

# VALUATION OF STOCK LOANS WITH REGIME SWITCHING

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**Abstract.** This paper is concerned with stock loan valuation in which the underlying stock price is dictated by geometric Brownian motion with regime switching. The stock loan pricing is quite different from that for standard American options because the associated variational inequalities may have infinitely many solutions. In addition, the optimal stopping time equals infinity with positive probability. Variational inequalities are used to establish values of stock loans and reasonable values of critical parameters such as loan sizes, loan rates and service fees in terms of certain algebraic equations. Numerical examples are included to illustrate the results.

**Key words.** Stock loan, regime switching, variational inequalities, optimal stopping, smooth fit

**AMS subject classifications.** 91B28, 60G35, 34B60

**Running head:** Stock Loan Valuation

**1. Introduction.** This paper is concerned with valuation of stock loans. A stock loan involves two parties, a borrower and a lender. The borrower owns one share of a stock and obtains a loan from the lender with the share as collateral. The borrower may regain the stock in the future by repaying the lender the principal plus interest or alternatively surrender the stock. Stock loan valuation has been attracting attentions of both academic researchers and lending institutions. When a stock holder prefers not to sell his stock due to either capital gain tax consideration or restrictions on sales of his stock, a stock loan is a viable alternative for raising cash. In addition, the loan can help the stock holder hedge against a market downturn. For example, if the stock price goes down, the borrower may forfeit the stock instead of repaying the loan. On the other hand, if the stock price goes up, the borrower can repay the loan and regain the stock.

In Xia and Zhou [16], the stock loan valuation is studied using a pure probabilistic approach with a classical geometric Brownian model. They pointed out that the variational inequality (VI) approach cannot be directly applied to stock loan pricing as in American option pricing because the associated VIs may have infinitely many solutions. In addition, the corresponding optimal stopping time equals infinity with positive probability. Nevertheless, the variational inequality approach is very useful for optimal stopping problems because it is associated with a set of sufficient conditions that are easy to verify. It is natural for models with regime switching, for instance. Moreover, the VI approach typically leads to partial differential equations that can be solved numerically.

In this paper, we first use variational inequalities to solve the stock loan pricing problem considered in [16]. We overcome the difficulty of possibly infinitely many solutions to the VIs by pinning down the *right* solution which is identical to the value function. Then, we carry this approach over to models in which the underlying stock price follows a geometric Brownian motion with regime switching. The model with

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regime switching stems from the need of more realistic models that better reflect random market environment. The market regime could reflect the state of the underlying marketplace and the general mood of investors which determines the overall volatility. The regime-switching model was first introduced by Hamilton [11] in 1989 to describe a regime-switching time series. Di Masi *et al.* [5] discuss mean-variance hedging for regime-switching European option pricing. To price regime-switching American and European options, Bollen [1] employs lattice method and simulation, whereas Buffington and Elliott [2] use risk-neutral pricing and derive a set of partial differential equations for the option price. Guo [7] and Shepp [15] use regime-switching to model option pricing with inside information; see also Guo [8, 9]. Duan *et al.* [6] establish a class of GARCH option models under regime switching. For the important issue of fitting the regime-switching model parameters, Hardy [12] develops maximum likelihood estimation using real data from the S&P 500 and TSE 300 indices. In addition to option pricing, regime-switching models have also been formulated and investigated for other problems; see Zhang [19] for the development of an optimal stock selling rule, Zhang and Yin [20] for applications in portfolio management, and Zhou and Yin [21] for a dynamic Markowitz problem.

In this paper, we use variational inequalities to establish sufficient conditions in terms of algebraic equations. The smooth-fit technique used in Guo and Zhang [10] in connection with stock selling is applied for the stock loan problem. We begin with the classical geometric Brownian motion model, then study the model with a single regime jump, and finally extend the results to incorporate the model with a two-state Markov chain.

To re-cap, the main contributions of this paper include: a) Development of the variational inequality approach for the stock loan valuation originally treated in Xia and Zhou [16] via a purely probabilistic method. The main difficulty associated with the VI approach is that the corresponding VIs do not possess the traditional uniqueness property. Such difficulty was overcome by establishing necessary conditions required to rule out all other solutions except the one identical to the value function, and b) The VI approach was extended to treat the valuation problem in more general setting involving models with regime switching. Delicate analysis was carried out to obtain closed-form solutions.

This paper is organized as follows. In the next section, we formulate the problem under consideration. In Section 3, we consider the case with no regime change and re-establish the results derived in [16] using variational inequalities. The case with only a single regime change and the case with a two-state Markov chain are considered in Sections 4 and 5 respectively. Sufficient conditions in each of these cases are obtained. Finally, in Section 6, we consider fair values of various parameters. We also report numerical examples in each section to illustrate these results.

**2. Problem Formulation.** Let  $r > 0$  denote the risk-free interest rate, and  $S_t^0$  the stock price (under risk-neutral setting) at time  $t$  satisfying the equation

$$\frac{dS_t^0}{S_t^0} = rdt + \sigma(\theta_t)dW_t, \quad S_0^0 = x,$$

where  $\theta_t \in \mathcal{M} = \{1, 2\}$  is a two-state Markov chain,  $\sigma(i)$ ,  $i = 1, 2$ , are constants and  $W_t$  is a standard Brownian motion independent of  $\theta_t$ . Here the stock under consideration pays no dividends.<sup>1</sup>

<sup>1</sup>The presence of dividends would add another dimension of complexity. Depending on whether the dividends are taken by the lender or accrued to the borrower's account the underlying stock

The stock loan problem is as follows. A client borrows amount  $q > 0$  from a bank with one share of the stock as collateral. After paying a service fee  $0 < c < q$  to the bank, the client receives the amount  $(q - c)$ . The client has the right (not obligation) to regain the stock at any time  $t \geq 0$  by repaying amount  $qe^{\gamma t}$ , where  $\gamma > r$  is the continuously compounding loan interest rate. The main goal of this paper is to evaluate the loan value defined in (1) below. Such an evaluation can in turn be used to determine the rational values of the parameters  $q$ ,  $c$ , and  $\gamma$ .

Let  $\mathcal{F}_t$  denote the filtration generated by  $(W, \theta)$  and  $\mathcal{T}$  denote the class of  $\mathcal{F}_t$ -stopping times. Our objective is to evaluate the loan value

$$(1) \quad v(x, i) = \sup_{\tau \in \mathcal{T}} E[e^{-r\tau}(S_\tau^0 - qe^{\gamma\tau})_+ I_{\{\tau < \infty\}} | S_0^0 = x, \theta_0 = i],$$

where  $x_+ = \max\{0, x\}$ .

Let  $\mu = \gamma - r > 0$  and  $S_t = e^{-\gamma t} S_t^0$ . Then,  $S_t$  satisfies the equation

$$(2) \quad \frac{dS_t}{S_t} = -\mu dt + \sigma(\theta_t) dW_t, \quad S_0 = x.$$

The corresponding value function can be written in terms of  $S_t$

$$v(x, i) = \sup_{\tau \in \mathcal{T}} E[e^{\mu\tau}(S_\tau - q)_+ I_{\{\tau < \infty\}} | S_0 = x, \theta_0 = i].$$

As in [16], it is easy to show that (a)  $v(0, i) = 0$ ; (b)  $v(x, i)$  is nondecreasing in  $x$ ; and (c)  $v(x, i)$  is convex, for each  $i \in \mathcal{M}$ .

REMARK 1.. Note that this corresponds to a perpetual American call option with a *negative* interest rate  $(-\mu)$ .

**3. The Model without Regime Switching.** To appreciate fully why the variational inequality approach does not work *directly* for the stock loan valuation problem, we first focus on the model with constant  $\theta_t$  (no jumps). In this case, both  $\sigma$  and  $v$  are independent of  $i \in \mathcal{M}$ . The corresponding value function is given by

$$v(x) = \sup_{\tau \in \mathcal{T}} E[e^{\mu\tau}(S_\tau - q)_+ I_{\{\tau < \infty\}} | S_0 = x],$$

subject to

$$\frac{dS_t}{S_t} = -\mu dt + \sigma dW_t, \quad S_0 = x.$$

Let  $\mathcal{A}$  denote the corresponding generator

$$\mathcal{A} = \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} - \mu x \frac{\partial}{\partial x}.$$

The associated VI has the form:

$$(3) \quad \max\{(\mu + \mathcal{A})f(x), (x - q)_+ - f(x)\} = 0, \quad f(0) = 0.$$

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loan models can be very different, whereas pricing of the latter is a more technically challenging problem. In [16] the former case is considered, and in [4] various structures of dividends disposition are investigated in great details. Here we omit the consideration of dividends to avoid unduly technicality as the main purpose here is to address the regime switching problem.

Typically, in the case with a positive interest rate (i.e.  $\mu < 0$ ), the above VI has a unique nearly smooth solution as prescribed in Øksendal [14] which can be proved to be identical to the value function. However, with a possibly negative interest rate ( $\mu > 0$ ), such uniqueness may not exist, as shown below.

To solve (1), one should solve the equation  $\mu f + \mathcal{A}f = 0$  on an interval  $[0, x_0]$  for some  $x_0 > q$  and then smooth-fit the solution with the function  $f(x) = x - q$  on  $(x_0, \infty)$ . Let us first consider the case when  $2\mu/\sigma^2 > 1$ .

