Optimal Pricing and Replenishment in a Single-Product Inventory System with Brownian Demand

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Abstract

We study an inventory system that supplies price-sensitive demand modeled by Brownian motion. The optimal pricing and inventory replenishment decisions under both long-run average and discounted objectives are derived, and related to or contrasted with previously known results. In addition, we emphasize the interplay between pricing and replenishment decisions, and the ways in which they react to the demand uncertainty. We show that the joint optimization of both decisions may result in significant profit improvement compared to the traditional method of making decisions separately or sequentially. We also show that multiple price changes result in only a limited profit improvement over the optimal single price.

Keywords: Joint pricing-replenishment decision, price sensitive demand, Brownian model.

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1 Introduction

In recent years, substantially more research has been done on joint inventory and pricing strategies. The focus has been on establishing the optimality of structural inventory and pricing policies. One of the key determinants of the complexity and generality of these models is the assumption about the demand processes. Various demand models have been proposed for periodic-review settings (e.g., additive demand, multiplicative demand, and other general demand models), and for continuous-review settings (e.g., Poisson and renewal demand models).

In this paper, the basic setting is a single-product continuous-review inventory system with a *Brownian demand model*. Brownian motion, with its continuous path, is particularly appropriate for modeling the demands of fast-moving products, such as sugar and coffee. It is a natural model when demand forecast involves Gaussian noises. It is a continuous-time analog to the normally distributed demand model in the discrete-time setting. The drift of the Brownian demand measures the demand rate; the diffusion term represents the demand variability. Both drift and diffusion terms depend on the price set by the firm.

We assume that inventory replenishment is instantaneous, which is appropriate when the lead time is insignificant relative to the length of the replenishment cycle. Thus, inventory is depleted by the demands in a continuous-time fashion and then replenished immediately when it drops to the reorder point. We assume, in our base model, that the demands must be satisfied immediately upon arrival. Consequently, the reorder point is zero, and the replenishment follows a simple order-up-to policy, with the order-up-to level denoted by S. We allow the pricing decisions to be dynamically adjusted within the replenishment cycle, but we do not adjust prices continuously over time, as most realistic pricing strategies do not involve too many different prices. Instead, we explicitly consider the number of prices to use. We divide S into N equal segments, N being a given integer. We charge one price when the inventory level falls in each segment. All N prices are optimally determined, jointly with the replenishment level S, so as to maximize the expected long-run average or the discounted profit.

The rest of the introductory section of this paper contains a literature review and a summary of the features of the Brownian demand model.

1.1 Literature Review

There has been substantial literature on joint pricing and inventory control. We refer the reader to three excellent survey papers: Yano and Gilbert (2003), Elmaghraby and Keskinocak (2003), and Chan et al. (2004). Our review below focuses on those papers which are closely related to our study.

Whitin (1955), Porteus (1985a), Rajan, Rakesh and Steinberg (1992), among others, study demands that are deterministic functions of price. Whitin (1955) connects pricing and inventory control in the EOQ framework, and Porteus (1985a) provides an explicit solution for the linear demand instance. Rajan, Rakesh and Steinberg (1992) investigates continuous pricing for perishable products for which demands may diminish as products age.

In periodic-review stochastic inventory and pricing models, several types of demand assumptions are made. The demand function in each period typically consists of a deterministic demand function and a random component. These two components can be additive (e.g., Chen and Simchi-Levi 2004a, Chen, Ray and Song 2006), or multiplicative (e.g., Song, Ray and Boyaci 2006), or both (e.g., Zabel 1972, Chen and Simchi-Levi 2004a,b). Some other models allow the random component to affect the demand in more general forms (e.g., Federgruen and Heching 1999, Polatoglu and Sahin 2000, Feng and Chen 2004, Huh and Janakiraman 2005).

Federgruen and Heching (1999) assume that the demand in each period depends on the price charged in the period and a random term. The dependence can be quite general, but every realization of demand function is assumed to be concave in price. The replenishment cost is linear, without a fixed setup cost. The authors show that a base-stock list-price policy is optimal for both average and discounted objectives. Earlier related works include those by Zabel (1972) and Thowsen (1975).

The above periodic-review models have been extended to include a replenishment setup cost. In the backordering setting, it was first conjectured by Thomas (1974), and then proved by Chen and Simchi-Levi (2004a), that the (s, S, p) policy is optimal for additive demand in a finite horizon. Chen and Simchi-Levi (2004b) further prove the optimality of the stationary (s, S, p) policy for both additive and multiplicative demand in an infinite horizon. Feng and Chen (2004) prove the optimality of (s, S, p) policy under more general demand functions, but restricting the prices to a finite set. Assuming lost sales, Polatoglu and Sahin (2000) obtain rather involved optimal policies under a general demand model and provide restrictive conditions under which the (s, S, p) policy is optimal. Chen, Ray and Song (2006) prove that the (s, S, p) policy is optimal under additive demand and lost sales. Huh and Janakiraman (2005) provide a novel approach for proving and generalizing many of the early results for both backorder and lost sales settings.

Continuous-review models are studied by Feng and Chen (2003), and Chen and Simchi-Levi (2006). Feng and Chen (2003) model the demand as a Poisson process with price-sensitive intensity.

Pricing and replenishment decisions are made upon finishing serving each demand, but the prices are restricted to a given finite set. An (s, S, p) policy is proven to be optimal. Chen and Simchi-Levi (2006) generalize this model by considering a compound renewal demand process with both the inter-arrival times and the size of the demand depending on the price. It is shown that the (s, S, p) policy is still optimal.

There are also works that involve production, for instance, Li (1988) considers a make-to-order production system with price-sensitive demand. Both production and demand are modeled by Poisson processes with controllable intensities, and the control of demand intensity is through pricing. A barrier policy is shown to be optimal: when the inventory level reaches an upper barrier, production stops; when the inventory level drops to zero, the demand stops (or the demand is lost). The optimal price is shown to be a non-increasing function of the inventory level.

1.2 Features of the Brownian Demand Model

First, the Brownian demand model allows us to explicitly and naturally bring out the impact of demand variability on the optimal pricing and replenishment decisions, whereas results along this line were previously limited to only a few numerical studies. Federgruen and Heching (1999) experimented with a multiplicative demand case, and found that the optimal base stock and the optimal prices are increasing in demand variability. Polatoglu and Sahin (2000) numerically examined an additive demand model with uniform random noise, and found that the order-up-to and reorder levels both tend to be higher when the demand variability increases, while the prices do not exhibit a clear monotone property. To the best of our knowledge, this paper is the first analytical examination of the impact of demand variability.

Second, with the Brownian demand model and the zero lead time assumption, the order-up-to replenishment policy is immediately seen as being optimal. This allows us to focus on how, given the order-up-to policy, the optimal order quantity and the optimal prices vary and interact with other model parameters, particularly the demand variability. This is very different from most of the literature, where the focus is on establishing the optimality of the order-up-to policy.

Third, using the Brownian demand model, we can derive an upper bound for the profit improvement generated from the use of the dynamic pricing strategy in comparison to a static strategy. We find that dynamic pricing results in only a limited profit improvement over the single price strategy (when both are optimally determined). The relative profit improvement, however, becomes more significant when the profit margin is low. This result is consistent with the numerical results in Feng and Chen (2004), and Chen, Ray and Song (2006).

The rest of this paper is organized as follows. In Section 2, we present a formal description of our model. In Section 3, we study the optimal pricing and replenishment decisions under the long-run average objective. We start by making these decisions separately, so as to highlight the comparisons with prior studies and known results, and then present the joint optimization model and demonstrate the profit improvement. We conclude the paper by pointing out several possible extensions in Section 4. In an online companion, we present parallel results under the discounted objective.

2 Model Description and Preliminary Results

We consider a continuous-review inventory model with a price-sensitive demand. The objective is to determine the inventory replenishment and pricing decisions that strike a balance between the sales revenue and the cost for holding and replenishing inventory over time, so as to maximize the expected long-run average or discounted profit.

2.1 The Demand Model

The cumulative demand up to time t is denoted as $D(t)$, and modeled by a diffusion process:

$$D(t) \;=\; \int_0^t \lambda(p_u)du + \int_0^t \sigma(\lambda(p_u))dB(u), \qquad t \geq 0, \tag{1}$$

where p_t is the price charged at time t, $\lambda(p_t)$ is the demand rate at time t, $\sigma(\lambda)$ is a positive function of λ measuring the variability of the demand (or the error of demand forecast) when demand rate is λ, and $B(t)$ denotes the standard Brownian motion.

Let $\mathcal{P}$ and $\mathcal{L}$ denote, respectively, the domain and the range of the demand rate function $\lambda(\cdot)$. Both are assumed to be intervals of $\Re_+$ (the set of nonnegative real numbers). The case of zero demand rate is not interesting in application. To avoid such cases, we assume that $\mathcal{L}$ is bounded below by a positive constant.

Assumption 1 (on demand rate) *The demand rate $\lambda(p)$ and its inverse $p(\lambda)$ are both strictly decreasing and twice continuously differentiable in the interior of $\mathcal{P}$ and $\mathcal{L}$, respectively. The revenue rate $r(\lambda) = p(\lambda)\lambda$ is strictly concave in λ.*

Many commonly-used demand functions satisfy the above assumption, including the following examples, where the parameters α, β and δ are all positive:

- The linear demand function $\lambda = \alpha - \beta p$, $p \in [0, \alpha/\beta]$: $r(\lambda) = \frac{\alpha}{\beta}\lambda - \frac{1}{\beta}\lambda^2$ is strictly concave;

- The exponential demand function $\lambda = \alpha e^{-\beta p}$, $p \geq 0$: $r(\lambda) = -\frac{1}{\beta}\lambda \log(\lambda/\alpha)$ is strictly concave;
- The power demand function $\lambda = \beta p^{-\delta}$, $p \geq 0$: $r(\lambda) = \lambda^{-\frac{1}{\delta}+1}\beta^{\frac{1}{\delta}}$ is strictly concave if $\delta \geq 1$.

Let $\rho(\lambda) = \sigma(\lambda)^2/\lambda$ be the relative measure of demand variability.

Assumption 2 (on demand variability) *The relative demand variability $\rho(\lambda)$ is convex and twice differentiable in the interior of $\mathcal{L}$.*

The following examples of demand variability satisfy the above assumption:

- Constant demand variability $\sigma(\lambda) = \sigma$: $\rho(\lambda) = \sigma^2/\lambda$;
- Linear demand variability $\sigma(\lambda) = \sigma\lambda$: $\rho(\lambda) = \sigma^2\lambda$;
- Square-root demand variability $\sigma(\lambda) = \sigma\sqrt{\lambda}$: $\rho(\lambda) = \sigma^2$.

Constant demand variability case is a continuous-time analog to the additive demand model in the periodic-review setting. The noise part in the demand model is usually due to factors other than price, such as forecast error and inventory accounting error; the former mainly depends on the information available, whereas the latter relies heavily on the prevailing technology. (For instance, the radio frequency identification (RFID) technology has demonstrated great advantages in improving inventory accounting.) Furthermore, in some empirical studies of demand price relations, the residual of the regression is usually found to be a sequence of normally distributed random variables independent of the price. Refer to, for example, Reiss and Wolak (2006), and Genesove and Mullin (1998). The latter studies the demand functions for the U.S. sugar industry.

In the linear demand variability case, $\sigma(\lambda) = \sigma\lambda$, and the demand model (1) can be written as

$$dD(t) \quad = \quad \lambda(p_t)\big(dt + \sigma dB(t)\big).$$

This is an analog to the multiplicative demand case in the periodic-review setting.

The square-root demand variability is motivated by approximating renewal processes. For example, a Poisson process with rate λ has mean λt and standard deviation $\sqrt{\lambda t}$ at time t.

