

Stochastic LQ Control via Semidefinite Programming*

David D. Yao[†], Shuzhong Zhang[‡] and Xun Yu Zhou[§]

(Original version: April, 1999; SIGEST version: September, 2003)

Abstract

We study stochastic linear-quadratic (LQ) optimal control problems over an infinite time horizon, allowing the cost matrices to be indefinite. We develop a systematic approach based on semidefinite programming (SDP). A central issue is the stability of the feedback control; and we show this can be effectively examined through the complementary duality of the SDP. Furthermore, we establish several implication relations among the SDP complementary duality, the (generalized) Riccati equation, and the optimality of the LQ control problem. Based on these relations, we propose a numerical procedure that provides a thorough treatment of the LQ control problem via primal–dual SDP: it identifies a stabilizing feedback control that is optimal or determines that the problem possesses no optimal solution. For the latter case, we develop an ϵ -approximation scheme that is asymptotically optimal.

Key words: stochastic LQ control, semidefinite programming, complementary duality, mean-square stability, generalized Riccati equation.

AMS subject classification: 93E20, 90C25, 93D15.

Running Head: Stochastic LQ Control via SDP

*Originally appeared in *SIAM Journal on Control and Optimization*, **40** (2001), 801-823. This version involves some editing and rewrite.

[†]Department of Industrial Engineering and Operations Research, Columbia University, New York, NY 10027; <yao@columbia.edu>. Research undertaken while on leave at the Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong. Supported in part by Hong Kong RGC Earmarked Grant CUHK4175/00E.

[‡]Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong <zhang@se.cuhk.edu.hk>. Research supported in part by RGC Earmarked Grants CUHK4233/01E and CUHK4175/00E.

[§]Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong <xyzhou@se.cuhk.edu.hk>. Research supported in part by RGC Earmarked Grants CUHK 4435/99E and CUHK4175/00E.

1 Introduction

Linear–quadratic (LQ) control is a control model where the system dynamics is linear in both the state and control variables and the cost functional is quadratic in the two variables. Specifically, in the deterministic case, it refers to the following dynamic optimization problem:

$$\text{(DLQ)} \quad \min \quad J(x_0, u(\cdot)) := \int_0^\tau [x(t)^T Q x(t) + u(t)^T R u(t)] dt \quad (1)$$

$$\text{s.t.} \quad \dot{x}(t) = Ax(t) + Bu(t) \quad (2)$$

$$x(0) = x_0 \in \mathbb{R}^n,$$

where τ is either finite or infinite, corresponding respectively to models in a finite time horizon and infinite time horizon. Here A, B, Q and R are constant matrices, with Q and R being symmetric matrices (in the finite horizon case, these matrices are all allowed to be time-varying); and the superscript T denotes the transpose of matrices and vectors.

Theoretically LQ control is one of the most important control problems in that, on one hand, it provides fundamental insights into more general problems and, on the other hand, it admits elegant, complete solutions. Practically it has wide applications. It has its roots in least-square estimation, which can be traced back to the era of calculus of variations. The very first attempt in tackling a deterministic LQ control was made by Bellman, Glicksberg, and Gross [6, Chapter 4] in 1958. However, Kalman has been widely credited for his pioneering work [19], published in 1960, in solving the problem in a linear state feedback control form. (A less well-known work was done by Letov [21], who obtained similar results at almost the same time.) Since then, the problem has been extensively studied and developed into a major research field in control theory. In the LQ theory, the ubiquitous Riccati equation (which is a backward ordinary differential equation for the finite horizon case and an algebraic equation for the infinite horizon case) plays a key role, and an optimal control can be represented explicitly via a solution to the Riccati equation. A systematic account on deterministic LQ theory can be found in the book by Anderson and Moore [4].

In the classical deterministic LQ control, one typically assumes that the matrices Q and R , the so-called cost weighting matrices, satisfy $Q \succeq 0$ and $R \succ 0$. Under this assumption, the LQ problem is well-posed and admits unique optimal control. The case when Q is allowed to be indefinite was investigated by Molinari [26]. On the other hand, $R \succeq 0$ is in fact necessary for the LQ problem to be well-posed (cf. [40, Chapter 6, Proposition 2.4]). The case when R is singular, termed the singular control problem, has attracted substantial research interests; see [12, 13, 17, 29, 30, 35]. Since the Riccati approach requires the nonsingularity of R , different techniques, such as impulsive distributions, control dimension reduction, and linear matrix inequality (LMI), have been utilized to handle the singular problem.

Due to the physical uncertainty present in most of the real-world control systems, it is natural to extend the LQ control to the stochastic case. Sometimes called the linear–quadratic–Gaussian (LQG) model, the stochastic LQ model employs white noise to represent the underlying uncertainty, resulting in a dynamic system governed by Itô’s stochastic differential equation. Wonham [36] is the first who solved a stochastic LQ problem making use of the stochastic Riccati equation. Subsequent development on the problem can be found in [5, 7, 14] and the references therein. It should be noted that in all those works, the positive definiteness/semidefiniteness of the cost weighting matrices, inherited from the deterministic case, was imposed as a pre-rendered assumption. Recent studies, starting from [9], have, however, made a case for studying LQ problems in which R is singular, or even *indefinite* in the stochastic case. In particular, a singular (or indefinite) R may naturally arise in a class of problems in which the control affects the diffusion part of the system dynamics, as we will demonstrate in the application example below. Further works on this so-called indefinite LQ control can be found in [11, 10, 1, 37, 18]. Moreover, this theory has been utilized to obtain analytical solutions to various continuous-time versions of the Nobel prizing winning Markowitz’s mean–variance portfolio selection problem in finance; see [41, 23, 22].

In this paper we consider the following stochastic LQ control problem in infinite time horizon:

$$\text{(LQ)} \quad \min \quad J(x_0, u(\cdot)) := \mathbb{E} \int_0^{+\infty} [x(t)^T Q x(t) + u(t)^T R u(t)] dt \quad (3)$$

$$\text{s.t.} \quad dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dW(t) \quad (4)$$

$$x(0) = x_0 \in \mathbb{R}^n.$$

Here $W(\cdot)$ is a one-dimensional standard Brownian motion (with $t \in [0, +\infty)$ and $W(0) = 0$), defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$; and $u(\cdot)$ denotes the (open-loop) control, which belongs to $L^2_{\mathcal{F}}(\mathbb{R}^m)$, the space of all \mathbb{R}^m -valued, \mathcal{F}_t -adapted measurable processes satisfying:

$$\mathbb{E} \int_0^{+\infty} \|u(t)\|^2 dt < +\infty.$$

(Allowing multi-dimensional Brownian motion will render no additional difficulty to the results below.) Note that the dynamics in (4) involve multiplicative noise in both the state and the control. Also note that we allow the cost matrices Q and R to be singular or even indefinite. Numerical examples where indefinite LQ problems are well-posed and solvable can be found in [1, 37]. However, a motivating application example is described in details as follows.

An application example

Consider the control of a portfolio in a market where there is one bond and one stock, with price dynamics governed, respectively, by

$$dP_0(t) = rP_0(t)dt, \quad P_0(0) = p_0,$$

and

$$dP_1(t) = P_1(t)[bdt + \sigma dW(t)], \quad P_1(0) = p_1.$$

Suppose an agent, with an initial endowment z_0 , wants to track a (stochastic) wealth trajectory (e.g., an index fund) $I(t)$ determined by the following equation:

$$dI(t) = I(t)[b_I dt + \sigma_I(t)dW(t)], \quad I(0) = i_0.$$

At any time $t \geq 0$ the total wealth of the agent is denoted by $z(t)$, of which the market value of the stock is denoted by $\pi(t)$. If we assume that the stock is traded continuously, and that there is no transaction cost, dividend payment and withdrawal for consumption, then $z(t)$ must satisfy the following (see, e.g., [40, Chapter 2, Section 3.2]):

$$dz(t) = [rz(t) + (b - r)\pi(t)]dt + \sigma\pi(t)dW(t), \quad z(0) = z_0.$$

The objective of the agent is to choose $\pi(\cdot)$ so as to minimize the following objective:

$$J(z(0), \pi(\cdot)) = \mathbf{E} \int_0^{+\infty} e^{-\rho t} |z(t) - I(t)|^2 dt,$$

where ρ is the discount rate.

To transform the above into the formulation in (3) and (4), define the state and control variables as follows:

$$(x(t), y(t)) = e^{-\frac{1}{2}\rho t}(z(t), I(t)), \quad u(t) = e^{-\frac{1}{2}\rho t}\pi(t).$$

Then, the state dynamics become:

$$\begin{aligned} dx(t) &= \left[\left(r - \frac{1}{2}\rho \right) x(t) + (b - r)u(t) \right] dt + \sigma u(t) dW(t), \quad x(0) = z_0, \\ dy(t) &= \left(b_I - \frac{1}{2}\rho \right) y(t) + \sigma_I y(t) dW(t), \quad y(0) = i_0. \end{aligned}$$

And, the objective function can be rewritten as follows:

$$\begin{aligned} J(x(0), y(0), u(\cdot)) &= \mathbf{E} \int_0^{+\infty} |x(t) - y(t)|^2 dt \\ &= \mathbf{E} \int_0^{+\infty} \begin{bmatrix} x(t) & y(t) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} dt. \end{aligned}$$

This way, we are in the framework of (3) and (4), with the control cost $R \equiv 0$, and the state cost $Q = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ being singular.

The classical approach

A classical tool in solving LQ control problems is the following (stochastic algebraic) Riccati equation, with the symmetric matrix P being the unknown:

$$A^T P + PA + Q + C^T PC - (PB + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T PC) = 0. \quad (5)$$

Suppose P^* is a solution to the above equation with $R + D^T P^* D \succ 0$ (positive definite). Then,

$$u^*(t) = -(R + D^T P^* D)^{-1}(B^T P^* + D^T P^* C)x^*(t) \quad (6)$$

is an optimal state feedback control for (LQ).

The classical theory has a rather serious limitation in treating the present indefinite LQ problem: in general, there is no easy way to solve the Riccati equation (5), in particular since the matrix inverse term also involves the unknown P . In fact, there is no guarantee for $R + D^T P^* D \succ 0$, except when $R \succ 0$, in which case $P^* \succeq 0$ follows as a solution to (5); see [1, Corollary 5.1].