In this case, the general solution to  $\mu f + \mathcal{A}f = 0$  is

$$(4) \quad f(x) = a_1 x + a_2 x^\beta,$$

for  $\beta = 2\mu/\sigma^2 > 1$  and some constants  $a_1$  and  $a_2$ . To paste  $a_1 x + a_2 x^\beta$  and  $x - q$  smoothly at  $x_0$ , we need

$$a_1 x_0 + a_2 (x_0)^\beta = x_0 - q,$$

$$a_1 + a_2 \beta (x_0)^{\beta-1} = 1.$$

For fixed  $0 \leq a_1 < 1$ , solve for  $a_2$  and  $x_0$  to obtain

$$\begin{aligned} x_0 &= \frac{\beta q}{(\beta - 1)(1 - a_1)}, \\ a_2 &= \frac{(\beta - 1)^{\beta-1}}{\beta^\beta} q^{1-\beta} (1 - a_1)^\beta. \end{aligned}$$

It is easy to see that  $x_0 > q$ , and it can be shown with some calculation that, for each  $0 \leq a_1 < 1$ ,

$$(5) \quad f(x) = \begin{cases} a_1 x + \frac{(\beta - 1)^{\beta-1}}{\beta^\beta} q^{1-\beta} (1 - a_1)^\beta x^\beta, & \text{if } x < x_0, \\ x - q, & \text{if } x \geq x_0, \end{cases}$$

is a solution to (1). Hence equation (1) has infinitely many nearly smooth solutions.

A key assumption imposed in [14, Theorem 10.4.1] requires  $\tau^* < \infty$  a.s., where  $\tau^* = \inf\{t : S_t > x_0\}$ . Recall that the expected rate in (2) is negative. This implies that, for  $S_0 < x_0$ , the probability that  $\tau^* = \infty$  is greater than zero! Therefore,

$$f(S_{\tau^*}) \neq (S_{\tau^*} - q)_+$$

with positive probability because  $S_{\tau^*}$  may never reach  $x_0$ . One therefore needs to modify the proof in [14] to relax the condition  $\tau^* < \infty$  a.s.

Let  $f$  be given by (3) parameterized by some  $a_1$ . Following the steps in [14, Theorem 10.4.1] and using Dynkin's formula and Fatou's lemma, we can show that

$$f(x) \geq v(x), \text{ for all } x > 0.$$

We next establish the opposite inequality for some particular value of  $a_1$ . If  $x > x_0$ , take  $\tau^* = 0$  which leads to  $f(x) = (x - q)_+ = v(x)$ . If  $x < x_0$ , take  $\tau^* = \inf\{t : S_t > x_0\}$ . Then on  $(0, x_0)$ ,  $(\mu + \mathcal{A})f(S_t) = 0$ . Again, using Dynkin's formula, we have, for each  $N$ ,

$$(6) \quad f(x) = E[e^{\mu(\tau^* \wedge N)} f(S_{\tau^* \wedge N})] \leq E[e^{\mu\tau^*} f(S_{\tau^*}) I_{\{\tau^* \leq N\}}] + E[e^{\mu N} f(S_N) I_{\{\tau^* > N\}}]$$

The first term on the right hand side converges to

$$E[e^{\mu\tau^*} f(S_{\tau^*}) I_{\{\tau^* < \infty\}}] = E[e^{\mu\tau^*} (S_{\tau^*} - q)_+ I_{\{\tau^* < \infty\}}].$$

We need conditions on  $a_1$  and  $a_2$  to make the second term in (4) go to zero. To this end, we estimate  $E[e^{\mu N} (S_N)^\kappa]$  for some  $\kappa \geq 1$ . It easy to see that

$$S_N = S_0 \exp \left\{ - \left( \mu + \frac{\sigma^2}{2} \right) N + \sigma W_N \right\}.$$

It follows that

$$E[e^{\mu N} (S_N)^\kappa] = S_0^\kappa E \left[ \exp \left\{ \mu N - \kappa \left( \mu + \frac{\sigma^2}{2} \right) N + \kappa \sigma W_N \right\} \right].$$

Note that

$$\exp \left\{ - \frac{\kappa^2 \sigma^2}{2} t + \kappa \sigma W_t \right\}$$

is a martingale. It follows that

$$\begin{aligned} E[e^{\mu N} (S_N)^\kappa] &= S_0^\kappa \exp \left\{ \left( \mu - \kappa \left( \mu + \frac{\sigma^2}{2} \right) + \frac{\kappa^2 \sigma^2}{2} \right) N \right\} \\ (7) \quad &= S_0^\kappa \exp \left\{ (\kappa - 1)(\kappa - \beta) \frac{\sigma^2 N}{2} \right\}. \end{aligned}$$

This means that the term  $E[e^{\mu N} (S_N)^\kappa]$  converges to 0 only for those  $1 < \kappa < \beta$ . Therefore, the linear term in (2) (which corresponds to  $\kappa = 1$ ) should be dropped, i.e.,  $a_1$  should be set to 0. Moreover, it is easy to see that, for  $0 \leq x \leq x_0$ ,

$$x^\beta \leq K x^\kappa, \text{ for some constant } K.$$

This implies that

$$E[e^{\mu N} (S_N)^\beta I_{\{\tau^* > N\}}] \rightarrow 0, \text{ as } N \rightarrow \infty.$$

Therefore, for  $2\mu/\sigma^2 > 1$ ,  $v(x) = f(x)$  which has the form (3) with  $a_1 = 0$ .

To complete the part with constant  $\theta_t$ , we consider the case when  $\beta = 2\mu/\sigma^2 \leq 1$ . First, note that the solution should be linear in  $x$ . Using (2), this is clear when  $\beta = 1$ . When  $0 < \beta < 1$ , recall the monotonicity and convexity of the value function. It follows that both the first and second derivatives  $f'(x)$  and  $f''(x)$  should be nonnegative. Recall  $f(x) = a_1 x + a_2 x^\beta$  given in (2). The condition  $f'(x) = a_1 + a_2 \beta x^{\beta-1} \geq 0$  implies that  $a_2 \geq 0$  because otherwise  $f'(x) \rightarrow -\infty$  as  $x \rightarrow 0^+$ . Moreover  $f''(x) = a_2 \beta(\beta-1) x^{\beta-2} \geq 0$  leads to  $a_2 \leq 0$ . Therefore,  $a_2 = 0$ . In what follows, we show that  $a_1 = 1$  or  $f(x) = x$ . It is clear that  $f(x) = x$  is a solution to the VI in (1). We can show that  $f(x) = x \geq v(x)$  for all  $x > 0$  as in [14, Theorem 10.4.1]. To establish the opposite inequality, note that

$$v(x) \geq E[e^{\mu N} (S_N - q)_+], \text{ for all } N.$$

If we can show  $E[e^{\mu N} (S_N - q)_+] \rightarrow S_0$ , then we have  $f(x) = x$ . To see this, we write

$$\begin{aligned} E[e^{\mu N} (S_N - q)_+] &= S_0 \int_{\{S_N > q\}} \exp \left( - \frac{\sigma^2 N}{2} + \sigma u \right) \Phi(u, 0, N) du - q H_N \\ &= S_0 \int_{\{S_N > q\}} \Phi(u, N\sigma, N) du - q H_N, \end{aligned}$$

where  $\Phi(u, m, \Sigma)$  is the Gaussian density function with mean  $m$  and variance  $\Sigma$  and  $H_N = e^{\mu N} P(S_N > q)$ . It is shown in Appendix that  $H_N \rightarrow 0$ . Using change of variable  $w = (u - N\sigma)/\sqrt{N}$  and noticing that

$$\{S_N > q\} = \left\{ w > \frac{1}{\sqrt{N}\sigma} \log \frac{q}{S_0} + \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right) \sqrt{N} \right\},$$

we have, as  $N \rightarrow \infty$ ,

$$(8) \ E[e^{\mu N} (S_N - q)_+] = S_0 \int_{A_N}^{\infty} \Phi(w, 0, 1) dw - q H_N \rightarrow \begin{cases} S_0, & \text{if } \frac{\mu}{\sigma} - \frac{\sigma}{2} < 0, \\ S_0/2, & \text{if } \frac{\mu}{\sigma} - \frac{\sigma}{2} = 0. \end{cases}$$

where

$$A_N = \frac{1}{\sqrt{N}\sigma} \log \frac{q}{S_0} + \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right) \sqrt{N}.$$

For  $2\mu/\sigma^2 < 1$ , this implies  $f(x) = x = v(x)$ . For  $2\mu/\sigma^2 = 1$ , the above approach does not work. We resort to a two point value problem approach used in [19] for stock selling. Given two numbers  $a$  and  $b$ . Suppose  $0 < a < q < b$ . For  $a < S_0 < b$ , let  $\tau_{a,b} = \inf\{t : S_t \notin (a, b)\}$ . Then,  $v(x) \geq h(x) := E[e^{\mu\tau_{a,b}} (S_{\tau_{a,b}} - q)_+]$ . It is shown in [19] that  $h(x)$  is a solution to the two point boundary value differential equation (TPBVDE)

$$(\mu + \mathcal{A})h(x) = 0, \quad h(a) = 0, \quad h(b) = b - q.$$

Under the condition  $2\mu = \sigma^2$ , the corresponding characteristic equation  $(\mu + \mathcal{A})h(x) = 0$  has multiple root 1. The general solution is

$$h(x) = C_1 x + C_2 x \log x.$$

Using the values at the boundaries to determine the values of  $C_1$  and  $C_2$ , we have

$$h(x) = -\frac{(b-q) \log a}{b(\log b - \log a)} x + \frac{b-q}{b(\log b - \log a)} x \log x.$$

Sending  $a \rightarrow 0$  yields

$$h(x) \rightarrow \frac{b-q}{b} x,$$

which converges to  $x$  as  $b \rightarrow \infty$ . Thus,  $v(x) = f(x) = x$  when  $2\mu/\sigma^2 = 1$ .

