Time-Inconsistent Stochastic Linear–Quadratic Control: Characterization and Uniqueness of Equilibrium

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Abstract

In this paper, we continue our study on a general time-inconsistent stochastic linear–quadratic (LQ) control problem originally formulated in [6]. We derive a necessary and sufficient condition for equilibrium controls via a flow of forward–backward stochastic differential equations. When the state is one dimensional and the coefficients in the problem are all deterministic, we prove that the explicit equilibrium control constructed in [6] is indeed unique. Our proof is based on the derived equivalent condition for equilibria as well as a stochastic version of the Lebesgue differentiation theorem. Finally, we show that the equilibrium strategy is unique for a mean–variance portfolio selection model in a complete financial market where the

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risk-free rate is a deterministic function of time but all the other market parameters are possibly stochastic processes.

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1 Introduction

Time inconsistency in dynamic decision making is often observed in social systems and daily life. The study on time inconsistency by economists dates back to Strotz [9] in the 1950s, who proposed the formulation of a time-inconsistent decision problem as a game between incarnations of the controller at different time instants.

The game formulation is fairly straightforward and easy to understand when the time setting is discrete. In a continuous-time setup, the formulation can be generalized in different ways. Yong [11] and Ekeland and Pirvu [3] define equilibrium controls in the class of feedback policies for problems involving hyperbolic discounting, and prove the existence of equilibria. Grenadier and Wang [4] investigate optimal stopping with, again, hyperbolic discounting. Björk and Murgoci [1] formulate a general Markovian stochastic control problem with time inconsistent terms, and establish sufficient conditions for equilibria through a generalized HJB equation system. They then present some special cases including a linear–quadratic (LQ) control problem in which solutions are constructed. Björk, Murgoci and Zhou [2] further derive analytically an equilibrium strategy for a mean–variance portfolio selection model with state-dependent risk aversion.

In our previous paper, [6], we formulate a general non-Markovian stochastic LQ control problem, where the objective functional includes terms leading to time-inconsistency, and derive a general *sufficient* condition for equilibria through a system of forward–backward stochastic differential equations (FBSDEs). Based on this condition, we construct explicitly an equilibrium control when the state is scalar-valued and all the coefficients are non-random. In contrast to the aforementioned works where an equilibrium control is defined within the class of feedback controls, we define our equilibrium via open-loop controls.

Most of the existing literature on game formulation of time-inconsistent problems has focused on existence of equilibria, and the only paper according to our best knowledge that mentions about the uniqueness is Vieille and Weibull [10], in which the authors show that the uniqueness does *not* hold in a discrete-time model. Uniqueness is important in both practice and theory. In applications, multiple equilibria lead to multiple value processes,[1] and there is an issue of the choice of the one to use and implement. Theoretically speaking, when a new, *weak* notion of a solution is introduced the uniqueness is always important, for it is one of the touchstones of the appropriateness of the new definition (non-uniqueness is a sign that the notion may be *too weak* to be useful). On the other hand, mathematically, proving uniqueness of a weaker notion is almost always challenging.[2]

In this paper, we take on the challenge of establishing the uniqueness of equilibrium control for the same time-inconsistent model formulated in [6]. First, we derive a general *necessary and sufficient* condition for equilibrium controls. A key step in the derivation is to prove a stochastic version of the Lebesgue differentiation theorem which is interesting in its own right and potentially useful for other stochastic control problems. Then, we focus on the case in which the state is one dimensional and the coefficients in the problem are all deterministic. Thanks to the derived equivalent condition for equilibria we show that the explicit equilibrium control constructed in [6] is indeed unique. Finally, we prove that the equilibrium strategy, again constructed in [6], is unique for a mean–variance portfolio selection model in a complete financial market where the risk-free rate is a deterministic function of time but all the other market parameters are possibly stochastic processes.

The rest of this paper is organized as follows. In Section 2, we recall the formulation of the time-inconsistent LQ control problem studied in our previous work [6]. We then derive an equivalent characterization of equilibrium controls in terms of the solution to a system of FBSDEs in Section 3. In Section 4 we prove that the equilibrium obtained in [6] is the unique one. Section 5 is devoted to the uniqueness for a mean–variance portfolio selection model. Finally, Section 6 concludes. Some technical derivations are placed in appendices.

[1] The value of an equilibrium control is the corresponding objective functional value.

[2] A good example is the uniqueness of viscosity solution for a nonlinear PDE; see [12].

2 Problem Formulation

Let $(W_t)_{0\le t\le T} = (W_t^1, \cdots, W_t^d)_{0\le t\le T}$ be a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(\mathcal{F}_t)$ the augmented filtration generated by (W_t).

We will use the same notation as in our previous paper [6], which we list here for the reader's convenience:

$$
\begin{aligned}
\mathbb{S}^l &: \quad \text{the set of symmetric } l\times l \text{ real matrices.}\\
L^2_{\mathcal{G}}(\Omega;\,\mathbb{R}^l) &: \quad \text{the set of random variables } \xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l))\\
&\quad\quad \text{with } \mathbb{E}\left[|\xi|^2\right] < +\infty.\\
L^\infty_{\mathcal{G}}(\Omega;\,\mathbb{R}^l) &: \quad \text{the set of essentially bounded random variables}\\
&\quad\quad \xi : (\Omega, \mathcal{G}) \to (\mathbb{R}^l, \mathcal{B}(\mathbb{R}^l)).\\
L^2_{\mathcal{G}}(t,\,T;\,\mathbb{R}^l) &: \quad \text{the set of } \{\mathcal{G}_s\}_{s\in[t,T]}\text{-adapted processes}\\
&\quad\quad f = \{f_s : t \le s \le T\} \text{ with } \mathbb{E}\left[\int_t^T |f_s|^2\, ds\right] < \infty.\\
L^\infty_{\mathcal{G}}(t,\,T;\,\mathbb{R}^l) &: \quad \text{the set of essentially bounded } \{\mathcal{G}_s\}_{s\in[t,T]}\text{-adapted processes.}\\
L^2_{\mathcal{G}}(\Omega;\,C(t,\,T;\,\mathbb{R}^l)) &: \quad \text{the set of continuous } \{\mathcal{G}_t\}_{s\in[t,T]}\text{-adapted processes}\\
&\quad\quad f = \{f_s : t \le s \le T\} \text{ with } \mathbb{E}\left[\sup_{s\in[t,T]} |f_s|^2\right] < \infty.
\end{aligned}
$$

We will often use vectors and matrices in this paper, where all vectors are column vectors. For a matrix M, define

$$
\begin{aligned}
M' &: \quad \text{transpose of a matrix } M.\\
|M| = \sqrt{\textstyle\sum_{i,j} m_{ij}^2} &: \quad \text{Frobenius norm of a matrix } M.
\end{aligned}
$$

The time-inconsistent LQ control model under consideration in this paper was introduced in [6]. Here we recall the formulation.

Let $T > 0$ be given and fixed. The controlled system is governed by the following stochastic differential equation (SDE) on $[0,T]$:

$$dX_s = [A_s X_s + B_s' u_s + b_s]ds + \sum_{j=1}^{d}[C_s^j X_s + D_s^j u_s + \sigma_s^j]dW_s^j; \quad X_0 = x_0, \tag{2.1}$$

where A is a bounded deterministic function on $[0,T]$ with value in $\mathbb{R}^{n\times n}$, B, C^j, D^j are all essentially bounded adapted processes on $[0,T]$ with values in $\mathbb{R}^{l\times n}$, $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times l}$, respectively, and b and σ^j are stochastic processes in $L^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$. The process $u \in L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^l)$ is the control, and $X \in L^2_{\mathcal{F}}(\Omega;\, C(0,\, T;\, \mathbb{R}^n))$ is the corresponding state process with initial value $x_0 \in \mathbb{R}^n$.