In this paper, we do not aim to find the most appropriate functional form of demand variability. This depends on particular industries and products, and is an empirical issue. We focus on analyzing inventory and pricing decisions under general form of demand variability. Where appropriate, we use the above three special functions as illustrative cases.

2.2 Cost Parameters

We assume that the holding cost is linear in the quantity held. Let h be the cost for holding one unit of inventory per unit of time. The replenishment cost is a function of the replenishment quantity, denoted as $c(S)$, which satisfies the following assumption.

Assumption 3 (on replenishment cost) *The replenishment cost function $c(S)$ is twice continuously differentiable and increasing in S for $S \in (0,\infty)$. The average cost $a(S) = c(S)/S$ is strictly convex in S, and $a(S) \to \infty$, as $S \to 0$.*

Consider a special case: $c(S) = K + cS^\delta$, $S > 0$, where $K, c, \delta > 0$. When $\delta = 1$, this is the most commonly used linear function with a setup cost K. The average cost function, $a(S) = \frac{K}{S} + cS^{\delta-1}$ is convex if $\delta \in (0,1] \cup [2,\infty]$. If $\delta \in (1,2)$, $a(S)$ is convex in the region where $a'(S) \leq 0$. (This region is of particular interest; refer to Section 3.1.) To see this, note that

$$a'(S) = \frac{1}{S^2}(-K + c(\delta-1)S^\delta),$$

and when $\delta \in (1,2)$ and $a'(S) \leq 0$, we have

$$a''(S) = -\frac{2-\delta}{S^3}\Big(-\frac{2}{2-\delta}K + c(\delta-1)S^\delta\Big) \geq -\frac{a'(S)(2-\delta)}{S} \geq 0.$$

2.3 Pricing and Replenishment Policies

Assume replenishment is instantaneous, i.e., zero lead time. We further assume that all demands will be supplied immediately upon arrival; i.e., no backorder is allowed, or there is an infinite backorder cost penalty. (Our results extend readily to the backorder case; refer to Section 4.2.)

The replenishment follows a continuous-review order-up-to policy. Specifically, whenever the inventory level drops to zero, it is brought up to S instantaneously via a replenishment, where S is a decision variable. We shall refer to the time between two consecutive replenishment epochs as a *cycle*.

We adopt the following dynamic pricing strategy. Let $N \geq 1$ be a given integer, and let $S = S_0 > S_1 > \cdots > S_{N-1} > S_N = 0$. Immediately after a replenishment at the beginning of a cycle, price p_1 is charged until the inventory drops to S_1; price p_2 is then charged until the inventory drops to S_2; ...; and finally when the inventory level drops to S_{N-1}, price p_N is charged until the inventory drops to $S_N = 0$, when another cycle begins. The same pricing strategy applies to all cycles. For simplicity, we set $S_n = S(N-n)/N$. That is, we divide the full inventory of S units into N equal segments, and price each segment with a different price as the inventory is depleted by demand.

In general, the segments are not necessarily equal, but our experiments show that the optimal segments (optimized jointly with prices and replenishment level) are quite even and the corresponding profit is almost the same as using equal segments. Furthermore, in practical situations when the prices are within a discrete set, our algorithm based on equal segments can effectively tell how many prices to use, when to change prices and what prices to change to. (See Example 9 for elaboration.)

In summary, the decision variables are: $(S, \mathbf{p})$, where $S \in \Re_+$, and $\mathbf{p} = (p_1, \ldots, p_N) \in \mathcal{P}^N$. Within a cycle, we shall refer to the time when the price p_n is applied as period n.

2.4 The Inventory Process

Without loss of generality, suppose at time zero the inventory is filled up to S. We focus on the first cycle which ends at the time when inventory reaches zero. Let $T_0 = 0$, and let T_n be the first time when inventory drops to S_n:

$$T_n \;:=\; \inf\{\, t \geq 0 \;:\; D(t) = nS/N \,\}, \qquad n = 1, 2, \ldots, N.$$

The length of period n is therefore $\tau_n := T_n - T_{n-1}$. Since T_n's are stopping times, by the strong Markov property of Brownian motion, τ_n is just the time during which S/N units of demand has occurred under the price p_n. That is,

$$\tau_n \;\stackrel{dist.}{=}\; \inf\{\, t \geq 0 \;:\; \lambda(p_n)t + \sigma(\lambda(p_n))B(t) = S/N \,\}. \tag{2}$$

Let $X(t)$ denote the inventory-level at t. We have,

$$X(t) \;=\; S - D(t), \qquad t \in [0, T_N).$$

Since our replenishment-pricing policy is the same for all cycles, $X(t)$ is a regenerative process with the replenishment epochs being its regenerative points.

We conclude this section with a lemma, which gives the first two moments and the generating function of the stopping time τ_n. (The proof is in the appendix.)

Lemma 1 *For the stopping time τ_n in (2), we have*

$$\begin{aligned}
\mathsf{E}[\tau_n] &= \frac{S}{N\lambda_n}, \\
\mathsf{E}[\tau_n^2] &= \frac{\sigma(\lambda_n)^2}{\lambda_n^2}\mathsf{E}[\tau_n] + \mathsf{E}^2[\tau_n] = \frac{\sigma(\lambda_n)^2 S}{N\lambda_n^3} + \frac{S^2}{N^2\lambda_n^2}, \\
\mathsf{E}[e^{-\gamma\tau_n}] &= \exp\Big[-\frac{\sqrt{\lambda_n^2 + 2\sigma(\lambda_n)^2\gamma} - \lambda_n}{\sigma(\lambda_n)^2}\frac{S}{N}\Big],
\end{aligned}$$

where $\lambda_n = \lambda(p_n) > 0$, and $\gamma > 0$ is a parameter.

3 Long-Run Average Objective

To optimize the long-run average profit, thanks to the regenerative structure of the inventory process, it suffices to focus on the first cycle. Recall that period n refers to the period in which the price p_n applies, and the inventory drops from $S_{n-1} = \frac{(N-n+1)S}{N}$ to $S_n = \frac{(N-n)S}{N}$. Applying integration by parts, and recognizing

$$dX(t) = -dD(t), \qquad X(T_{n-1}) = S_{n-1}, \quad X(T_n) = S_n,$$

we have

$$\begin{aligned}
\int_{T_{n-1}}^{T_n} X(t)dt &= T_n S_n - T_{n-1}S_{n-1} - \int_{T_{n-1}}^{T_n} t dX(t) \\
&= T_n S_n - T_{n-1}S_{n-1} - \int_{T_{n-1}}^{T_n} T_{n-1} dX(t) + \int_{T_{n-1}}^{T_n} (t - T_{n-1}) dD(t) \\
&= \tau_n S_n + \int_{T_{n-1}}^{T_n} (t - T_{n-1}) \big[\lambda(p_n)dt + \sigma(\lambda(p_n))dB(t)\big].
\end{aligned}$$

A simple change of variable yields

$$\int_{T_{n-1}}^{T_n} (t - T_{n-1})dt \;=\; \int_0^{\tau_n} u du \;=\; \frac{\tau_n^2}{2},$$

whereas

$$\mathsf{E}\Big[\int_{T_{n-1}}^{T_n} (t - T_{n-1})dB(t)\Big] \;=\; \mathsf{E}\Big[\int_{T_{n-1}}^{T_n} t dB(t)\Big] - \mathsf{E}\big[T_{n-1}\big]\mathsf{E}\big[B(T_n) - B(T_{n-1})\big] \;=\; 0,$$

follows from the martingale property of $B(t)$ and the optional stopping theorem.

Let $v_n(S, p_n)$ denote the expected profit (sales revenue minus inventory holding cost) during period n. Then, making use of the above derivation, along with Lemma 1, we have

$$\begin{aligned}
v_n(S, p_n) &= \frac{p_n S}{N} - \mathsf{E}\Big[\int_{T_{n-1}}^{T_n} hX(t)dt\Big] \\
&= \frac{p_n S}{N} - h\mathsf{E}[\tau_n]S_n - \frac{1}{2}h\lambda(p_n)\mathsf{E}[\tau_n^2] \\
&= \frac{p_n S}{N} - \frac{hS^2(N - n + \frac{1}{2})}{N^2\lambda(p_n)} - \frac{h\rho(\lambda(p_n))S}{2N\lambda(p_n)}, \qquad (3)
\end{aligned}$$

Note in the above expression, the first term is the sales revenue from period n, the second term is the inventory holding cost attributed to the deterministic part of the demand (i.e., the drift part of the Brownian motion), and the last term is the additional holding cost incurred by demand uncertainty.

For ease of analysis, below we shall often use $\{\mu_n = \lambda(p_n)^{-1}, n = 1, \ldots, N\}$ as decision variables and denote $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N) \in \mathcal{M}^N$ and $\mathcal{M} = \{\lambda^{-1} : \lambda \in \mathcal{L}\}$. Then, the long-run average objective can be written as follows:

$$V(S,\boldsymbol{\mu}) = \frac{\sum_{n=1}^{N} v_n(S,p_n) - c(S)}{\frac{S}{N}\sum_{n=1}^{N}\mu_n} = \frac{\sum_{n=1}^{N}\Big[p(\frac{1}{\mu_n}) - \frac{hS}{N}(N-n+\frac{1}{2})\mu_n - \frac{1}{2}h\mu_n\rho(\frac{1}{\mu_n}) - a(S)\Big]}{\sum_{n=1}^{N}\mu_n}. \quad (4)$$

The additional holding cost rate due to demand uncertainty is represented by $\frac{h\sum_{n=1}^{N}\mu_n\rho(\frac{1}{\mu_n})}{2\sum_{n=1}^{N}\mu_n}$.

For the special case when $N=1$, the price and the demand rate are both constants in a cycle (and hence constant throughout the horizon). The objective function (4) takes a simpler form. For comparison with some classical work, we use λ as the decision variable and denote the long-run average profit under single-price policy as $V(S,\lambda)$. Then,

$$V(S,\lambda) \;=\; \frac{v_1(S,p(\lambda)) - c(S)}{S/\lambda} \;=\; r(\lambda) - \frac{hS}{2} - \lambda a(S) - \frac{h\rho(\lambda)}{2}. \quad (5)$$

The classical EOQ model only consists of the second and third terms in (5), which are the average inventory cost if the demand is deterministic with a constant rate. Whitin (1955) and Porteus (1985a) considered the price-sensitive EOQ model, which involves the first three terms in (5). Our model gives rise to an additional holding cost due to demand variability.

3.1 Optimal Replenishment with Fixed Prices

In this case, the set of prices $\mathbf{p}$, or equivalently, $\boldsymbol{\mu}$ is given, and the firm's problem is

$$\max_{S>0} \quad V(S,\boldsymbol{\mu}).$$

Under Assumption 3, $V(S,\boldsymbol{\mu})$ is strictly concave in S. Note that $V(S,\boldsymbol{\mu}) \to -\infty$ as $S\to\infty$ or $S\to 0$ (the latter is due to Assumption 3 that $a(S)\to\infty$ as $S\to 0$). Thus, the unique optimal replenishment level is determined by the first-order condition:

$$S^* \;=\; {a'}^{-1}\Big(-\frac{h}{N^2}\sum_{n=1}^{N}(N-n+\frac{1}{2})\mu_n\Big), \quad (6)$$

where ${a'}^{-1}$ is well-defined since $a(\cdot)$ is strictly convex and $a'(\cdot)$ is strictly increasing under Assumption 3. In practice, the average cost as a function of quantity is usually first decreasing (due to economy of scale) and then increasing (due to capacity or other technological restrictions). However, at the optimal replenishment level, we have $a'(S^*)<0$ for any fixed prices. This observation helps to reduce the search space when the replenishment level is optimized jointly with pricing decisions (see Section 3.3).