**4. The Model with a Single Regime Switching.** In this section, we consider the case when  $\theta_t$  has a single jump. This is the simplest model with stochastic volatility. Such model is used in [17] to capture volatility smile in connection with option pricing. Without loss of generality, we assume the generator of  $\theta_t$  has the form

$$Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ 0 & 0 \end{pmatrix}, \quad \text{with } \lambda_1 > 0.$$

This means that state 2 is the absorbing state. To compute the value function  $v(x, i)$ , we treat the following two cases.

**Case 1:**  $2\mu/\sigma^2(2) \leq 1$ . First, we show that

$$v(x, 1) = v(x, 2) = x.$$

Let  $\mathcal{A}$  denote the generator of  $(S_t, \theta_t)$  given by

$$\begin{aligned}\mathcal{A}f(x, \cdot)(1) &= \frac{\sigma^2(1)}{2} x^2 \frac{\partial^2 f(x, 1)}{\partial x^2} - \mu x \frac{\partial f(x, 1)}{\partial x} + \lambda_1(f(x, 2) - f(x, 1)) \\ \mathcal{A}f(x, \cdot)(2) &= \frac{\sigma^2(2)}{2} x^2 \frac{\partial^2 f(x, 2)}{\partial x^2} - \mu x \frac{\partial f(x, 2)}{\partial x}.\end{aligned}$$

The associated VIs have the following form:

$$(9) \quad \max\{(\mu + \mathcal{A})f(x, \cdot)(i), (x - q)_+ - f(x, i)\} = 0, \quad f(0, i) = 0, \quad i = 1, 2.$$

Using the result in the last section and recall that  $\theta_t = 2$  is absorbing, we have

$$f(x, 2) = v(x, 2) = x.$$

It remains to show

$$f(x, 1) = v(x, 1) = x.$$

Clearly,  $f(x, 1) = f(x, 2) = x$  gives a solution to (1). It follows that  $f(x, 1) = x \geq v(x, 1)$ . To establish the opposite inequality, it suffices to find a sequence of stopping times  $\tau_N < \infty$ , a.s. such that

$$E[e^{\mu\tau_N}(S_{\tau_N} - q)_+] \rightarrow x, \quad \text{as } N \rightarrow \infty.$$

Take  $\tau_N = N$ ,  $N = 1, 2, \dots$ , and let

$$c_i^N(t, x) = E[e^{\mu(N-t)}(S_N - q)_+ | S_t = x, \theta_u = i, t \leq u \leq N],$$

which is the expected payoff with  $\tau_N$  and constant  $\theta_t$  on  $[t, N]$ . Then,  $0 \leq c_i^N(t, x) \leq x$ . We can show as in (6) that, if  $2\mu/\sigma^2(2) < 1$ ,

$$c_2^N(t, x) \rightarrow x \text{ as } N \rightarrow \infty \text{ for all } t > 0.$$

By conditioning on the jump time of  $\theta_t$ , we have, as in [17],

$$\begin{aligned}& E[e^{\mu N}(S_N - q)_+ | S_0 = x, \theta_0 = 1] \\ &= \int_0^N e^{\mu t} \left( \int_{-\infty}^{\infty} c_2^N(t, x e^u) \Phi(u, m_1 t, \Sigma_1 t) du \right) \lambda_1 e^{-\lambda_1 t} dt + e^{-\lambda_1 N} c_1^N(0, x),\end{aligned}$$

where  $m_1 = -\mu - \sigma^2(1)/2$  and  $\Sigma_1 = \sigma^2(1)$ . Recall that  $0 \leq c_1^N(0, x) \leq x$ . The second term  $e^{-\lambda_1 N} c_1^N(0, x) \rightarrow 0$ , as  $N \rightarrow \infty$ . Also recall that, for  $2\mu/\sigma^2(2) < 1$ ,  $c_2^N(t, x) \rightarrow x$  for all  $t$ . We have

$$\begin{aligned}& \lim_{N \rightarrow \infty} E[e^{\mu N}(S_N - q)_+ | S_0 = x, \theta_0 = 1] \\ &= \int_0^{\infty} e^{\mu t} \left( \int_{-\infty}^{\infty} x e^u \Phi(u, m_1 t, \Sigma_1 t) du \right) \lambda_1 e^{-\lambda_1 t} dt \\ &= x \int_0^{\infty} \lambda_1 e^{-\lambda_1 t} dt = x.\end{aligned}$$

Therefore,  $f(x, 1) = v(x, 1) = x$  for  $2\mu/\sigma^2(2) < 1$ . If  $2\mu/\sigma^2(2) = 1$ , we adopt the two point boundary value approach. Given  $0 < a < q < b$ , let  $\tau_{a,b} = \inf\{t : S_t \notin (a, b)\}$ . Then,  $\tau_{a,b} < \infty$ , a.s. Let

$$h(x, i) = E[e^{\mu\tau_{a,b}}(S_{\tau_{a,b}} - q)_+ | S_0 = x, \theta_0 = i].$$

Then  $h$  must satisfy the equations

$$\begin{cases} (\mu + \mathcal{A})h(x, \cdot)(i) = 0, & i = 1, 2 \\ h(a, i) = 0, & h(b, i) = x - q. \end{cases}$$

Solve these equations to obtain

$$\begin{cases} h(x, 1) = A_1 x^{\beta_1} + A_2 x^{\beta_2} + \phi(x) \\ h(x, 2) = B_1 x + B_2 x \log x, \end{cases}$$

where  $A_i, B_i, i = 1, 2$ , are constants and

$$(10) \quad \begin{cases} \beta_1 = \frac{1}{\sigma^2(1)} \left\{ \mu + \frac{\sigma^2(1)}{2} + \sqrt{\left(\mu - \frac{\sigma^2(1)}{2}\right)^2 + 2\sigma^2(1)\lambda_1} \right\} > 1, \\ \beta_2 = \frac{1}{\sigma^2(1)} \left\{ \mu + \frac{\sigma^2(1)}{2} - \sqrt{\left(\mu - \frac{\sigma^2(1)}{2}\right)^2 + 2\sigma^2(1)\lambda_1} \right\} < 1, \end{cases}$$

and  $\phi(x)$  is a special solution

$$\phi(x) = \left[ B_1 + \frac{1}{\lambda_1} \left( \frac{\sigma^2(1)}{2} - \mu \right) B_2 \right] x + B_2 x \log x.$$

Using the two point boundary values, we have

$$\begin{cases} B_1 = -\frac{(b-q)\log a}{b(\log b - \log a)}, \\ B_2 = \frac{b-q}{b(\log b - \log a)}. \end{cases}$$

Sending  $a \rightarrow 0$ , we have  $\phi(x) \rightarrow (b-q)x/b$ . Similarly, using

$$(11) \quad \begin{cases} A_1 a^{\beta_1} + A_2 a^{\beta_2} + \phi(a) = 0, \\ A_1 b^{\beta_1} + A_2 b^{\beta_2} + \phi(b) = b - q, \end{cases}$$

and dividing both sides of the first equation by  $a^{\beta_2}$ , we have

$$A_1 a^{\beta_1 - \beta_2} + A_2 - \frac{(b-q)\log a}{b(\log b - \log a)} a^{1-\beta_2} + \frac{b-q}{b(\log b - \log a)} a^{1-\beta_2} \log a = 0.$$

Recall that  $\beta_1 > 1$  and  $\beta_2 < 1$ . Sending  $a \rightarrow 0$  yields  $A_2 \rightarrow 0$ . Also send  $a \rightarrow 0$  in the second equation in (3) to obtain  $A_1 \rightarrow 0$ . Therefore, as  $a \rightarrow 0$ ,

$$h(x, 1) \rightarrow \frac{b-q}{b} x.$$

This implies that  $f(x, 1) = x = \lim_{b \rightarrow \infty} h(x, 1) \leq v(x, 1)$ .

**Case 2:**  $2\mu/\sigma^2(2) > 1..$  Recall that state 2 is absorbing. Using the results with constant  $\theta_t$ , we have

$$(12) \quad f(x, 2) = v(x, 2) = \begin{cases} B_2 x^{\alpha_2} & \text{if } x < x_2, \\ x - q & \text{if } x \geq x_2, \end{cases}$$

where  $\alpha_2 = 2\mu/\sigma^2(2)$ ,  $x_2 = q\alpha_2/(\alpha_2 - 1)$ , and

$$B_2 = \frac{(\alpha_2 - 1)^{\alpha_2 - 1}}{\alpha_2^{\alpha_2}} q^{1 - \alpha_2}.$$

With  $f(x, 2)$  given in (4), we solve the equation

$$(13) \quad f(x, 1) + \frac{\sigma^2(1)}{2} x^2 \frac{\partial^2 f(x, 1)}{\partial x^2} - \mu x \frac{\partial f(x, 1)}{\partial x} + \lambda_1 (f(x, 2) - f(x, 1)) = 0, \quad f(0, 1) = 0,$$

on the continuation region  $(0, x_1)$  for some  $x_1$ . We consider two separate subcases.