A traditional method for solving the Riccati equation (in the case of $D = 0$) is to consider the so-called associated Hamiltonian matrix ([8]). In this case it is known that the Riccati equation has a solution if and only if the associated Hamiltonian matrix admits no pure imaginary eigenvalues, a condition that can be verified using a Routh-Hurwitz type test on a set of polynomial inequalities involving the given matrices. If the associated Hamiltonian matrix passes this test, then the solution to the Riccati equation can be constructed using the eigenvectors of the associated Hamiltonian matrix. This procedure, however, does not apply when R is indefinite, as the matrix inverse in (5), and hence the Riccati equation and the associated Hamiltonian matrix themselves, may not be well-defined.

The SDP resolution

To overcome this difficulty, our idea here is to use semidefinite programming (SDP) as a unifying approach to solve the stochastic LQ control problem, generally in the absence of the positive definiteness/semidefiniteness of the cost matrices R and Q . SDP is a relatively new optimization tool developed in recent years following the seminal work of Nesterov and Nemirovski [27]. It relates intimately to the so-called linear matrix inequalities (LMI) (see [38]). Pioneered by Yakubovich [39] and Willems [34], a vast literature has since appeared, applying the LMI approach to both deterministic and stochastic systems; refer to [8] for a systematic exposition and detailed literature review. However, the definiteness of R has remained a predominant assumption in the research literature. In a recent work, [1], the relationship between the Riccati equation in (5) and the associated LMI has been examined under the assumption that

$R + D^T P D \succ 0$ for some solution P of the Riccati equation (while R itself is allowed to be indefinite). This assumption, however, is hard to verify *a priori* as P is unknown; and even when this assumption is not satisfied, the corresponding LQ problem may still possess a meaningful optimal control; see Example 6.1 below.

In contrast, our focus here is to develop a direct connection between the LQ control problem and the duality theory of SDP. In particular, we demonstrate that to extend beyond the confines of the classical LQ theory (with positive definite cost matrices) the central issue is *stability*; and stability is intimately related to the *complementary duality* of the SDP associated with the LQ problem. We establish several equivalence relations between the stability/optimality of the LQ problem and the duality of the SDP, and demonstrate that a new class of optimal controls can be constructed based on the dual SDP. Furthermore, exploiting the primal–dual structure of the SDP also leads to powerful and efficient computational means, based on the newly developed primal-dual interior-point techniques (refer to, e.g., [31]), to solving the LQ problem. In short, while the LMI approach is a primal-only method, our primal-dual SDP approach applies to a more general class of problems, generates new theoretical results, and leads to practical computational algorithms.

Briefly, the rest of the paper is organized as follows. In §2, we introduce several regularity conditions relating to stability, and present the preliminaries of SDP. The main results of the paper are presented in the next two sections: We establish first in §3, the connection between stability and the dual SDP; and then in §4, several implication relations among the optimality of LQ control problem, the complementary duality of the SDP, and a generalized version of the Riccati equation (5) involving a matrix pseudo-inverse. A synthesis of these results is presented in §5, along with a computational procedure that provides a systematic treatment of the LQ control problem via SDP. Several examples are collected in §6 to illustrate some of the key technical issues involved in the SDP approach. For problems that do not possess an attainable optimal control, an ϵ -approximation scheme is developed in §7, which achieves asymptotic optimality. Brief concluding remarks are summarized in §8.

2 Preliminaries

2.1 Regularity Conditions

Since we are concerned with an infinite horizon in (LQ), we need to address the general issue of *stability*.

Definition 2.1 (i) An open-loop control $u(\cdot)$ is called (*mean-square*) *stabilizing* at x_0 , if the corresponding state $x(\cdot)$ of (4) with the initial state x_0 satisfies $\lim_{t \rightarrow +\infty} \mathbf{E}[x(t)^T x(t)] = 0$.

(ii) A feedback control $u(t) = Kx(t)$, where K is a constant matrix, is called *stabilizing*, if for every initial state x_0 , the solution to the following equation

$$\begin{cases} dx(t) = (A + BK)x(t)dt + (C + DK)x(t)dW(t), \\ x(0) = x_0, \end{cases}$$

satisfies $\lim_{t \rightarrow +\infty} \mathbb{E}[x(t)^T x(t)] = 0$.

(iii) The system in (4) is called (mean-square) *stabilizable* if there exists a stabilizing feedback control of the form $u(t) = Kx(t)$ where K is a constant matrix.

Definition 2.2 (i) An open-loop control $u(\cdot) \in L^2_{\mathcal{F}}(\mathbb{R}^m)$ is called admissible (at x_0) if it is stabilizing at x_0 . The set of all admissible controls at x_0 is denoted as $U_{ad}^{x_0}$.

(ii) An admissible pair $(x^*(\cdot), u^*(\cdot))$ is called *optimal* (at x_0) if $u^*(\cdot)$ achieves the infimum of $J(x_0, u(\cdot))$ over $u(\cdot) \in U_{ad}^{x_0}$.

(iii) The control problem (LQ) is called *well-posed* (at x_0) if

$$-\infty < \inf_{u(\cdot) \in U_{ad}^{x_0}} J(x_0, u(\cdot)) < +\infty;$$

(LQ) is called *attainable* (at x_0) if it is well-posed (at x_0) and there exists an optimal admissible control.

Unless explicitly stated otherwise, we shall assume throughout the paper that the system under consideration, (4), is mean-square stabilizable. Note that this is a very mild regularity condition; in particular, it is implied by the well-posedness of (LQ) when $Q \succ 0$ and $R \succeq 0$. Indeed, the well-posedness of (LQ) yields that there is at least one control whose cost is finite. As a result, under that control, $\lim_{t \rightarrow +\infty} \mathbb{E}[x(t)^T Q x(t)] = 0$. Thus the control must be stabilizing due to the nonsingularity of Q .

On the other hand, to appreciate why the admissible controls have to be stabilizing, consider in the case when $Q \succ 0$ and $R \succeq 0$. In order for the cost objective in (3) to be finite it is necessary (as shown above) that the corresponding control must be stabilizing. In general, a non-stabilizing control is ill-behaved, and hence should be excluded.

Attainability of an optimal admissible control is another issue, even when the problem is well-posed. Consider the following simple (deterministic) LQ problem:

$$\begin{aligned} \min \quad & \int_0^\infty x(t)^2 dt \\ \text{s.t.} \quad & dx(t) = [-x(t) + u(t)]dt \\ & x(0) = 1. \end{aligned}$$

Clearly, this problem has an infimum value zero. However, there is no control that attains the zero cost. In general, deciding whether or not a problem has an attainable optimal control is as hard as solving the (LQ) problem, especially for large problems.

2.2 Semidefinite Programming

SDP is a special form of the so-called *conic optimization problem*, which is in essence to optimize a linear function over the intersection of two closed convex sets: one being an affine subspace and the other, a cone. In SDP, the cone is formed by positive semidefinite matrices in the linear space of symmetric matrices. Similar to linear programming, an SDP problem can be cast in various ways. In standard form, an SDP is the following convex optimization problem:

$$\begin{aligned}
 (SDP)_p \quad & \min \quad \langle C, X \rangle \\
 & \text{s.t.} \quad \langle A_i, X \rangle = b_i, \text{ for } i = 1, \dots, m \\
 & \quad \quad X \succeq 0
 \end{aligned}$$

where C and A_i are $n \times n$ symmetric matrices, $b_i \in \mathbb{R}^m$ is a vector, $i = 1, \dots, m$, and $\langle X, Y \rangle := \sum_{i,j} X_{ij}Y_{ij}$ denotes the matrix inner-product.

The above has an associated dual, which is also an SDP problem:

$$\begin{aligned}
 (SDP)_d \quad & \max \quad b^T y \\
 & \text{s.t.} \quad \sum_{i=1}^m y_i A_i + Z = C \\
 & \quad \quad Z \succeq 0.
 \end{aligned}$$

A conic optimization problem is said to satisfy the *Slater condition* if its feasible region has a non-empty intersection with the interior of the cone. In our standard primal–dual SDP problems stated above, the Slater condition has the following characterization. For the primal problem $(SDP)_p$, the Slater condition is equivalent to the existence of a primal feasible solution X^0 such that $X^0 \succ 0$. Similarly, for the dual problem $(SDP)_d$, the Slater condition is the existence of a dual feasible solution (y^0, Z^0) with $Z^0 \succ 0$.

For conic optimization problems, a well-developed duality theory exists; see e.g. [38, 24, 27]. Key points of the theory can be highlighted as follows:

- The *weak duality* always holds, i.e. any feasible solution to the primal (minimization) problem always possesses an objective value that is no less than the (dual) objective value of any dual feasible solution (the dual being a maximization problem).
- In contrast, the *strong duality* — that the optimal values of the primal and dual problems coincide — holds if there exists a pair of *complementary optimal solutions* X^* and (y^*, Z^*) , namely, they satisfy $X^*Z^* = 0$. If, furthermore, it holds that $X^* + Z^* \succ 0$, then this pair of optimal solutions is called *strictly complementary*.
- Unlike linear programming, SDP may fail to satisfy the strong duality, let alone strict complementarity. It is known, however, that if the primal problem is feasible and the dual satisfies the Slater condition, then the primal problem must have a non-empty and

compact optimal solution set. Moreover, if both the primal and the dual satisfy the Slater condition, then both must have non-empty and compact optimal solution sets, and the strong duality holds.

In the SDP literature, the Slater type regularity conditions are mostly assumed in order to avoid pathological cases. In Luo, Sturm and Zhang [24, 25], extensive analysis can be found addressing the issue of regularity in the context of duality theory, and the related issue of how to detect the duality status numerically.