The way that the demand variability and the holding cost impact the optimal replenishment level can be readily derived from (6).

Proposition 1 *With prices fixed,*

(i) the optimal replenishment level S^* is decreasing in h and in p_n (for any n), and independent of the form of demand variability;

(ii) the optimal profit is decreasing in h, and decreases when $\rho(\lambda)$ increases for all $\lambda \in \mathcal{L}$.

Proof. Part (i) is obvious from (6). Part (ii) is also obvious since the objective in (4) is decreasing in h, and decreases when $\rho(\lambda)$ increases for all λ, for any decisions. ∎

The interpretation of this proposition is quite intuitive, except that the optimal replenishment level is independent of $\sigma(\lambda)$. The latter results from the fact that the demand variability impacts on the average profit function through the additional holding cost term, $h\rho(\lambda)/2$, which does not affect the replenishment decision. This will not be the case under a discounted objective, as will be shown in Section 4.1.

Example 1 Consider $c(S) = K + cS$, where $K, c > 0$. The first-order condition in (6) leads to the familiar EOQ formula:

$$S^* = \sqrt{\frac{2K\lambda_a}{h}}, \qquad \text{where} \qquad \lambda_a = \Big(\sum_{n=1}^{N} \frac{2N - 2n + 1}{N^2} \mu_n \Big)^{-1}. \tag{7}$$

In the standard EOQ model, the demand rate is taken as the average demand per time unit, which in our setting becomes

$$\bar{\lambda} \;=\; \Big(\frac{1}{N} \sum_{n=1}^{N} \mu_n \Big)^{-1}. \tag{8}$$

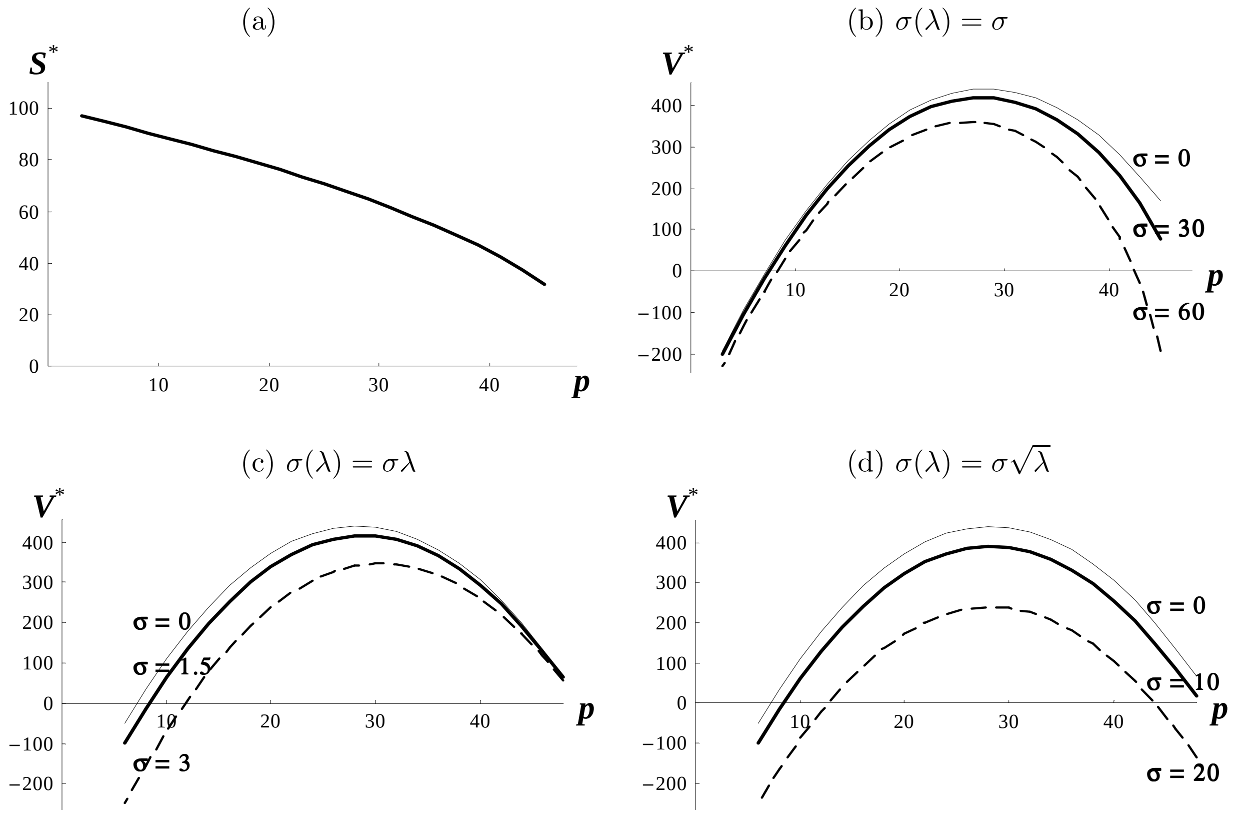
Suppose that the fixed prices satisfy $p_1 \leq p_2 \leq \cdots \leq p_N$ (or $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$), i.e., a lower price is charged when the inventory level is higher (we will see this price pattern in Theorem 1). Higher weights are given to smaller μ_n's in (7), while μ_n's are equally weighted in (8). Hence, $\bar{\lambda} \leq \lambda_a$. This implies that the standard EOQ with a time-average demand rate may lead to a replenishment level lower than optimally desirable. ∎

Example 2 Let $c(S) = 100 + 5S$, $\lambda(p) = 50 - p$, $h = 1$, and $N = 1$. In Figure 1(a), the EOQ quantity is shown as a function of the single fixed price, $S^* = 10\sqrt{100 - 2p}$.

Figure 1(b) illustrates the optimal profit as a function of the price in the case of $\sigma(\lambda) = \sigma$. The peak of each V^* curve is reached when the price and the replenishment level are jointly optimized. When the price deviates away from the optimum, the profit falls dramatically. When the price is jointly optimized, we can see that the optimal price decreases slightly as σ increases. This will be formally investigated in Section 3.3.

Figure 1(c) and (d) illustrate the case of $\sigma(\lambda) = \sigma\lambda$ and $\sigma(\lambda) = \sigma\sqrt{\lambda}$, respectively. The shape of the profit curves are similar to that in (b), but the optimal price in (c) increases in σ, while the optimal price in (d) seems invariant to σ. These will be investigated in Section 3.3. ∎

Figure 1: Optimal replenishment level with a fixed price



3.2 Optimal Pricing with a Fixed Replenishment Level

In this case, the decision variables are the set of prices $\mathbf{p}$, or equivalently, $\boldsymbol{\mu}$. Specifically, the problem is

$$\max_{\boldsymbol{\mu} \in \mathcal{M}^N} \quad V(S, \boldsymbol{\mu}) = \frac{\sum_{n=1}^N \left[p(\frac{1}{\mu_n}) - \frac{hS}{N}(N - n + \frac{1}{2})\mu_n - \frac{1}{2}h\mu_n\rho(\frac{1}{\mu_n}) - a(S) \right]}{\sum_{n=1}^N \mu_n}. \tag{9}$$

Notice the following fact. Suppose $g(x) = xf(x^{-1})$, where $f(x)$ is twice differentiable for $x > 0$. Then $g''(x) = x^{-3}f''(x^{-1})$. Based on this relation, the strict concavity of the revenue function $r(\lambda) = \lambda p(\lambda)$ in Assumption 1 implies that $p(\frac{1}{\mu})$ is strictly concave in μ; and the convexity of $\rho(\lambda)$ in Assumption 2 implies that $\mu\rho(\frac{1}{\mu})$ is convex in μ.

Thus, the numerator of the objective is strictly concave in $\boldsymbol{\mu}$. The ratio of a concave function over a positive linear function is known to be pseudo-concave (Mangasarian 1970). Hence, $V(S, \boldsymbol{\mu})$ is pseudo-concave in $\boldsymbol{\mu}$. Furthermore, it is strictly pseudo-concave in the sense that

$$\nabla_\mu V(S, \boldsymbol{\mu}_1)(\boldsymbol{\mu}_2 - \boldsymbol{\mu}_1) \leq 0 \qquad \Rightarrow \qquad V(S, \boldsymbol{\mu}_2) < V(S, \boldsymbol{\mu}_1).$$

Hence, the optimal $\boldsymbol{\mu}$ must be unique. (This follows from a straightforward extension of Mangasarian 1970 from pseudo-concavity to strict pseudo-concavity.) Suppose the optimality is achieved at an interior point. Then the optimal $\boldsymbol{\mu}$ is uniquely determined by the first-order condition:

$$\frac{\partial \widetilde{v}_n(S, \mu_n)}{\partial \mu_n} \quad = \quad \frac{\sum_{k=1}^N \widetilde{v}_k(S, \mu_k) - c(S)}{\sum_{k=1}^N \mu_k}, \qquad n = 1, \ldots, N, \tag{10}$$

where $\widetilde{v}_n(S, \mu_n) = v_n(S, p(\frac{1}{\mu_n}))$. Note that the right side of the above does not depend on n. This implies that the optimal pricing must be such that the marginal profit is the same across all periods (assuming an interior optimum).

The following proposition describes the basic optimal pricing pattern.

Theorem 1 *For any replenishment quantity S, the optimal prices are increasing over the periods, i.e., $p_1^* \leq p_2^* \leq \cdots \leq p_N^*$.*

Proof. Since $p_n = p(\frac{1}{\mu_n})$ is increasing in μ_n, it suffices to show that the optimal μ_n^* is increasing in n. But this follows immediately from (9), since given any $(\mu_1, \ldots, \mu_N)$, rearranging these variables (but not changing their values) in increasing order will maximize the term $\sum_{n=1}^N n\mu_n$ in the numerator, while all other terms remain unchanged. ∎

In other words, within each cycle, the optimal prices are increasing over the periods. Intuitively, when the inventory is higher at the beginning of a cycle, we charge a lower price to induce higher demand so as to reduce the inventory holding cost.

Next, we derive an upper bound to quantify the potential benefit of multiple price changes over a single price. To provide intuition of what enhances and what limits the profit improvement, we note that the holding cost attributed to the deterministic part of the demand is the *only* motive for varying prices over the periods. In the numerator of (9), the additional holding cost term $-\sum \frac{1}{2} h \mu_n \rho(\frac{1}{\mu_n})$ and the revenue term $\sum p(\frac{1}{\mu_n})$ are, on the contrary, suggesting not varying prices, since they are both concave in $\boldsymbol{\mu}$. These two terms limit the extend to which the optimal prices vary, and thus limit the potential profit improvement. The following lemma and the proposition formalize this intuition.

Lemma 2 *For fixed S, let $\boldsymbol{\mu}^*$ be the optimal N prices. Then for $1 \le m < n \le N$,*

$$\mu_n^* - \mu_m^* \quad \le \quad \frac{S(n-m)}{N\left(G_1 + G_2/h\right)},$$

where $G_1 = \inf\left\{ d^2\left(\frac{1}{2}\mu\rho(\frac{1}{\mu})\right)/d\mu^2 \;:\; \mu \in [\mu_1^*, \mu_N^*] \right\}$ and $G_2 = \inf\left\{ -d^2 p(\frac{1}{\mu})/d\mu^2 \;:\; \mu \in [\mu_1^*, \mu_N^*] \right\}$.

This lemma indicates that the optimal dynamic pricing may not involve marked price changes under some situations. Note that G_1 and G_2 measure the concavity of $-\sum \frac{1}{2}\mu_n\rho(\frac{1}{\mu_n})$ and $\sum p(\frac{1}{\mu_n})$, respectively. When $\sigma(\lambda) = \sigma$, $G_1 = \sigma^2$; when $\sigma(\lambda) = \sigma\lambda$ or $\sigma\sqrt{\lambda}$, $G_1 = 0$. Consistent with our previous intuition, increasing either G_1 or G_2 results in a smaller bound.