**Subcase 1:**  $q < x_1 \leq x_2..$  We first solve equation (5) on  $(0, x_1)$ . Note that  $f(x, 2) = B_2 x^{\alpha_2}$  on this interval. A special solution to (5) can be given as

$$\phi(x) = A_0 x^{\alpha_2},$$

where

$$A_0 = -\lambda_1 B_2 \left( \mu - \lambda_1 + \frac{\sigma^2(1)}{2} \alpha_2 (\alpha_2 - 1) - \mu \alpha_2 \right)^{-1},$$

assuming the denominator does not equal zero.

Let  $\beta_1 > 1$  and  $\beta_2 < 1$  be as given in (2). In view of the estimate in (5), we drop the term involving  $x^{\beta_2}$ . The general solution to (5) is given by

$$f(x, 1) = A_1 x^{\beta_1} + A_0 x^{\alpha_2}, \quad \text{for } x \in (0, x_1).$$

Smoothly fit this solution at  $x_1$  with  $f(x, 1) = x - q$  on  $(x_1, \infty)$  to obtain

$$\begin{cases} A_1 x_1^{\beta_1} + A_0 x_1^{\alpha_2} = x_1 - q, \\ A_1 \beta_1 x_1^{\beta_1 - 1} + A_0 \alpha_2 x_1^{\alpha_2 - 1} = 1. \end{cases}$$

Eliminating  $A_1$  from both equations leads to

$$(14) \quad A_0 (\beta_1 - \alpha_2) x_1^{\alpha_2} = (\beta_1 - 1) x_1 - \beta_1 q.$$

Therefore,

$$A_1 = \frac{1 - A_0 \alpha_2 x_1^{\alpha_2 - 1}}{\beta_1 x_1^{\beta_1 - 1}}.$$

Being a solution to the VIs (1), it also requires, for all  $x > 0$ ,

$$f(x, 1) \geq (x - q)_+ \quad \text{and}$$

$$(\mu + \mathcal{A})f(x, \cdot)(1) \leq 0.$$

An equivalent condition for the preceding second inequality is

$$\mu(x - q) - \mu x + \lambda_1(B_2 x^{\alpha_2} - (x - q)) \leq 0 \text{ on } (x_1, x_2).$$

This is also equivalent to the following inequality:

$$(15) \quad \lambda_1(B_2 x_1^{\alpha_2} - (x_1 - q)) \leq \mu q.$$

THEOREM 2.. Let  $x_1$  be a solution to (6) with  $q < x_1 \leq x_2$  and  $\lambda_1(B_2 x_1^{\alpha_2} - (x_1 - q)) \leq \mu q$ . Let

$$f(x, 1) = \begin{cases} A_1 x^{\beta_1} + A_0 x^{\alpha_2} & \text{if } x < x_1, \\ x - q & \text{if } x \geq x_1, \end{cases}$$

$$f(x, 2) = \begin{cases} B_2 x^{\alpha_2} & \text{if } x < x_2, \\ x - q & \text{if } x \geq x_2. \end{cases}$$

Assume  $f(x, 1) \geq (x - q)_+$ , for all  $x > 0$ . Then

$$f(x, i) = v(x, i), \quad i = 1, 2,$$

and the stopping time

$$(16) \quad \tau^* = \inf \left\{ t : (S_t, \theta_t) \notin (0, x_1) \times \{1\} \cup (0, x_2) \times \{2\} \right\}$$

is optimal.

*Proof.* It suffices to show  $f(x, 1) \leq v(x, 1)$  because the opposite inequality can be obtained using Dynkin's formula and Fatou's lemma as in [14, Theorem 10.4.1].

If  $\theta_0 = 1$  and  $x \geq x_1$  (or  $\theta_0 = 2$  and  $x \geq x_2$ ), take  $\tau^* = 0$ . Then

$$f(x, i) = (x - q)_+ = E[e^{\mu\tau^*} (S_{\tau^*} - q)_+].$$

If  $\theta_0 = 1$  and  $x < x_1$  (or  $\theta_0 = 2$  and  $x < x_2$ ), take  $\tau^*$  as in (8). Then, for  $t < \tau^*$ , we have  $(\mu + \mathcal{A})f(S_t, \cdot)(\theta_t) = 0$ . Applying Dynkin's formula, we have, for each  $N$ ,

$$\begin{aligned} f(x, 1) &= E[e^{\mu(\tau^* \wedge N)} f(S_{\tau^* \wedge N}, \theta_{\tau^* \wedge N})] \\ &\leq E[e^{\mu\tau^*} f(S_{\tau^*}, \theta_{\tau^*}) I_{\{\tau^* \leq N\}}] + E[e^{\mu N} f(S_N, \theta_N) I_{\{\tau^* > N\}}]. \end{aligned}$$

The first term on the right hand side converges to

$$E[e^{\mu\tau^*} f(S_{\tau^*}, \theta_{\tau^*}) I_{\{\tau^* < \infty\}}] = E[e^{\mu\tau^*} (S_{\tau^*} - q)_+ I_{\{\tau^* < \infty\}}],$$

as  $N \rightarrow \infty$ . It remains to show the second term goes to zero. For  $\kappa \geq 1$ , we write

$$S_t^\kappa = S_0^\kappa Y_t \exp \left\{ \int_0^t \left( -\kappa\mu + (\kappa^2 - \kappa) \frac{\sigma^2(\theta_s)}{2} \right) ds \right\},$$

where

$$Y_t = \exp \left\{ \int_0^t -\frac{\kappa^2 \sigma^2(\theta_s)}{2} ds + \int_0^t \kappa \sigma(\theta_s) dW_s \right\}$$

is a martingale. Recall that  $2\mu/\sigma^2(i) > 1$  for  $i = 1, 2$ . There exists  $\delta > 0$  such that

$$\frac{\sigma^2(i)}{2} < \mu - \delta, \text{ for } i = 1, 2.$$

It follows that

$$\begin{aligned} E[e^{\mu N} S_N^\kappa] &= S_0^\kappa E \left( Y_N \exp \left\{ \int_0^N \left( (\kappa - 1) \left( \kappa \frac{\sigma^2(\theta_s)}{2} - \mu \right) \right) ds \right\} \right) \\ &\leq S_0^\kappa E \left( Y_N \exp \left\{ \int_0^N ((\kappa - 1) (\kappa(\mu - \delta) - \mu)) ds \right\} \right) \\ &= S_0^\kappa \exp \{ (\kappa - 1) (\kappa(\mu - \delta) - \mu) N \} \rightarrow 0, \end{aligned}$$

for  $1 < \kappa < \mu/(\mu - \delta)$ . Moreover, there exists a constant  $K$  such that, for  $x \in (0, \max(x_1, x_2))$ ,

$$x^{\beta_1} \leq Kx^\kappa \text{ and } x^{\alpha_2} \leq Kx^\kappa.$$

This implies that

$$E[e^{\mu N} f(S_N, \theta_N) I_{\{\tau^* > N\}}] \rightarrow 0.$$

The proof is completed.  $\square$

**Subcase 2:**  $q < x_2 < x_1$ . In this case,  $f(x, 2)$  consists of two pieces on  $(0, x_1)$ . Using the vector form of the first order differential equation, we can find a special solution to (5) given by  $\phi(x) = F(\log x)$  where

$$F(z) = \frac{2\lambda_1}{\sigma^2(1)(\beta_1 - \beta_2)} \int_0^z \left( -e^{\beta_1(z-u)} + e^{\beta_2(z-u)} \right) f(e^u, 2) du,$$

where  $\beta_1 > 1$  and  $\beta_2 < 1$  are given in (2). The general solution to (5), after dropping the term  $x^{\beta_2}$ , is given by

$$f(x, 1) = A_1 x^{\beta_1} + \phi(x).$$

Similarly as in the previous subcase, we obtain

$$\begin{cases} A_1 x_1^{\beta_1} + \phi(x_1) = x_1 - q, \\ A_1 \beta_1 x_1^{\beta_1-1} + \phi'(x_1) = 1. \end{cases}$$

These lead to

$$(17) \quad \beta_1 \phi(x_1) - x_1 \phi'(x_1) = (\beta_1 - 1)x_1 - \beta_1 q,$$

and

$$A_1 = \frac{1 - \phi(x_1)}{\beta_1 x_1^{\beta_1-1}}.$$

Moreover, it is easy to check

$$(\mu + \mathcal{A})(x - q) \leq 0 \text{ for } x > x_1.$$

THEOREM 3.. Let  $x_1$  be a solution to (9) with  $q < x_2 < x_1$ . Let

$$f(x, 1) = \begin{cases} A_1 x^{\beta_1} + \phi(x) & \text{if } x < x_1, \\ x - q & \text{if } x \geq x_1, \end{cases}$$

$$f(x, 2) = \begin{cases} B_2 x^{\alpha_2} & \text{if } x < x_2, \\ x - q & \text{if } x \geq x_2. \end{cases}$$

Assume  $f(x, 1) \geq (x - q)_+$ , for all  $x > 0$ . Then

$$f(x, i) = v(x, i), \quad i = 1, 2,$$

and

$$\tau^* = \inf\{t : (S_t, \theta_t) \notin (0, x_1) \times \{1\} \cup (0, x_2) \times \{2\}\}$$

is optimal.