The linkage between the LQ control problem and the SDP is best understood in the deterministic setting. Consider the deterministic version of (LQ) as follows:

$$\begin{aligned} \min \quad & \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt \\ \text{s.t.} \quad & \dot{x}(t) = Ax(t) + Bu(t) \\ & x(0) = x_0 \in \mathfrak{R}^n. \end{aligned}$$

Assume $Q \succeq 0$ and $R \succ 0$. Then, the optimal solution to the above problem is:

$$u^*(t) = -R^{-1} B^T P^* x^*(t),$$

where P^* is a nonnegative definite solution to the Riccati equation:

$$Q + A^T P + PA - PBR^{-1}B^T P = 0.$$

It turns out that solutions to the Riccati equation can be found through solving the following SDP:

$$\begin{aligned} \max \quad & \langle I, P \rangle \\ \text{s.t.} \quad & \begin{bmatrix} R, & B^T P \\ PB, & Q + A^T P + PA \end{bmatrix} \succeq 0 \\ & P \in \mathcal{S}^{n \times n}, \end{aligned}$$

where $\mathcal{S}^{n \times n}$ denotes the space of n by n symmetric matrices. Note that the above is an SDP problem because the constraint set can be viewed as an intersection between a linear subspace and the cone of positive semidefinite matrices. The dual problem is:

$$\begin{aligned} \min \quad & \langle R, Z_b \rangle + \langle Q, Z_n \rangle \\ \text{s.t.} \quad & I + Z_u^T B^T + B Z_u + Z_n A^T + A Z_n = 0 \\ & Z := \begin{bmatrix} Z_b, & Z_u \\ Z_u^T, & Z_n \end{bmatrix} \succeq 0. \end{aligned}$$

It is interesting to note that both the primal and dual SDP's above are well-defined even in the presence of the singularity of R and Q . In particular, there is no matrix inverse involved. Similarly, the primal and dual SDP's for the stochastic problem in (LQ) can be written as follows:

$$\begin{aligned}
\text{(P)} \quad & \max \quad \langle I, P \rangle \\
& \text{s.t.} \quad \mathcal{L}(P) \succeq 0 \\
& \quad \quad P \in \mathcal{S}^{n \times n},
\end{aligned}$$

where

$$\mathcal{L}(P) := \begin{bmatrix} R + D^T P D, & B^T P + D^T P C \\ P B + C^T P D, & Q + C^T P C + A^T P + P A \end{bmatrix}; \quad (7)$$

and

$$\begin{aligned}
\text{(D)} \quad & \min \quad \langle R, Z_B \rangle + \langle Q, Z_N \rangle \\
& \text{s.t.} \quad I + Z_U^T B^T + B Z_U + Z_N A^T + A Z_N \\
& \quad \quad + C Z_N C^T + D Z_U C^T + C Z_U^T D^T + D Z_B D^T = 0 \\
& \quad \quad Z := \begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix} \succeq 0.
\end{aligned}$$

In the above forms, (P) and (D) are said to satisfy the Slater condition, if there exist primal and dual feasible solutions, P^0 and Z^0 , such that $\mathcal{L}(P^0) \succ 0$ and $Z^0 \succ 0$, respectively. On the other hand, a primal optimal solution P^* and a dual optimal solution Z^* are called complementary optimal solutions if $\mathcal{L}(P^*)Z^* = 0$. Furthermore, they are called strictly complementary if $\mathcal{L}(P^*) + Z^* \succ 0$.

As a least condition in order to apply the SDP approach, we assume throughout the paper that the feasible set of (P) is nonempty. This assumption is satisfied automatically if the Riccati equation (5) has a solution P^* with $R + D^T P^* D \succ 0$ (which is the key assumption in [1]) by virtue of the well-known Schur lemma. It is also satisfied when $Q \succeq 0$ and $R \succeq 0$. Therefore, without this assumption, the original LQ problem cannot be solved by either the Riccati or the SDP approach.

3 Stability

Since we require admissible controls to be stabilizing, we need first to address the issue of stability, which, as it will become evident below, relates closely to the dual SDP.

The following results from [1, Theorems 2.1, 5.2] will be used later.

Proposition 3.1 The following conditions are equivalent.

- (i) System (4) is mean-square stabilizable.
- (ii) Problem (D) satisfies the Slater condition.
- (iii) There exists a matrix K and a symmetric matrix Y such that

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T \prec 0, Y \succ 0. \quad (8)$$

In this case the feedback $u(t) = Kx(t)$ is a stabilizing control.

(iv) There exists a matrix K such that for any X there exists a unique solution Y to the following equation

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T + X = 0. \quad (9)$$

Moreover, if $X \succ 0$ (resp. $X \succeq 0$) then $Y \succ 0$ (resp. $Y \succeq 0$). Furthermore, in this case the feedback $u(t) = Kx(t)$ is a stabilizing control.

(v) There exist a matrix X and a positive definite matrix $Y \succ 0$ such that

$$\begin{bmatrix} AY + YA^T + BX + X^T B^T & CY + DX \\ YC^T + X^T D^T & -Y \end{bmatrix} \prec 0. \quad (10)$$

In this case, the feedback $u(t) = XY^{-1}x(t)$ is a stabilizing control.

Note that the last equivalent condition above is an LMI condition, based on which the mean-square stabilizability can be verified numerically (cf. [8, 15]). Moreover, to check whether or not a given feedback control $u(t) = Kx(t)$ is stabilizing, it suffices to check if the LMI's in (8) have a feasible solution, which again can be carried out numerically.

Define:

$$F(P) := A^T P + PA + Q + C^T P C - (PB + C^T P D)(R + D^T P D)^+ (B^T P + D^T P C). \quad (11)$$

Here, M^+ stands for the pseudo-inverse of a matrix M (refer to [28]). Note that when M is a positive semidefinite matrix, M^+ satisfies the following properties:

$$M^+ \succeq 0, \quad (M^+)^T = M^+, \quad M^+ M = M M^+;$$

$$M M^+ M = M, \quad M^+ M M^+ = M^+.$$

Clearly, the equation $F(P) = 0$ generalizes the classical Riccati equation (5). Hence, we shall refer to it below as the *generalized Riccati equation*.

The following extended Schur's lemma ([2]) plays an important technical role.

Lemma 3.2 Let matrices $M = M^T, N$ and $S = S^T$ be given with appropriate dimensions. Then the following three conditions are equivalent:

- (i) $M - NS^+N^T \succeq 0, S \succeq 0$, and $N(I - SS^+) = 0$.
- (ii) $\begin{bmatrix} M & N \\ N^T & S \end{bmatrix} \succeq 0$.
- (iii) $\begin{bmatrix} S & N^T \\ N & M \end{bmatrix} \succeq 0$.

Theorem 3.3 If a feasible solution of (P), P^* , is such that $F(P^*) = 0$, and the feedback control

$$u(t) = -(R + D^T P^* D)^+(B^T P^* + D^T P^* C)x(t) \quad (12)$$

is stabilizing, then there exist complementary optimal solutions of (P) and (D). In particular, P^* is optimal to (P); and there exists a complementary dual optimal solution Z^* , such that $Z_N^* \succ 0$.

Proof. Denote $K := -(R + D^T P^* D)^+(B^T P^* + D^T P^* C)$. By the stability assumption of the control (12) and Proposition 3.1-(iv), the following equation

$$(A + BK)Y + Y(A + BK)^T + (C + DK)Y(C + DK)^T + I = 0$$

has a positive solution. Let it be $Y^* \succ 0$. Furthermore, let

$$Z_N^* = Y^*, Z_U^* = K Z_N^* \text{ and } Z_B^* = K(Z_U^*)^T. \quad (13)$$

By this construction, we can easily verify that

$$\begin{bmatrix} Z_B^* & Z_U^* \\ (Z_U^*)^T & Z_N^* \end{bmatrix} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & Z_N^* \end{bmatrix} \begin{bmatrix} I & 0 \\ K^T & I \end{bmatrix} \succeq 0.$$

Moreover, by direct verification, we know

$$I + (Z_U^*)^T B^T + B Z_U^* + Z_N^* A^T + A Z_N^* + C Z_N^* C^T + D Z_U^* C^T + C(Z_U^*)^T D^T + D Z_B^* D^T = 0.$$

Therefore, Z^* is a feasible solution of (D). Moreover, using extended Schur's lemma (Lemma 3.2), we have

$$\begin{aligned} & \mathcal{L}(P^*) \begin{bmatrix} Z_B^* & Z_U^* \\ (Z_U^*)^T & Z_N^* \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -K^T & I \end{bmatrix} \begin{bmatrix} R + D^T P^* D & 0 \\ 0 & F(P^*) \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_B^* & Z_U^* \\ (Z_U^*)^T & Z_N^* \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -K^T & I \end{bmatrix} \begin{bmatrix} (R + D^T P^* D)(Z_B^* - K(Z_U^*)^T) & R(Z_U^* - K Z_N^*) \\ F(P^*)(Z_U^*)^T & F(P^*) Z_N^* \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where in the first equation the decomposition of $\mathcal{L}(P^*)$ into the product of three matrices (the Schur decomposition) is possible because $\mathcal{L}(P^*) \succeq 0$, since P^* is a feasible solution to (P); hence, Lemma 3.2 can be invoked.

Therefore, the above establishes that P^* and Z^* are complementary solutions; in particular, P^* is optimal to (P) and Z^* is optimal to (D). The last statement of the theorem follows from the fact that $Z_N^* = Y^* \succ 0$. \square

Note that the assumption in the above theorem, that the control in (12) is stabilizing, is not automatically satisfied even in the case when $R + D^T P^* D \succ 0$; see Example 6.2. The following result shows, however, that we can possibly obtain a stabilizing feedback control via the *dual* SDP.

Theorem 3.4 Let $Z = \begin{bmatrix} Z_B & Z_U \\ Z_U^T & Z_N \end{bmatrix}$ be a feasible solution of (D) with $Z_N \succ 0$, then the feedback control $u(t) = Z_U Z_N^{-1} x(t)$ is stabilizing.

Proof. First of all, by feasibility of Z to (D) along with Schur's lemma we have

$$Z_B \succeq Z_U Z_N^{-1} Z_U^T.$$

Then,

$$\begin{aligned} 0 &= I + Z_U^T B^T + B Z_U + Z_N A^T + A Z_N + C Z_N C^T + D Z_U C^T + C Z_U^T D^T + D Z_B D^T \\ &\succeq I + Z_U^T B^T + B Z_U + Z_N A^T + A Z_N + C Z_N C^T + D Z_U C^T + C Z_U^T D^T + D Z_U Z_N^{-1} Z_U^T D^T \\ &\succ Z_U^T B^T + B Z_U + Z_N A^T + A Z_N + (C Z_N + D Z_U) Z_N^{-1} (Z_N C^T + Z_U^T D^T). \end{aligned}$$

Applying Schur's lemma again, we conclude that Proposition 3.1-(v) holds with $X = Z_U$ and $Y = Z_N \succ 0$. Hence $u(t) = Z_U Z_N^{-1} x(t)$ is stabilizing. \square

4 Optimality

Here we establish the relationship among the optimality of the original LQ problem, the primal/dual SDP problems, and the generalized Riccati equation.