Theorem 2 *For fixed S, let $\boldsymbol{\mu}^*$ be the optimal N prices, and let V_N^* be the optimal average profit defined in (9). Then,*

$$V_N^* - V_1^* \quad \le \quad \frac{hS^2\left(1 - N^{-2}\right)}{12\bar{\mu}\left(G_1 + G_2/h\right)},$$

where $\bar{\mu} = \frac{1}{N}\sum_{n=1}^N \mu_n^*$, $G_1 = \inf\left\{ d^2\left(\frac{1}{2}\mu\rho(\frac{1}{\mu})\right)/d\mu^2 \;:\; \mu \in [\mu_1^*, \mu_N^*] \right\}$ and $G_2 = \inf\left\{ -d^2 p(\frac{1}{\mu})/d\mu^2 \;:\; \mu \in [\mu_1^*, \mu_N^*] \right\}$.

Proof. We consider a feasible single-price policy that charges a price $p(\bar{\mu}^{-1})$, and compare this with the optimal N-price policy. We show the latter has a lower average revenue, a higher additional holding cost due to demand variability, and the same average ordering cost. Hence, for the optimal N-price policy to yield a higher profit, it must have a lower average holding cost attributed to the deterministic part of the demand. The difference between these two holding cost terms corresponding to the two policies constitute an upper bound on the profit improvement from the single-price policy to the N-price policy.

Specifically, the average profit of charging a single price $p(\bar{\mu}^{-1})$ is

$$V_1 \quad = \quad \frac{p(\frac{1}{\bar{\mu}}) - \frac{hS}{2}\bar{\mu} - \frac{1}{2}h\bar{\mu}\rho(\bar{\mu}^{-1}) - a(S)}{\bar{\mu}}.$$

We have

$$\begin{aligned} & V_N^* - V_1^* \\ \le \;& V_N^* - V_1 \\ = \;& \frac{\sum_{n=1}^N \left[p(\frac{1}{\mu_n^*}) - p(\frac{1}{\bar{\mu}})\right] - \frac{hS}{N}\sum_{n=1}^N \left[(N - n + \frac{1}{2})\mu_n^* - \frac{N}{2}\bar{\mu})\right] - \frac{hN}{2}\left[\frac{1}{N}\sum_{n=1}^N \mu_n^*\rho(\frac{1}{\mu_n^*}) - \bar{\mu}\rho(\frac{1}{\bar{\mu}})\right]}{\sum_{n=1}^N \mu_n^*}. \end{aligned}$$

The first term in the numerator, $\sum_{n=1}^N \left[p(\frac{1}{\mu_n^*}) - p(\frac{1}{\bar{\mu}})\right] \le 0$, since $p(\frac{1}{\mu})$ is concave in μ; and the last term in the numerator $\frac{1}{N}\sum_{n=1}^{N}\mu_n^*\rho(\frac{1}{\mu_n^*}) - \bar{\mu}\rho(\frac{1}{\bar{\mu}}) \geq 0$, since $\mu\rho(\frac{1}{\mu})$ is convex in μ. Thus,

$$
\begin{aligned}
V_N^* - V_1^* &\leq \frac{-\frac{hS}{N}\sum_{n=1}^{N}\left[(N-n+\frac{1}{2})\mu_n^* - \frac{N}{2}\bar{\mu})\right]}{\sum_{n=1}^{N}\mu_n^*} \\
&= \frac{hS}{N}\left(\frac{\sum_{n=1}^{N} n\mu_n^*}{\sum_{n=1}^{N}\mu_n^*} - \frac{N+1}{2}\right) \\
&= \frac{hS}{N}\left(\frac{2\mu_1^* + 4\mu_2^* + \cdots + 2N\mu_N^* - (N+1)(\mu_1^* + \cdots + \mu_N^*)}{2\sum_{n=1}^{N}\mu_n^*}\right) \\
&= \frac{hS}{N}\left(\frac{(N-1)(\mu_N^* - \mu_1^*) + (N-3)(\mu_{N-1}^* - \mu_2^*) + \ldots}{2N\bar{\mu}}\right)
\end{aligned}
$$

where in the last line, the series ends with $\mu_{N/2+1}^* - \mu_{N/2}^*$ if N is even, and ends with $2\Big(\mu_{(N+3)/2}^* - \mu_{(N-1)/2}^*\Big)$ if N is odd.

Applying Lemma 2 and the identity:

$$
(N-1)^2 + (N-3)^2 + \cdots + \Big(N + 1 - 2\Big\lfloor\frac{N}{2}\Big\rfloor\Big)^2 = \frac{(N-1)N(N+1)}{6},
$$

we have

$$
\begin{aligned}
V_N^* - V_1^* &\leq \frac{hS^2}{\bar{\mu}\,(G_1 + G_2/h)}\left(\frac{(N-1)^2 + (N-3)^2 + \cdots + \big(N+1-2\lfloor\frac{N}{2}\rfloor\big)^2}{2N^3}\right) \\
&= \frac{hS^2}{\bar{\mu}\,(G_1 + G_2/h)}\frac{(N-1)N(N+1)}{12N^3} \\
&= \frac{hS^2\left(1 - N^{-2}\right)}{12\bar{\mu}\,(G_1 + G_2/h)}.
\end{aligned}
$$

■

The bound in the above theorem indicates that the N-price policy cannot improve much over the single-price policy when the replenishment quantity S and the holding cost rate h are low, and the convexity of $\rho(\lambda)$ and the concavity of $r(\lambda)$ are high. These parameters affect the above bound in the same way as they do to the price increment bound in Lemma 2.

Finally, we study how the optimal prices vary with demand variability, holding cost rate, and the replenishment level. Since demand variability affects the average profit through the additional holding cost $\frac{h\sum\mu_n\rho(\frac{1}{\mu_n})}{2\sum\mu_n}$, the impact of demand variability and holding cost on the optimal decisions would depend on the functional form of $\sigma(\lambda)$. We examines three special cases in the next proposition. As expected, different forms of demand variability leads to different monotone properties.

Proposition 2 *Suppose the replenishment level is fixed.*

(i) *If* $\sigma(\lambda) = \sigma$, *then* p_N^* *is decreasing in* σ, *and* p_1^* *is decreasing in* h;

(ii) *If* $\sigma(\lambda) = \sigma\lambda$, *then all of the prices* $p_1^*, \ldots, p_N^*$ *are increasing in* σ, *and* p_N^* *is increasing in* h;

(iii) *If* $\sigma(\lambda) = \sigma\sqrt{\lambda}$, *then the optimal prices are independent of* σ, p_1^* *is decreasing in* h, *and* p_N^* *is increasing in* h; *if* $N = 1$, p^* *is invariant to* h;

(iv) *For general* $\sigma(\lambda)$ *that satisfies Assumption 2, if* $a'(S) < 0$, *then* p_1^* *is decreasing in* S; *the optimal profit is decreasing in* h, *and decreases when* $\rho(\lambda)$ *increases for all* $\lambda \in \mathcal{L}$.

Federgruen and Heching (1999) numerically examined a periodic-review model with multiplicative demand, and found that the optimal prices are increasing in demand variability. The authors explain that a higher demand uncertainty can be reduced by increasing the price, and thus reducing the safety stock cost and shortage costs. Recall that our demand model with $\sigma(\lambda) = \sigma\lambda$ is an analog to the multiplicative demand. The result in Proposition 2(ii) that prices are increasing in σ is consistent with Federgruen and Heching (1999), although the additional holding cost here is caused by inventory fluctuations rather than the safety stock.

In a periodic-review setting, if the demand uncertainty is independent of the price (i.e., additive demand model), then a higher demand uncertainty cannot be reduced by adjusting prices, but the relative demand uncertainty can be reduced by decreasing the price and increasing the mean demand. This intuition also applies to our model. When $\sigma(\lambda) = \sigma$, the additional holding cost is $\frac{h\sigma^2 \sum \mu_n^2}{2\sum \mu_n}$. If σ increases, in order to balance out the increase of this cost, intuitively we need to decrease $\boldsymbol{\mu}$ and at the same time decrease the spread of $\boldsymbol{\mu}$ (i.e., $\mu_N - \mu_1$). The composite effect is mixed except for μ_N^*. This is formalized in Proposition 2(i).

Example 3 This example illustrates Proposition 2 and Theorem 1. In Figure 2–4, we optimize four prices and plot the optimal prices. For comparison, the optimal price for the single-price problem is also shown as p^* in these figures.

1. Effect of demand variability:
   (a) Let $p(\lambda) = 50 - 10\lambda, \lambda \in [0.5, 5]$, $c(S) = 500 + 5S$, $S = 100$, $h = 2$, and $\sigma(\lambda) = \sigma$. For $N = 4$, the optimal prices are plotted in for various values of σ in Figure 2(a). The highest price is decreasing in σ while the others are not.
   (b) Let $p(\lambda) = 50 - 2\lambda, \lambda \in [0.5, 25]$, $c(S) = 500 + 5S$, $S = 100$, $h = 2$, and $\sigma(\lambda) = \sigma\lambda$. The optimal prices are all increasing in σ, as shown in Figure 2(b). Intuitively, to balance out the increase of the additional holding cost $\frac{h\sigma^2}{2\sum \mu_n}$ due to the increase in σ, we need to increase $\boldsymbol{\mu}$.
   (c) Same setting as in (b) except that $\sigma(\lambda) = \sigma\sqrt{\lambda}$. In this case, the impact of demand variability is to reduce the average profit by a constant, and hence, optimal prices are invariant to σ,

shown in Figure 2(c).

Figure 2: Effect of demand variability on optimal prices with a fixed replenishment level

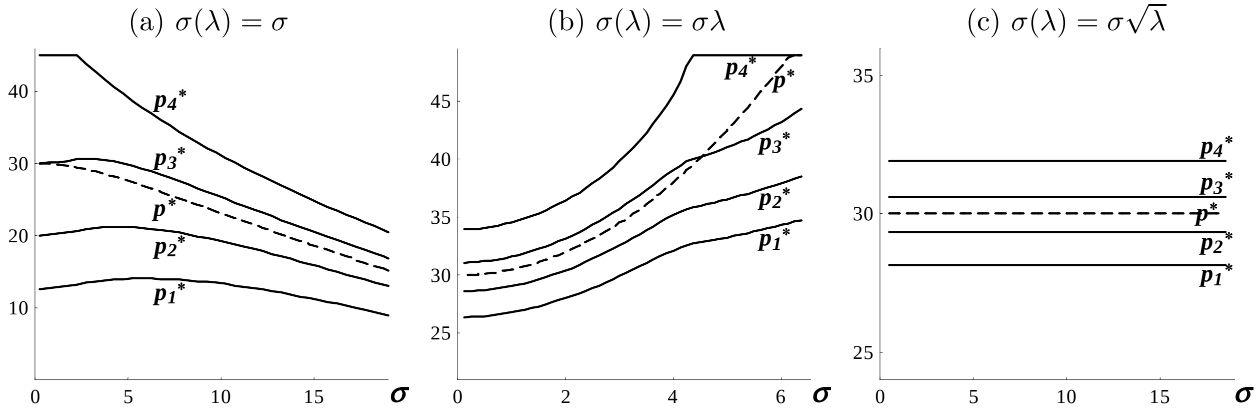
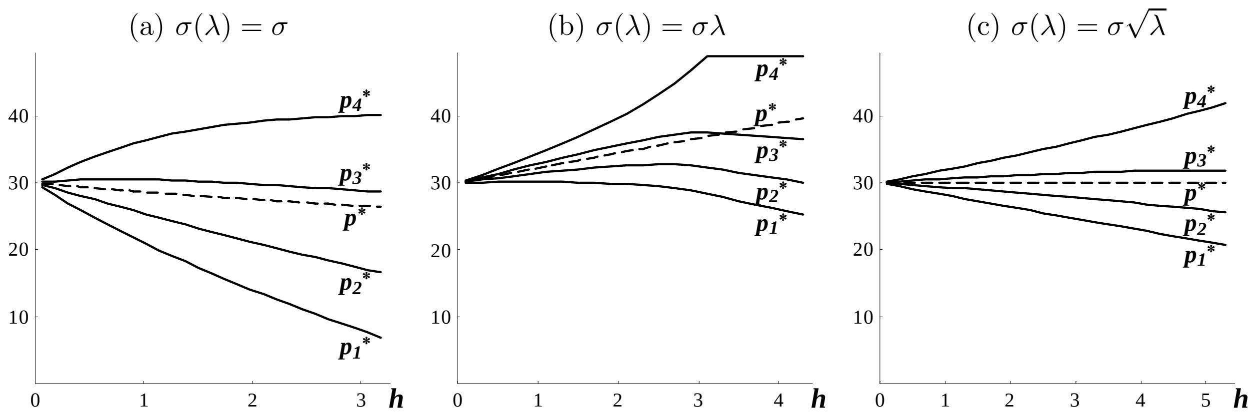