*Proof.* Note that

$$F(z) \leq \frac{2\lambda_1}{\sigma^2(1)(\beta_1 - \beta_2)} \int_0^z e^{\beta_2(z-u)} f(e^u, 2) du.$$

Using (4), for each  $0 < \kappa < \mu/(\mu - \delta)$ , we can show by direct computation that there exists  $K$  such that

$$\phi(x) \leq Kx^\kappa,$$

for  $x \in (0, \max(x_1, x_2))$ . The rest of the proof is similar to that of Theorem 2.  $\square$

EXAMPLE 4.. In this example we take

$$\mu = 0.03, \quad \sigma^2(1) = 0.02, \quad \sigma^2(2) = 0.04, \quad \lambda_1 = 2, \quad q = 1.$$

Here  $2\mu/\sigma^2(2) = 1.5 > 1$ , and it is easy to verify that all the conditions in Theorem 2 are satisfied. We plot the corresponding value functions in Figure 1 in which  $x_1 = 2.851$  and  $x_2 = 3$ . We then vary  $\lambda_1$ . The results are summarized in Table 1. Recall that  $1/\lambda_1$  is the mean time for  $\theta_t$  to remain in state 1. We expect that as  $\lambda_1$  gets smaller, the corresponding  $x_1$  should converge to  $x_1^0 = \alpha_1 q / (\alpha_1 - 1) = 1.5$  (In this example,  $\alpha_1 = 2\mu/\sigma^2(1)$ ), a threshold level when there is no jump and  $\theta_t = 1$  (for all  $t$ ) and as  $\lambda_1$  gets bigger,  $x_1$  should be closer to  $x_2$ . These are confirmed in Table 1.

Economically, the table shows that the sooner the regime is likely to switch to a more volatile state the more valuable the underlying stock loan (hence the higher the threshold level to regain the stock).

| $\lambda_1$ | 0.001 | 0.01  | 0.1   | 1     | 2     | 3     | 4     | 5     |
|-------------|-------|-------|-------|-------|-------|-------|-------|-------|
| $x_1$       | 1.554 | 1.855 | 2.433 | 2.794 | 2.851 | 2.878 | 2.893 | 2.905 |

Table 1.  $x_1$  with varying  $\lambda_1$ .

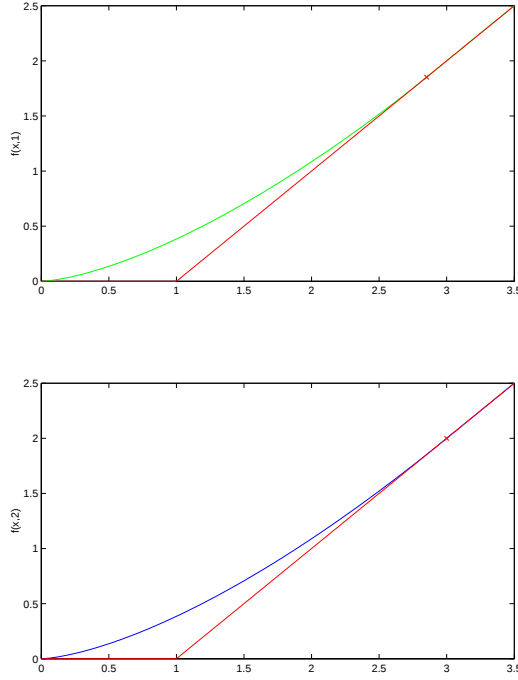


FIG. 1. Value functions with a single jump.

**5. The Model with a General Two-State Regime Switching.** In this section, we consider the model (2) in which  $\theta_t$  is a two-state Markov chain generated by  $Q = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}$  with both  $\lambda_1 > 0$  and  $\lambda_2 > 0$ . The generator for the pair  $(S_t, \theta_t)$  has the form:

$$\begin{aligned} \mathcal{A}f(x, \cdot)(1) &= \frac{\sigma^2(1)}{2} x^2 \frac{\partial^2 f(x, 1)}{\partial x^2} - \mu x \frac{\partial f(x, 1)}{\partial x} + \lambda_1 (f(x, 2) - f(x, 1)), \\ \mathcal{A}f(x, \cdot)(2) &= \frac{\sigma^2(2)}{2} x^2 \frac{\partial^2 f(x, 2)}{\partial x^2} - \mu x \frac{\partial f(x, 2)}{\partial x} + \lambda_2 (f(x, 1) - f(x, 2)). \end{aligned}$$

The associated VIs are given by

$$(18) \quad \max \left\{ (\mu + \mathcal{A})f(x, \cdot)(i), (x - q)_+ - f(x, i) \right\} = 0, \quad f(0, i) = 0, \quad i = 1, 2.$$

We begin by solving the equations  $(\mu + \mathcal{A})f(x, \cdot)(i) = 0$ ,  $i = 1, 2$ . Using  $(\mu + \mathcal{A})f(x, \cdot)(1) = 0$ , we write  $f(x, 2)$  in terms of  $f(x, 1)$  and its derivatives. Substituting it into the second equation  $(\mu + \mathcal{A})f(x, \cdot)(2) = 0$ , we obtain a fourth order ordinary differential equation. Its characteristic equation is given by

$$g_1(\beta)g_2(\beta) = \lambda_1\lambda_2,$$

where

$$g_i(\beta) = \frac{\sigma^2(i)}{2} \beta^2 - \left( \mu + \frac{\sigma^2(i)}{2} \right) \beta + \mu - \lambda_i, \quad i = 1, 2.$$

Let  $\psi(\beta) = g_1(\beta)g_1(\beta) - \lambda_1\lambda_2$ . Then,  $\psi(1) = 0$ ,  $\psi(\infty) = \psi(-\infty) = \infty$ . Let  $\eta_1 < 1$  and  $\eta_2 > 1$  denote the roots of  $g_1$ . Then both  $\psi(\eta_1)$  and  $\psi(\eta_2)$  equal  $-\lambda_1\lambda_2 < 0$ . It is easy to see that

$$\psi'(1) = \lambda_1 \left( \mu - \frac{\sigma^2(2)}{2} \right) + \lambda_2 \left( \mu - \frac{\sigma^2(1)}{2} \right) = (\lambda_1 + \lambda_2) \left( \mu - \left( \nu_1 \frac{\sigma^2(1)}{2} + \nu_2 \frac{\sigma^2(2)}{2} \right) \right),$$

where  $\nu_1 = \lambda_2/(\lambda_1 + \lambda_2)$  and  $\nu_2 = \lambda_1/(\lambda_1 + \lambda_2)$  are the stationary distributions of  $\theta_t$ .

Let  $\beta_i$ ,  $i = 1, 2, 3, 4$ , denote the zeros of  $\psi(\beta)$ . The intermediate value property implies that they are real numbers. We consider the following three cases.

Case 1:  $\psi'(1) < 0$  (or  $\gamma < r + (\nu_1\sigma^2(1) + \nu_2\sigma^2(2))/2$ ). In this case,

$$\beta_1 < \eta_1 < \beta_2 < \beta_3 = 1 < \eta_2 < \beta_4.$$

Case 2:  $\psi'(1) = 0$  (or  $\gamma = r + (\nu_1\sigma^2(1) + \nu_2\sigma^2(2))/2$ ). In this case,  $\beta = 1$  is a multiple root. Therefore,

$$\beta_1 < \eta_1 < \beta_2 = \beta_3 = 1 < \eta_2 < \beta_4.$$

Case 3:  $\psi'(1) > 0$  (or  $\gamma > r + (\nu_1\sigma^2(1) + \nu_2\sigma^2(2))/2$ ). In this case,

$$\beta_1 < \eta_1 < \beta_2 = 1 < \beta_3 < \eta_2 < \beta_4.$$

Next we show that the value functions  $v(x, i) = x$  in the first two cases. Note that  $f(x, i) = x$  is a solution to the VIs. Therefore,  $v(x, i) \leq x$ . It remains to show the opposite inequalities in these two cases.