When time evolves to $t \in [0,T]$, we need to consider the controlled system starting from t and state $x_t \in L^2_{\mathcal{F}_t}(\Omega;\, \mathbb{R}^n)$:

$$dX_s = [A_s X_s + B_s' u_s + b_s]ds + \sum_{j=1}^{d}[C_s^j X_s + D_s^j u_s + \sigma_s^j]dW_s^j, \quad X_t = x_t. \tag{2.2}$$

For any control $u \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^l)$, there exists a unique solution $X^{t,x_t,u} \in L^2_{\mathcal{F}}(\Omega;\, C(t,T;\mathbb{R}^n))$. At t with the system state $X_t = x_t$, our aim is to minimize

$$\begin{aligned} J(t,x_t;u) \;&\triangleq\; \frac{1}{2}\mathbb{E}_t \int_t^T \left[\langle Q_s X_s, X_s\rangle + \langle R_s u_s, u_s\rangle\right] ds + \frac{1}{2}\mathbb{E}_t[\langle G X_T, X_T\rangle] \\ &\quad -\frac{1}{2}\langle h\mathbb{E}_t\left[X_T\right], \mathbb{E}_t\left[X_T\right]\rangle - \langle \mu_1 x_t + \mu_2, \mathbb{E}_t\left[X_T\right]\rangle \end{aligned} \tag{2.3}$$

over $u \in L^2_{\mathcal{F}}(t,\, T;\, \mathbb{R}^l)$, where $X = X^{t,x_t,u}$, and $\mathbb{E}_t\left[\cdot\right] = \mathbb{E}\left[\cdot|\mathcal{F}_t\right]$. In the above Q and R are both positive semi-definite and essentially bounded adapted processes on $[0,T]$ with values in $\mathbb{S}^n$ and $\mathbb{S}^l$ respectively, G, h, μ_1, μ_2 are constants in $\mathbb{S}^n$, $\mathbb{S}^n$, $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$ respectively, and moreover G is positive semi-definite.

We define an *equilibrium (control)* in the following manner. Given a control u^*, for any $t \in [0,T)$, $\varepsilon > 0$ and $v \in L^2_{\mathcal{F}_t}(\Omega;\, \mathbb{R}^l)$, define

$$u_s^{t,\varepsilon,v} = u_s^* + v\mathbf{1}_{s\in[t,t+\varepsilon)}, \quad s \in [t,T]. \tag{2.4}$$

Definition 2.1 *Let $u^* \in L^2_{\mathcal{F}}(0,\, T;\, \mathbb{R}^l)$ be a given control and X^* be the state process corresponding to u^*. The control u^* is called an equilibrium if*

$$\liminf_{\varepsilon\downarrow 0} \frac{J(t,X_t^*;u^{t,\varepsilon,v}) - J(t,X_t^*;u^*)}{\varepsilon} \geq 0,$$

where $u^{t,\varepsilon,v}$ is defined by (2.4), for any $t \in [0,T)$ and $v \in L^2_{\mathcal{F}_t}(\Omega;\, \mathbb{R}^l)$.

Notice that here we have changed lim in [6] to lim inf in this definition, resulting in a weaker definition. As a result, the sufficient condition derived in [6] is also sufficient for this new definition. On the other hand, the uniqueness result to be established for the new definition will also imply the uniqueness for the old one. For these reasons, the above definition appears to be more appropriate.

3 Necessary and Sufficient Condition of Equilibrium Controls

In our previous paper, [6], a sufficient condition is derived via the second-order expansion in the local spike variation, in the same spirit of proving the stochastic Pontryagin's maximum principle [7, 8, 12]. In this section, we present a general necessary and sufficient condition for equilibria. This condition is made possible by a stochastic Lebesgue differentiation theorem involving conditional expectation. The latter theorem, interesting in its own right, is new according to our best knowledge.

To proceed, we start with some relevant known result from our previous paper [6]. Let u^* be a fixed control and X^* be the corresponding state process. For any $t \in [0,T)$, define in the time interval $[t,T]$ the processes $(p(\cdot;t),(k^j(\cdot;t))_{j=1,\cdots,d}) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^n)\times(L^2_{\mathcal{F}}(t,T;\mathbb{R}^n))^d$ as the unique solution to

$$
\left\{
\begin{aligned}
&dp(s;t) = -[A_s'p(s;t) + \textstyle\sum_{j=1}^d (C_s^j)'k^j(s;t) + Q_sX_s^*]ds \\
&\qquad\quad + \textstyle\sum_{j=1}^d k^j(s;t)dW_s^j, \quad s\in[t,T], \\
&p(T;t) = GX_T^* - h\mathbb{E}_t\left[X_T^*\right] - \mu_1 X_t^* - \mu_2.
\end{aligned}
\right. \tag{3.1}
$$

Furthermore, define $(P(\cdot;t),(K^j(\cdot;t))_{j=1,\cdots,d}) \in L^2_{\mathcal{F}}(t,T;\mathbb{S}^n)\times(L^2_{\mathcal{F}}(t,T;\mathbb{S}^n))^d$ as the unique solution to

$$
\left\{
\begin{array}{rl}
dP(s;t) = & -\Big\{A_s'P(s;t)+P(s;t)A_s \\
& +\sum_{j=1}^d[(C_s^j)'P(s;t)C_s^j+(C_s^j)'K^j(s;t)+K^j(s;t)C_s^j]+Q_s\Big\}ds \\
& +\sum_{j=1}^d K^j(s;t)dW_s^j, \quad s\in[t,T], \\
P(T;t) = & G.
\end{array}
\right.
\tag{3.2}
$$

The following estimate under local spike variation is reproduced from [6, Proposition 3.1].

Proposition 3.1 *For any $t\in[0,T)$, $\varepsilon>0$ and $v\in L^2_{\mathcal{F}_t}(\Omega;\mathbb{R}^l)$, define $u^{t,\varepsilon,v}$ by (2.4). Then*

$$
J(t,X_t^*;u^{t,\varepsilon,v})-J(t,X_t^*;u^*)=\mathbb{E}_t\int_t^{t+\varepsilon}\left(\langle\Lambda(s;t),v\rangle+\frac{1}{2}\langle H(s;t)v,v\rangle\right)ds+o(\varepsilon)
\tag{3.3}
$$

where $\Lambda(s;t)\triangleq B_sp(s;t)+\sum_{j=1}^d(D_s^j)'k^j(s;t)+R_su_s^*$ *and* $H(s;t)\triangleq R_s+\sum_{j=1}^d(D_s^j)'P(s;t)D_s^j$.

In view of Proposition 3.1 and the fact that $H(s;t)\succeq 0$, it is straightforward to get the following characterization of an equilibrium.

Corollary 3.2 *A control $u^*\in L^2(0,T,\mathbb{R}^l)$ is an equilibrium if and only if*

$$
\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[\Lambda(s;t)\right]ds=0,\quad a.s.,\quad \forall t\in[0,T).
\tag{3.4}
$$

The next result provides a key property for the solution to the BSDE (3.1), and represents the process $\Lambda(s;t)$ in a special form.

Proposition 3.3 *For any given pair of state and control processes (X^*,u^*), the solution to (3.1) satisfies $k(s;t_1)=k(s;t_2)$ for a.e. $s\geq\max(t_1,t_2)$. Moreover, there exist $\lambda_1\in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l),\lambda_2\in L^\infty_{\mathcal{F}}(0,T;\mathbb{R}^{l\times n})$ and $\xi\in L^2(\Omega;C(0,T;\mathbb{R}^n))$, such that $\Lambda(s;t)$ has the representation*

$$
\Lambda(s;t)=\lambda_1(s)+\lambda_2(s)\xi_t.
$$

Proof: Define the function $\psi(\cdot)$ as the unique solution for the matrix-valued ordinary differential equation (ODE)

$$d\psi(t) = \psi(t)A(t)'dt, \quad \psi(T) = I_n,$$

where I_n denotes the $n \times n$ identity matrix. It is clear that $\psi(\cdot)$ is invertible, and both $\psi(\cdot)$ and $\psi(\cdot)^{-1}$ are bounded.

Let $\hat{p}(s;t) = \psi(s)p(s;t) + h\mathbb{E}_t[X_T^*] + \mu_1 X_t^* + \mu_2$ and $\hat{k}^j(s;t) = \psi(s)k^j(s;t)$ for $j = 1, \cdots, d$. Then on the interval $[t,T]$, $(\hat{p}(\cdot;t), \hat{k}(\cdot;t))$ satisfies

(3.5)
$$\begin{cases} d\hat{p}(s;t) = -\left[\sum_{j=1}^d \psi(s)(C_s^j)'\psi(s)^{-1}\hat{k}^j(s;t) + \psi(s)Q_s X_s^*\right] ds + \sum_{j=1}^d \hat{k}^j(s;t)dW_s^j, \\ \hat{p}(T;t) = GX_T^*. \end{cases}$$

Notice that neither the terminal condition nor the coefficients of this equation depend on t; so it can be taken as a BSDE on the entire time interval $[0,T]$. Denote its solution as $(\hat{p}(s), \hat{k}(s))$, $s \in [0,T]$. It then follows from the uniqueness of the solution to BSDE that $(\hat{p}(s;t), \hat{k}(s;t)) = (\hat{p}(s), \hat{k}(s))$ at $s \in [t,T]$ for any $t \in [0,T]$. As a result, $k(s;t) = \psi(s)^{-1}\hat{k}(s) := k(s)$, proving the first claim of the proposition.