Theorem 4.1 If (LQ) is attainable at any $x_0 \in \mathfrak{R}^n$, then (P) must have an optimal solution P^* satisfying $F(P^*) = 0$.

Proof. First, note that since (LQ) has a finite optimal value with respect to any initial value, it must have a quadratic representation

$$\inf_{u(\cdot) \in U_{ad}^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0, \quad \forall x_0 \in \mathfrak{R}^n;$$

see [4, p.21] (note that the proof there, which is for the deterministic case, readily extends to the stochastic case).

Suppose for the time being that the matrix M is a feasible solution of (P), the validity of which will be proved later. Let $u^*(\cdot)$ be the optimal control and $x^*(\cdot)$ be the corresponding state from the initial x_0 . Then for any feasible solution P of (P), applying Itô's formula (see, e.g., [40, p.36]) yields

$$\begin{aligned}
& d(x^*(t)^T P x^*(t)) \\
&= \left[(Ax^*(t) + Bu^*(t))^T P x^*(t) + x^*(t)^T P (Ax^*(t) + Bu^*(t)) \right. \\
&\quad \left. + (Cx^*(t) + Du^*(t))^T P (Cx^*(t) + Du^*(t)) \right] dt + \{\dots\} dW(t) \\
&= \left[x^*(t)^T (A^T P + PA + C^T P C) x^*(t) + 2u^*(t)^T (B^T P + D^T P C) x^*(t) + u^*(t)^T D^T P D u^*(t) \right] dt \\
&\quad + \{\dots\} dW(t).
\end{aligned}$$

Integrating the above over $[0, \infty)$, taking expectations, and using the fact that $E[x^*(t)^T P x^*(t)] \rightarrow 0$ (as $u^*(\cdot)$ is stabilizing), we obtain:

$$\begin{aligned}
0 = & x_0^T P x_0 + E \int_0^\infty [x^*(t)^T (A^T P + PA + C^T P C) x^*(t) + 2u^*(t)^T (B^T P + D^T P C) x^*(t) \\
& + u^*(t)^T D^T P D u^*(t)] dt.
\end{aligned} \tag{14}$$

Completion of square yields:

$$\begin{aligned}
& J(x_0, u^*(\cdot)) \\
&= E \int_0^\infty [x^*(t)^T Q x^*(t) + u^*(t)^T R u^*(t)] dt \\
&= x_0^T P x_0 + \\
&\quad E \int_0^\infty \left\{ [u^*(t) - K x^*(t)]^T (R + D^T P D) [u^*(t) - K x^*(t)] + x^*(t)^T F(P) x^*(t) \right\} dt, \tag{15}
\end{aligned}$$

where $K := -(R + D^T P D)^+ (B^T P + D^T P C)$. Since P is feasible to (P), we have $R + D^T P D \succeq 0$ and $F(P) \succeq 0$ by extended Schur's lemma. This means that

$$x_0^T M x_0 \equiv J(x_0, u^*(\cdot)) \geq x_0^T P x_0, \tag{16}$$

for any P feasible to (P). Hence M must be optimal to (P). On the other hand, taking $P = M$ in (15) and noting $J(x_0, u^*(\cdot)) = x_0^T M x_0$, we conclude that $x(t)^T F(M) x(t) = 0$ for all $t \in [0, \infty)$. Since x_0 can be chosen arbitrarily it follows that $F(M) = 0$. The desired result thus follows.

What remains is to show the primal feasibility of M . To this end we consider a perturbation on the problem (LQ). That is, we keep all the data A, B, C and D unchanged, and let $R_\epsilon = R + \epsilon I$ and $Q_\epsilon = Q + \epsilon I$ where $\epsilon > 0$ is a small positive number.

Under the perturbation ($\epsilon > 0$), the corresponding SDP's,

$$(P_\epsilon) \quad \max \quad \langle I, P \rangle$$

$$\text{s.t.} \quad \begin{bmatrix} R + \epsilon I + D^T P D, & B^T P + D^T P C \\ P B + C^T P D, & Q + \epsilon I + C^T P C + A^T P + P A \end{bmatrix} \succeq 0$$

$$P \in \mathcal{S}^{n \times n}$$

and

$$(D_\epsilon) \quad \min \quad \langle R + \epsilon I, Z_B \rangle + \langle Q + \epsilon I, Z_N \rangle$$

$$\text{s.t.} \quad I + Z_U^T B^T + B Z_U + Z_N A^T + A Z_N$$

$$+ C Z_N C^T + D Z_U C^T + C Z_U^T D^T + D Z_B D^T = 0$$

$$\begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix} \succeq 0$$

both satisfy the Slater condition (the former does because the feasible set of (P) is assumed to be nonempty, and the latter because of the mean-square stabilizability assumption and Proposition 3.1-(ii)), and therefore complementary optimal solutions exist ([38]). Observe that the feasible set of (D_ϵ) is independent of ϵ . Take any dual feasible solution Z^0 . By weak duality, we have

$$\text{tr } P = \langle I, P \rangle \leq \langle R + \epsilon I, Z_B^0 \rangle + \langle Q + \epsilon I, Z_N^0 \rangle. \quad (17)$$

Let \hat{P} be a feasible solution of (P), which exists by our assumption. Certainly, \hat{P} is feasible to (P_ϵ) for all $\epsilon \geq 0$. Theorem 5.5 in [1] asserts that for $\epsilon > 0$, the unique optimal solution for (P_ϵ) , denoted by P_ϵ^* , dominates any other feasible solutions. Hence, $\hat{P} \preceq P_\epsilon^*$ for all $\epsilon > 0$.

This, together with (17), implies in particular that P_ϵ^* are contained in a compact set, with $0 \leq \epsilon \leq \epsilon_0$ ($\epsilon_0 > 0$ is a pre-determined constant). Now, take a convergent subsequence such that

$$\lim_{i \rightarrow \infty} P_{\epsilon_i}^* = P_0^*$$

with $\epsilon_i \rightarrow 0$ as $i \rightarrow \infty$.

Clearly, P_0^* is a feasible solution of (P) since the feasible region of (P_ϵ) monotonically shrinks as $\epsilon \downarrow 0$. We now show that $P_0^* = M$. Define the perturbed cost function

$$J_\epsilon(x_0, u(\cdot)) = \mathbb{E} \int_0^\infty [x(t)^T Q_\epsilon x(t) + u(t)^T R_\epsilon u(t)] dt.$$

Similar to (15), we can show

$$J_\epsilon(x_0, u(\cdot)) = x_0^T P_\epsilon^* x_0 + \mathbb{E} \int_0^\infty \{ [u(t) - K_\epsilon x(t)]^T (R_\epsilon + D^T P_\epsilon^* D) [u(t) - K_\epsilon x(t)] \\ + x(t)^T F_\epsilon(P_\epsilon^*) x(t) \} dt,$$

for any $u(\cdot) \in U_{ad}^{x_0}$, where $K_\epsilon := -(R_\epsilon + D^T P_\epsilon^* D)^+ (B^T P_\epsilon^* + D^T P_\epsilon C)$ and F_ϵ is the “perturbed” Riccati operator with Q and R in (11) replaced by Q_ϵ and R_ϵ , respectively. In [1], Theorems 5.4 and 5.5, it is stated that if the problem (P_ϵ) is strictly feasible (which is the case here) with an optimal solution P_ϵ^* , then the corresponding feedback control $u(t) = K_\epsilon x(t)$ must be stabilizing (and hence admissible), and P_ϵ^* must be the unique optimal solution to (P_ϵ) satisfying the corresponding Riccati equation $F_\epsilon(P) = 0$. Thus $u(t) = K_\epsilon x(t)$ must be optimal, and

$$\inf_{u(\cdot) \in U_{ad}^{x_0}} J_\epsilon(x_0, u(\cdot)) = x_0^T P_\epsilon^* x_0,$$

which further yields

$$x_0^T P_{\epsilon_i}^* x_0 = \inf_{u(\cdot) \in U_{ad}^{x_0}} J_{\epsilon_i}(x_0, u(\cdot)) \geq \inf_{u(\cdot) \in U_{ad}^{x_0}} J(x_0, u(\cdot)) = x_0^T M x_0.$$

Letting $\epsilon_i \rightarrow 0$, we obtain

$$x_0^T P_0^* x_0 \geq x_0^T M x_0.$$

On the other hand, (16) gives rise to the opposite inequality since P_0^* is feasible to (P) . Thus we have $M = P_0^*$. This establishes that M is indeed a primal feasible solution. \square

An important implication, which is the contrapositive of the above theorem is this: If (P) has no optimal solution, or if (P) has optimal solutions but none of them satisfies the generalized Riccati equation $F(P) = 0$, then (LQ) has no attainable optimal control; in particular, it does not have any optimal feedback control.

A natural question to ask at this point is whether or not the converse of the above statement is true. Namely, if (P) admits an optimal solution P^* satisfying $F(P^*) = 0$, then does (LQ) have an attainable optimal control? Recall that in the finite horizon case an optimal feedback control is represented as $u^*(t) = -(R + D^T P^* D)^{-1} (B^T P^* + D^T P^* C) x^*(t)$ (cf. [9, Theorem 3.2]). However in the present case $R + D^T P^* D$ may be singular, therefore we naturally expect that an optimal feedback control should be

$$u^*(t) = -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C) x^*(t). \quad (18)$$

The following result establishes that this control is indeed optimal if it is stabilizing. (Recall that in the infinite horizon case, stability is essential.)

Theorem 4.2 If a feasible solution of (P) , P^* , is such that $F(P^*) = 0$, and the feedback control $u^*(t)$ in (18) is stabilizing, then it must be optimal for (LQ) .