**Case 1:**  $\gamma < r + (\nu_1\sigma^2(1) + \nu_2\sigma^2(2))/2$ . We use the two point boundary value approach. Given  $0 < a < q < b$ , let  $h(x, i) = E[e^{\mu\tau_{a,b}}(S_{\tau_{a,b}} - q)_+]$ , where  $\tau_{a,b} = \inf\{t : S_t \notin (a, b)\}$ . It is easy to see that

$$(19) \quad 0 \leq h(x, i) \leq v(x, i) \leq x.$$

As shown in [19],  $h$  is a  $C^2$  solution to the following TPBVDE

$$\begin{cases} (\mu + \mathcal{A})h(x, \cdot)(i) = 0, \\ h(a, i) = 0, \quad h(b, i) = b - q. \end{cases}$$

Note that

$$h(x, 2) = -\frac{1}{\lambda_1} \left( (\mu - \lambda_1)h(x, 1) + \frac{\sigma^2(1)}{2} x^2 \frac{\partial^2 h(x, 1)}{\partial x^2} - \mu x \frac{\partial h(x, 1)}{\partial x} \right).$$

There are constants  $A_i$ ,  $i = 1, 2, 3, 4$ , such that the general solutions

$$\begin{cases} h(x, 1) = A_1 x^{\beta_1} + A_2 x^{\beta_2} + A_3 x + A_4 x^{\beta_4}, \\ h(x, 2) = \kappa_1 A_1 x^{\beta_1} + \kappa_2 A_2 x^{\beta_2} + A_3 x + \kappa_4 A_4 x^{\beta_4}, \end{cases}$$

where  $\kappa_i = -g_1(\beta_i)/\lambda_1$ ,  $i = 1, 2, 3, 4$ . Note that  $\kappa_3 = 1$  and  $\kappa_i$  are independent of the choice of  $(a, b)$ . Note also that the coefficients  $A_i$  are functions of  $(a, b)$ . In view of

(2),  $A_i = A_i(a, b)$  are bounded on  $0 < a < b < \infty$ . Moreover, since  $g_1$  is a quadratic function with zeros  $\eta_1 < 1$  and  $\eta_2 > 1$ , it follows that

$$\kappa_1 < \kappa_2 \text{ and } \kappa_4 < 1.$$

Using the boundary conditions at  $a$ , we have

$$a^{-\beta_1} h(a, 1) = A_1 + A_2 a^{\beta_2 - \beta_1} + A_3 a^{1 - \beta_1} + A_4 a^{\beta_4 - \beta_1} = 0.$$

Sending  $a \rightarrow 0^+$ , we have  $A_1 \rightarrow 0$ . Similarly, recall that  $\kappa_1 \neq \kappa_2$  and note that

$$a^{-\beta_2} (h(a, 2) - \kappa_1 h(a, 1)) = (\kappa_2 - \kappa_1) A_2 + (1 - \kappa_1) a^{1 - \beta_2} + (\kappa_4 - \kappa_1) A_4 a^{\beta_4 - \beta_2} = 0$$

implies  $A_2 \rightarrow 0$  as  $a \rightarrow 0^+$ . With a little abuse of notation, we keep using  $A_3$  and  $A_4$  as their limits. Then the boundary conditions at  $b$  give

$$\begin{cases} A_3 b + A_4 b^{\beta_4} = b - q, \\ A_3 b + \kappa_4 A_4 b^{\beta_4} = b - q. \end{cases}$$

Dividing both sides of the first equation by  $b^{\beta_4}$  and sending  $b \rightarrow \infty$  yields  $A_4 \rightarrow 0$ . Similarly, multiplying the first equation by  $\kappa_4$  and subtracting the second equation leads to

$$(\kappa_4 - 1) A_3 b = (\kappa_4 - 1) (b - q).$$

Send  $b \rightarrow \infty$  to obtain  $A_3 \rightarrow 1$ . Hence, sending  $a \rightarrow 0^+$  and then  $b \rightarrow \infty$ , we have

$$h(x, i) \rightarrow x, \quad i = 1, 2.$$

It follows that  $v(x, i) = x, i = 1, 2$ .

**Case 2:**  $\gamma = r + (\nu_1 \sigma^2(1) + \nu_2 \sigma^2(2))/2..$  In this case, note that  $\beta = 1$  is a multiple root. Therefore, the general solution has the form

$$\begin{cases} h(x, 1) = A_1 x^{\beta_1} + A_2 x + A_3 x \log x + A_4 x^{\beta_4}, \\ h(x, 2) = \kappa_1 A_1 x^{\beta_1} + A_2 x + A_3 \left( x \log x + \frac{1}{\lambda_1} \left( \mu - \frac{\sigma^2(1)}{2} \right) x \right) + \kappa_4 A_4 x^{\beta_4}. \end{cases}$$

We can show also  $\kappa_1 \neq 1$  and  $\kappa_4 \neq 1$ . Following the similar procedure as in Case 1, we have

$$A_1 \rightarrow 0 \text{ and } A_3 \rightarrow 0 \text{ as } a \rightarrow 0^+,$$

and then

$$A_2 \rightarrow 1 \text{ and } A_4 \rightarrow 0 \text{ as } b \rightarrow \infty.$$

Therefore, we have  $h(x, i) \rightarrow x$ , which implies  $v(x, i) = x, i = 1, 2$ .

**Case 3:**  $\gamma > r + (\nu_1 \sigma^2(1) + \nu_2 \sigma^2(2))/2..$  We consider the following three subcases.

**Subcase 1:**  $q < x_1 < x_2$ . In view of (5), we consider the solution on  $(0, x_1)$  to be of the form

$$\begin{cases} f(x, 1) = A_3 x^{\beta_3} + A_4 x^{\beta_4}, \\ f(x, 2) = \kappa_3 A_3 x^{\beta_3} + \kappa_4 A_4 x^{\beta_4}, \end{cases}$$

where  $\kappa_i = -g_1(\beta_i)/\lambda_1$ ,  $i = 3, 4$ . Then, on  $(x_1, x_2)$ ,  $f(x, 1) = x - q$ , and

$$f(x, 2) = B_1 x^{\gamma_1} + B_2 x^{\gamma_2} + \phi(x),$$

where

$$(20) \quad \begin{cases} \gamma_1 = \frac{1}{\sigma^2(2)} \left\{ \mu + \frac{\sigma^2(2)}{2} + \sqrt{\left( \mu - \frac{\sigma^2(2)}{2} \right)^2 + 2\sigma^2(2)\lambda_2} \right\} > 1, \\ \gamma_2 = \frac{1}{\sigma^2(2)} \left\{ \mu + \frac{\sigma^2(2)}{2} - \sqrt{\left( \mu - \frac{\sigma^2(2)}{2} \right)^2 + 2\sigma^2(2)\lambda_2} \right\} < 1, \end{cases}$$

and a special solution (assuming  $\mu \neq \lambda_2$ )

$$\phi(x) = x + \frac{\lambda_2 q}{\mu - \lambda_2}.$$

The smooth-fit conditions at  $x_1$  and  $x_2$  require

$$\begin{cases} A_3 x_1^{\beta_3} + A_4 x_1^{\beta_4} = x_1 - q, \\ A_3 \beta_3 x_1^{\beta_3} + A_4 \beta_4 x_1^{\beta_4} = x_1, \\ \kappa_3 A_3 x_1^{\beta_3} + \kappa_4 A_4 x_1^{\beta_4} = B_1 x_1^{\gamma_1} + B_2 x_1^{\gamma_2} + \phi(x_1), \\ \kappa_3 A_3 \beta_3 x_1^{\beta_3} + \kappa_4 A_4 \beta_4 x_1^{\beta_4} = B_1 \gamma_1 x_1^{\gamma_1} + B_2 \gamma_2 x_1^{\gamma_2} + x_1 \phi'(x_1), \\ B_1 x_2^{\gamma_1} + B_2 x_2^{\gamma_2} + \phi(x_2) = x_2 - q, \\ B_1 \gamma_1 x_2^{\gamma_1} + B_2 \gamma_2 x_2^{\gamma_2} + x_2 \phi'(x_2) = x_2. \end{cases}$$

Eliminating  $A_i$  and  $B_i$ , we obtain

$$(21) \quad \begin{pmatrix} x_1^{-\gamma_1} \\ x_1^{-\gamma_2} \end{pmatrix} F_1(x_1) = \begin{pmatrix} x_2^{-\gamma_1} \\ x_2^{-\gamma_2} \end{pmatrix} F_2(x_2),$$

where

$$\begin{aligned} F_1(x) &= \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \left[ \begin{pmatrix} \kappa_3 & \kappa_4 \\ \kappa_3 \beta_3 & \kappa_4 \beta_4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \beta_3 & \beta_4 \end{pmatrix}^{-1} \begin{pmatrix} x - q \\ x \end{pmatrix} - \begin{pmatrix} \phi(x) \\ x \phi'(x) \end{pmatrix} \right], \\ F_2(x) &= \begin{pmatrix} 1 & 1 \\ \gamma_1 & \gamma_2 \end{pmatrix}^{-1} \begin{pmatrix} x - q - \phi(x) \\ x - x \phi'(x) \end{pmatrix}. \end{aligned}$$

We can prove the next theorem as in Theorem 2.