Figure 3: Effect of holding cost on optimal prices with a fixed replenishment level



2. Effect of holding cost:
   (a) Same setting as in part 1(a) of this example, and we fix $\sigma(\lambda) = 5$. The optimal prices are shown in Figure 3(a). The lowest price is decreasing in h while the others are not.
   (b) Same setting as in 1(b), with $\sigma(\lambda) = 3\lambda$. The highest price is increasing in h, while the others are not, shown in Figure 3(b).
   (c) Same setting as in 1(c). Figure 3(c) shows that p_N^* is increasing in h, and p_1^* is decreasing in h. If a single price is optimized, the optimal price p^* is invariant to h.

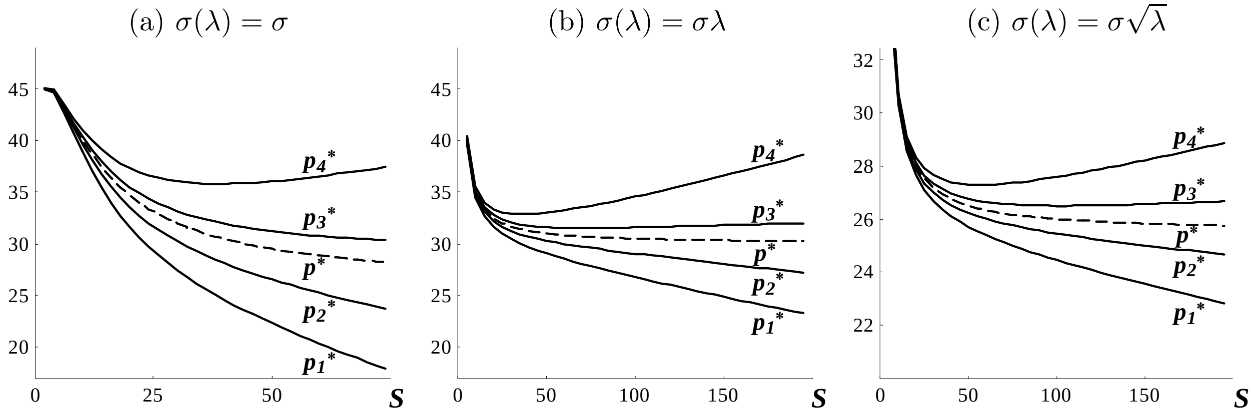
   In all three cases, the spread of the prices increases significantly with the increase in h.

3. Effect of replenishment level: Figure 4(a)-(c) show the effect of replenishment level on the

optimal prices under the three forms of demand variability. In all three cases, the lowest price is decreasing in S. (Note that $a'(S) < 0$ for all S considered in Figure 4.) The spread of the prices also increases in S.

All of the above examples show that $p_1^* \leq \cdots \leq p_N^*$. ∎

Figure 4: Effect of replenishment level on optimal prices with a fixed replenishment level



3.3 Joint Pricing-Replenishment Optimization

In this section, we consider the problem of deciding both prices and inventory to maximize the long-run average profit. Complications exist in the behavior of the optimal decisions. So we consider the single price case and the case of multiple price changes in sequence.

3.3.1 Single Price

In this case, the firm's problem is

$$\max_{S>0,\, \lambda \in \mathcal{L}} \qquad V(S,\lambda) = r(\lambda) - \frac{hS}{2} - \lambda a(S) - \frac{h\rho(\lambda)}{2}. \tag{11}$$

The first-order conditions are given by

$$\lambda a'(S) = -\frac{h}{2} \qquad \text{and} \qquad r'(\lambda) - \frac{h}{2}\rho'(\lambda) \quad = \quad a(S). \tag{12}$$

The difficulty is that $V(S,\lambda)$ may not be concave in (S,λ), and it may even have multiple local maxima (see Example 4 below).

Example 4 Consider $\lambda(p) = 50 - p$, $c(S) = 500 + 2S$, $\sigma(\lambda) = 0.2$, and $h = 1$. The optimal S for any fixed λ is given by $S(\lambda) = 10\sqrt{10\lambda}$. Substituting this EOQ into the first-order condition for λ

gives

$$50 - 2\lambda + \frac{1}{50\lambda^2} = \frac{5\sqrt{10}}{\sqrt{\lambda}} + 2.$$

However, simply solving the above equation yields three stationary points: $\lambda_1 = 0.0162$, $\lambda_2 = 0.101$, and $\lambda_3 = 22.33$. Using the second-order condition, it can be verified that λ_1 and λ_3 are local maxima, while λ_2 is a local minimum. Furthermore, $V(S(\lambda_1), \lambda_1) = -4.48$, $V(S(\lambda_2), \lambda_2) = -5.41$, and $V(S(\lambda_3), \lambda_3) = 423.8$. Hence $\lambda^* = \lambda_3 = 22.33$ and $S^* = 149.42$. ∎

We can further prove that when the demand variability is constant, the demand function and the replenishment cost function are both linear, a solution satisfying the first-order conditions and yielding a positive profit is the global optimum. To save space, the proof is omitted.

The monotonicity of the joint optimum is explored in the following proposition. Since the optimal solutions may not be unique, in lieu of increasing and decreasing, the relevant properties are *ascending* and *descending*, respectively (refer to Topkis 1978).

Proposition 3 *Suppose a single price and the replenishment level are jointly optimized.*

(i) If $\sigma(\lambda) = \sigma$, then p^* is descending in σ, and S^* is ascending in σ and descending in h;

(ii) If $\sigma(\lambda) = \sigma\lambda$, then p^* is ascending in σ and h, and S^* is descending in σ and h;

(iii) If $\sigma(\lambda) = \sigma\sqrt{\lambda}$, then p^* and S^* are independent of σ, p^* is ascending in h, and S^* is descending in h;

(iv) For general $\sigma(\lambda)$, the optimal profit is decreasing in h, and decreases when $\rho(\lambda)$ increases for all $\lambda \in \mathcal{L}$.

Figure 1 also serves to illustrate the effect of the demand variability in the above proposition. In Figure 1(b), a higher σ leads to a lower p^*, which in turn, based on Figure 1(a), leads to a higher S^*. This is stated in Proposition 3(i). In Figure 1(c), a higher σ results in a higher p^*, which in turn leads to a lower S^*. This is given in Proposition 3(ii). In Figure 1(d) the optimal price is invariant to σ, and thus S^* is unaffected by σ either. This is consistent with Proposition 3(iii).

The literature has also seen the differences in the monotonicity properties between the additive demand model and multiplicative demand model. (Refer to Federgruen and Heching 1999, and Polatoglu and Sahin 2000. See also Section 1.2).

Comparing the monotonicity of p^* in Proposition 3 with Proposition 2 applied to a single price setting, we see that the monotonicity of p^* with respect to the demand variability are the same no matter S is fixed or jointly optimized. However, that p^* is decreasing in h in Proposition 2(i) no longer holds when S is jointly optimized, and that p^* is invariant to h in Proposition 2(iii) no longer holds either. Two forces are present. A higher holding cost reduces the optimal replenishment level, leading to a higher unit cost – a force to increase price. Another force is that, if $\sigma(\lambda) = \sigma$, a lower price can offset the additional holding cost; while if $\sigma(\lambda) = \sigma\lambda$, a higher price can offset the additional holding cost. So the composite effect is mixed for $\sigma(\lambda) = \sigma$, but unambiguous in the other two cases, as stated in Proposition 3(ii) and (iii).

As an application of Proposition 3, we compare the joint optimization here with the sequential optimization scheme that is usually followed in practice: the marketing/sales department first makes the pricing decision, and then the purchase department decides the replenishment quantity based on demand projection as a consequence of the pricing decision. For instance, the marketing/sales department solves the problem:

$$\lambda^0 \quad = \quad \arg\max_{\lambda \in \mathcal{L}} \{ \ r(\lambda) \ \},$$

and sets $p^0 = p(\lambda^0)$. Then, the purchase department takes λ^0 as given and solves the problem:

$$S^0 \quad = \quad \arg\min_{S>0} \left\{ \ \lambda^0 a(S) + hS/2 \ \right\}. \tag{13}$$

Clearly, the sequential decision procedure does not take demand variability into account. Clearly, the optimal price and inventory level found by the sequential scheme only satisfy the first-order condition for S in (12).

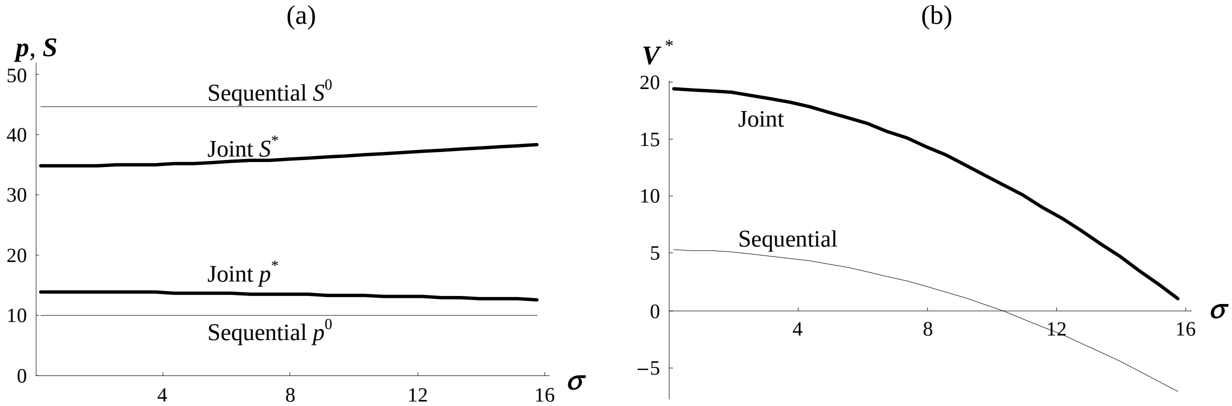
When $\sigma(\lambda) = \sigma$, the first-order condition for optimal price, $r'(\lambda) - a(S) + \frac{h\sigma^2}{2\lambda^2} = 0$, holds only when the demand variability happens to be $\widehat{\sigma} = \lambda^0\sqrt{2a(S^0)/h}$. Applying Proposition 3, we can see that if $\sigma < \widehat{\sigma}$, then the sequential decision underprices and overstocks; and if $\sigma > \widehat{\sigma}$, then the sequential decision overprices and understocks.