Next,

$$p(s;t) = \psi(s)^{-1}\hat{p}(s) - \psi(s)^{-1}(h\mathbb{E}_t[X_T^*] + \mu_1 X_t^* + \mu_2) = p(s) + \psi(s)^{-1}\xi_t,$$

where $\xi_t := -h\mathbb{E}_t[X_T^*] - \mu_1 X_t^* - \mu_2$ defines the process $\xi \in L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R}^n))$ and $p(s) := \psi(s)^{-1}\hat{p}(s)$ defines the process $p \in L^2_{\mathcal{F}}(\Omega; C(0,T;\mathbb{R}^n))$. Consequently,

$$\begin{aligned} \Lambda(s;t) &= B_s p(s;t) + \sum_{j=1}^d (D_s^j)'k^j(s;t) + R_s u_s^* \\ &= B_s p(s) + \sum_{j=1}^d (D_s^j)'k^j(s) + R_s u_s^* + B_s\psi(s)^{-1}\xi_t \\ &= \lambda_1(s) + \lambda_2(s)\xi_t, \end{aligned}$$

where $\lambda_1(s) := B_s p(s) + \sum_{j=1}^d (D_s^j)'k^j(s) + R_s u_s^*$ and $\lambda_2(s) := B_s\psi(s)^{-1}$. *Q.E.D.*

We now set out to derive our general necessary and sufficient condition for equilibrium controls. Although (3.4) already provides a characterizing condition, it is nevertheless not very useful because it involves a limit. It is tempting to expect that the limit therein is $\Lambda(t;t)$, in the spirit of the Lebesgue differentiation theorem.[3] However, one needs to be very careful since in (3.4) the conditional expectation with respect to $\mathcal{F}_t$ is involved. The following general result can be regarded as a *stochastic* Lebesgue differentiation theorem. While it serves our purpose in this paper, it is of interest in its own right and may be potentially useful for (among others) various stochastic control problems.

Lemma 3.4 *Let $Y \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$ be a given process. If $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds = 0,\ a.e. t \in [0,T), a.s.,$ then $Y_t = 0,\ a.e. t \in [0,T), a.s.$.*

Proof: Since $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)$ is a separable space, it follows from the (deterministic) Lebesgue differentiation theorem that there is a countable dense subset $\mathcal{D} \subset L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)\cap L^\infty_{\mathcal{F}_T}(\Omega;\mathbb{R}^l)$, such that for almost all t, we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\langle Y_s,\eta\rangle\right]ds = \mathbb{E}\left[\langle Y_t,\eta\rangle\right], \quad \forall \eta \in \mathcal{D}, \tag{3.6}$$

and $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[Y_s^2\right]ds = \mathbb{E}\left[Y_t^2\right]$.

For any $\eta \in \mathcal{D}$, define $\eta_s = \mathbb{E}_s[\eta]$. Then $\mathbb{E}\left[\langle Y_s,\eta\rangle\right] = \mathbb{E}\left[\langle Y_s,\eta_s\rangle\right]$. We have the following estimates:

$$\begin{aligned}
\left|\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\langle Y_s,\eta_s-\eta_t\rangle\right]ds\right| &\le \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\sqrt{\int_t^{t+\varepsilon}\mathbb{E}\left[Y_s^2\right]ds\int_t^{t+\varepsilon}\mathbb{E}\left[(\eta_s-\eta_t)^2\right]ds} \\
&= \lim_{\varepsilon\downarrow 0}\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[Y_s^2\right]ds}\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[(\eta_s-\eta_t)^2\right]ds} \\
&\le \lim_{\varepsilon\downarrow 0}\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[Y_s^2\right]ds}\sqrt{\sup_{s\in[t,t+\varepsilon]}\mathbb{E}\left[(\eta_s-\eta_t)^2\right]} \\
&\le 2\lim_{\varepsilon\downarrow 0}\sqrt{\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[Y_s^2\right]ds}\sqrt{\mathbb{E}\left[(\eta_{t+\varepsilon}-\eta_t)^2\right]} = 0,
\end{aligned}$$

where the last inequality is due to Doob's martingale inequality as η_s is a square-integrable

[3] A simple version of this theorem states that if φ is an integrable real function on $[0,T]$, then $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\varphi(s)ds = \varphi(t)$ a.e. $t \in [0,T]$.

martingale. Hence for any $\eta \in \mathcal{D}$,

$$\begin{aligned}
\mathbb{E}\left[\langle Y_t, \eta_t\rangle\right] &= \mathbb{E}\left[\langle Y_t, \eta\rangle\right] \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\langle Y_s, \eta\rangle\right]ds \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\langle Y_s, \eta_s\rangle\right]ds \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\langle Y_s, \eta_t\rangle\right]ds \\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[\langle \mathbb{E}_t\left[Y_s\right], \eta_t\rangle\right]ds \\
&= \lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\langle\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds, \eta_t\rangle\right].
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds\right)^2\right] &\leq \mathbb{E}\left[\int_t^{t+\varepsilon}\frac{1}{\varepsilon^2}ds\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]^2ds\right] \\
&= \frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]^2ds\right] \\
&\leq \frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[Y_s^2\right]ds,
\end{aligned}$$

and $\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}\left[Y_s^2\right]ds = \mathbb{E}\left[Y_t^2\right]$, there exists a constant $\delta_t > 0$, such that

$$\mathbb{E}\left[\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds\right)^2\right] < 2\mathbb{E}\left[Y_t^2\right], \quad \forall\,\varepsilon \in (0, \delta_t).$$

This implies that $\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds$ is uniformly integrable in $\varepsilon \in (0, \delta_t)$. Hence

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds\right|\right] = \mathbb{E}\left[\lim_{\varepsilon\downarrow 0}\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds\right|\right] = 0.$$

Since η is essentially bounded, so is η_t; hence there exists a constant $c > 0$ such that

$$\begin{aligned}
\left|\mathbb{E}\left[\langle\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds, \eta_t\rangle\right]\right| &\leq c\mathbb{E}\left[\left|\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds\right|\right] \\
&\to 0,
\end{aligned}$$

implying

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\left[\langle\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\mathbb{E}_t\left[Y_s\right]ds, \eta_t\rangle\right] = 0.$$

Thus $\mathbb{E}\left[\langle Y_t, \eta\rangle\right] = 0,\ a.e.t \in [0, T]$ for any $\eta \in \mathcal{D}$, which implies

$$Y_t = 0,\ a.e.t \in [0, T],\ a.s..$$

Q.E.D.

We are now in the position to present the main result of this section.

Theorem 3.5 *Given a control $u^* \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^l)$, let X^* be the corresponding state process and $(p(\cdot;t),k(\cdot;t)) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^n) \times (L^2_{\mathcal{F}}(t,T;\mathbb{R}^n))^d$ be the unique solution to the BSDE (3.1). Then u^* is an equilibrium control if and only if*

$$\Lambda(t;t) = 0, \; a.s., \; a.e. \; t \in [0,T]. \tag{3.7}$$

Proof: Recall that we have the representation $\Lambda(s;t) = \lambda_1(s) + \lambda_2(s)\xi_t$. Since λ_2 is essentially bounded and ξ is continuous, we have

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbb{E}_t \left[\frac{1}{\varepsilon} \int_t^{t+\varepsilon} |\lambda_2(s)(\xi_s - \xi_t)| ds \right] &\leq c \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t \left[|\xi_s - \xi_t| \right] ds \\ &= 0, \end{aligned}$$

where the last equality is because $\mathbb{E}_t[|\xi_s - \xi_t|]$ is a continuous function of s and vanishes at $s = t$.