**THEOREM 5..** *Let  $x_1$  and  $x_2$  be solutions to (4) with  $q < x_1 \leq x_2$ . Let*

$$f(x, 1) = \begin{cases} A_3x^{\beta_3} + A_4x^{\beta_4} & \text{if } x < x_1, \\ x - q & \text{if } x \geq x_1, \end{cases}$$

$$f(x, 2) = \begin{cases} \kappa_3A_3x^{\beta_3} + \kappa_4A_4x^{\beta_4} & \text{if } x < x_1, \\ B_1x^{\gamma_1} + B_2x^{\gamma_2} + \phi(x) & \text{if } x_1 \leq x < x_2, \\ x - q & \text{if } x \geq x_2. \end{cases}$$

*Assume  $f(x, i) \geq (x - q)_+$ , for all  $x > 0$ , and  $\lambda_1(f(x, 2) - (x - q)) \leq \mu q$  on  $(x_1, x_2)$ . Then*

$$f(x, i) = v(x, i), \quad i = 1, 2,$$

*and the stopping time*

$$(22) \quad \tau^* = \inf \left\{ t : (S_t, \theta_t) \notin (0, x_1) \times \{1\} \cup (0, x_2) \times \{2\} \right\}$$

*is optimal.*

**Subcase 2:**  $q < x_2 < x_1$ .. As in the previous subcase, consider  $f(x, i)$  on  $(0, x_1)$  to be of the form

$$\begin{cases} f(x, 1) = A_3x^{\beta_3} + A_4x^{\beta_4}, \\ f(x, 2) = \kappa_3A_3x^{\beta_3} + \kappa_4A_4x^{\beta_4}. \end{cases}$$

On  $(x_2, x_1)$ ,

$$\begin{cases} f(x, 1) = x - q, \text{ and} \\ f(x, 2) = B_1x^{\tilde{\gamma}_1} + B_2x^{\tilde{\gamma}_2} + \tilde{\phi}(x), \end{cases}$$

where

$$(23) \quad \begin{cases} \tilde{\gamma}_1 = \frac{1}{\sigma^2(1)} \left\{ \mu + \frac{\sigma^2(1)}{2} + \sqrt{\left( \mu - \frac{\sigma^2(1)}{2} \right)^2 + 2\sigma^2(1)\lambda_1} \right\} > 1, \\ \tilde{\gamma}_2 = \frac{1}{\sigma^2(1)} \left\{ \mu + \frac{\sigma^2(1)}{2} - \sqrt{\left( \mu - \frac{\sigma^2(1)}{2} \right)^2 + 2\sigma^2(1)\lambda_1} \right\} < 1, \end{cases}$$

and a special solution (assuming  $\mu \neq \lambda_1$ )

$$\tilde{\phi}(x) = x + \frac{\lambda_1 q}{\mu - \lambda_1}.$$

Smooth-fitting these pieces at  $x_1$  and  $x_2$  gives

$$\begin{cases} \kappa_3 A_3 x_2^{\beta_3} + \kappa_4 A_4 x_2^{\beta_4} = x_2 - q, \\ \kappa_3 A_3 \beta_3 x_2^{\beta_3} + \kappa_4 A_4 \beta_4 x_2^{\beta_4} = x_2, \\ A_3 x_2^{\beta_3} + A_4 x_2^{\beta_4} = B_1 x_2^{\tilde{\gamma}_1} + B_2 x_2^{\tilde{\gamma}_2} + \tilde{\phi}(x_2), \\ A_3 \beta_3 x_2^{\beta_3} + A_4 \beta_4 x_2^{\beta_4} = B_1 \tilde{\gamma}_1 x_2^{\tilde{\gamma}_1} + B_2 \tilde{\gamma}_2 x_2^{\tilde{\gamma}_2} + x_2 \tilde{\phi}'(x_2), \\ B_1 x_1^{\tilde{\gamma}_1} + B_2 x_1^{\tilde{\gamma}_2} + \tilde{\phi}(x_1) = x_1 - q, \\ B_1 \tilde{\gamma}_1 x_1^{\tilde{\gamma}_1} + B_2 \tilde{\gamma}_2 x_1^{\tilde{\gamma}_2} + x_1 \tilde{\phi}'(x_1) = x_1. \end{cases}$$

Similarly, we obtain by eliminating  $A_i$  and  $B_i$

$$(24) \quad \begin{pmatrix} x_1^{-\tilde{\gamma}_1} \\ x_1^{-\tilde{\gamma}_2} \end{pmatrix} \tilde{F}_1(x_1) = \begin{pmatrix} x_2^{-\tilde{\gamma}_1} \\ x_2^{-\tilde{\gamma}_2} \end{pmatrix} \tilde{F}_2(x_2),$$

where

$$\begin{aligned} \tilde{F}_1(x) &= \begin{pmatrix} 1 & 1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_2 \end{pmatrix}^{-1} \begin{pmatrix} x - q - \tilde{\phi}(x) \\ x - x\tilde{\phi}'(x) \end{pmatrix}, \\ \tilde{F}_2(x) &= \begin{pmatrix} 1 & 1 \\ \tilde{\gamma}_1 & \tilde{\gamma}_2 \end{pmatrix}^{-1} \left[ \begin{pmatrix} 1 & 1 \\ \beta_3 & \beta_4 \end{pmatrix} \begin{pmatrix} \kappa_3 & \kappa_4 \\ \kappa_3 \beta_3 & \kappa_4 \beta_4 \end{pmatrix}^{-1} \begin{pmatrix} x - q \\ x \end{pmatrix} - \begin{pmatrix} \tilde{\phi}(x) \\ x\tilde{\phi}'(x) \end{pmatrix} \right]. \end{aligned}$$

We can prove the next theorem.

**THEOREM 6..** *Let  $x_1$  and  $x_2$  be solutions to (7) with  $q < x_2 \leq x_1$ . Let*

$$\begin{aligned} f(x, 1) &= \begin{cases} A_3 x^{\beta_3} + A_4 x^{\beta_4} & \text{if } x < x_2, \\ B_1 x^{\tilde{\gamma}_1} + B_2 x^{\tilde{\gamma}_2} + \tilde{\phi}(x) & \text{if } x_2 \leq x < x_1, \\ x - q & \text{if } x \geq x_1, \end{cases} \\ f(x, 2) &= \begin{cases} \kappa_3 A_3 x^{\beta_3} + \kappa_4 A_4 x^{\beta_4} & \text{if } x < x_2, \\ x - q & \text{if } x \geq x_2. \end{cases} \end{aligned}$$

*Assume  $f(x, i) \geq (x - q)_+$ , for all  $x > 0$ , and  $\lambda_2(f(x, 1) - (x - q)) \leq \mu q$  on  $(x_2, x_1)$ . Then*

$$f(x, i) = v(x, i), \quad i = 1, 2,$$

and the stopping time

$$(25) \quad \tau^* = \inf \left\{ t : (S_t, \theta_t) \notin (0, x_1) \times \{1\} \cup (0, x_2) \times \{2\} \right\}$$

is optimal.

**Subcase 3:**  $x_1 = x_2$ . In this subcase,

$$\begin{aligned} f(x, 1) &= \begin{cases} A_3 x^{\beta_3} + A_4 x^{\beta_4} & \text{if } x < x_1 \\ x - q & \text{if } x \geq x_1, \end{cases} \\ f(x, 2) &= \begin{cases} \kappa_3 A_3 x^{\beta_3} + \kappa_4 A_4 x^{\beta_4} & \text{if } x < x_1 \\ x - q & \text{if } x \geq x_1. \end{cases} \end{aligned}$$

The smooth-fit conditions are given by

$$\begin{cases} A_3 x_1^{\beta_3} + A_4 x_1^{\beta_4} = x_1 - q, \\ A_3 \beta_3 x_1^{\beta_3} + A_4 \beta_4 x_1^{\beta_4} = x_1, \\ \kappa_3 A_3 x_1^{\beta_3} + \kappa_4 A_4 x_1^{\beta_4} = x_1 - q, \\ \kappa_3 A_3 \beta_3 x_1^{\beta_3} + \kappa_4 A_4 \beta_4 x_1^{\beta_4} = x_1. \end{cases}$$

Therefore,  $A_3 = \kappa_3 A_3$  and  $A_4 = \kappa_4 A_4$ . These imply that  $f(x, 1) = f(x, 2)$ . Hence,  $\sigma^2(1) = \sigma^2(2)$  from the VIs (1). This corresponds to the case when there is no jump.

EXAMPLE 7.. In this example, we take

$$\mu = 0.03, \sigma^2(1) = 0.02, \sigma^2(2) = 0.04, \lambda_1 = \lambda_2 = 2, q = 1.$$

These parameters fall into Case 3 discussed earlier. Moreover, noting that  $\sigma^2(1) < \sigma^2(2)$ , we expect accordingly that  $x_1 < x_2$ . We then use the setting of Subcase 1 to solve for the value functions. In this case,  $x_1 = 2.009$ ,  $x_2 = 2.060$ . These functions are plotted in Figure 2. In this example, all the conditions of Theorem 5 are satisfied.

We next examine the monotonicity of these threshold levels (or, equivalently, the values of the underlying stock loans) with varying  $\sigma^2(1)$ ,  $\lambda_1$ , and  $q$ . It can be seen from Table 2, when varying  $\sigma^2(1)$ , that the pair  $(x_1, x_2)$  increase in  $\sigma^2(1)$ . This suggests that bigger  $\sigma^2(1)$  implies a higher payoff and therefore a higher threshold levels.

Then we vary  $\lambda_1$ . The results in Table 3 indicate that the pair  $(x_1, x_2)$  also increase. This is because a larger  $\lambda_1$  implies shorter time for  $\theta_t$  to stay in state 1. The corresponding average volatility increases in  $\lambda_1$ .

Finally, we vary  $q$  in Table 4. Both  $x_1$  and  $x_2$  increase in  $q$ . This is because a larger  $q$  implies a larger amount of money borrowed from the bank which needs to be matched with higher threshold levels.

