Stochastic Orders, Submodularity, and Asymptotic Optimality

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Stochastic Ordering

- X and Y are two r.v.'s, with d.f.'s $F(x)$ and $G(x)$. The stochastic ordering, $X \geq_{\mathrm{st}} Y$, is defined as

$$\mathsf{P}(X \geq x) \geq \mathsf{P}(Y \geq x); \quad \text{or} \quad \bar{F}(x) \geq \bar{G}(x), \quad \forall x.$$

- Coupling: Since $\bar{F}(x) \geq \bar{G}(x)$ iff $F^{-1}(x) \geq G^{-1}(x)$, and

$$X \stackrel{\mathrm{d}}{=} F^{-1}(U), \quad Y \stackrel{\mathrm{d}}{=} G^{-1}(U), \quad \text{where} \quad U \sim \text{ uniform } [0,1],$$

we know $X \geq_{\mathrm{st}} Y$ iff $F^{-1}(U) \geq G^{-1}(U)$ for any $U \in [0,1]$.

- From the above, we also have, $X \geq_{\mathrm{st}} Y$ iff $\mathsf{E}h(X) \geq \mathsf{E}h(Y)$ for any increasing (i.e., non-decreasing) function $h(x)$.

- If $X \geq_{\mathrm{st}} Y$, and W is independent of X and Y, then $X+W \geq_{\mathrm{st}} Y+W$. Without independence, this will not hold.

- For example, X and Y are both Bernoulli (i.e., binary) r.v.'s, $\mathsf{P}(X=1)=\frac{1}{2}$, $\mathsf{P}(Y=1)=\frac{1}{3}$; Hence, $X \geq_{\mathrm{st}} Y$. Let W be a version of Y that's independent of X, whereas $W \equiv Y$. Then,

$$X+W = 0,1,2, \quad \text{w.p.} \quad \frac{1}{4}, \frac{1}{2}, \frac{1}{4};$$

whereas

$$Y+W = 0,2, \quad \text{w.p.} \quad \frac{2}{3}, \frac{1}{3}.$$

Hence, $X+W \geq_{\mathrm{st}} Y+W$ does *not* hold.

Rearrangement Inequalities

- If $x \geq y$ and $a \geq b$, then $ax + by \geq ay + bx$.

- Suppose $x \geq y$ are the processing times of two jobs, then $2x + y \geq x + 2y$, i.e., the SPT rule minimizes the delay:

$$x + (x + y) \geq y + (y + x).$$

- Now, suppose the processing times are random: $X \geq_{\text{st}} Y$. Then,

$$X + (X + Y) \not\geq_{\text{st}} Y + (Y + X),$$

since X (Y) and $X + Y$ are not independent.

- Hence, what is needed is some order: $X \succeq Y$, such that E$g(X, Y) \geq$ E$g(Y, X)$, for all bivariate function $g(x, y)$, such that

$$x \geq y \quad \Rightarrow \quad g(x,y) \geq g(y,x).$$

- The above in fact defines what's known as the likelihood ratio ordering, $X \geq_{\rm lr} Y$:

$$\mathsf{P}(X=x)\mathsf{P}(Y=y) \geq \mathsf{P}(X=y)\mathsf{P}(Y=x), \qquad \forall x \geq y,$$

  which is stronger than the stochastic ordering, i.e., $X \geq_{\rm lr} Y \Rightarrow X \geq_{\rm st} Y$.

- For example, suppose X and Y follow exponential distributions. Then, $X \geq_{\rm lr} Y$ if $\mathsf{E}(X) \geq \mathsf{E}(Y)$.

- Another example: M/M/1 queue, with traffic intensity $\rho < 1$. Let X denote the total number of jobs in the system in equilibrium. Then, X is increasing in ρ, in the sense of likelihood ratio ordering.

Submodular Functions

- A function $f : \mathcal{R}^2 \mapsto \mathcal{R}$ is submodular, if

  $$f(x_1, y_2)+f(x_2, y_1) \geq f(x_1, y_1)+f(x_2, y_2); \qquad x_1 \leq x_2,\ y_1 \leq y_2;$$

  i.e., off-diagonal dominance. If f is twice differentiable, then $\frac{\partial^2 f}{\partial x \partial y} \leq 0$.

- Implication: Let $x^*(y) := \arg\min_x f(x, y)$. Then, $y_1 \leq y_2$ implies $x^*(y_1) \leq x^*(y_2)$; i.e., monotone optimizers, or the isotone property.