Proof. For any admissible control $u(\cdot) \in U_{ad}^{x_0}$, an argument similar to that in proving (15) leads to

$$\begin{aligned}
& J(x_0, u(\cdot)) \\
&= \mathbb{E} \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt \\
&= x_0^T P^* x_0 \\
&\quad + \mathbb{E} \int_0^\infty \left\{ [u(t) - Kx(t)]^T (R + D^T P^* D) [u(t) - Kx(t)] + x(t)^T F(P^*) x(t) \right\} dt \\
&= x_0^T P^* x_0 \\
&\quad + \mathbb{E} \int_0^\infty [u(t) - Kx(t)]^T (R + D^T P^* D) [u(t) - Kx(t)] dt, \tag{19}
\end{aligned}$$

where $K := -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C)$. Since $u^*(t) = Kx^*(t)$ is stabilizing, the above shows that $u^*(\cdot)$ must be optimal. \square

As mentioned earlier, the stability of the control in (18) can be examined numerically via the LMIs as stipulated in Proposition 3.1-(iii). On the other hand, by Theorem 3.3, in order for this control to be stabilizing, it is *necessary* that there exist complementary solutions P^* and Z^* of (P) and (D), respectively, with $Z_N^* \succ 0$. It is interesting that under these (weaker) conditions, one can prove the existence of an explicitly representable optimal feedback control of (LQ) in (22) below (which is not necessarily in the same form as the control in (18)).

First we need a lemma.

Lemma 4.3 Suppose (P) and (D) have complementary optimal solutions, P^* and Z^* , respectively. Then, $F(P^*) = 0$.

Proof. We have the following decomposition:

$$\mathcal{L}(P^*) = \begin{bmatrix} I, & 0 \\ -K^T, & I \end{bmatrix} \begin{bmatrix} R + D^T P^* D, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K \\ 0, & I \end{bmatrix}, \tag{20}$$

where $K := -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C)$. From the relation $\mathcal{L}(P^*)Z^* = 0$, it follows that

$$\begin{aligned}
& \begin{bmatrix} R + D^T P^* D, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_B^*, & Z_U^* \\ (Z_U^*)^T, & Z_N^* \end{bmatrix} \\
&= \begin{bmatrix} (R + D^T P^* D)(Z_B^* - K(Z_U^*)^T), & (R + D^T P^* D)(Z_U^* - K Z_N^*) \\ F(P^*)(Z_U^*)^T, & F(P^*)Z_N^* \end{bmatrix} \\
&= \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}. \tag{21}
\end{aligned}$$

Therefore,

$$F(P^*)(Z_U^*)^T = 0, \quad F(P^*)Z_N^* = 0$$

and

$$Z_U^*F(P^*) = 0, \quad Z_N^*F(P^*) = 0.$$

On the other hand, the dual nonnegativity constraint, together with the extended Schur's lemma, yields

$$Z_B^* - Z_U^*(Z_N^*)^+(Z_U^*)^T \succeq 0,$$

and

$$Z_U^*(I - Z_N^*(Z_N^*)^+) = 0.$$

Multiplying $F(P^*)$ on both sides of the dual equality constraint and making use of the above relations, we obtain

$$\begin{aligned} 0 &= F(P^*)(I + CZ_N^*C^T + DZ_U^*C^T + C^T Z_U^*D + DZ_B^*D^T)F(P^*) \\ &\succeq F(P^*)(I + CZ_N^*C^T + DZ_U^*C^T + C^T Z_U^*D + DZ_U^*(Z_N^*)^+(Z_U^*)^T D^T)F(P^*) \\ &= F(P^*)^2 + F(P^*)[CZ_N^* + DZ_U^*](Z_N^*)^+[Z_N^*C^T + (Z_U^*)^T D^T]F(P^*) \\ &\succeq F(P^*)^2. \end{aligned}$$

This means, $F(P^*) = 0$. □

Theorem 4.4 Assume that solving (P) and (D) yields complementary optimal solutions P^* and Z^* , with $Z_N^* \succ 0$. Then $F(P^*) = 0$, and (LQ) has an attainable optimal feedback control given by

$$u^*(t) = Z_U^*(Z_N^*)^{-1}x^*(t). \quad (22)$$

Proof. First, that $F(P^*) = 0$ is seen from Lemma 4.3 (even without the assumption $Z_N^* \succ 0$).

Next, for any feasible solution P of (P) and any (stabilizing) control $u(\cdot) \in U_{ad}^{x_0}$, along with the corresponding state $x(\cdot)$, an argument similar to the one that proved (15) leads to

$$\begin{aligned} &J(x_0, u(\cdot)) \\ &= \mathbf{E} \int_0^\infty [x(t)^T Q x(t) + u(t)^T R u(t)] dt \\ &= x_0^T P x_0 \\ &\quad + \mathbf{E} \int_0^\infty \left\{ [u(t) - Kx(t)]^T (R + D^T P D) [u(t) - Kx(t)] + x(t)^T F(P) x(t) \right\} dt, \quad (23) \end{aligned}$$

where $K := -(R + D^T P D)^+(B^T P + D^T P C)$. Since P is feasible to (P), we have $F(P) \succeq 0$.

This means that

$$J(x_0, u(\cdot)) \geq x_0^T P x_0, \quad \forall u(\cdot) \in U_{ad}^{x_0},$$

for any P feasible to (P).

Note that as yet we cannot conclude that the control in (18) is optimal, since we do not know whether or not the control is stabilizing, whereas (23) holds only for stabilizing controls. To get around, let us define a feedback control $u^*(t) = Z_U^*(Z_N^*)^{-1}x^*(t)$, which, following Theorem 3.4, is stabilizing. Hence (23) with $u(\cdot) = u^*(\cdot)$ and $P = P^*$ yields

$$J(x_0, u^*(\cdot)) = x_0^T P^* x_0 + \mathbb{E} \int_0^\infty [u^*(t) - K^* x^*(t)]^T (R + D^T P^* D) [u^*(t) - K^* x^*(t)] dt, \quad (24)$$

with $K^* := -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C)$. Next, we show that

$$\begin{aligned} & [u^*(t) - K^* x^*(t)]^T (R + D^T P^* D) [u^*(t) - K^* x^*(t)] \\ \equiv & [u^*(t) - Z_U^*(Z_N^*)^{-1}x^*(t)]^T (R + D^T P^* D) [u^*(t) - Z_U^*(Z_N^*)^{-1}x^*(t)] \\ = & 0. \end{aligned} \quad (25)$$

To this end, apply the complementary duality. From the relation $\mathcal{L}(P^*)Z^* = 0$, it follows that

$$\begin{aligned} & \begin{bmatrix} R + D^T P^* D, & 0 \\ 0, & F(P^*) \end{bmatrix} \begin{bmatrix} I, & -K^* \\ 0, & I \end{bmatrix} \begin{bmatrix} Z_B^*, & Z_U^* \\ (Z_U^*)^T, & Z_N^* \end{bmatrix} \\ = & \begin{bmatrix} (R + D^T P^* D)(Z_B^* - K^*(Z_U^*)^T), & (R + D^T P^* D)(Z_U^* - K^* Z_N^*) \\ F(P^*)(Z_U^*)^T, & F(P^*)Z_N^* \end{bmatrix} \\ = & \begin{bmatrix} (R + D^T P^* D)Z_B^* + (B^T P^* + D^T P^* C)(Z_U^*)^T, & (R + D^T P^* D)Z_U^* + (B^T P^* + D^T P^* C)Z_N^* \\ F(P^*)(Z_U^*)^T, & F(P^*)Z_N^* \end{bmatrix} \\ = & \begin{bmatrix} 0, & 0 \\ 0, & 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$(R + D^T P^* D)Z_U^* = -(B^T P^* + D^T P^* C)Z_N^*. \quad (26)$$

Repeatedly using the above identity, we get

$$\begin{aligned} 0 &= [u^*(t) - Z_U^*(Z_N^*)^{-1}x^*(t)]^T (R + D^T P^* D) [u^*(t) - Z_U^*(Z_N^*)^{-1}x^*(t)] \\ &= u^*(t)^T (R + D^T P^* D) u^*(t) - 2u^*(t)^T (R + D^T P^* D) Z_U^*(Z_N^*)^{-1}x^*(t) \\ &\quad + x^*(t)^T (Z_N^*)^{-1} (Z_U^*)^T (R + D^T P^* D) Z_U^*(Z_N^*)^{-1}x^*(t) \\ &= u^*(t)^T (R + D^T P^* D) u^*(t) + 2u^*(t)^T (B^T P^* + D^T P^* C) x^*(t) \\ &\quad + x^*(t)^T (Z_N^*)^{-1} (Z_U^*)^T (R + D^T P^* D) (R + D^T P^* D)^+ (R + D^T P^* D) Z_U^*(Z_N^*)^{-1}x^*(t) \\ &= u^*(t)^T (R + D^T P^* D) u^*(t) + 2u^*(t)^T (B^T P^* + D^T P^* C) x^*(t) \\ &\quad + x^*(t)^T (P^* B + C^T P^* D) (R + D^T P^* D)^+ (B^T P^* + D^T P^* C) x^*(t) \\ &= [u^*(t) - K^* x^*(t)]^T (R + D^T P^* D) [u^*(t) - K^* x^*(t)]. \end{aligned}$$

This proves (25). It then follows from (23) and (24) that

$$J(x_0, u^*(\cdot)) = x_0^T P^* x_0 \leq J(x_0, u(\cdot)), \quad \forall u(\cdot) \in U_{ad}^{x_0}.$$

Hence, $u^*(\cdot)$ is optimal. \square

A sufficient condition for Theorem 4.4 to hold is the *strict complementarity* of the SDP's (P) and (D), as the following proposition asserts.

Proposition 4.5 Suppose that (P) and (D) have strictly complementary optimal solutions P^* and Z^* respectively, i.e. $\mathcal{L}(P^*)Z^* = 0$ and $\mathcal{L}(P^*) + Z^* \succ 0$. Then, $Z_N^* \succ 0$.

Proof. Following Lemma 4.3 and (20) we have

$$\mathcal{L}(P^*) = (H^{-1})^T \text{diag}(R + D^T P^* D, 0) H^{-1}$$

where

$$H = \begin{bmatrix} I, & K \\ 0, & I \end{bmatrix} \text{ and } K = -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C).$$

Let $\bar{Z}^* := H^{-1} Z^* (H^{-1})^T$.