In summary, the higher the volatility or the loan size the more valuable the stock loan contract.

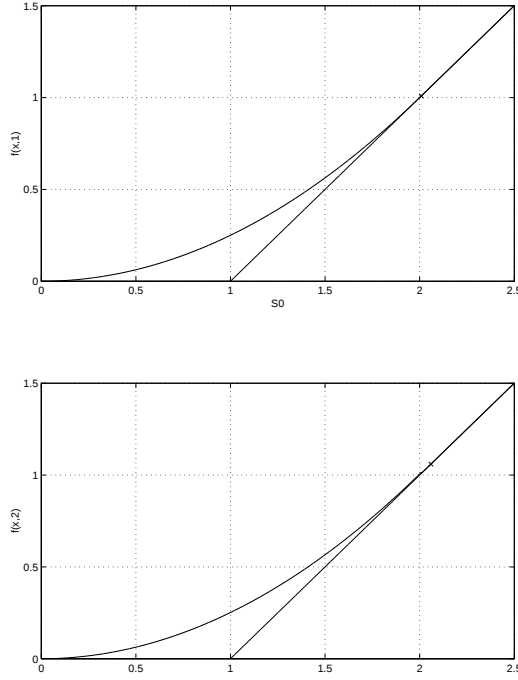


FIG. 2. Value functions with a two-state Markov chain.

|               |                |                |                |                |                |
|---------------|----------------|----------------|----------------|----------------|----------------|
| $\sigma^2(1)$ | 0.001          | 0.005          | 0.01           | 0.02           | 0.03           |
| $(x_1, x_2)$  | (1.419, 1.587) | (1.536, 1.669) | (1.673, 1.779) | (2.009, 2.060) | (2.049, 2.219) |

Table 2.  $(x_1, x_2)$  with varying  $\sigma(1)$ .

|              |                |                |                |                |                |
|--------------|----------------|----------------|----------------|----------------|----------------|
| $\lambda_1$  | 1              | 2              | 3              | 4              | 5              |
| $(x_1, x_2)$ | (1.740, 1.842) | (2.009, 2.060) | (2.045, 2.148) | (2.167, 2.258) | (2.291, 2.355) |

Table 3.  $(x_1, x_2)$  with varying  $\lambda_1$ .

|              |                |                |                |                |                |
|--------------|----------------|----------------|----------------|----------------|----------------|
| $q$          | 0.2            | 0.4            | 0.6            | 0.8            | 1              |
| $(x_1, x_2)$ | (0.390, 0.420) | (0.783, 0.818) | (1.175, 1.224) | (1.578, 1.636) | (2.009, 2.060) |

Table 4.  $(x_1, x_2)$  with varying  $q$ .

**6. Fair Values of the Parameters.** In this section, we consider the fair values for  $\gamma$ ,  $q$ , and  $c$ . Clearly, the borrower initially pays an amount  $S_0 - (q - c)$  in exchange of the right implied by the stock loan. Hence  $S_0 - (q - c) > 0$  and, moreover, the fair values of  $q$ ,  $c$ , and  $\gamma$  should satisfy the condition that the value of the stock loan is exactly  $(S_0 - q + c)$ , i.e.,  $v(S_0, i) = S_0 - q + c$ .

We use the model with the two-state Markov chain, and consider the following three cases.

**Case a.**  $\gamma \leq r + (\nu_1 \sigma^2(1) + \nu_2 \sigma^2(2))/2$ . In this case,  $v(S_0, i) = S_0$ . The equation  $v(S_0, i) = S_0 - q + c$  implies that the amount the borrower receives is  $q - c = 0$ . In

other words, in this case the lender is not interested in doing the business because the loan interest rate is too low. This is clearly not an interesting case.

**Case b.**  $\gamma > r + (\nu_1\sigma^2(1) + \nu_2\sigma^2(2))/2$  and  $v(S_0, i) = S_0 - q$ .

This is not an interesting case either because the optimal time to regain the stock is  $\tau^* = 0$ , i.e., as soon as the borrower received the loan, he needs to regain the stock back right away. He does so because the terms of the loan are not favorable to him (e.g., the loan size is too small and/or the loan rate is too high) so there is no actual exchange initially.

**Case c.**  $\gamma > r + (\nu_1\sigma^2(1) + \nu_2\sigma^2(2))/2$  and  $v(S_0, i) > S_0 - q$ .

In this case, both the lender and the borrower have enough incentives to do the business. The fair values for  $q$ ,  $c$  and  $\gamma$  should meet the following conditions:

$$\gamma > r + \frac{\nu_1\sigma^2(1) + \nu_2\sigma^2(2)}{2}, \quad S_0 < x_1 \text{ if } \theta_0 = 1 \text{ and } S_0 < x_2 \text{ if } \theta_0 = 2,$$

$$c = v(S_0, i) - S_0 + q.$$

For example, if we take

$$q = 5, \quad r = 0.05, \quad \gamma = 0.15, \quad \sigma(1) = 0.15, \quad \sigma(2) = 0.4, \quad \lambda_1 = \lambda_2 = 4,$$

then  $x_1 = 7.945$  and  $x_2 = 9.300$ . If  $\theta_0 = 1$ , the dependence of service fee  $c$  on  $S_0$  is given in Table 5.

| $S_0$ | 2     | 3     | 4     | 5     | 6     | 7     |
|-------|-------|-------|-------|-------|-------|-------|
| $c$   | 3.143 | 2.346 | 1.647 | 1.052 | 0.563 | 0.188 |

Table 5. Dependence of  $c$  on  $S_0$  when  $\theta_0 = 1$ .

Similarly, if  $\theta_0 = 2$ , the dependence of  $c$  on  $S_0$  is given in Table 6.

| $S_0$ | 2     | 3     | 4     | 5     | 6     | 7     |
|-------|-------|-------|-------|-------|-------|-------|
| $c$   | 3.147 | 2.354 | 1.662 | 1.075 | 0.598 | 0.233 |

Table 6. Dependence of  $c$  on  $S_0$  when  $\theta_0 = 2$ .

It can be seen from these two tables that the service fees decrease in  $S_0$ . This makes perfect sense as when  $S_0$  increases, the loan-to-value decreases (recall that the loan size is fixed at  $q = 5$ ) and hence the service fee gets less (since the bank bears less risk). Notice that this decrease is very rapid. On the other hand, the service fees in Table 6 are uniformly higher than those in Table 5 because the corresponding volatility  $\sigma(2)$  is greater than  $\sigma(1)$  hence the loans corresponding to the former are more valuable.

**7. Conclusion.** In this paper, we considered the valuation of stock loans under regime switching models. A key assumption is that the market mode  $\theta_t$  is observable at each time  $t$ . It would be interesting to study the case when  $\theta_t$  is not completely measurable. In this connection, Wonham filter can be used to come up with conditional probability of  $\theta_t = i$  given the historical price  $S(u)$ ,  $u \leq t$ . Extension along this direction will make the results more useful in practice. Finally, stock loan contracts are relatively new financial product and have yet reached the stage of being traded in major exchanges as stock options. Should market data become available, it would

be interesting to compare the theoretical loan value obtained in this paper with its market value. Further research topics could emerge from these studies.

**8. Appendix.** In the section, we give a technical lemma used in the paper. Let

$$S_N = S_0 \exp \left\{ - \left( \mu + \frac{\sigma^2}{2} \right) N + \sigma W_N \right\}.$$

LEMMA 1.

$$e^{\mu N} P(S_N > q) \rightarrow 0, \text{ as } N \rightarrow \infty.$$

*Proof.* Note that

$$P(S_N > q) = \int_{\{S_N > q\}} \Phi(u, 0, N) du = \int_{\frac{1}{\sigma} \log \frac{q}{S_0} + (\frac{\mu}{\sigma} + \frac{\sigma}{2})N}^{\infty} \Phi(u, 0, N) du.$$

With a change of variable  $w = u/\sqrt{N}$ , we have

$$P(S_N > q) = \int_{F_N}^{\infty} \Phi(w, 0, 1) dw,$$

where

$$F_N = \frac{1}{\sqrt{N}\sigma} \log \frac{q}{S_0} + \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) \sqrt{N}.$$

Using the inequality (see Chow and Teicher [3, p. 49])

$$\int_x^{\infty} e^{-\frac{u^2}{2}} du < \frac{1}{x} e^{-\frac{x^2}{2}},$$

for all  $x > 0$ , we have

$$P(S_N > q) < \frac{1}{\sqrt{2\pi}F_N} \exp \left( -\frac{1}{2}(F_N)^2 \right).$$

It follows that

$$e^{\mu N} P(S_N > q) < \frac{1}{\sqrt{2\pi}F_N} \exp \left( -\frac{1}{2}G_N \right)$$

where

$$G_N = \frac{1}{N\sigma^2} \left( \log \frac{q}{S_0} \right)^2 + \frac{2}{\sigma} \left( \log \frac{q}{S_0} \right) \left( \frac{\mu}{\sigma} + \frac{\sigma}{2} \right) + \left( \frac{\mu}{\sigma} - \frac{\sigma}{2} \right)^2 N.$$

It is easy to see  $F_N \rightarrow \infty$  as  $N \rightarrow \infty$ . Therefore, the right hand side of the above inequality goes to 0 as  $N \rightarrow \infty$ .  $\square$

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