma \downarrow z} EE(\gamma) = (z - x_0) \left(-e^{-\beta(T)} N\left(\frac{1}{2}\sqrt{\beta(T)}\right) + N\left(\frac{3}{2}\sqrt{\beta(T)}\right) \right).$$
(4.21)

and

$$\lim_{\gamma \to -\infty} EE(\gamma) = z - x_0, \quad \lim_{\gamma \uparrow x_0} EE(\gamma) = 0.$$
(4.22)

Proof. These results are immediate by noticing that

$$EE(\gamma) - ES(\gamma) = W_{\gamma}(T) - x_0,$$

and then combining Theorem 4.1 and Proposition 4.1.

We end this section by proposing a new portfolio selection model: For any given a > 0, to choose γ and the corresponding stopped Markowitz strategy so as to

maximise
$$P(\tau_{\gamma}(T) \leq T)$$

subject to $EE(\gamma) \leq a$, (4.23)
 $\gamma > z \text{ or } \gamma < x_0$.

An optimal solution to the above achieves a Pareto optimality between the goal-reaching probability and the shortfall risk. Since both the analytical forms of $P(\tau_{\gamma}(T) \leq T)$ and $EE(\gamma)$ have been derived in this paper, (4.23) is a very easy one-dimensional optimisation problem.

5 Conclusions

This paper introduces some revised Markowitz strategies and investigates their properties in terms of the goal-reaching probabilities, expected discounted terminal payments, and expected shortfall. The results suggest that the continuous-time mean-variance model produce sensible investment solutions beyond the original mean-variance criterion, and Markowitz's efficient strategies could serve as building blocks to construct more models and portfolios that address different investment problems.

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