It then follows

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;t)] ds = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;s)] ds.$$

Now, if (3.7) holds, then

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;t)] ds = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;s)] ds = 0.$$

Conversely, if (3.4) holds, then $\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \mathbb{E}_t[\Lambda(s;s)] ds = 0$, leading to (3.7) by virtue of Lemma 3.4. *Q.E.D.*

4 Uniqueness When State is One-dimensional and Coefficients Are Deterministic

In our previous paper [6], when the state variable is scalar-valued, i.e., $n = 1$, and all the coefficients are deterministic, an explicit equilibrium is constructed essentially based on the equivalent condition (3.7) (although we were not yet able to prove it there). In this section, we will prove that in the same setting the equilibrium is actually unique, thanks again to (3.7).

Throughout this section we assume that $n = 1$ and all the parameters $A, B, b, C^j, D^j, \sigma^j, Q$ and R are deterministic function of t. In this case the controlled system reduces to

$$dX_s = [A_s X_s + B_s' u_s + b_s]ds + [C_s X_s + D_s u_s + \sigma_s]' dW_s; \quad X_0 = x_0, \tag{4.1}$$

where

$$C := (C^1, \cdots, C^d)', \ D := ((D^1)', \cdots, (D^d)')', \ \sigma := (\sigma^1, \cdots, \sigma^d)'.$$

Accordingly, the BSDE (3.1) is simplified to (also noting that $k(s;t) \equiv k(s)$)

$$\begin{cases} dp(s;t) = -[A_s p(s;t) + C_s' k(s) + Q_s X_s^*]ds + k(s)' dW_s, \quad s \in [t, T], \\ p(T;t) = G X_T^* - h \mathbb{E}_t[X_T^*] - \mu_1 X_t^* - \mu_2, \end{cases} \tag{4.2}$$

whereas the corresponding $\Lambda(s;t)$ is now in the form

$$\Lambda(s;t) = B_s p(s;t) + D_s' k(s) + R_s u_s^*.$$

In [6], an equilibrium control was constructed through the solution of the following system of ODEs (where we suppress subscripts s for notational simplicity):

$$
(4.3) \quad \begin{cases} 0 = \dot{M} + (2A + |C|^2)M + Q \\ \quad -M(B' + C'D)(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C], \ s \in [0, T], \\ M_T = G; \end{cases}
$$

$$
(4.4) \quad \begin{cases} 0 = \dot{N} + 2AN \\ \quad -NB'(R + MD'D)^{-1}[(M - N - \Gamma^{(1)})B + MD'C], \ s \in [0, T], \\ N_T = h; \end{cases}
$$

$$
(4.5) \quad \begin{cases} \dot{\Gamma}^{(1)} = -A\Gamma^{(1)}, \quad s \in [0, T], \\ \Gamma^{(1)}_T = \mu_1; \end{cases}
$$

$$
(4.6) \quad \begin{cases} 0 = \dot{\Phi} + \{A - [(M - N)B' + MC'D](R + MD'D)^{-1}B\}\Phi + (M - N)b \\ \quad +C'M\sigma - [(M - N)B' + MC'D](R + MD'D)^{-1}MD'\sigma, \ s \in [0, T], \\ \Phi_T = -\mu_2. \end{cases}
$$

If this system of equations admits a solution $(M, N, \Gamma^{(1)}, \Phi)$, then the feedback control law

$$
(4.7) \qquad u^*_s = \alpha_s X^*_s + \beta_s
$$

defines an equilibrium, where

$$
(4.8) \quad \begin{aligned} \alpha_s &\triangleq -(R_s + M_s D'_s D_s)^{-1}[(M_s - N_s - \Gamma^{(1)}_s)B_s + M_s D'_s C_s], \\ \beta_s &\triangleq -(R_s + M_s D'_s D_s)^{-1}(\Phi_s B_s + M_s D'_s \sigma_s); \end{aligned}
$$

see [6, Theorem 4.4]. Moreover, the existence of solution to (4.3)–(4.6) is studied in [6].

The next theorem provides that the control constructed above is the *only* equilibrium.

Theorem 4.1 *If (4.3)–(4.6) admits a solution $(M, N, \Gamma^{(1)}, \Phi)$, then there is a unique equilibrium control.*

Proof: Suppose there is another equilibrium state–control pair (X, u). Then, with a slight abuse of notation, equation (3.1), with X^* replaced by X, admits a unique solution $(p(\cdot;t), k(\cdot))$ satisfying $\Lambda(s;s) \equiv B_s p(s;s) + D'_s k(s) + R_s u_s = 0$ for a.e. $s \in [0,T]$.

Define

$$\bar{p}(s;t) := p(s;t) - (M_s X_s - N_s \mathbb{E}_t[X_s] - \Gamma_s^{(1)} X_t + \Phi_s),$$

$$\bar{k}(s) := k(s) - M_s(C_s X_s + D_s u_s + \sigma_s).$$

The equilibrium condition for (X, u) yields

$$B_s\left[\bar{p}(s;s) + (M_s - N_s - \Gamma_s^{(1)})X_s + \Phi_s\right] + D'_s\left[\bar{k}(s) + M_s(C_s X_s + D_s u_s + \sigma_s)\right] + R_s u_s = 0.$$

Since $R_s + D'_s M_s D_s$ is invertible, we solve for u_s in the above equation to obtain the following expression

$$\begin{aligned} u_s = \; & -(R_s + D'_s M_s D_s)^{-1}\left[B_s \bar{p}(s;s) + D'_s \bar{k}(s)\right. \\ & \left. +(B_s(M_s - N_s - \Gamma_s^{(1)}) + D'_s M_s C_s)X_s + B_s\Phi_s + D'_s M_s \sigma_s\right]. \end{aligned} \tag{4.9}$$

On the other hand, we can show that $(\bar{p}(\cdot;t), \bar{k}(\cdot))$ satisfies the following BSDE (details are relegated to Appendix A):

(4.10)

$$\begin{cases} d\bar{p}(s;t) = & -\big(A\bar{p}(s;t) + C'\bar{k}(s) - [C'MD + MB'][R + D'MD]^{-1}[B\bar{p}(s;s) + D'\bar{k}(s)] \\ & +NB'[R + D'MD]^{-1}\mathbb{E}_t\left[B\bar{p}(s;s) + D'\bar{k}(s)\right]\big)\, ds + \bar{k}(s)' dW_s, \quad s \in [t,T], \\ \bar{p}(T;t) = 0, \end{cases}$$

where we suppress the subscript s for A, B, C, D, M, N, R, and we have used the equations for $M, N, \Gamma^{(1)}, \Phi$. Moreover, it is easy to prove that $\mathbb{E}\left[\int_0^T |\bar{k}(s)|^2 ds\right] < +\infty$ and $\sup_{t\in[0,T]} \mathbb{E}\left[\sup_{s\geq t} |\bar{p}(s;t)|^2\right] < +\infty$.

We will prove in the next theorem that equation (4.10) admits at most one solution in the space $\mathcal{L}_1 \times \mathcal{L}_2$, where

$$\mathcal{L}_1 := \left\{ X(\cdot;\cdot) : X(\cdot;t) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}),\ \sup_{t\in[0,T]} \mathbb{E}\left[\sup_{s\geq t} |X(s;t)|^2\right] < +\infty \right\},$$

and

$$\mathcal{L}_2 := \left\{ Y(\cdot;\cdot) : Y(\cdot;t) \in L^2_{\mathcal{F}}(t,T;\mathbb{R}^d),\ \sup_{t\in[0,T]} \mathbb{E}\left[\int_t^T |Y(s;t)|^2 ds\right] < +\infty \right\}.$$

Hence $\bar{p}(s;t) \equiv 0$ and $\bar{k}(s) \equiv 0$.

Finally, plugging $\bar{p} \equiv \bar{k} \equiv 0$ into (4.9), we find that u has exactly the same form of feedback control as that of u^*; see (4.7). This proves that u and u^* lead to an identical control. *Q.E.D.*

It remains to prove the uniqueness of solution for (4.10). Indeed we will do it for a more general equation

$$\text{(4.11)} \quad \begin{cases} d\bar{p}(s;t) = & -f\left(s, \bar{p}(s;t), \bar{p}(s;s), \mathbb{E}_t\left[l_1(s)\bar{p}(s;s)\right], \bar{k}(s;t), \mathbb{E}_t\left[l_2(s)\bar{k}(s;t)\right]\right) ds \\ & \qquad +\bar{k}(s;t)' dW_s, \quad s \in [t,T], \\ \bar{p}(T;t) = & 0, \end{cases}$$

where l_1 and l_2 are two essentially bounded, adapted vector processes with suitable dimensions, and $f(s,\cdots\cdot)$ is a deterministic function satisfying uniform Lipschitz condition in all variables except s.