- In general, $f : \mathcal{R}^n \mapsto \mathcal{R}$ is submodular if for any $v, w \in \mathcal{R}^n$,

  $$f(v) + f(w) \geq f(v \vee w) + f(v \wedge w); \qquad \vee := \max,\ \wedge := \min\,.$$

- $f : 2^E \mapsto \mathcal{R}$, where $E = \{1, \ldots, n\}$, is submodular if:

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B), \qquad A, B \subseteq E.$$

(If $A \cap B = \emptyset$, then the above is simply subadditivity.) Interpretation: "diminishing marginals" —

$$f(A \cup B) - f(A) \leq f(B) - f(A \cap B).$$

(Note: $A \cup B - A = B - A \cap B$, and $B \subseteq A \cup B$.)

- The set selection problem:

$$\min_{A \subseteq E} [f(A) - d(A)];$$

where $f(A)$, cost for "coverage", is submodular (economies of scale); and $d(A) = \sum_{i \in A} d_i$, "fees" collected from coverage. Without submodularity, the problem is hard.

Sequential Inspection

- A batch of N units. Each unit is defective w.p. Θ, itself a random variable with a known distribution.

- Non-defective and defective units have lifetimes X and Y, respectively; $X \geq_{\rm st} Y$. Suppose $\Theta = \theta$, a constant; and exactly n units in the batch are inspected (and the defectives repaired). Then, the expected warranty (or after-sales service) cost is

  $$\phi(n,\theta) := \mathsf{E}[C(X_1 + \cdots + X_n + Z_{n+1}(\theta) + \cdots + Z_N(\theta))],$$

  where $Z(\theta) = Y$ w.p. θ and $= X$ w.p. $1-\theta$; and $C(t)$ is decreasing and convex in t.

- Inspection and repair are perfect, and costs c_i and c_r per unit.

- Assume

$$c_r \leq \mathsf{E}C(X_1 + \cdots + X_{N-1} + Y) - \mathsf{E}C(X_1 + \cdots + X_N);$$

hence (since $C(\cdot)$ is decreasing and convex),

$$c_r \leq \mathsf{E}C(X_1 + \cdots + X_{n-1} + Y) - \mathsf{E}C(X_1 + \cdots + X_n),$$

for any $n \leq N$; i.e., there's enough incentive to repair any identified defect.

- Sequential inspection. Let D_n be the number of defectives identified among n inspected units. The optimal policy is characterized by a sequence of threshold values,

$$d_{n_0} \leq \ldots \leq d_n \leq \ldots \leq d_{n_k},$$

such that it is optimal to stop inspection at the first n that satisfies $D_n < d_n$.

Dynamic Programming

- $V_n(d)$: the expected total remaining cost, following an optimal policy, after n units are inspected and d units are found defective. Let $\Theta_n(d) := [\Theta | D_n = d]$ be the conditional defective rate.

- Let $\Phi_n(d) := \mathsf{E}[\phi(n, \Theta_n(d))]$ (stop inspection), and

$$\begin{aligned} \Psi_n(d) \quad := \quad & c_i + [c_r + V_{n+1}(d+1)]\mathsf{P}[D_{n+1} = d+1 | D_n = d] \\ & + V_{n+1}(d)\mathsf{P}[D_{n+1} = d | D_n = d] \end{aligned}$$

  (continue inspecting one more unit).

- Then, $V_n(d) = \min\{\Phi_n(d), \Psi_n(d)\}$, for $0 \le n \le N-1$ and $V_N(d) = \Phi_N(d)$; and $V_0(0)$ is the optimal cost for the original problem.

K-Submodularity

- Key: the warranty cost $\phi(n, \Theta)$ is (stochastically) K-submodular, with $K = c_r$, in the following sense:

$$\begin{aligned}&[\mathsf{E}\phi(n-1, \Theta') + \mathsf{E}\phi(n, \Theta)] - [\mathsf{E}\phi(n-1, \Theta) + \mathsf{E}\phi(n, \Theta')] \\ &\quad \geq c_r \mathsf{E}[\Theta' - \Theta],\end{aligned}$$

  for all n, provided $\Theta' \geq_{\mathrm{st}} \Theta$.

- $g(x, y)$, is called K-submodular, if for some $K \geq 0$, we have,

$$\begin{aligned}& [g(x_1, y_2) + g(x_2, y_1)] - [g(x_1, y_1) + g(x_2, y_2)] \\ \geq \quad & K(x_2 - x_1)(y_2 - y_1),\end{aligned}$$

  for all $x_1 \leq x_2$ and $y_1 \leq y_2$.