It is readily seen that $\bar{Z}_N^* = Z_N^*$. Moreover, $\text{diag}(R + D^T P^* D, 0) = H^T \mathcal{L}(P^*) H$ and $\bar{Z}^* = H^{-1} Z^* (H^{-1})^T$ are nonnegative definite and they are complementary to each other. We shall further show that they stay strictly complementary. To see this we first note that the range space of $\mathcal{L}(P^*)$, $\text{range}(\mathcal{L}(P^*))$, and the range space of Z^* , $\text{range}(Z^*)$, form an orthogonal decomposition of the whole space. Clearly, $\text{range}(H^T \mathcal{L}(P^*) H)$ has the same dimension as that of $\text{range}(\mathcal{L}(P^*))$; and $\text{range}(\bar{Z}^*)$ has the same dimension as that of $\text{range}(Z^*)$. Finally, $\text{range}(H^T \mathcal{L}(P^*) H)$ and $\text{range}(\bar{Z}^*)$ remain orthogonal to each other. Hence they span the whole space too. Therefore,

$$H^T \mathcal{L}(P^*) H + \bar{Z}^* \succ 0.$$

As we noted before, the last diagonal block of the above matrix is Z_N^* . Hence, $Z_N^* \succ 0$, as the proposition stipulates. \square

If P^* and Z^* are strictly complementary optimal solutions with

$$R + D^T P^* D \succ 0, \tag{27}$$

and $Z_N^* \succ 0$, then by (26), the feedback control in (22) coincides with the one in (12). But if $R + D^T P^* D$ is singular, then these two controls can indeed be *different*; see Example 6.1.

5 Synthesis

To summarize the results we have so far obtained, consider the following statements:

- (a) (LQ) is attainable at any $x_0 \in \mathfrak{R}^n$.

(b) (P) has an optimal solution P^* that satisfies:

- (i) the generalized stochastic Riccati equation $F(P) = 0$;
- (ii) the corresponding feedback control $u^*(t)$ of (18) is stabilizing.

(c) (P) and (D) have complementary optimal solutions P^* and Z^* , with $Z_N^* \succ 0$.

The following is a summary of our main results:

Theorem 5.1 The following implications hold:

- **(a)** \Rightarrow **(b(i))**: Theorem 4.1.
- **(b)** \Rightarrow **(a)**, with the control in (18) being optimal: Theorem 4.2.
- **(b)** \Rightarrow **(c)**: Theorem 3.3.
- **(c)** \Rightarrow **(a)**, with the control in (22) being optimal: Theorem 4.4.

Some remarks are in order. Among the above statements, **(a)** is a direct statement about the solution of the original problem (LQ); **(b)** and **(c)**, on the other hand, provide two computational approaches to (LQ) via SDP — note that they both imply **(a)** with the respective optimal feedback controls explicitly given. There are differences, however, between the two; in particular, they have different requirements, and lead to different controls. Computationally, **(c)** appears to have an edge over **(b)**, as most SDP solvers are based on primal–dual interior point methods. This implies that the iterative solutions produced by such a solver will likely to converge to the analytic centers of the primal and dual optimal sets respectively, which are known to be “maximally complementary” to each other. Therefore, if there is indeed any dual optimal solution with $Z_N^* \succ 0$, then the solver will return such a solution. In this respect, checking $Z_N^* \succ 0$ is much easier than verifying the stabilizing condition in **(b(ii))**.

Furthermore, Theorem 5.1 also reveals the relationship between **(b)** and **(c)**: **(b)** implies **(c)**, whereas **(c)** implies **(b(i))** via **(a)**. That **(c)** cannot imply the stabilizing condition in **(b(ii))** is in itself an interesting fact, which suggests that the two controls in (18) and (22) are in general intrinsically different. Even when the former is stabilizing, and hence both controls are optimal — since we then have **(b)** \Rightarrow **(c)** \Rightarrow **(a)** — they can still be different (except for the special case of (27), where **(b)** and **(c)** become equivalent); see Example 6.1.

If **(b(ii))** is satisfied, then **(b)** is reduced to **(b(i))**. Consequently, **(a)**, **(b)** and **(c)** are equivalent. Hence we have the following result.

Corollary 5.2 Suppose that the control in (18) is stabilizing, Then, **(a)**, **(b)** and **(c)** are equivalent.

An important special case is when

$$Q \succ 0, \quad R \succ 0. \quad (28)$$

In this case, (P) satisfies the Slater condition because $P = 0$ is a strictly feasible solution, and so does (D) because of the mean-square stabilizability assumption of the original LQ problem. Therefore, **(c)** holds. On the other hand, by [1], Corollary 5.1, the control in (18) must be stabilizing. Hence Corollary 5.2 stipulates that **(a)** and **(b)** must hold true as well.

Corollary 5.3 Suppose (28) holds. Then, all the three statements **(a)**, **(b)**, and **(c)** hold true.

Based on the results obtained earlier, it is possible to develop a computational procedure as follows to provide a complete treatment of the stochastic LQ control problem, and in particular to answer if the problem has an optimal feedback control representable as in (18) or in (22). The procedure only involves solving some LMIs, an SDP and its dual, for which numerical algorithms have been extensively developed (among others see [16, 32, 33]). Insofar as the complementary primal SDP solution satisfies the generalized Riccati equation (see Lemma 4.3), the procedure can also be viewed as a numerical approach to solving the Riccati equation.

Step 1. Check if the feasible set of (P) is nonempty (which is an LMI condition). If not, then stop: the LQ problem cannot be solved by either the SDP approach or by the Riccati equation; else continue.

Step 2. Check if (D) satisfies the Slater condition, which amounts to solving a system of strict LMIs. If not, then stop: the LQ problem is not mean-square stabilizable according to Proposition 3.1-(ii) and hence ill-posed; else continue.

Step 3. At this point we know that (P) is feasible and (D) satisfies the Slater condition, and hence (P) has an optimal solution (see, e.g., [24, Theorem 5]). Check if there is any optimal solution of (P) that satisfies $F(P) = 0$. If not, then stop: the LQ problem has no attainable optimal feedback control according to Theorem 4.1; else continue.

Step 4. Check if the control in (18) is stabilizing (which can be checked by LMIs according to Proposition 3.1-(iii)). If yes, then stop: the control is optimal; else continue.

Step 5. Check if (P) and (D) have complementary optimal solutions P^* and Z^* with $Z_N^* \succ 0$. If yes, then stop: the control $u^*(t) = Z_U^*(Z_N^*)^{-1}x^*(t)$ is optimal; otherwise (LQ) cannot be solved by our SDP approach, neither can it be solved by any other existing method.

Notice that in practical implementation one might as well start solving (P) and (D) by means of a primal–dual interior point code (e.g., that of using the homogeneous self-dual embedding technique; see [32]), i.e., running Step 5 first. If the result turns out to be positive, then Steps 1–4 are not necessary. On the other hand, even if the result is negative, the algorithm will still tell the feasibility of (P) and (D). This makes it easier to carry out Step 1, followed by subsequent steps.

6 Examples

The first example below demonstrates how the LQ control problem can be solved by the SDP approach developed here, even in the presence of the singularity of $R + D^T P^* D$. It also shows that optimal stabilizing controls can be obtained by both SDP approaches in (b) and (c) of Theorem 5.1, leading to different optimal controls in (18) and (22), respectively.

Example 6.1 Let $m = n = 1$; $A = C = -1$, $B = D = 1$; $Q = 1$ and $R = -1$. Namely, the problem is this:

$$\begin{aligned} \min \quad & \mathbb{E} \int_0^\infty [x(t)^2 - u(t)^2] dt \\ \text{s.t.} \quad & dx(t) = [-x(t) + u(t)]dt + [-x(t) + u(t)]dW(t) \\ & x(0) = x_0. \end{aligned}$$

This system is mean-square stabilizable, as $u(t) = \alpha x(t)$ is stabilizing for any α with $|\alpha| < 1$. To see this, applying Itô's formula to the system (1) under the above feedback control, we obtain

$$d\mathbb{E}[x(t)^2] = (\alpha^2 - 1)\mathbb{E}[x(t)^2]dt, \quad \mathbb{E}[x(0)^2] = x_0^2.$$

Hence

$$\mathbb{E}[x(t)^2] = e^{(\alpha^2 - 1)t} x_0^2, \tag{29}$$

which converges to 0 as $t \rightarrow +\infty$.

Now, the primal SDP is

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & \begin{bmatrix} -1 + p & 0 \\ 0 & 1 - p \end{bmatrix} \succeq 0. \end{aligned}$$

The above has an optimal solution $p^* = 1$ (the only feasible solution), which also satisfies the generalized Riccati equation $F(p) = 1 - p = 0$. (Note that singularity occurs in this solution). The feedback control given by (18) reduces to $u^*(t) = 0$, which is stabilizing as shown above ($\alpha = 0 < 1$). Therefore, all the tests in Steps 1–4 of the computational procedure presented in

§5 are passed. Consequently, $u^*(t) = 0$ is *one* optimal control of the LQ problem. Moreover, the corresponding objective value is

$$\mathbf{E} \int_0^\infty [x^*(t)]^2 dt = x_0^2 \int_0^\infty e^{-t} dt = x_0^2, \quad (30)$$

where the first equality is due to (29) with $\alpha = 0$.

