Characterizing All Optimal Controls for An Indefinite Stochastic Linear Quadratic Control Problem

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***Abstract*— This paper is concerned with a stochastic linear quadratic (LQ) control problem in the infinite time horizon, with indefinite state and control weighting matrices in the cost function. It is shown that the solvability of this problem is equivalent to the existence of a so-called static stabilizing solution to a generalized algebraic Riccati equation. Moreover, another algebraic Riccati equation is introduced and all the possible optimal controls, including the ones in state feedback form, of the underlying LQ problem are explicitly obtained in terms of the two Riccati equations.**

***Index Terms*— Indefinite stochastic LQ control, solvability, generalized algebraic Riccati equation, static stabilizing solution**

I. Introduction

Linear quadratic (LQ) control is one of the most fundamental and widely used tools in modern engineering. Recently, there has been extensive research (see [1], [2], [3], [6], [7] and the references therein) on the so-called *indefinite* stochastic LQ problems where the control affects not only the drift component of the systems but also the diffusion part, and the control and state weighting matrices in the cost function are singular or even indefinite. This class of problems arise naturally in many practical situations especially in finance; see [3], [4], [7], [8] for example.

In [2] an infinite stochastic LQ problem in the infinite time horizon was studied. To accommodate the possible matrix singularity due to the indefiniteness of the problem, a so-called generalized algebraic Riccati equation (GARE) involving the Moore–Penrose pseudo inverse was introduced to study the well-posedness and solvability of the LQ problem. On the other hand, to cope with the stability issue due to the infinite time horizon, two types of solutions to the GARE, namely, stabilizing solution and static stabilizing solution, were introduced. It was then shown that the solvability of the underlying LQ problem is equivalent to the existence of a stabilizing solution to the GARE. Moreover, all optimal controls are represented as the sum of two parts, one linear state feedback part and one nonhomogeneous part. However, each of the two parts involves an unknown process, hence the optimal controls have not really been obtained especially in view of the implementation of them.

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This paper is a continuation of [2], aiming to greatly improve the results there. Specifically, we will show that the solvability of the LQ problem is necessary and sufficient for the existence of a *static* stabilizing solution to the GARE. In addition, we will again represent any optimal control as the sum of a linear state feedback part and a nonhomogeneous part. However, we will identify the unknown process in the feedback part, as presented in [2] and mentioned above, via an *additional* algebraic Riccati equation introduced in this paper. One should note that it is the feedback part that matters as far as the implementation is concerned.

The rest of the paper is organized as follows. In Section II the LQ problem is formulated and some preliminaries are presented. Section III is devoted to the equivalence between the solvability of the LQ problem and the existence of a static stabilizing solution to GARE. In Section IV all the optimal controls are identified explicitly. Finally, an illustrative example is given in Section V.

II. Problem Formulation and Preliminaries

Let $(\Omega, \mathcal{F}, \mathcal{P}; \mathcal{F}_t)$ be a given standard filtered probability space with a standard scalar Brownian motion $w(t)$ on $[0, +\infty)$ (with $w(0) = 0$). The Brownian motion is assumed to be one-dimensional only for simplicity; there is no essential difficulty with the multi-dimensional case. Consider the following controlled Itô stochastic differential equation

$$\left\{ \begin{array}{l} dx(t) = [Ax(t) + Bu(t)]dt + [Cx(t) + Du(t)]dw(t), \\ x(0) = x_0 \in \Re^n, \end{array} \right. \tag{1}$$

where A, B, C and D are matrices of sizes $n \times n$, $n \times m$, $n \times n$ and $n \times m$, respectively. The associated cost function is as follows:

$$\begin{array}{rl} J(x_0; u(\cdot)) = & E \int_0^{+\infty} [x(t)^T Q x(t) + 2u(t)^T S x(t) \\ & \qquad + u(t)^T R u(t)] \, dt \end{array} \tag{2}$$

where Q and R are symmetric matrices and S a matrix, all of appropriate sizes. Throughout this paper, the superscript "T" denotes the transpose of a matrix.