- The K-submodularity of $\phi(n, \Theta)$ implies:
  - if it is optimal to continue inspection when $D_n = d$, then it is also optimal to continue inspection when $D_n = d + 1$; following from $\Psi_n(d) - \Phi_n(d)$ decreasing in d;
  - if it is optimal to stop inspection when $D_n = d$, then it is also optimal to stop inspection when $D_{n+1} = d$; following from $\Psi_n(d) - \Phi_n(d)$ increasing in n.

- Recall, $g(x, y)$ is K-submodular if for $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$\begin{aligned} & [g(x_1, y_2) + g(x_2, y_1)] - [g(x_1, y_1) + g(x_2, y_2)] \\ \geq \quad & K(x_2 - x_1)(y_2 - y_1), \end{aligned}$$

That is, for the values of $g(x, y)$ on any four corner points of a rectangle: the off-diagonal sum is greater than the diagonal sum by at least K times the area of the rectangle. Obviously, K-submodularity specializes to submodularity with $K = 0$.

- $g(x, y)$ is K-submodular if and only if $Kxy + g(x, y)$ is submodular. Let $x^*(y) = \arg\min_x[Kxy + g(x, y)]$, for a given y. Then, $x^*(y)$ is increasing in y, if $g(x, y)$ is K-submodular.

- Example. A batch of $N = 30$ units; Θ uniform on $[5\%, 30\%]$; X uniform on [70, 110]; Y uniform on [30, 70]; $c_i = 0.5$ and $c_r = 1$; a cumulative, pro-rata rebate warranty for the batch, $C(T) = (cN)[NW - T]^+/(NW)$, with the unit price $c = 100$ and a warranty period $W = 82$ per unit.

- The optimal thresholds are:

$$d_n = 0,\ n \leq 8; \quad d_9 = d_{10} = d_{11} = 1; \quad d_{12} = d_{13} = d_{14} = 2;$$
$$d_{15} = d_{16} = 3; \quad d_{17} = 4;\ d_{18} = 5;\ d_{19} = 6;\ d_{20} = 8;\ d_{21} = 12.$$

For instance, as long as there is at least one defect among 11 inspected units, inspection should continue; whereas inspection can be terminated if there are fewer than 4 defectives among 17 inspected units. Under no circumstances should inspection continue beyond $n_1 = 22$ inspected units ($n_0 = 0$).

- Following this optimal policy,
  - the expected number of inspected units is 14.79, uncovering an average of 2.76 defective units.
  - the expected total cost is 12.964.
  - the expected total costs under zero-inspection and full-inspection policies are 36.41 and 20.25, respectively.

The Inventory Function

- Let D denote demand, a random variable with distribution function $F(x)$.

- Let $H(x) := \mathsf{E}[x - D]^+$ be the inventory function; where $[x]^+ := \max\{x, 0\}$. We have

$$H(x) = \int_0^Q (x - t)dF(t) = xF(x) - \int_0^x tdF(t).$$

Integrating by parts leads to:

$$H(x) = \int_0^x F(t)dt.$$

Note, $H'(x) = F(x)$, and $H''(x) = f(x) := F'(x)$.

- When $F(x) = \Phi(x)$, the standard normal d.f.,

$$H(x) = x\Phi(x) - \int_0^x t\phi(t)dt = x\Phi(x) + \phi(x).$$

- Hence, $H'(x) = \Phi(x) \approx 1$ when $x \geq 3$ and $H'(x) \approx 0$ when $x \leq -3$. That is,

$$H(x) \approx x, \quad x \geq 3; \qquad H(x) \approx 0, \quad x \leq -3.$$

- Hence, the only non-linear part of $H(x)$ is $x \in (-3, +3)$.

- In general, when $D \sim N(\mu, \sigma)$, i.e., $D = \mu + \sigma Z$, we have

$$\mathsf{E}[x - D]^+ = \mathsf{E}[x - \mu - \sigma Z]^+ = \sigma \mathsf{E}\left[\frac{x-\mu}{\sigma} - Z\right] = \sigma H\left(\frac{x-\mu}{\sigma}\right),$$

and its nonlinear part is with $x \in (\mu - 3\sigma, \mu + 3\sigma)$.