When $\sigma(\lambda) = \sigma\lambda$ or $\sigma(\lambda) = \sigma\sqrt{\lambda}$, the first-order condition for optimal price always leads to $r'(\lambda) > 0$, which implies that the sequential decision always underprices and overstocks.

Numerically, the sequential decision can lead to significant profit loss, as illustrated in the following examples.

Example 5 Let $\sigma(\lambda) = \sigma$, $\lambda(p) = 20 - p$, $c(S) = 100 + 5S$ and $h = 1$. Suppose the marketing/sales department sets price at $p^0 = 10$ in order to maximize the revenue rate. Then, the purchase department minimizes the operating cost using the EOQ model: $S^0 = 20\sqrt{5} \approx 44.7$. The threshold $\widehat{\sigma} = \lambda^0\sqrt{2a(S^0)/h} \approx 38$. In Figure 5 we compare the sequential decision with the joint decision. The parameter range is chosen such that the best decision will be able to achieve a positive profit. Substantial profit loss is observed: when $\sigma = 0$, the sequential decision both underprices and

Figure 5: Sequential vs. joint pricing and replenishment decisions



overstocks by 28%, resulting 73% profit loss compared to the joint decision; when $\sigma = 10$, the sequential decision underprices by 25% and overstocks by 22%, making almost no profit. ∎

3.3.2 N Prices

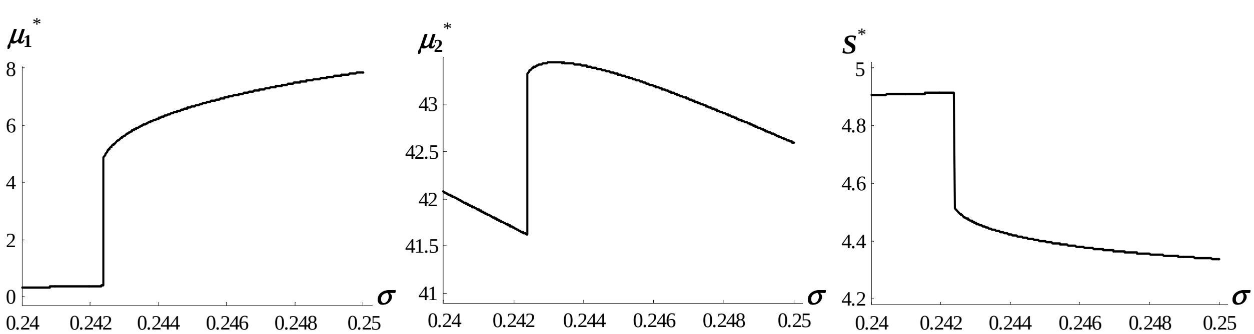
The problem is

$$\max_{S>0,\, \boldsymbol{\mu}\in\mathcal{M}^N} \qquad V(S,\boldsymbol{\mu}) = \frac{\sum_{n=1}^{N}\left[p(\frac{1}{\mu_n}) - \frac{hS}{N}(N-n+\frac{1}{2})\mu_n - \frac{1}{2}h\mu_n\rho(\frac{1}{\mu_n}) - a(S)\right]}{\sum_{n=1}^{N}\mu_n}. \tag{14}$$

As in the single price case, the objective function $V(S,\boldsymbol{\mu})$ may not be jointly concave. However, the monotonicity in Proposition 2 and Proposition 3 need not hold here, as evident from the following example.

Example 6 Let $\sigma(\lambda) = \sigma$, $p(\lambda) = 10 - 10^{-3}\lambda + \lambda^{-1}$, $c(S) = 50 + S^2$ and $h = 0.2$. (Note the term λ^{-1} in $p(\lambda)$ only adds a constant to the objective function to ensure that the average profit is positive.) Consider $N = 2$. The optimal solutions are plotted in Figure 6. Two observations emerge

Figure 6: Complications in the behavior of joint optimal solutions



from the figure. First, there exist jumps in the optimal policy, due to multiple local maxima. For example, for $\sigma = 0.243$, there are at least two local maximizers: $(S = 4.917, \mu_1 = 0.401, \mu_2 = 41.51)$ and $(S = 4.464, \mu_1 = 5.637, \mu_2 = 43.44)$. The first yields an objective value $V = 0.2639$, which is slightly higher than the second one ($V = 0.2638$). For $\sigma = 0.244$, the two local maximizers are slightly different: $(S = 4.918, \mu_1 = 0.448, \mu_2 = 41.33)$ and $(S = 4.425, \mu_1 = 6.248, \mu_2 = 43.41)$. However, the objective value corresponds to the second one ($V = 0.261908$) is slightly better than the first ($V = 0.261903$). Hence, the optimal policy exhibits discontinuity when σ varies from 0.243 to 0.244. Our second observation is that when the optimal solutions are continuous in σ, there exists a range in which both μ_1 and μ_2 are increasing in σ, while S is decreasing in σ. This is in sharp contrast with the results in the single-price case. ∎

Despite the lack of monotonicity, the result in Theorem 1 (which holds for any replenishment level) continues to hold here, i.e., $p_1^* \le p_2^* \le \cdots \le p_N^*$. Based on Theorem 2, we develop a bound on the profit improvement as the number of prices (N) increases.

Theorem 3 *Let $(S_N^*, \boldsymbol{\mu}^*)$ be the optimal joint pricing-replenishment decision, and V_N^* be the corresponding optimal profit in (14). Then,*

$$V_N^* - V_1^* \quad \le \quad \frac{h {S_N^*}^2 \left(1 - N^{-2}\right)}{12 \bar{\mu} \left(G_1 + G_2/h\right)},$$

where $\bar{\mu} = \frac{1}{N} \sum_{n=1}^{N} \mu_n^*$, $G_1 = \inf \left\{ d^2\left(\frac{1}{2}\mu\rho(\frac{1}{\mu})\right)/d\mu^2 \;:\; \mu \in [\mu_1^*, \mu_N^*] \right\}$ and $G_2 = \inf \left\{ -d^2 p(\frac{1}{\mu})/d\mu^2 \;:\; \mu \in [\mu_1^*, \mu_N^*] \right\}$.

Proof. Let $V_N^*(S)$ denote the optimal profit when S is given (i.e., pricing decision only). If S happens to be fixed at S_N^*, then the profit is V_N^*, i.e., $V_N^* = V_N^*(S_N^*)$. Applying Theorem 2, we have the desired bound immediately:

$$V_N^* - V_1^* \;=\; V_N^*(S_N^*) - V_1^*(S_1^*) \;\le\; V_N^*(S_N^*) - V_1^*(S_N^*) \;\le\; \frac{h {S_N^*}^2 \left(1 - N^{-2}\right)}{12 \bar{\mu} \left(G_1 + G_2/h\right)}.$$

∎

The shortfall of the above bound is that it involves the solution to the N-price problem. Heuristically, we can use the solution to the joint single-price and replenishment problem, denoted by (S_1^*, μ^*), in the upper bound. Specifically, replace S_N^* by S_1^*, replace $\bar{\mu}$ by μ^*, replace G_1 by $G_1^0 = d^2\left(\frac{1}{2}\mu^*\rho(\frac{1}{\mu^*})\right)/d\mu^2$, replace G_2 by $G_2^0 = -d^2 p(\frac{1}{\mu^*})/d\mu^2$, and replace $1 - N^{-2}$ by $3/4$ to tighten the bound. That is, the bound in Theorem 3 can be evaluated heuristically as follows:

$$\frac{h S_1^{*2}}{16 \mu^* \left(G_1^0 + G_2^0/h\right)}. \tag{15}$$

Finally, we numerically examine the profit improvement of using multiple price changes over a single price. To save space, we only present the case when $\sigma(\lambda) = \sigma$.

Example 7 Let $\sigma(\lambda) = \sigma$, $c(S) = K + cS$ and $p(\lambda) = a - b\lambda$, where a, b, c, K are all positive parameters. The optimization problem in (14) becomes

$$\max_{S>0,\, \boldsymbol{\mu} \in [b/a,\infty)^N} \qquad V(S, \boldsymbol{\mu}) = \frac{\sum_{n=1}^{N} \left[a - \frac{b}{\mu_n} - \frac{hS}{N}(N - n + \frac{1}{2})\mu_n - \frac{h\sigma^2}{2}\mu_n^2 - \frac{K}{S} - c \right]}{\sum_{n=1}^{N} \mu_n}.$$

Applying a change of variables, $\boldsymbol{\mu} = b\widetilde{\boldsymbol{\mu}}$ and $S = K\widetilde{S}$, we can rewrite the above problem as follows:

$$\max_{\widetilde{S}>0,\, \widetilde{\boldsymbol{\mu}} \in [1/a,\infty)^N} \qquad V(\widetilde{S}, \widetilde{\boldsymbol{\mu}}) = \frac{\sum_{n=1}^{N} \left[a - c - \frac{1}{\widetilde{\mu}_n} - \frac{Khb\widetilde{S}}{N}(N - n + \frac{1}{2})\widetilde{\mu}_n - \frac{hb^2\sigma^2}{2}\widetilde{\mu}_n^2 - \frac{1}{\widetilde{S}} \right]}{b \sum_{n=1}^{N} \widetilde{\mu}_n}. \quad (16)$$

Clearly, the above expression indicates that there are four degrees of freedom in terms of independent parameters: $(N,\, a - c,\, Khb,\, hb^2\sigma^2)$. Specifically, the four degrees of freedom are determined by N, either a or c, and two from (K, h, b, σ). In the numerical studies reported here and below, we choose to vary (N, c, h, σ) while fixing (K, a, b).

Figure 7: Effect of the number of price changes.