Set

$$L^2_{\mathcal{F}}(\Re^k) := \left\{ \begin{array}{l} \phi(\cdot) : [0, +\infty) \times \Omega \mapsto \Re^k | \ \phi(\cdot) \text{ is } \mathcal{F}_t\text{-adapted}, \\ \text{measurable}, E \int_0^{+\infty} \|\phi(t, \omega)\|^2 \, dt < +\infty, \end{array} \right.$$

which is a Hilbert space with the inner product $E\int_0^{+\infty}\phi(t)^T\psi(t)\,dt$ for $\phi(\cdot),\,\psi(\cdot)\in L^2_{\mathcal{F}}(\Re^k)$. Define $\mathcal{U}(x_0)\subset L^2_{\mathcal{F}}(\Re^m)$, the set of admissible controls (at x_0), as the collection of such $u(\cdot)\in L^2_{\mathcal{F}}(\Re^m)$ that the corresponding solution $x(\cdot)\equiv x(\cdot;x_0,u(\cdot))$ of (1) satisfies $x(\cdot)\in L^2_{\mathcal{F}}(\Re^n)$.

The (indefinite) stochastic LQ optimal control problem can be stated as follows:

Problem (LQ) *For a given $x_0\in\Re^n$, find $u(\cdot)\in\mathcal{U}(x_0)$ so that the cost function (2) is minimized.*

If it holds that

$$\inf_{u(\cdot)\in\mathcal{U}(x_0)} J(x_0;u(\cdot))>-\infty,$$

then we say that Problem (LQ) is well-posed at $x_0\in\Re^n$. If there exists a $\bar{u}(\cdot)\in\mathcal{U}(x_0)$ such that

$$J(x_0;\bar{u}(\cdot))=\inf_{u(\cdot)\in\mathcal{U}(x_0)} J(x_0;u(\cdot))>-\infty, \tag{3}$$

then we say that (LQ) is solvable or attainable at $x_0\in\Re^n$. If there is only one optimal control satisfying (3), then (LQ) is called uniquely solvable or attainable at $x_0\in\Re^n$.

Another important term related to the LQ problem is the stability. System (1) is called (mean-square) stabilizable if there exists a feedback control $u(\cdot)=Kx(\cdot)$ with a *constant* matrix K, such that the corresponding solution $x(\cdot)$ of the system (1), for any initial state $x_0\in\Re^n$, satisfies $\lim_{t\to+\infty}E[x(t)^Tx(t)]=0$. In this case, the matrix K is called a (mean-square) stabilizing feedback operator and the feedback control $u(\cdot)=Kx(\cdot)$ is called a (mean-square) stabilizing control.

The following standard assumption is imposed throughout this paper.

Assumption 1: System (1) is stabilizable.

Under Assumption 1, we can further assume, without loss of generality, that the uncontrolled system of (1) (i.e., the system (1) with $u(t)\equiv 0$) is (mean-square) stable. Indeed, let K be a stabilizing feedback operator and put $u(\cdot)=Kx(\cdot)+v(\cdot)$ in (1). Then (1) is turned to

$$dx(t)=[A_1x(t)+Bv(t)]dt+[C_1x(t)+Dv(t)]dw(t), \tag{4}$$

where $A_1=A+BK$ and $C_1=C+DK$. So the system (4) with $v(t)\equiv 0$ is stable.

Based on the above argument, we assume throughout this paper:

Assumption 2: The uncontrolled system of (1) is stable.

The following technical lemma, proved in [6], is useful in the sequel.

Lemma 1: $\mathcal{U}(x_0)=L^2_{\mathcal{F}}(\Re^m)$ for any $x_0\in\Re^n$.

Next, we introduce the Riccati equation necessary for solving Problem (LQ). For any matrix M, there exists ([5]) a unique matrix $M^\dagger$ satisfying

$$\begin{array}{ll} MM^\dagger M=M, & M^\dagger MM^\dagger=M^\dagger, \\ (M^\dagger M)^T=M^\dagger M, & (MM^\dagger)^T=MM^\dagger. \end{array}$$

$M^\dagger$ is called the (Moore–Penrose) pseudo inverse of M. When M is invertible, its pseudo inverse coincides with the conventional inverse, i.e., $M^\dagger=M^{-1}$.