- $H(x)$ can be very well approximated by a small number of linear pieces.

- Preselect the following points on the x-axis:

$$-3 := z_0 < z_1 < \cdots < z_p < z_{p+1} := 3.$$

- For $x \in [z_\ell, z_{\ell+1}]$, write

$$x = z_\ell + \lambda(z_{\ell+1} - z_\ell),$$

for some $\lambda \in [0, 1]$. Then,

$$H(x) \approx H(z_\ell) + \lambda[H(z_{\ell+1}) - H(z_\ell)],$$

which is nothing more than linear interpolation.

- In general, for any $x \in (-3, +3)$, we write

$$x = \sum_{\ell=0}^{p} \lambda_{\ell+1}(z_{\ell+1} - z_\ell),$$

and

$$H(x) \approx \sum_{\ell=0}^{p} \lambda_{\ell+1}[H(z_{\ell+1}) - H(z_\ell)].$$

These, along with $H(x) = x$ for $x \geq 3$ and $H(x) = 0$ for $x \leq -3$, approximate the H function by $p + 2$ linear pieces.

Asymptotics

- Recall

$$\lim_{x\to+\infty} \frac{H(x)}{x} = \lim_{x\to+\infty} [\Phi(x) + \frac{\phi(x)}{x}] = 1,$$

which we shall write as $H(x) \sim x$ below. Also note that

$$\lim_{x\to+\infty} H(-x) = \lim_{x\to+\infty} [-x\Phi(-x)+\phi(x)] = \lim_{x\to+\infty} [-x\bar{\Phi}(x)+\phi(x)] = 0,$$

where $\bar{\Phi}(x) := 1 - \Phi(x)$, and we have used the fact that $\bar{\Phi}(x) \sim \frac{\phi(x)}{x}$.

- For the Poisson variate with mean a, denoted $N(a)$, we can make use of the infinite divisibility of the Poisson distribution along with the central limit theorem to derive: for $a \to \infty$,

$$\mathsf{E}[a - N(a)]^+ \sim \mathsf{E}[a - (a + \sqrt{a}Z)]^+ = \sqrt{a}H(0) = \sqrt{\frac{a}{2\pi}}.$$

- Similarly, for $0 < \beta < 1$,

$$\begin{aligned} \mathsf{E}[a - N(\beta a)]^+ &\sim \mathsf{E}[a - (\beta a + \sqrt{\beta a}Z)]^+ \\ &= \sqrt{\beta a}\Big[\frac{a - \beta a}{\sqrt{\beta a}} - Z\Big]^+ \\ &= \sqrt{\beta a}H\Big(\frac{a - \beta a}{\sqrt{\beta a}}\Big). \end{aligned}$$

Hence,

$$\mathsf{E}[a - N(\beta a)]^+ \sim (1 - \beta)a, \qquad \beta < 1.$$

- When $\beta > 1$, the argument of $H(\cdot)$ becomes negative; hence,

$$\mathsf{E}[a - N(\beta a)]^{+} \sim \sqrt{\beta a} H\Big(- \frac{(\beta - 1)a}{\sqrt{\beta a}} \Big) \sim 0, \qquad \beta > 1.$$

A Dynamic Pricing Problem

- FG inventory of n units; time horizon: t; maximize revenue.
- Poisson demand, price-dependent intensity:

$$\lambda(p) = Ae^{-Bp}, \quad \text{or} \quad p(\lambda) = \frac{1}{B} \ln \frac{A}{\lambda}; \qquad A, B > 0.$$

- Optimal control solution:

$$J^*(n,t) = \frac{1}{B} \ln \sum_{j=0}^{n} \frac{(\lambda_0 t)^j}{j!}, \qquad \text{with} \quad \lambda_0 := \frac{A}{e},$$

and the optimal pricing is a feedback control law:

$$p^*(k,s) = J^*(k, t-s) - J^*(k-1, t-s) + \frac{1}{B}.$$

- Suppose we solve $\max_\lambda \lambda p(\lambda)$ to obtain

$$\lambda_1 = \frac{A}{e}(= \lambda_0), \qquad \text{and} \qquad p_1 = 1/B;$$

with

$$J^1(n,t) := p_1 \mathsf{E}[N(\lambda_1 t) \wedge n] = \frac{1}{B}\mathsf{E}[N(\lambda_1 t) \wedge n].$$

- Alternative solution: $\lambda_2 = \lambda_1 \wedge \frac{n}{t}$ (to ensure $\mathsf{E}N(\lambda_2 t) \le n$); with $p_2 := \frac{1}{B}\ln(\frac{A}{\lambda_2})$, and

$$J^2(n,t) := p_2 \mathsf{E}[N(\lambda_2 t) \wedge n] = \frac{1}{B}\ln\frac{A}{\lambda_2}\mathsf{E}[N(\lambda_2 t) \wedge n].$$

- Question: How do J^1 and J^2 compare with J^*?