Theorem 4.2 *Equation (4.11) admits at most one solution $(\bar{p}, \bar{k})$ in the space $\mathcal{L}_1 \times \mathcal{L}_2$.*

Proof: Suppose there are two solutions $(\bar{p}^{(1)}, \bar{k}^{(1)})$ and $(\bar{p}^{(2)}, \bar{k}^{(2)})$ in the space $\mathcal{L}_1 \times \mathcal{L}_2$. Define $\bar{p}(s;t) \stackrel{\triangle}{=} \bar{p}^{(1)}(s;t) - \bar{p}^{(2)}(s;t), \bar{k}(s;t) \stackrel{\triangle}{=} \bar{k}^{(1)}(s;t) - \bar{k}^{(2)}(s;t)$ and

$$\begin{aligned} \Delta f(s;t) \ \stackrel{\triangle}{=}\ & f(s, \bar{p}^{(1)}(s;t), \bar{p}^{(1)}(s;s), \mathbb{E}_t\left[l_1(s)\bar{p}^{(1)}(s;s)\right], \bar{k}^{(1)}(s;t), \mathbb{E}_t\left[l_2(s)\bar{k}^{(1)}(s;t)\right]) \\ & -f(s, \bar{p}^{(2)}(s;t), \bar{p}^{(2)}(s;s), \mathbb{E}_t\left[l_1(s)\bar{p}^{(2)}(s;s)\right], \bar{k}^{(2)}(s;t), \mathbb{E}_t\left[l_2(s)\bar{k}^{(2)}(s;t)\right]). \end{aligned}$$

Then $|\Delta f(s;t)| \leq c_1\left(|\bar{p}(u;t)| + |\bar{k}(u;t)| + |\bar{p}(u;u)| + \mathbb{E}_t\left[|\bar{p}(u;u)|\right] + \mathbb{E}_t\left[|\bar{k}(u;t)|\right]\right)$ for some constant c_1, and

$$d\bar{p}(s;t) = \Delta f(s;t)dt + \bar{k}(s;t)' dW_s, \quad \bar{p}(T;t) = 0.$$

For any $t \in [0,T]$, $s \in [t,T]$, by Itô's formula, we have

$$|\bar{p}(s;t)|^2 + \int_s^T |\bar{k}(u;t)|^2 du = 2\int_s^T \bar{p}(u;t)\Delta f(u;t)du - 2\int_s^T \bar{p}(u;t)\bar{k}(u;t)' dW_u.$$

Thus

$$\begin{aligned}
& \mathbb{E}\left[|\bar{p}(s;t)|^2\right] + \mathbb{E}\left[\int_s^T |\bar{k}(u;t)|^2 du\right] \\
\leq \; & c_1 \mathbb{E}\left[\int_s^T |\bar{p}(u;t)| \left(|\bar{p}(u;t)| + |\bar{k}(u;t)| + |\bar{p}(u;u)| + \mathbb{E}_t\left[|\bar{p}(u;u)|\right] + \mathbb{E}_t\left[|\bar{k}(u;t)|\right]\right) du\right] \\
\leq \; & c_2 \mathbb{E}\left[\int_s^T \left(|\bar{p}(u;t)|^2 + |\bar{p}(u;u)|^2\right) du\right] + \frac{1}{2}\mathbb{E}\left[\int_s^T |\bar{k}(u;t)|^2 du\right],
\end{aligned}$$

where we have used the inequality $cxy \leq c^2x^2 + \frac{1}{4}y^2$ for any nonnegative c, x, y. Consequently, there exists $c_3 > 0$ such that

$$\mathbb{E}\left[|\bar{p}(s;t)|^2\right] + \mathbb{E}\left[\int_s^T |\bar{k}(u;t)|^2 du\right] \leq c_3 \mathbb{E}\left[\int_s^T \left(|\bar{p}(u;t)|^2 + |\bar{p}(u;u)|^2\right) du\right]. \tag{4.12}$$

Furthermore, for any $s \in [t,T]$, we have

$$\begin{aligned}
\mathbb{E}\left[|\bar{p}(s;t)|^2 + \int_s^T |\bar{k}(u;t)|^2 du\right] \; & \leq \; c_3(T-t)\left[\sup_{u\in[t,T]} \mathbb{E}\left[|\bar{p}(u;t)|^2\right] + \sup_{u\in[t,T]} \mathbb{E}\left[|\bar{p}(u;u)|^2\right]\right] \\
& \leq \; 2c_3(T-t) \sup_{t\leq u\leq s\leq T} \mathbb{E}\left[|\bar{p}(s;u)|^2\right].
\end{aligned}$$

Hence

$$\sup_{t\leq u\leq s\leq T} \mathbb{E}\left[|\bar{p}(s;u)|^2\right] \leq 2c_3(T-t) \sup_{t\leq u\leq s\leq T} \mathbb{E}\left[|\bar{p}(s;u)|^2\right]. \tag{4.13}$$

Now take $\delta \in (0, 1/(4c_3))$. Then for any $t \in [T-\delta, T]$, we have

$$\sup_{t\leq u\leq s\leq T} \mathbb{E}\left[|\bar{p}(s;u)|^2\right] \;\; \leq \;\; \frac{1}{2} \sup_{t\leq u\leq s\leq T} \mathbb{E}\left[|\bar{p}(s;u)|^2\right],$$

which implies $\sup_{t\leq u\leq s\leq T} \mathbb{E}\left[|\bar{p}(s;u)|^2\right] = 0$. It follows that $\bar{p}(s;u) = 0, a.s.$ almost everywhere in $\{(s,u) : t \leq u \leq s \leq T\}$.

For $t \in [T-2\delta, T-\delta]$ and $s \in [T-\delta, T]$, since $\bar{p}(u,u) = 0$ for any $u \in [s,T]$, we have by (4.12) that

$$\mathbb{E}\left[|\bar{p}(s;t)|^2\right] + \mathbb{E}\left[\int_s^T |\bar{k}(u;t)|^2 du\right] \leq c_3 \mathbb{E}\left[\int_s^T |\bar{p}(u;t)|^2 du\right]. \tag{4.14}$$

Grownwall's inequality then leads to $\bar{p}(s;t) = \bar{k}(s;t) = 0$.

For $t \in [T-2\delta, T-\delta]$ and $s \in [t, T-\delta]$, noting $\bar{p}(T-\delta;t) = 0$, we can apply the previous argument for the region $t \in [T-\delta, T]$ and $s \in [t,T]$ to deduce that $\bar{p}(s;t) = \bar{k}(s;t) = 0$.

We can then repeat the same analysis in a backward manner to $t \in [T-3\delta, T-2\delta]$ and so on until we reach time $t = 0$. *Q.E.D.*

5 Uniqueness of Mean-Variance Equilibrium Strategies in A Complete Market with Random Parameters

Following [6], as an application of the time-inconsistent LQ theory, we study the continuous-time Markowitz mean–variance portfolio selection model in a complete market with random model coefficients. We aim to establish the uniqueness of the equilibrium strategy. The model is mathematically a special case of the general LQ problem formulated earlier in this paper, with $n = 1$ naturally. However, since some coefficients are allowed to be random, the uniqueness result of the previous section is not applicable here.

We use the same setup of [6]. The wealth equation is governed by the SDE

$$\begin{cases} dX_s = r_s X_s ds + \theta_s' u_s ds + u_s' dW_s, \qquad s \in [t, T], \\ X_t = x_t, \end{cases} \tag{5.1}$$

where r is the (bounded) deterministic interest rate function, and θ is the essentially bounded stochastic risk premium process.

The objective at time t with state $X_t = x_t$ is to minimize

$$\begin{aligned} J(t, x_t; u) &\triangleq \frac{1}{2}\mathrm{Var}_t(X_T) - (\mu_1 x_t + \mu_2)\mathbb{E}_t[X_T] \\ &= \frac{1}{2}\left(\mathbb{E}_t[X_T^2] - (\mathbb{E}_t[X_T])^2\right) - (\mu_1 x_t + \mu_2)\mathbb{E}_t[X_T] \end{aligned} \tag{5.2}$$

with $\mu_1 \geq 0$. As noted in [6], there are two sources of time-inconsistency in this model, one from the variance term and the other from the state-dependent tradeoff between the mean and the variance.