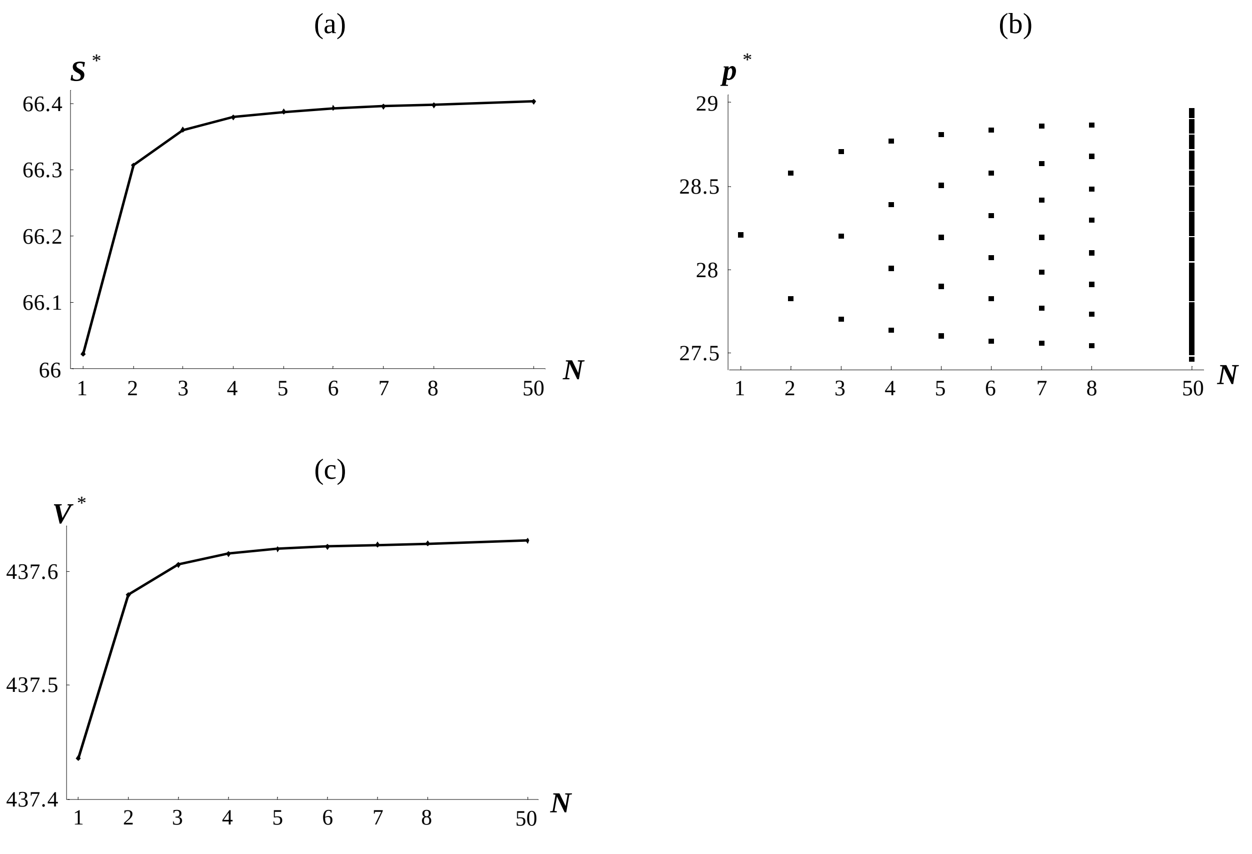
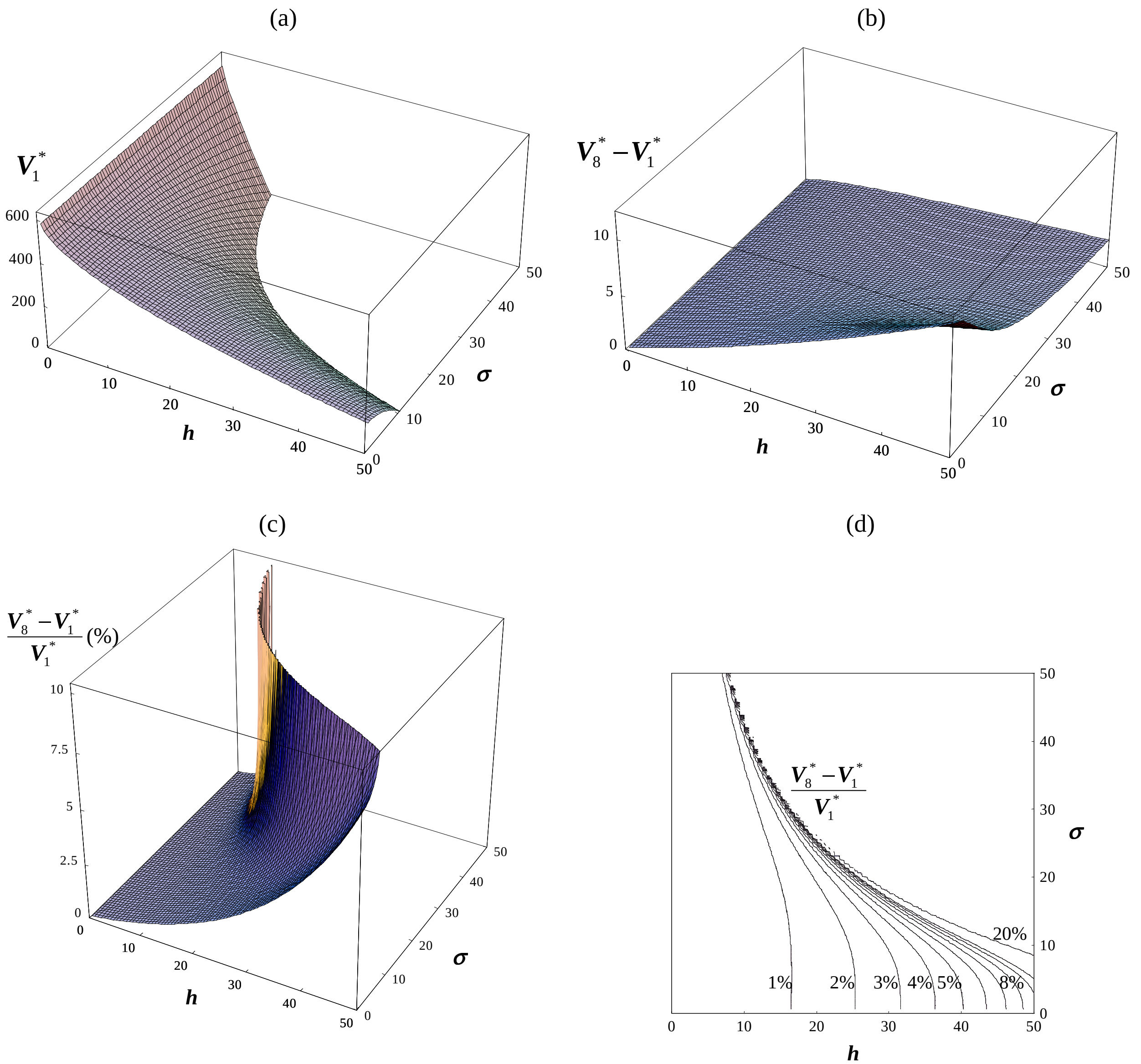


Figure 7 shows the optimal replenishment levels, prices, and profit values corresponding to different values of N, the number of price changes. We see that as N increases the optimal prices are inter-leaved (e.g., the optimal single price is sandwiched between the two-price solutions, which, in turn, are each sandwiched between a neighboring pair of the three-price solutions). This inter-leaving property seems to persist in all examples we have studied. Figure 7(c) shows that N has a decreasing marginal effect on profit. Using two prices already achieves most of the potential profit improvement, and beyond $N = 8$, the marginal improvement is essentially nil. ∎

Figure 8: Profit improvement of multiple price changes over a single price



Example 8 We continue with the last example, but focus on comparing the optimal profits under $N = 1$ and $N = 8$. We consider the parameter values $c \in (0, 30], h \in (0, 50], \sigma \in (0, 50]$, while fixing the others at $K = 100$, $a = 50$, $b = 1$.

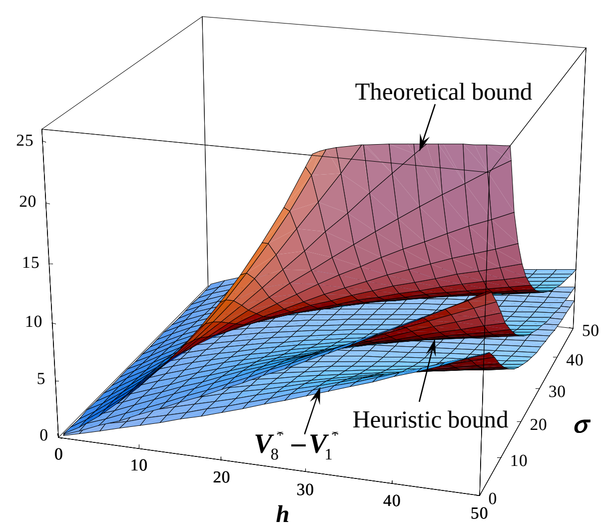
The results reported in Figure 8 are for $c = 1$. Similar results are observed for $c = 5, 10, 20, 30$.

Figure 8(a) shows the optimal profit corresponding to a single price. The profit is clearly decreasing in h and σ (Proposition 3). Furthermore, the negative effect of σ is larger when h is larger and vice versa, suggesting that the profit function is submodular in (h, σ). Intuitively, since the effect of σ shows up in the profit function through the additional holding cost, the higher the h (resp. σ) value, the more sensitive is the profit to σ (resp. h).

Figure 8(b) shows the absolute improvement in profit when using 8 prices ($V_8^* - V_1^*$). The improvement is increasing in h and decreasing in σ. Intuitively, as the inventory holding cost increases, the right trade-off among revenue, holding cost and replenishment cost becomes more important, thus more pricing options over time is more beneficial. However, pricing becomes less effective as demand variability increases.

Note that when the profit under a single price V_1^* approaches zero, the absolute improvement $(V_8^* - V_1^*)$ does not diminish. In particular, while V_1^* decreases in h, the improvement increases in h. Indeed, the *relative* improvement $(= \frac{V_8^* - V_1^*}{V_1^*})$ is increasing in h, and approaching infinity when V_1^* is close to zero, as demonstrated in Figure 8(c) and (d).

Figure 9: Bounds on the profit improvement



To conclude this example, we show in Figure 9 the profit improvement in comparison with the theoretical bound in Theorem 3 and heuristic bound in (15). The heuristic bound can be seen in this example as giving a good estimate of the maximum potential improvement. ∎

3.3.3 The Algorithm: Fractional Programming

To solve the fractional optimization problem in (14), we can instead solve the following:

$$\max_{S>0,\,\boldsymbol{\mu}\in\mathcal{M}^N} V_\eta \;:=\; \sum_{n=1}^{N}\left[p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{1}{2}h\mu_n\rho\Big(\frac{1}{\mu_n}\Big) - a(S)\right] - \eta\sum_{n=1}^{N}\mu_n, \qquad (17)$$

where η is a parameter. When the optimal objective value of the problem in (17) is zero, the corresponding solution is the optimal solution to the original problem in (14), and the corresponding η is the optimal value of the original problem. To see the equivalence, write the optimal value in (14) as $V^* = A^*/B^*$, where A and B denote the numerator and denominator in (14) respectively. When $\eta = V^*$, the optimal value of (17) is zero. On the other hand, suppose there exists an η which yields a zero objective value in (17), specifically, $A^* - \eta B^* = 0$, then for any feasible A, B, we have $A - \eta B \leq 0$, which implies $A/B \leq \eta = A^*/B^*$.

The algorithm described below takes advantage of the separability of the objective function with respect to $\boldsymbol{\mu}$ when S and η are given. Specifically, the sub-problems are:

$$\max_{\mu_n\in\mathcal{M}} g_n(\mu_n) := p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{1}{2}h\mu_n\rho\Big(\frac{1}{\mu_n}\Big) - a(S) - \eta\mu_n, \quad n = 1,\ldots,N. \quad (18)$$

Under Assumption 1, the objective in (18) is concave in μ_n.

`Algorithm for solving (14)`

1. Initialize $\eta = \eta_0$, $S = S_0$, and $\mu_n = \mu$ for $n = 1,\ldots,N$.
   (For instance, $\eta_0 = 0$; $S_0 = S^*$ and $\mu_n = \mu^*$ for all n,
   with (S^*, μ^*) being the optimal solution to the single-price problem.)
   Set ε at small positive values (according to the required precision).

2. Solve the N single-dimensional concave function maximization problems in (18).

3. Update S following a specified stepsize or use equation (6).
   If the difference between the new and the old S values is smaller than ε,
   go to step 4; otherwise, return to step 2.

4. $V_\eta =$ sum of the objective values of the N sub-problems.
   If $|V_\eta| \leq \varepsilon$, stop. Otherwise,
   if $V_\eta > \varepsilon$, increase η;
   if $V_\eta < -\varepsilon$, decrease η.
   Go to step 2.

In step 2, there are ways to accelerate the search procedure. First, we can take the advantage of the monotonicity of μ_n^* in n following Theorem 1. Second, we have $g_n(\mu_n^*) \le g_{n+1}(\mu_n^*) \le g_{n+1}(\mu_{n+1}^*)$, for $n = 1, \ldots, N-1$. Hence, if we find μ_1^* such that $g_1(\mu_1^*) > \varepsilon/N$, then we can be sure that $V_\eta > \varepsilon$, and to bypass the rest of the algorithm to increase η directly and proceed to the next loop. Similarly, if we first solve for μ_N^* and find that $g_N(\mu_N^*) < -\varepsilon/N$, we can decrease η directly. Third, we can make use of the monotonicity of μ_n^* in S according to Proposition 2 (iv). This is particularly useful when the stepsize for updating S in step 3 is small.