- Gallego and van Ryzin, *Management Science* (1994), showed, in a more general setting, that

$$J^2/J^* \to 1 \quad \text{when} \quad n \to \infty, \quad \text{or} \quad t \to \infty.$$

- In this special case – exponential price function, we can get more explicit results.

Dynamic Pricing: Asymptotic Performance

- Consider

$$n \to \infty, \quad t \to \infty, \quad \frac{n}{t} \to c > 0.$$

And, w.l.g., assume $B = 1$.

- Case 1: $c > \lambda_0$. Write $\lambda_0 t := \beta n$, i.e., $\beta := \lambda_0 / c < 1$.

$$\begin{aligned} J^* = \ln \sum_{j=0}^{n} \frac{(\beta n)^j}{j!} &= \beta n + \ln \sum_{j=0}^{n} \frac{(\beta n)^j}{j!} e^{-\beta n} \\ &\sim \beta n + \ln \mathsf{P}[\beta n + \sqrt{\beta n} Z \le n] \\ &= \beta n + \ln \Phi\left(\frac{(1-\beta)n}{\sqrt{\beta n}}\right) \sim \beta n. \end{aligned}$$

On the other hand, since $\lambda_1 t = \lambda_2 t = \beta n$, we have

$$\begin{aligned} J^1 = J^2 &= \mathsf{E}[N(\beta n) \wedge n] \\ &= n - \mathsf{E}[n - N(\beta n)]^+ \\ &\sim n - (1-\beta)n = \beta n. \end{aligned}$$

- Case 2: $c = \lambda_0$. Since $\lambda t = ct \sim n$, we have

$$\begin{aligned} J^* &= n + \ln \sum_{j=0}^{n} \frac{(n)^j}{j!} e^{-n} \\ &\sim n + \ln \mathsf{P}[n + \sqrt{n} Z \leq n] \\ &= n + \ln \Phi(0) \sim n. \end{aligned}$$

Furthermore, we have $\lambda_1 = \lambda_2 = \lambda_0$ and

$$J^1 = J^2 = \mathsf{E}[N(n) \wedge n] = n - \mathsf{E}[n - N(n)]^+$$

$$\sim \quad n - \sqrt{\frac{n}{2\pi}}.$$

- Case 3: $c < \lambda_0$. Here $\beta := \lambda_0/c > 1$. Write

$$\sum_{j=0}^{n} \frac{(\beta n)^j}{j!} = \frac{(\beta n)^n}{n!}\Big[1 + \sum_{j=1}^{n} \frac{n(n-1)\cdots(n-j+1)}{(\beta n)^j}\Big],$$

where the summation is dominated by the following geometric series:

$$\sum_{j=0}^{n} \frac{1}{\beta^j} \uparrow \frac{\beta}{\beta - 1}.$$

Hence, this part is finite, i.e., not growing with n. Hence,

$$J^* \sim \ln \frac{(\beta n)^n}{n!} \sim n(\ln \beta + 1) - \frac{1}{2}\ln n.$$

In this case,

$$J^1 = n - \mathsf{E}[n - N(\beta n)]^+ \sim n;$$

and

$$J^2 = (\ln \frac{A}{c})\{n - \mathsf{E}[n - N(n)]^+\} \sim (\ln \beta + 1)[n - \sqrt{\frac{n}{2\pi}}].$$

- In summary,

$$\begin{aligned}
&\text{Case 1 } (\beta < 1): \quad && J^1 = J^2 = J^* \sim \beta n; \\
&\text{Case 2 } (\beta = 1): \quad && J^1 = J^2 \sim n - \sqrt{\frac{n}{2\pi}}, \quad J^* \sim n; \\
&\text{Case 3 } (\beta > 1): \quad && J^1 \sim n, \quad J^2 \sim (\ln \beta + 1)[n - \sqrt{\frac{n}{2\pi}}], \\
& && J^* \sim n(\ln \beta + 1) - \frac{1}{2}\ln n.
\end{aligned}$$