Next, we can obtain additional — in fact, infinitely many more — optimal controls by virtue of (c). Indeed, the dual SDP in this case is:

$$\begin{aligned} \min \quad & -z_b + z_n \\ \text{s.t.} \quad & 1 + z_b - z_n = 0, \\ & z := \begin{bmatrix} z_b & z_u \\ z_u & z_n \end{bmatrix} \succeq 0. \end{aligned}$$

It can be directly verified that the above has multiple optimal solutions:

$$(z_b, z_u, z_n) = (z_b, z_u, 1 + z_b),$$

parameterized by (z_u, z_b) with

$$z_b \geq 0, \quad z_u^2 \leq z_b(1 + z_b). \quad (31)$$

In particular, note that the above ensures $z_n = 1 + z_b > 0$. Furthermore, these (parameterized) solutions are all complementary to the primal optimal solution $p^* = 0$. Hence, the test in Step 5 of the numerical procedure is passed, which gives rise to (multiple) optimal controls

$$u^*(t) = z_u z_n^{-1} x^*(t) \equiv \alpha x^*(t).$$

Notice that the feedback gain α satisfies

$$|\alpha| = \frac{z_u}{1 + z_b} \leq \frac{\sqrt{z_b(1 + z_b)}}{1 + z_b} = \sqrt{\frac{z_b}{1 + z_b}} < 1$$

where the first inequality follows from (31). Therefore, these controls are indeed stabilizing by (29). Finally, the optimal cost corresponding to these controls is:

$$\begin{aligned} J &= \mathbf{E} \int_0^\infty (1 - \alpha^2) [x(t)]^2 dt \\ &= x_0^2 (1 - \alpha^2) \int_0^\infty e^{(\alpha^2 - 1)t} dt \\ &= x_0^2, \end{aligned}$$

which coincides with (30).

The next example illustrates two points: First, when the primal SDP solution satisfies the Riccati equation (i.e., **(b(i))** holds), and moreover (27) holds, the resulting feedback control may still not be stabilizing (i.e., **(b(ii))** fails). When this does happen, the complementarity condition in (c) fails. Second, there indeed exist well-posed stochastic control problems which, however, do not have any attainable optimal control. This calls for approximation methods, which will be presented in the next section.

Example 6.2 Suppose $m = n = 1$; $A = C = 0$, $B = D = 1$; $Q = 4$ and $R = -1$. The control system is as follows:

$$\begin{aligned} \min \quad & \mathbf{E} \int_0^\infty [4x(t)^2 - u(t)^2] dt \\ \text{s.t.} \quad & dx(t) = u(t)dt + u(t)dW(t) \\ & x(0) = x_0. \end{aligned}$$

Consider a feedback control $u(t) = -kx(t)$. Applying Itô's lemma yields

$$d[x(t)]^2 = (k^2 - 2k)[x(t)]^2 - 2k[x(t)]^2 dW(t).$$

Clearly, such a feedback control is stabilizing if and only if $k^2 - 2k < 0$, or, $0 < k < 2$. In particular, this implies that $u(t) = -x(t)$ is stabilizing while $u(t) = -2x(t)$ is not.

For any $0 < k < 2$, it follows that $\mathbf{E} \int_0^\infty [x(t)]^2 dt = x_0^2 / (2k - k^2)$. Therefore, the control $u(t) = -kx(t)$ has a cost

$$(4 - k^2) \mathbf{E} \int_0^\infty [x(t)]^2 dt = (1 + \frac{2}{k}) x_0^2.$$

As $k \uparrow 2$ we see that the cost can be arbitrarily close to $2x_0^2$. Nevertheless, this optimum is not attainable when $x_0 \neq 0$.

In terms of the corresponding SDPs, the primal reads

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & \begin{bmatrix} -1 + p & p \\ p & 4 \end{bmatrix} \succeq 0. \end{aligned}$$

This problem has only one feasible solution $p^* = 2$, which is necessarily the optimal solution. It clearly satisfies the Riccati equation:

$$4 + \frac{p^2}{1 - p} = 0.$$

The control in (18) is hence:

$$u^*(t) = -\frac{p^*}{-1 + p^*} x^*(t) = -2x^*(t),$$

which is *not* stabilizing as we discussed before.

Since **(b(ii))** does not hold, we expect **(c)** to fail as well, as **(b)** and **(c)** are equivalent under (27). So, let us now examine the dual:

$$\begin{aligned} \min \quad & 4z_n - z_b \\ \text{s.t.} \quad & 1 + 2z_u + z_b = 0, \\ & z := \begin{bmatrix} z_b & z_u \\ z_u & z_n \end{bmatrix} \succeq 0. \end{aligned}$$

This problem is strictly feasible, since the original LQ problem is stabilizable. For instance, $(z_b, z_u, z_n) = (1, -1, 2)$ is a strictly feasible solution. Hence ([38, Theorem 3.1]), the infimum of

the dual objective value must coincide with the supremum of the primal objective value, which is 2. This means, should the dual optimal solution z^* exist, it must satisfy $4z_n^* - z_b^* = 2$, or $z_n^* = (2 + z_b^*)/4$. This, along with $z_u^* = -(1 + z_b^*)/2$, leads to

$$z_b^* z_n^* = z_b^* \cdot \frac{2 + z_b^*}{4} < \frac{(1 + z_b^*)^2}{4} = (z_u^*)^2,$$

which violates $z \succeq 0$. Consequently, the dual does not have an attainable optimal solution, and the complementary duality fails.

The third example below illustrates a situation opposite to Example 6.2: **(b(i))** fails while **(b(ii))** holds. Namely, when $R + D^T P^* D = 0$, the optimal primal solution P^* may *not* satisfy the generalized Riccati equation $F(P) = 0$ (and *vice versa*), even when the corresponding control is stabilizing. In this case, Theorem 4.1 shows that the LQ problem has no attainable optimal control.

Example 6.3 Let $m = n = 1$; $A = -1$, $B = 1$, $C = D = 0$; $Q = 1$ and $R = 0$. This is actually a deterministic system that is mean-square stabilizable, as $u(t) = 0$ is stabilizing. The corresponding SDP reads:

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & \begin{bmatrix} 0, & p \\ p, & 1 - 2p \end{bmatrix} \succeq 0. \end{aligned}$$

The above has a unique feasible solution $p^* = 0$, which is hence optimal too. However, the generalized Riccati equation in this case is $F(p) = 1 - 2p = 0$, which has a unique solution $p = \frac{1}{2}$. Therefore, the two solutions are completely different.

Moreover, notice that while **(b(i))** fails in this case, **(b(ii))** does hold: $u^*(t) = 0$ is indeed stabilizing. On the other hand, in view of Theorem 5.1, **(c)** \Rightarrow **(a)** \Rightarrow **(b(i))**, we expect **(c)** to fail. Indeed, the dual SDP is

$$\begin{aligned} \min \quad & z_n \\ \text{s.t.} \quad & 1 + 2z_u - 2z_n = 0, \\ & z := \begin{bmatrix} z_b, & z_u \\ z_u, & z_n \end{bmatrix} \succeq 0. \end{aligned}$$

This dual problem is strictly feasible, and it has an infimum equal to 0, which is the supremum of the primal. However, the dual optimal solution is not attainable, because whenever $z_n = 0$ we must have $z_u = 1/2$, and hence it is impossible to have $z \succeq 0$.

7 ϵ -Approximation

Examples 6.2 and 6.3 have illustrated that the LQ control problem could be well-posed, but still there exists no attainable optimal control. When this happens we propose to consider (LQ_ϵ) ,

obtained by keeping all the data A, B, C and D in (LQ) unchanged, and letting $R_\epsilon = R + \epsilon I$ and $Q_\epsilon = Q + \epsilon I$ with $\epsilon > 0$; such a perturbation was already considered in the proof of Theorem 4.1. Recall that the associated SDPs for (LQ $_\epsilon$) are

$$(P_\epsilon) \quad \max \quad \langle I, P \rangle$$

$$\text{s.t.} \quad \begin{bmatrix} R + \epsilon I + D^T P D, & B^T P + D^T P C \\ P B + C^T P D, & Q + \epsilon I + C^T P C + A^T P + P A \end{bmatrix} \succeq 0$$

$$P \in \mathcal{S}^{n \times n}$$

and

$$(D_\epsilon) \quad \min \quad \langle R + \epsilon I, Z_B \rangle + \langle Q + \epsilon I, Z_N \rangle$$

$$\text{s.t.} \quad I + Z_U^T B^T + B Z_U + Z_N A^T + A Z_N$$

$$+ C Z_N C^T + D Z_U C^T + C Z_U^T D^T + D Z_B D^T = 0$$

$$\begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix} \succeq 0.$$

Assuming that the LQ is stabilizable and (P) is feasible, both (P $_\epsilon$) and (D $_\epsilon$) satisfy the Slater condition.

Theorem 7.1 Suppose (LQ) is well-posed. Let $J_\epsilon^*(x_0)$ and $J^*(x_0)$ be the optimal values of (LQ $_\epsilon$) and (LQ), respectively. Then,

$$\lim_{\epsilon \downarrow 0} J_\epsilon^*(x_0) = J^*(x_0).$$

Proof. Let the optimal solution of (P $_\epsilon$) be P_ϵ^* . In the proof of Theorem 4.1, we proved that

$$u^\epsilon(t) = -(R_\epsilon + D^T P_\epsilon^* D)^+ (B^T P_\epsilon^* + D^T P_\epsilon^* C) x^\epsilon(t) \quad (32)$$

is optimal for (LQ $_\epsilon$), with the corresponding optimal objective value equal to $J_\epsilon^*(x_0) = x_0^T P_\epsilon^* x_0$.

Following the same argument as in the proof of Theorem 4.1, we know that P_ϵ^* is contained in a compact set, with $0 < \epsilon \leq \epsilon_0$. Moreover, since by definition $J_\epsilon^*(x_0)$ decreases monotonically as $\epsilon \downarrow 0$, so does P_ϵ^* . Therefore, P_ϵ^* itself also converges as $\epsilon \downarrow 0$.

What remains is to show that $x_0^T P_0^* x_0$ is equal to the true infimum of (LQ), now denoted as $J^*(x_0)$. To this end, first note that

$$x_0^T P_\epsilon^* x_0 = J_\epsilon^*(x_0) \geq J^*(x_0),$$

where the inequality is due to the positive perturbation in (P $_\epsilon$). Letting $\epsilon \rightarrow 0$, we obtain

$$x_0^T P_0^* x_0 \geq J^*(x_0).$$

On the other hand, since P_0^* is feasible to (P) (see the proof of Theorem 4.1), it follows from (23) that

$$J^*(x_0) \equiv \inf_{u(\cdot) \in U_{ad}^{x_0}} J(x_0, u(\cdot)) \geq x_0^T P_0^* x_0.$$

This completes the proof. \square

The above theorem says that the objective value achieved by the perturbed problem is asymptotically optimal. The next result is concerned with the asymptotic optimality of the feedback control.