The generalized algebraic Riccati equation (GARE, for short), introduced in [2], is as follows:

$$\left\{\begin{array}{l} \mathcal{M}(P)-\mathcal{L}(P)^T\mathcal{N}(P)^\dagger\mathcal{L}(P)=0, \\ [I-\mathcal{N}(P)^\dagger\mathcal{N}(P)]\mathcal{L}(P)=0, \\ \mathcal{N}(P)\geq 0, \end{array}\right. \tag{5}$$

where

$$\left\{\begin{array}{l} \mathcal{M}(P):=A^TP+PA+C^TPC+Q, \\ \mathcal{N}(P):=R+D^TPD, \\ \mathcal{L}(P):=S+B^TP+D^TPC. \end{array}\right. \tag{6}$$

Definition 1: A solution P to the GARE (5) is called stabilizing if for any initial state $x_0\in\Re^n$ there exist an $\mathcal{F}_t$-adapted, $m\times n$ matrix process $Y(\cdot)$ and a $z(\cdot)\in L^2_{\mathcal{F}}(\Re^m)$ such that the following control is admissible:

$$\begin{array}{rl} u(t)= & -\Big\{\mathcal{N}(P)^\dagger\mathcal{L}(P)+[I-\mathcal{N}(P)^\dagger\mathcal{N}(P)]Y(t)\Big\}x(t) \\ & +[I-\mathcal{N}(P)^\dagger\mathcal{N}(P)]z(t) \end{array} \tag{7}$$

where $x(\cdot)\in L^2_{\mathcal{F}}(\Re^n)$ is the solution of (1) under the above control with the initial state x_0. Moreover, P is called static stabilizing if there exists a constant, non-random matrix Y such that for any initial state $x_0\in\Re^n$ the following feedback control is admissible:

$$u(t)=-\{\mathcal{N}(P)^\dagger\mathcal{L}(P)+[I-\mathcal{N}(P)^\dagger\mathcal{N}(P)]Y\}x(t), \tag{8}$$

where $x(\cdot)\in L^2_{\mathcal{F}}(\Re^n)$ is the solution of (1) under the above control with the initial state x_0.

Remark 1: The terms of stabilizing and static stabilizing solutions to the GARE (5) were first introduced in [2]. Obviously, a solution P to the GARE (5) is stabilizing if and only if for any initial state $x_0\in\Re^n$ there exists some $z(\cdot)\in L^2_{\mathcal{F}}(\Re^m)$ such that the following control is admissible:

$$u(t)=-\mathcal{N}(P)^\dagger\mathcal{L}(P)x(t)+[I-\mathcal{N}(P)^\dagger\mathcal{N}(P)]z(t), \tag{9}$$

where $x(\cdot)\in L^2_{\mathcal{F}}(\Re^n)$ is the solution of (1) under the above control with the initial x_0.

III. Solvability of LQ problem

In this section we characterize the solvability of Problem (LQ) by the existence of a *static* stabilizing solution to the GARE (5). First we need the following technical lemma.

Lemma 2: If Problem (LQ) is solvable at any $x_0\in\Re^n$, then GARE (5) has a solution P such that, for any optimal control $\bar{u}(\cdot)$ at x_0, there exists some $z(\cdot)\in L^2_{\mathcal{F}}(\Re^m)$ satisfying

$$\bar{u}(t)=-\mathcal{N}(P)^\dagger\mathcal{L}(P)\bar{x}(t)+[I-\mathcal{N}(P)^\dagger\mathcal{N}(P)]z(t), \tag{10}$$

where $\bar{x}(\cdot)$ is the corresponding optimal state trajectory with the initial x_0.

Proof: By Theorem 4.1 in [2], GARE (5) must have a stabilizing solution P. Hence with this P the desired result follows from Remark 1. ■

Theorem 1: Problem (LQ) is solvable at any initial state $x_0\in\Re^n$ if and only if GARE (5) admits a static stabilizing solution P with $P^T=P$.

Proof: The "if" part is standard; it follows from Theorem 2.1 of [2] along with the fact that a static stabilizing solution is, by definition, a stabilizing solution. Now we prove the "only if" part. Assume that Problem (LQ) is solvable at any initial state $x_0 \in \Re^n$. By Theorem 4.1 in [2], GARE (5) must have a stabilizing solution P (but not necessarily *static* stabilizing). Define

$$\begin{cases} A_1 := A - B\mathcal{N}(P)^\dagger \mathcal{L}(P), \\ C_1 := C - D\mathcal{N}(P)^\dagger \mathcal{L}(P), \\ B_1 := B[I - \mathcal{N}(P)^\dagger \mathcal{N}(P)], \\ D_1 := D[I - \mathcal{N}(P)^\dagger \mathcal{N}(P)]. \end{cases} \quad (11)$$

Then consider the following controlled system

$$\begin{cases} dx(t) = [A_1 x(t) + B_1 \xi(t)]dt + [C_1 x(t) + D_1 \xi(t)]dw(t), \\ x(0) = x_0 \in \Re^n, \end{cases} \quad (12)$$

where $\xi(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$ is the control. Denote by $\Delta(x_0)$ the set of all admissible controls $\xi(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$ for the system (12) whose corresponding solution $x(\cdot)$ satisfies $x(\cdot) \in L^2_{\mathcal{F}}(\Re^n)$.