In [6, Section 5], we constructed an equilibrium through the solutions (M, U), $(\Gamma^{(1)}, \gamma^{(1)})$,

$(\Gamma^{(2)}, \gamma^{(2)})$, and $(\Gamma^{(3)}, \gamma^{(3)})$ to BSDEs:

$$
\left\{
\begin{array}{rcl}
dM_s &=& -[2r_s M_s + (\theta_s M_s + U_s)'\alpha_s]ds + U_s' dW_s, \quad M_T = 1, \\[2ex]
d\Gamma_s^{(1)} &=& -r_s \Gamma_s^{(1)} ds + (\gamma_s^{(1)})' dW_s, \quad \Gamma_T^{(1)} = \mu_1, \\[2ex]
d\Gamma_s^{(2)} &=& -[r_s \Gamma_s^{(2)} + (\theta_s M_s + U_s)'\beta_s]ds + (\gamma_s^{(2)})' dW_s, \quad \Gamma_T^{(2)} = -\mu_2, \\[2ex]
d\Gamma_s^{(3)} &=& -[r_s \Gamma_s^{(3)} + (\theta_s M_s + U_s)'\beta_s]ds + (\gamma_s^{(3)})' dW_s, \quad \Gamma_T^{(3)} = 0,
\end{array}
\right.
\tag{5.3}
$$

where

$$
\begin{aligned}
\alpha_s &\triangleq -M_s^{-1}\left(-\theta_s \Gamma_s^{(1)} + U_s - \gamma_s^{(1)}\right), \\
\beta_s &\triangleq -M_s^{-1}\left[\theta_s (\Gamma_s^{(2)} - \Gamma_s^{(3)}) + \gamma_s^{(2)}\right].
\end{aligned}
\tag{5.4}
$$

In this case, the BSDE (3.1) for $p(\cdot;t)$ corresponding to a given strategy (control) u^* with the wealth (state) process X^* specializes to

$$
\left\{
\begin{array}{l}
dp(s;t) = -r_s p(s;t) ds + k(s)' dW_s, \\[2ex]
p(T;t) = X_T^* - \mathbb{E}_t[X_T^*] - \mu_1 X_t^* - \mu_2,
\end{array}
\right.
\tag{5.5}
$$

and the corresponding $\Lambda(s;t)$ is

$$
\Lambda(s;t) = p(s;t)\theta_s + k(s).
$$

It is proved in [6, Proposition 5.1] that the system of BSDEs (5.3) admits a unique solution with both M and M^{-1} being *bounded*, and $U \cdot W$ a BMO martingale. Furthermore, the feedback strategy

$$
u_s^* = \alpha_s X_s^* + \beta_s
\tag{5.6}
$$

defines a control in the space $L^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$, which is an equilibrium strategy for the mean–variance investment problem.

We now claim that the equilibrium above is unique.

For any $q > 1$, define

$$
\mathcal{L}_3(q) := \left\{ X(\cdot;\cdot) : X(\cdot;t) \in L^q_{\mathcal{F}}(\Omega; C(t,T;\mathbb{R})) \ \ \forall\, t \in [0,T] \right\},
$$

and

$$\mathcal{L}_4(q) := \left\{ Y(\cdot) : Y \text{ is adapted and } \mathbb{E}\left[\left(\int_t^T |Y(s)|^2 ds \right)^{q/2} \right] < +\infty \right\}.$$

Theorem 5.1 *There is a unique equilibrium strategy for the mean–variance problem (5.1)–(5.2), which is identical to the one generated from the feedback law (5.6).*

Proof: Suppose there is another equilibrium wealth–strategy pair (X, u). Then equation (5.5), with X^* replaced by X, admits a unique solution $(p(\cdot; t), k(\cdot))$ satisfying $\Lambda(s; s) \equiv p(s; s)\theta_s + k(s) = 0$ for a.e. $s \in [0, T]$.

It is proved in [6] that M, M^{-1}, $\Gamma^{(1)}$, $\Gamma^{(2)}$ and $\Gamma^{(3)}$ are all bounded, and $\gamma^{(2)} \cdot W$ and $U \cdot W$ are both BMO martingales. In particular, since $U \cdot W$ is a BMO martingale, it follows from the John–Nirenberg inequality (see Kazamaki [5, Theorem 2.2, p.29]) that there exists $\varepsilon > 0$ such that $\mathbb{E}\left[e^{\varepsilon \int_0^T |U_s|^2 ds} \right] < +\infty$. Thus $\mathbb{E}\left[\left(\int_0^T |U_s|^2 ds \right)^q \right] < +\infty$ for any $q > 0$.

Define

$$\begin{aligned}
\bar{p}(s; t) &:= p(s; t) - \left[M_s X_s + \Gamma_s^{(2)} - \mathbb{E}\left(M_s X_s + \Gamma_s^{(3)} \right) - \Gamma_s^{(1)} X_t \right], \\
\bar{k}(s) &= k(s) - \left(M_s u_s + U_s X_s + \gamma_s^{(2)} \right).
\end{aligned}$$

It is easy to check that $\bar{p} \in \mathcal{L}_3(2)$. On the other hand, $k \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d), Mu + \gamma^{(2)} \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^d)$, and for any $q \in (1, 2)$,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T |U_s X_s|^2 ds \right)^{q/2} \right] &\le \mathbb{E}\left[\sup_{s \in [0,T]} |X_s|^q \left(\int_0^T |U_s|^2 ds \right)^{q/2} \right] \\
&\le \left(\mathbb{E}\left[\sup_{s \in [0,T]} |X_s|^2 \right] \right)^{q/2} \left(\mathbb{E}\left[\left(\int_0^T |U_s|^2 ds \right)^{q/(2-q)} \right] \right)^{1-q/2} \\
&< +\infty.
\end{aligned}$$

These, together with the fact that $L^2_{\mathcal{F}}(0, T; \mathbb{R}^d) \subset \mathcal{L}_4(q)\ \forall q \in (1, 2)$, imply $\bar{k} \in \mathcal{L}_4(q)$ for $q \in (1, 2)$.

Furthermore, the equivalent condition gives

$$\bar{p}(s; s)\theta_s + \bar{k}(s) + \theta_s[\Gamma_s^{(2)} - \Gamma_s^{(3)} - \Gamma_s^{(1)} X_s] + [M_s u_s + U_s X_s + \gamma_s^{(2)}] = 0.$$

Solving for u_s we obtain

$$
(5.7) \qquad \begin{aligned} u_s = & -M_s^{-1}\left[(U_s-\theta_s\Gamma_s^{(1)})X_s+\theta_s\bar{p}(s;s)+\bar{k}(s)+\theta_s(\Gamma_s^{(2)}-\Gamma_s^{(3)})+\gamma_s^{(2)}\right] \\ = & \ \alpha_s X_s+\beta_s-M_s^{-1}[\theta_s\bar{p}(s;s)+\bar{k}(s)]. \end{aligned}
$$

Next, we can derive the following BSDE that $(\bar{p}(\cdot;t),\bar{k}(\cdot))$ satisfies (details are placed in Appendix B)

$$
(5.8) \qquad \begin{cases} d\bar{p}(s;t) = & -\left\{r_s\bar{p}(s;t)-(\theta_s+U_sM_s^{-1})'[\theta_s\bar{p}(s;s)+\bar{k}(s)]\right. \\ & \left. +\mathbb{E}_t\left[(\theta_s+U_sM_s^{-1})'[\theta_s\bar{p}(s;s)+\bar{k}(s)]\right]\right\}ds+\bar{k}(s)'dW_s,\ s\in[t,T], \\ \bar{p}(T;t) = & \ 0. \end{cases}
$$

We will prove in the next theorem that this equation admits at most one solution $(\bar{p},\bar{k})$ in the space $\mathcal{L}_3(q)\times\mathcal{L}_4(q)$ for some $q\in(1,2)$, leading to $\bar{p}\equiv 0$ and $\bar{k}\equiv 0$. Consequently, we have $u_s=\alpha_sX_s+\beta_s$. In other words, u_s has exactly the same feedback form as u_s^*. This establishes the uniqueness. *Q.E.D.*

Theorem 5.2 *For any $q\in(1,2)$, equation (5.8) admits at most one solution $(\bar{p},\bar{k})\in\mathcal{L}_3(q)\times\mathcal{L}_4(q)$.*