Theorem 7.2 The feedback control $u^\epsilon(\cdot)$ constructed by (32) is asymptotically optimal for (LQ), namely,

$$\lim_{\epsilon \downarrow 0} J(x_0, u^\epsilon(\cdot)) = J^*(x_0).$$

Proof. Denote by $J_\epsilon(x_0, u(\cdot))$ the cost of the perturbed problem (LQ $_\epsilon$) under an admissible control $u(\cdot) \in U_{ad}^{x_0}$ w.r.t. the initial state x_0 . Then for any $\eta > 0$, there is an ϵ_0 such that when $0 < \epsilon < \epsilon_0$:

$$\begin{aligned} J^*(x_0) &\leq J(x_0, u^\epsilon(\cdot)) \\ &\leq J_\epsilon(x_0, u^\epsilon(\cdot)) \\ &= J_\epsilon^*(x_0) \\ &\leq J^*(x_0) + \eta, \end{aligned}$$

where the last inequality is due to Theorem 7.1. This proves our claim. \square

Consider Example 6.2. With perturbation, the corresponding primal SDP becomes

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & \begin{bmatrix} -1 + \epsilon + p, & p \\ p, & 4 + \epsilon \end{bmatrix} \succeq 0. \end{aligned}$$

Solving this problem yields

$$p_\epsilon^* = \frac{4 + \epsilon + \sqrt{(4 + \epsilon)^2 + 4(4 + \epsilon)(-1 + \epsilon)}}{2}.$$

Clearly, $p_\epsilon^* = 2 + O(\sqrt{\epsilon})$, and hence the optimal value of (P_ϵ) , $p_\epsilon^* x_0^2$, converges to $2x_0^2$ as $\epsilon \downarrow 0$.

8 Conclusions

We have developed a systematic approach to the stochastic LQ control problem based on primal–dual SDP, allowing indefinite cost matrices. We have shown that, in addition to its obvious computational advantage, the SDP duality theory provides critical qualitative information about the LQ control problem, in particular, regarding issues such as stability and optimality.

Among the three statements presented in §5, the strongest is **(b)**, which consists of two parts: (i) the optimal solution to the primal SDP satisfies the generalized Riccati equation, and

(ii) the corresponding feedback control is stabilizing. It implies **(c)**: the SDP complementary duality, which, in turn, implies **(a)**: the existence of an optimal control to the LQ problem. Both **(b)** and **(c)** hence provide useful computational approaches to solving the LQ problem.

Conversely, our results also provide new insight as to when the LQ problem does *not* possess an optimal solution: Since, **(b(i))** is implied by **(a)**, if no primal SDP solution satisfies the generalized Riccati equation, then the LQ problem has no attainable optimal control. (For such problems we have developed an ϵ -approximation scheme that yields asymptotic optimal solutions.) However, as the SDP in general possesses multiple optimal solutions, and most SDP solvers usually return a single optimal solution, this result is of more theoretical, as opposed to computational, interest. This limitation is reflected in Step 3 of the procedure outlined in §5: it requires checking if there is *any* optimal solution of (P) satisfying $F(P) = 0$.

Finally, the gap alluded to in the last step of the same procedure points to an open problem: whether our SDP approach might fail to find an optimal control (a counter-example), or this is simply impossible (a proof).

References

- [1] M. Ait Rami and X. Y. Zhou, *Linear matrix inequalities, Riccati equations, and indefinite stochastic linear quadratic controls*, IEEE Trans. Aut. Contr., AC-45 (2000) 1131-1143.
- [2] A. Albert, *Conditions for positive and nonnegative definiteness in terms of pseudo-inverses*, SIAM J. Appl. Math., 17 (1969) 434-440.
- [3] F. Alizadeh, *Interior point methods in semidefinite programming with applications to combinatorial optimization problems*, SIAM J. Optim., 5 (1995) 13-51.
- [4] B.D.O. Anderson and J.B. Moore, *Optimal Control - Linear Quadratic Methods*, Prentice-Hall, New Jersey, 1989.
- [5] M. Athens, *Special issues on linear-quadratic-Gaussian problem*, IEEE Trans. Auto. Contr., AC-16 (1971) 527-869.
- [6] R. Bellman, I. Glicksberg and O. Gross, *Some Aspects of the Mathematical Theory of Control Processes*, Rand Co., Santa Monica-California, 1958.
- [7] A. Bensoussan, *Lecture on stochastic control, part I*, Lecture Notes in Math., 972 (1983) 1-39.
- [8] S. Boyd, L. El Ghaoui, E. Feron and V. Balakrishnan, *Linear Matrix Inequality in Systems and Control Theory*, SIAM, 1994.
- [9] S. Chen, X. Li and X.Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Contr. Optim., 36 (1998) 1685-1702.
- [10] S. Chen and J. Yong, *Stochastic linear quadratic optimal control problems*, Appl. Math. Optim., 43 (2001) 21-45.

- [11] S. Chen and X.Y. Zhou, *Stochastic linear quadratic regulators with indefinite control weight costs. II*, SIAM J. Contr. Optim., 39 (2000) 1065-1081.
- [12] D.J. Clements and B.D.O. Anderson, *Singular Optimal Control: The Linear-Quadratic Problem*, Springer-Verlag, Berlin Heidelberg, 1978.
- [13] D.J. Clements, B.D.O. Anderson and P.J. Moylan, *Matrix inequality solution to linear-quadratic singular control problems*, IEEE Trans. Auto. Contr., AC-22 (1977) 55-57.
- [14] M.H.A. Davis, *Linear Estimation and Stochastic Control*, Chapman and Hall, London, 1977.
- [15] L. El Ghaoui, R. Nikoukhah and F. Delebecque, *LMITOOL: A front-end for LMI optimization, user's guide*, 1995. Available via anonymous ftp to [ftp.ensta.fr](ftp://ftp.ensta.fr/pub/elghaoui/lmitool), under [/pub/elghaoui/lmitool](ftp://ftp.ensta.fr/pub/elghaoui/lmitool).
- [16] K. Fujisawa, M. Fukuda, M. Kojima and K. Nakata, *Numerical evaluation of SDPA*, in "High Performance Optimization", J.B.G. Frenk *et al.* (eds.), Kluwer, Boston/Dordrecht/London, 1999, pp. 267-301.
- [17] M.L.J. Hautus and L.M. Silverman, *System structure and singular control*, Lin. Algebra Appl., 50 (1983) 369-402.
- [18] Y. Hu and X.Y. Zhou, *Indefinite stochastic Riccati equations*, SIAM J. Contr. Optim., 42 (2003) 123-137.
- [19] R. E. Kalman, *Contributions to the theory of optimal control*, Bol. Soc. Math. Mexicana, 5 (1960) 102-119.
- [20] M. Kohlmann and X. Y. Zhou, *Relationship between backward stochastic differential equations and stochastic controls: An LQ approach*, SIAM J. Contr. Optim., 38 (2000) 1392-1407.
- [21] A.M. Letov, *Analytic design of regulators*, Avtomat. i Telemekh., 4 (1960) 436-446; 5 (1960) 561-571; 6 (1960) 661-669 (in Russian). English transl. in Automat. Remote Contr., 21 (1960).
- [22] X. Li, X.Y. Zhou and A.E.B. Lim, *Dynamic mean-variance portfolio selection with no-shorting constraints*, SIAM J. Contr. Optim., 40 (2001) 1540-1555.
- [23] A.E.B. Lim and X. Zhou, *Mean-variance portfolio selection with random parameters*, Math. Oper. Res., 27 (2002) 101-120.
- [24] Z.Q. Luo, J.F. Sturm and S. Zhang, *Duality results for conic convex programming*, Report 9719/A, Econometric Institute, Erasmus University Rotterdam, 1997.
- [25] Z.Q. Luo, J.F. Sturm and S. Zhang, *Conic convex programming and self-dual embedding*, Optim. Methods and Software, 14 (2000) 169-218.
- [26] B.P. Molinari, *The time-invariant linear-quadratic optimal control problem*, Automatica, 13 (1977) 347-357.
- [27] Yu. Nesterov and A. Nemirovski, *Interior Point Polynomial Methods in Convex Programming*, SIAM, Philadelphia, 1994.
- [28] R. Penrose, *A generalized inverse of matrices*, Proc. Cambridge Philos. Soc., 52 (1955) 17-19.

- [29] J.M. Schumacher, *The role of the dissipation matrix in singular optimal control*, Syst. Contr. Lett., 2 (1983) 262-266.
- [30] R.E. Skelton, T. Iwasaki and K. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, London, 1998.
- [31] J.F. Sturm, *Primal-Dual Interior Point Approach to Semidefinite Programming*, Ph.D thesis, Tinbergen Institute Series 156, Erasmus University Rotterdam, 1997.
- [32] J.F. Sturm, *Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones*, Optimization Methods and Software, 11-12 (1999) 625-653.
- [33] K.C. Toh, M.J. Todd and R.H. Tütüncü, *SDPT3 — A MATLAB software package for semidefinite programming*, Optim. Methods and Software, 11-12 (1999) 545-581.
- [34] J.C. Willems, *Least square stationary optimal control and the algebraic Riccati equation*, IEEE Trans. Aut. Contr., AC-16 (1971) 621-634.
- [35] J.C. Willems, A. Kitapci and L.M. Silverman, *Singular optimal control: a geometric approach*, SIAM J. Contr. Optim., 24 (1986) 323-337.
- [36] W. M. Wonham, *On a matrix Riccati equation of stochastic control*, SIAM J. Contr., 6 (1968) 312-326.
- [37] H. Wu and X.Y. Zhou, *Stochastic frequency characteristics*, SIAM J. Contr. Optim., 40 (2001) 557-576.
- [38] L. Vandenberghe and S. Boyd, *Semidefinite Programming*, SIAM Review 38 (1996) 49-95.
- [39] V.A. Yakubovich, *The solution of certain linear matrix inequalities in automatic control theory*, Soviet Math. Dokl., 5 (1964) 620-623.
- [40] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer, New York, 1999.
- [41] X. Y. Zhou and D. Li, *Continuous-time mean-variance portfolio selection: A stochastic LQ framework*, Appl. Math. Optim., 42 (2000) 19-33.