Note that one does not know a priori if the controlled system (12) is stabilizable or not (even under the assumption that the original system (1) is stabilizable). Therefore we are not able to apply relevant results in stochastic LQ literature where stabilizability is typically assumed. However, it follows from Lemma 2 that, for any $x_0 \in \Re^n$, $\Delta(x_0)$ is nonempty, for at least the process $z(\cdot)$ identified in Lemma 2 for an optimal control $\bar{u}(\cdot)$ is an element of $\Delta(x_0)$.

Let Φ and Ψ be any given positive definite matrices. Using the same argument as in the last part of the proof of Theorem 5.1 in [6], we can prove that the following (new) algebraic Riccati equation

$$\begin{aligned} 0 = \ & A_1^T \Pi + \Pi A_1 + C_1^T \Pi C_1 + \Psi \\ & -(\Pi B_1 + C_1^T \Pi D_1)(\Phi + D_1^T \Pi D_1)^{-1}(B_1^T \Pi + D_1^T \Pi C_1) \end{aligned} \quad (13)$$

admits a maximal solution, which is a positive semidefinite matrix solution Π such that

$$\xi(t) = -(\Phi + D_1^T \Pi D_1)^{-1}(B_1^T \Pi + D_1^T \Pi C_1)x(t) \quad (14)$$

is a stabilizing feedback control for the controlled system (12). Plugging the above control in the system (12), one easily sees that the control in the form (8) with

$$Y := (\Phi + D_1^T \Pi D_1)^{-1}(B_1^T \Pi + D_1^T \Pi C_1)$$

is admissible for the original system (1). This proves the desired claim. ■

Remark 2: In Theorem 4.1 of [2] it is proved that the solvability of (LQ) implies the existence of a stabilizing solution to the GARE (5). Moreover, in [2], the process $Y(\cdot)$ in (7) is not specified while here it is explicitly given in the form (8) through the new Riccati equation (13). Hence the above result improves that of [2].

IV. Optimal Controls

In this section we shall explicitly represent all the optimal controls of Problem (LQ) when it is solvable.

Theorem 2: Assume that Problem (LQ) is solvable at any initial state $x_0 \in \Re^n$ and P is the static stabilizing solution to the GARE (5). Then a control $\bar{u}(\cdot)$ is optimal at x_0 if and only if there is some $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$ such that

$$\begin{aligned} \bar{u}(t) = \ & -\Big\{\mathcal{N}(P)^\dagger \mathcal{L}(P) + [I - \mathcal{N}(P)^\dagger \mathcal{N}(P)]Y\Big\}x(t) \\ & +[I - \mathcal{N}(P)^\dagger \mathcal{N}(P)]v(t), \quad \forall t \in [0, +\infty), \end{aligned} \quad (15)$$

with

$$Y = (\Phi + D_1^T \Pi D_1)^{-1}(B_1^T \Pi + D_1^T \Pi C_1),$$

where $\bar{x}(\cdot)$ is the corresponding solution of (1) under $\bar{u}(\cdot)$ with the initial x_0, and Π is the maximal solution of the Riccati equation (13) as specified in the proof of Theorem 1 with Φ and Ψ given to be positive definite.

Proof: We first prove the "if" part. The key is to show that the control given by (15), for any $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$, must be admissible (i.e., the corresponding $\bar{x}(\cdot) \in L^2_{\mathcal{F}}(\Re^n)$). Following the proof of Theorem 1, define

$$\begin{cases} A_2 := A_1 - B_1(\Phi + D_1^T \Pi D_1)^{-1}(B_1^T \Pi + D_1^T \Pi C_1), \\ C_2 := C_1 - D_1(\Phi + D_1^T \Pi D_1)^{-1}(B_1^T \Pi + D_1^T \Pi C_1), \end{cases} \quad (16)$$

where A_1, B_1, C_1 and D_1 are defined in (11). Consider the controlled system

$$\begin{cases} dx(t) = [A_2 x(t) + B_1 v(t)]dt + [C_2 x(t) + D_1 v(t)]dw(t), \\ x(0) = x_0 \in R^n, \end{cases} \quad (17)$$

where $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$ is the control. By the construction of Π and the proof of Theorem 1, $v(t) \equiv 0$ is a stabilizing control for the system (17). Hence it follows from Lemma 1 that $L^2_{\mathcal{F}}(\Re^m)$ is the set of admissible controls for the system (17) at any initial state. This in turn implies that, by definition of admissibility, the control given by (15) is admissible for the original control system (1). Hence the assumption of Theorem 2.1 in [2] is satisfied, which yields that $u(\cdot)$ must be optimal.