Proof: Fix t. Taking $\mathbb{E}_t[\cdot]$ on both sides of the integral form of (5.8) and noticing that $\int_s^t \bar{k}\cdot W$ is a martingale, we get

$$
\mathbb{E}_t[\bar{p}(s;t)] = \int_s^T r_\nu \mathbb{E}_t[\bar{p}(\nu;t)]\,d\nu,
$$

which implies $\mathbb{E}_t[\bar{p}(s;t)]=0$ for any $s\geq t$. In particular, taking $s=t$, we have $\bar{p}(t;t)=0$. Hence equation (5.8) reduces to

$$
(5.9) \qquad \begin{cases} d\bar{p}(s;t) = -\left\{r_s\bar{p}(s;t)-(\theta_s+U_sM_s^{-1})'\bar{k}(s)+\mathbb{E}_t\left[(\theta_s+U_sM_s^{-1})'\bar{k}(s)\right]\right\}ds+\bar{k}(s)'dW_s, \\ \bar{p}(T;t)=0. \end{cases}
$$

As r is deterministic and bounded, we can discount $\bar{p}(s;t)$ by $e^{-\int_s^T r_v dv}$ to remove the term $-r_s\bar{p}(s;t)$ on the right hand side of the above equation; thus henceforth we assume

$r \equiv 0$ without loss of generality. Define $\tilde{p}(s;t) := \bar{p}(s;t) - \int_s^T \mathbb{E}_t\left[(\theta_v + U_v M_v^{-1})'\bar{k}(v)\right] dv$. Then $\tilde{p}(T;t) = 0$ and

$$d\tilde{p}(s;t) = (\theta_s + U_s M_s^{-1})'\bar{k}(s)ds + \bar{k}(s)'dW_s.$$

For any $\bar{q} \in (1,q)$, denote $\hat{q} = q/\bar{q}$, and $1/\hat{p} + 1/\hat{q} = 1$. Then

$$\begin{aligned}
& \mathbb{E}\left[\sup_{s\in[t,T]}\left|\int_s^T \mathbb{E}_t\left[\left(\theta_\nu + U_\nu M_\nu^{-1}\right)'\bar{k}(\nu)\right]d\nu\right|^{\bar{q}}\right] \\
\leq\ & \mathbb{E}\left[\left(\int_t^T \left|\left(\theta_\nu + U_\nu M_\nu^{-1}\right)'\bar{k}(\nu)\right| d\nu\right)^{\bar{q}}\right] \\
\leq\ & c_0\mathbb{E}\left[\left(\int_t^T |\theta_\nu'\bar{k}(\nu)|d\nu\right)^{\bar{q}}\right] + c_0\mathbb{E}\left[\left(\int_t^T M_\nu^{-1}|U_\nu'\bar{k}(\nu)|d\nu\right)^{\bar{q}}\right] \\
\leq\ & c_1\mathbb{E}\left[\left(\int_t^T |\bar{k}(\nu)|^2 d\nu\right)^{\bar{q}/2}\right] + c_2\mathbb{E}\left[\left(\int_t^T |U_\nu|^2 d\nu\right)^{\bar{q}/2}\left(\int_t^T |\bar{k}(\nu)|^2 d\nu\right)^{\bar{q}/2}\right] \\
\leq\ & c_3 + c_2\left(\mathbb{E}\left[\left(\int_t^T |U_\nu|^2 d\nu\right)^{\bar{q}\hat{p}/2}\right]\right)^{1/\hat{p}}\left(\mathbb{E}\left[\left(\int_t^T |\bar{k}(\nu)|^2 d\nu\right)^{q/2}\right]\right)^{1/\hat{q}} \\
<\ & +\infty,
\end{aligned}$$

where c_0, c_1, c_2 and c_3 are proper constants. On the other hand, it is assumed that $\bar{p} \in \mathcal{L}_3(q)$. So it follows that $\mathbb{E}\left[\sup_{s\in[t,T]}|\tilde{p}(s;t)|^{\bar{q}}\right] < +\infty$.

Define $\xi = \mathcal{E}(-(\theta_s + U_s M_s^{-1})\cdot W)_T \equiv e^{-\frac{1}{2}\int_0^T |\theta_s + U_s M_s^{-1}|^2 ds - \int_0^T (\theta_s + U_s M_s^{-1})' dW_s}$. Since $UM^{-1}\cdot W$ is a BMO martingale, $\mathbb{E}[\xi] = 1$; so it can be used to define a new probability measure $\mathbb{Q}$ by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \xi$, under which $\hat{W}_s = W_s + \int_0^s (\theta_v + U_v M_v^{-1})dv$ is a standard Brownian motion. Furthermore,

$$d\tilde{p}(s;t) = \bar{k}(s)'d\hat{W}_s, \quad \tilde{p}(T;t) = 0.$$

Applying Itô's formula, we obtain

$$\begin{aligned}
dM_s^{-1} &= -M_s^{-2}dM_s + M_s^{-3}U_s^2 ds \\
&= M_s^{-1}\left\{\left[\theta\left(\frac{\Gamma_s^{(1)}}{M_s} - 1\right)\frac{U_s}{M_s} + \frac{\Gamma_s^{(1)}|\theta_s|^2}{M}\right]ds - \frac{U_s'}{M_s}dW_s\right\}.
\end{aligned}$$

Hence

$$M_T^{-1} = M_0^{-1}\exp\left(-\int_0^T\left[\frac{U_s'\theta_s}{M_s} - \Gamma_s^{(1)}\frac{|\theta_s|^2}{M_s} + \frac{1}{2}\frac{|U_s|^2}{M_s^2} - \Gamma_s^{(1)}\frac{U_s'\theta_s}{M_s^2}\right]ds - \int_0^T \frac{U_s'}{M_s}dW_s\right).$$

Comparing ξ and M_T^{-1}, we deduce

$$\xi M_T = M_0 \exp\left(-\int_0^T \Gamma_s^{(1)}|\theta_s|^2 \frac{1}{M_s} ds\right) \exp\left(-\int_0^T \Gamma_s^{(1)} \frac{\theta_s'}{M_s}\frac{U_s}{M_s} ds\right) \exp\left(-\frac{1}{2}\int_0^T |\theta_s|^2 ds - \int_0^T \theta_s dW_s\right).$$

It is clear that $M_0 e^{-\int_0^T \Gamma_s^{(1)}|\theta_s|^2 \frac{1}{M_s} ds}$ is bounded, and $e^{-\frac{1}{2}\int_0^T |\theta_s|^2 ds - \int_0^T \theta_s' dW_s} \in L^{\bar{q}}$ for any $\bar{q} > 1$. Moreover, for any $\bar{q} > 1$ and any $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\left(e^{-\int_0^T \Gamma_s^{(1)} \frac{\theta_s}{M_s}\frac{U_s}{M_s} ds}\right)^{\bar{q}}\right] \leq C\mathbb{E}\left[e^{\varepsilon\int_0^T |U_s|^2 ds}\right].$$

We have shown previously that there exists $\varepsilon > 0$ such that $\mathbb{E}\left[e^{\varepsilon\int_0^T |U_s|^2 ds}\right] < +\infty$. Therefore $e^{-\int_0^T \Gamma_s^{(1)} \frac{\theta_s}{M_s}\frac{U_s}{M_s} ds} \in L^{\bar{q}}$. This in turn proves $\xi M_T \in L^{\bar{q}}$. However, M^{-1} is bounded, so $\xi \in L^{\bar{q}}$ for any $\bar{q} > 1$.

Now for any $\bar{q} \in (1, q)$ and $\hat{q} \in (1, \bar{q})$, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\sup_{s\in[t,T]} |\tilde{p}(s;t)|^{\hat{q}}] &= \mathbb{E}\left[\sup_{s\in[t,T]} |\tilde{p}(s;t)|^{\hat{q}}\xi\right] \\
&\leq \left(\mathbb{E}\left[\sup_{s\in[t,T]} |\tilde{p}(s;t)|^{\bar{q}}\right]\right)^{\hat{q}/\bar{q}} \left(\mathbb{E}\left[\xi^{\bar{q}/(\bar{q}-\hat{q})}\right]\right)^{(\bar{q}-\hat{q})/\bar{q}} \\
&< +\infty,
\end{aligned}$$

which implies that $\tilde{p}(\cdot;t)$ is a $\mathbb{Q}$-martingale, and hence $\tilde{p} \equiv 0$ and $\bar{k} \equiv 0$. Since $\bar{p}(s;t) = \tilde{p}(s;t) + \int_s^T \mathbb{E}_t\left[(\theta_v + U_v M_v^{-1})'\bar{k}(v)\right] dv$, we conclude $\bar{p} \equiv 0$.

Q.E.D.

6 Concluding Remarks

Equilibrium control is an alternative and weak notion of solution to a dynamic control problem when the traditional time-consistency is absent. The uniqueness results we establish in this paper (if only for some special cases) justify, from an important aspect, not only the game formulation for the time-inconsistent dynamic decision making, but also our definition of equilibria over the set of open-loop (instead of feedback) controls. They also shed a light on the search of conditions for uniqueness of more general problems.

Since equilibria are defined via local perturbation for the game formulation, unlike the optimal solution for a time-consistent problem, they do not inherently lead to the same value process. The uniqueness of the solution does indeed imply the uniqueness of the value process, which in turn addresses concerns such as "why an equilibrium is defined this way", or "which one to choose if there are multiple solutions".

We realize that in this paper the uniqueness has been established only for some special classes of the LQ control problem. For general time-inconsistent LQ or even non-LQ problems, existence and uniqueness of equilibria remain outstanding research problems.

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A Derivation of (4.10)

We write (4.9) as $u_s = \alpha_s X_s + \beta_s + V_s$ where $V_s := -(R_s + M_s D_s' D_s)^{-1}[B_s \bar{p}(s;s) + D_s' \bar{k}(s)]$ and α_s and β_s are given by (4.8). The equations for $M, N, \Gamma^{(1)}, \Phi$ can be rewritten as

(A.1) $$0 = \dot{M} + (2A + |C|^2)M + Q + M(B' + C'D)\alpha,\ s \in [0,T], \qquad M_T = G;$$

(A.2) $$0 = \dot{N} + 2AN + NB'\alpha,\ s \in [0,T], \qquad N_T = h;$$

(A.3) $$\dot{\Gamma}^{(1)} = -A\Gamma^{(1)},\ \ s \in [0,T], \qquad \Gamma_T^{(1)} = \mu_1;$$

(A.4) $$\begin{cases} 0 = \dot{\Phi} + A\Phi + [(M-N)B' + MC'D]\beta + (M-N)b + C'M\sigma,\ s \in [0,T], \\ \\ \Phi_T = -\mu_2. \end{cases}$$

Hence (the subscript s is suppressed)

$$\begin{aligned} d(MX) &= [M(AX + B'u + b) - XQ - XM(2A + C^2 + (B' + C'D)\alpha)]ds \\ &\quad + M(CX + Du + \sigma)' dW_s \\ &= \left[M(B'\beta + B'V + b) - XQ - XM(A - B'\alpha + C^2 + (B' + C'D)\alpha)\right] ds \\ &\quad + M(CX + Du + \sigma)' dW_s \\ &= \left[M(B'\beta + B'V + b) - XQ - XM(A + C^2 + C'D\alpha)\right] ds \\ &\quad + M(CX + Du + \sigma)' dW_s. \end{aligned}$$

Similarly,

$$\begin{aligned} d(N\mathbb{E}_t[X_s]) &= [N\mathbb{E}_t[AX + B'u + b] - N(2A + B'\alpha)\mathbb{E}_t[X_s]]ds \\ &= [N(B'\beta + B'\mathbb{E}_t[V_s] + b) - N(A - B'\alpha + B'\alpha)\mathbb{E}_t[X_s]]ds \\ &= [N(B'\beta + B'\mathbb{E}_t[V] + b) - NA\mathbb{E}_t[X_s]]ds; \\ d(\Gamma_s^{(1)} X_t) &= -A\Gamma_s^{(1)} X_t ds. \end{aligned}$$

So

$$d(MX - N\mathbb{E}_t[X_s] - \Gamma^{(1)} X_t + \Phi) = \zeta^{(1)} ds + (\zeta^{(2)})' dW_s$$

where $\zeta^{(2)} = M(CX + Du + \sigma)$ and

$$\begin{aligned}
\zeta^{(1)} &= M(B'\beta + B'V + b) - XQ - XM(A + C^2 + C'D\alpha) \\
&\quad -N(B'\beta + B'\mathbb{E}_t[V] + b) + NA\mathbb{E}_t[X_s] \\
&\quad +A\Gamma_s^{(1)}X_t \\
&\quad -A\Phi - [(M-N)B' + MC'D]\beta - (M-N)b - C'M\sigma \\
&= [-Q - M(A + C^2 + C'D\alpha)]X \quad + NA\mathbb{E}_t[X_s] \quad + A\Gamma_s^{(1)}X_t \\
&\quad +(MB'V - NB'\mathbb{E}_t[V_s]) - A\Phi - MC'(D\beta + \sigma).
\end{aligned}$$

However, $\bar{p}(s;t) = p(s;t) - [M_sX_s - N_s\mathbb{E}_t[X_s] - \Gamma_s^{(1)}X_t + \Phi_s]$, we deduce

$$\begin{aligned}
d\bar{p} &= dp - \zeta^{(1)}ds - (\zeta^{(2)})'dW_s \\
&= -[A_sp(s;t) + C_s'k_s + Q_sX_s + \zeta_s^{(1)}]ds + [k_s - \zeta^{(2)}]'dW_s \\
&= \zeta_s^{(3)}ds + \bar{k}_s'dW_s,
\end{aligned}$$

where

$$\begin{aligned}
\zeta^{(3)} &= -A_s[p(s;t) - MX + N\mathbb{E}_t[X_s] + \Gamma_s^{(1)}X_t - \Phi] \\
&\quad -C_s'(k_s - CMX - MD\alpha X - MD\beta - M\sigma) \\
&\quad -(MB'V - NB'\mathbb{E}_t[V_s]) \\
&= -A_s\bar{p}(s;t) - C'\bar{k}_s - C'MDV - (MB'V - NB'\mathbb{E}_t[V_s]) \\
&= -A_s\bar{p}(s;t) - C'\bar{k}_s - (C'MD + MB')V + NB'\mathbb{E}_t[V_s]).
\end{aligned}$$

This proves (4.10).

B Derivation of (5.8)

We write (5.7) as $u_s = \alpha_sX_s + \beta_s + V_s$, where $V_s := -M_s^{-1}[\theta_s\bar{p}(s;s) + \bar{k}(s)]$ and α_s and β_s are given by (5.4).

Making use of (5.3), we can compute

$$\begin{aligned}
d[MX] &= [M(rX + \theta'u) - X(2rM + (M\theta + U)'\alpha) + u'U]ds + [Mu + XU]'dW_s \\
&= [-rMX + (\theta M + U)'(\beta + V)]ds + [Mu + XU]'dW_s; \\
d\mathbb{E}_t[MX] &= \mathbb{E}_t[-rMX + (\theta M + U)'(\beta + V)]\,ds; \\
d\Gamma_s^{(1)}X_t &= -r\Gamma_s^{(1)}X_tds,
\end{aligned}$$

where we have used the fact that $\gamma^{(1)} \equiv 0$, which can be seen from the BSDE for $\Gamma^{(1)}$.

Since

$$\bar{p}(s;t) = p(s;t) - M_s X_s - \Gamma_s^{(2)} + \mathbb{E}_t \left[M_s X_s + \Gamma_s^{(3)} \right] + \Gamma_s^{(1)} X_t, \ \bar{k}(s) = k(s) - M_s u_s - X_s U_s - \gamma_s^{(2)},$$

we derive $d\bar{p}(s;t) = \zeta_s^{(4)} ds + (\zeta^{(5)})'_s dW_s$, where $\zeta^{(5)} = k(s) - [M_s u_s + X_s U_s + \gamma^{(2)}]$, and

$$\begin{aligned}
\zeta^{(4)} &= -rp(s;t) \quad - [-rMX + (\theta M + U)'(\beta + V)] \quad + [r\Gamma^{(2)} + (\theta M + U)'\beta] \\
&\quad + \mathbb{E}_t [-rMX + (\theta M + U)'(\beta + V)] \quad - \mathbb{E}_t \left[r\Gamma^{(3)} + (\theta M + U)'\beta \right] \quad - r\Gamma_s^{(1)} X_t \\
&= -r[p(s;t) - MX - \Gamma^{(2)} + \mathbb{E}_t \left[MX + \Gamma^{(3)} \right] + \Gamma^{(1)} X_t] \\
&\quad - (\theta M + U)'V + \mathbb{E}_t [(\theta M + U)'V] \\
&= -r\bar{p}(s;t) - (\theta M + U)'V + \mathbb{E}_t [(\theta M + U)'V].
\end{aligned}$$

This proves (5.8).