Conversely, if $\bar{u}(\cdot)$ is an optimal control at x_0 with the corresponding state trajectory $\bar{x}(\cdot)$, then by Lemma 2, $\bar{u}(\cdot)$ can be represented by (10) where $z(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$. Now define

$$v(t) = z(t) + (\Phi + D_1^T \Pi D_1)^{-1}(B_1^T \Pi + D_1^T \Pi C_1)\bar{x}(t). \quad (18)$$

It is clear that $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$. Combining (10) and (18) yields that we have the representation (15). ■

Remark 3: It was shown in Theorem 4.1 of [2] that any optimal control must be of the form (7) for some (unknown) $\mathcal{F}_t$-adaptable $m \times n$ order matrix process $Y(\cdot)$ and $z(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$. The above theorem gives the mathematical formula for all optimal controls while one feedback law is provided through taking $v(\cdot) = 0$ in (15). From (15) it also follows immediately that, assuming that it is solvable at any initial state $x_0 \in \Re^n$, Problem (LQ) is either uniquely solvable at any initial state, or solvable with at least two different optimal controls at any initial state.

V. An Example

Here we consider Example 6.2 in [2], which was stated to be attainable, but all possible optimal controls were not explicitly given.

Example 1: Consider a two-dimensional LQ problem with the following data in the system dynamics

$$A = \begin{bmatrix} -10, & 5 \\ 0, & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -1, & 0 \\ 0, & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix};$$

and with the following cost weighting matrices

$$Q = \begin{bmatrix} 54, & -10 \\ -10, & 0 \end{bmatrix}, \quad S = 0, \quad R = \begin{bmatrix} -1, & 0 \\ 0, & 0 \end{bmatrix}.$$

It was shown in [2] that

$$P = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix}$$

is a static feedback stabilizing solution to the GARE (5).

Obviously, $\mathcal{N}(P) = 0$, and then

$$A_1 = A, \quad B_1 = B, \quad C_1 = C, \quad D_1 = D.$$

Now we choose

$$\Phi = \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 19, & -5 \\ -5, & 7 \end{bmatrix}.$$

The corresponding Riccati equation (13) admits the maximal solution

$$\Pi = \begin{bmatrix} 1, & 0 \\ 0, & 7 \end{bmatrix}.$$

By Theorem 2, all optimal controls are given as follows:

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 7 \end{bmatrix} \bar{x}(t) + v(t),$$

with $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$. Moreover, one feedback law is

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 7 \end{bmatrix} \bar{x}(t).$$

Next, we would like to see how the choice of Φ and Ψ might affect the form of the optimal controls. Take the following matrices

$$\Phi_\varepsilon = \begin{bmatrix} 1+\varepsilon, & 0 \\ 0, & 1+\varepsilon \end{bmatrix}, \quad \Psi_k = \begin{bmatrix} 19k, & -5k \\ -5k, & 7k \end{bmatrix},$$

parameterized by ε and k with $|\varepsilon| < \frac{1}{2}$ and $0 < k < 18$. Both Φ_ε and Ψ_k are positive definite and the corresponding Riccati equation (13) admits the maximal solution

$$\Pi_{\varepsilon,k} = \begin{bmatrix} k, & 0 \\ 0, & (1+\varepsilon)(3 + \sqrt{9 + \frac{7}{1+\varepsilon}}) \end{bmatrix}.$$

In this case, it follows from Theorem 2 that all optimal controls of the original LQ problem can also be given as follows:

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 3 + \sqrt{9 + \frac{7}{1+\varepsilon}} \end{bmatrix} \bar{x}(t) + v(t),$$

with $v(\cdot) \in L^2_{\mathcal{F}}(\Re^m)$. Hence, a different optimal feedback law is

$$\bar{u}(t) = \begin{bmatrix} 0, & 0 \\ 0, & 3 + \sqrt{9 + \frac{7}{1+\varepsilon}} \end{bmatrix} \bar{x}(t).$$

It is interesting to note that the optimal controls above do not dependent on the parameter k.

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