In step 3, if we are in the early stage of the algorithm, i.e., when V_η is still substantially away from zero, then the updating of S needs to cover a wide range so as not to miss the true optimum. This can be done by using large stepsize first and then reduce them gradually. When we are at the late stage of fine-tuning V_η, we can update S to nearby values.

In step 3, we can use (6) as an updating scheme. This is in fact the coordinate assent method: we alternate between optimizing $\boldsymbol{\mu}$ for fixed S and optimizing S for fixed $\boldsymbol{\mu}$. This procedure is guaranteed to converge to a local maximum, but not necessary the global maximum (see Luenberger 1984). So it is useful once the algorithm enters the region containing the true optimum without other local maxima.

It can be verified that the optimal objective value in (17) is strictly decreasing in η, so there is a unique zero-crossing point at which V_η is zero. Thus, in step 4, we can update η following a standard line search algorithm, such as the bi-section or the golden ratio.

The above algorithm with a slight modification appears more suitable for applications. In practice, it is often relevant to ask how many prices to use, when to change prices and what prices to change to. In addition, the prices and the replenishment level may be restricted to some given (discrete) sets of values. It is readily verified that the equivalence between (14) and (17) still holds under discrete sets. To solve the discrete version of (17), we simply modify step 2 of the algorithm such that N single-dimensional concave maximization problems are solved, and modify step 3 such that the updating scheme for S is restricted to the discrete set. Using a large N, the solution can effectively answer the questions posed above, as evident from the following example.

Example 9 Let $\lambda(p) = 50 - p$, $c(S) = 100 + S$, $h = 1$, $\sigma(\lambda) = 10$. Suppose the prices have to be integer valued, and S has to be a multiple of 5. Using the algorithm (adapted as above), and choosing $N = 140$, we find the optimal policy is to order 70 units for each cycle, charge a price of \$25 until the inventory level drops to 67 units, charge \$26 until inventory drops to 19, and charge \$27 until the inventory runs out. The policy uses only three prices (despite the choice of a large N value) and yields a profit of 528.745.

In other words, the equal partitioning of $[0, S]$ (into 140 segments) does not prevent the algorithm to find the optimal partitioning (of three uneven segments). ∎

4 Extensions

We have demonstrated the effectiveness of the Brownian motion model for price-sensitive demand, in particular in making optimal pricing and replenishment decisions, quantifying the profit improvement, and bringing out the impact of demand variability. The Brownian motion model also enables us to work out explicit analytical solutions, which facilitate making connections to and comparisons with, wherever applicable, previously known results in both deterministic and stochastic settings.

In the rest of this paper, we elaborate on three possible extensions of our model and results developed above.

4.1 Discounted Objective: An Example

Many results in Section 3 continue to hold if the long-run average objective is replaced by a discounted objective. While we relegate the technical details to an online companion of this paper, we present below a simple example to highlight the contrasts between the two cases.

Let $\gamma > 0$ be the discount rate. Under a single price policy, the discounted profit starting from zero inventory takes the form (equality (34) in the online companion),

$$V_\gamma(S, \lambda) \quad = \quad \frac{r(\lambda)}{\gamma} - \frac{S(h/\gamma + a(S))}{1 - e^{-b(\lambda)S}} + \frac{h\lambda}{\gamma^2},$$

where $b(\lambda) = 2\gamma/(\sqrt{\lambda^2 + 2\sigma(\lambda)^2\gamma} + \lambda)$. Unlike the average objective case, the additional cost due to demand variability, defined as the difference between $V_\gamma(S, \lambda)$ with a positive $\sigma(\lambda)$ and $V_\gamma(S, \lambda)$ with $\sigma(\lambda)$ set to zero, now depends on S.

Consider optimizing the replenishment level with a fixed price, that is

$$\max_{S>0} \; V_\gamma(S, \lambda).$$

The first-order condition is

$$[h/\gamma + a(S) + Sa'(S)](e^{bS} - 1) \quad = \quad bS[h/\gamma + a(S)]. \tag{19}$$

From Proposition 1, under the average objective and fixed price, the demand variability has no effect on the optimal replenishment level. In contrast, the demand variability will raise the replenishment level under the discounted objective. (See the proof in the online companion.) The following example illustrates this effect.

Suppose $c(S) = K + cS$, where $K, c > 0$. When γ is small, $b(\lambda)$ is also small and the first-order condition (19) can be written as

$$(h/\gamma + c)(bS + \tfrac{1}{2}b^2S^2 + o(b^2)) \;=\; bS(h/\gamma + K/S + c).$$

Then,

$$S^*_\gamma \;\approx\; \sqrt{\frac{2K\gamma}{b(h+c\gamma)}}.$$

Approximating b by Taylor series when γ is small: $b \approx \frac{\gamma}{\lambda} - \frac{\sigma(\lambda)^2\gamma^2}{2\lambda^3}$, we have

$$S^*_\gamma \;\approx\; \sqrt{\frac{2K\big(\lambda + \frac{1}{2}\rho(\lambda)\gamma\big)}{h+c\gamma}}.$$

Porteus (1985b) pointed out that the EOQ solution under a discounted objective can be approximated by $\sqrt{\frac{2K\lambda}{h+c\gamma}}$. Our result generalizes this result by incorporating the effect of demand variability.

4.2 Full Backordering

Our model can be extended to allow backordering, in which case the replenishment takes the form of a $(-s, S)$ policy: a replenishment order is issued whenever the number of backorders has reached s, to bring the inventory level back to S. Hence, the replenishment quantity is $S + s$. (As before, zero lead time is assumed.) The pricing policy is modified accordingly: equally divide S and s into N and M (positive integers) segments, respectively, such that

$$S = S_0 > S_1 > \cdots > S_N = 0 > S_{N+1} > \cdots > S_{N+M} = -s,$$

and apply price p_n until the net inventory (inventory on hand net the backorders) falls to S_n, $n = 1, \ldots, N + M$.

Assuming linear backordering cost, most of the results we have derived for the no-backorder case will continue to hold, with suitable modifications. For instance, the optimal pricing will satisfy the monotonicity:

$$p^*_1 \le p^*_2 \le \cdots \le p^*_N, \qquad p^*_{N+1} \ge p^*_{N+2} \ge \cdots \ge p^*_{N+M}.$$

That is, the optimal prices increase as the on-hand inventory is depleted, and decrease as the backorders accumulate. This pattern is consistent with the findings in Chen and Simchi-Levi (2006) for a periodic-review system.

The bound on the profit improvement is (assuming $M = N$):

$$V^*_N - V^*_1 \le \frac{\left(1 - N^{-2}\right)}{12}\left[\frac{hS^2}{\bar{\mu}_h\,(G_1 + G_2/h)} + \frac{h's^2}{\bar{\mu}'_h\,(G_1 + G_2/h')}\right],$$

where h' is the cost for holding one backorder per unit of time, $\bar{\mu}_h = \frac{1}{N}\sum_{n=1}^{N}\mu_n^*$, $\bar{\mu}'_h = \frac{1}{N}\sum_{n=N+1}^{2N}\mu_n^*$, G_1 and G_2 denote, as before, the lower bounds for $d^2\big(\frac{1}{2}\mu\rho(\frac{1}{\mu})\big)/d\mu^2$ and $-d^2p(\frac{1}{\mu})/d\mu^2$ over the spread of $\boldsymbol{\mu}^*$, respectively.

5 Appendix

Proof of Lemma 1. (This lemma follows from well-known results regarding optional stopping applied to Brownian motion; refer to, e.g., Karlin and Taylor (1975), and Ross (1996). The proof here is included for completeness.) Omit the subscript n, and write $T := \tau_n$, $x := S/N$. We have

$$\lambda T + \sigma B(T) = x. \tag{20}$$

This, combined with $\mathsf{E}B(T) = 0$, which follows from applying optional stopping to $B(t)$, leads to $\mathsf{E}[T] = x/\lambda$.

To derive the second moment of T, we have

$$\sigma^2[B^2(T) - T] = (x - \lambda T)^2 - \sigma^2 T.$$

Applying optional stopping again, to the martingale $B^2(t) - t$, we have

$$\mathsf{E}[(x - \lambda T)^2] = \sigma^2 \mathsf{E}[T],$$

which can be expressed as

$$\lambda^2 \mathsf{Var}(T) = \sigma^2 \mathsf{E}[T].$$

This establishes

$$\mathsf{E}[T^2] \;=\; \frac{\sigma^2}{\lambda^2}\mathsf{E}[T] + \mathsf{E}^2[T] \;=\; \frac{\sigma^2 x}{\lambda^3} + \frac{x^2}{\lambda^2}.$$

Now consider $\mathsf{E}[e^{-\gamma T}]$, where $\gamma > 0$ is a constant (discount rate). Write

$$-\gamma T = aB(T) - \frac{1}{2}a^2 T - bx, \tag{21}$$

where a and b are parameters to be determined. Then, applying optional stopping to the exponential martingale, $e^{aB(t)-\frac{1}{2}a^2 t}$, we have

$$\mathsf{E}[e^{-\gamma T}] = e^{-bx} \cdot \mathsf{E}[e^{aB(T)-\frac{1}{2}a^2 T}] = e^{-bx}.$$

The power b can be derived as follows. Substituting (20) into the right hand side of (21), we have

$$-\gamma T = aB(T) - \frac{1}{2}a^2 T - b\sigma B(T) - b\lambda T.$$

Equating the coefficients of T and $B(T)$ on both sides yields:

$$a = b\sigma, \qquad \frac{1}{2}a^2 + b\lambda = \gamma,$$

leading to

$$\sigma a^2 + 2\lambda a - 2\gamma\sigma = 0.$$

Hence,

$$a = \frac{\sqrt{\lambda^2 + 2\sigma^2\gamma} - \lambda}{\sigma};$$

and

$$\mathsf{E}[e^{-\gamma T}] = e^{-bx} = \exp[-\frac{\sqrt{\lambda^2 + 2\sigma^2\gamma} - \lambda}{\sigma^2}x].$$

∎

Proof of Lemma 2. By Theorem 1, $\mu_n^* \geq \mu_m^*$ for $n > m$. If $\mu_m^* = \mu_n^*$, then the desired inequality obviously holds. If $\mu_m^* < \mu_n^*$, consider the following alternative pricing policy: $(\mu_1^*, \ldots, \mu_m^* + \delta, \ldots, \mu_n^* - \delta, \ldots, \mu_N^*)$, where $0 < \delta < \frac{\mu_n^* - \mu_m^*}{2}$. Since this alternative cannot be better than the optimum, we have

$$\begin{aligned} & p(\tfrac{1}{\mu_m^*+\delta}) + p(\tfrac{1}{\mu_n^*-\delta}) - \tfrac{hS}{N}\left[(N-m+\tfrac{1}{2})(\mu_m^*+\delta) + (N-n+\tfrac{1}{2})(\mu_n^*-\delta)\right] \\ & -h\left[f(\mu_m^*+\delta) + f(\mu_n^*-\delta)\right] \\ \leq \; & p(\tfrac{1}{\mu_m^*}) + p(\tfrac{1}{\mu_n^*}) - \tfrac{hS}{N}\left[(N-m+\tfrac{1}{2})\mu_m^* + (N-n+\tfrac{1}{2})\mu_n^*\right] - h\left[f(\mu_m^*) + f(\mu_n^*)\right], \end{aligned}$$

where $f(\mu) := \frac{1}{2}\sigma(\frac{1}{\mu})^2\mu^2$. Taylor's expansion with straightforward algebra simplifies the inequality to

$$\tfrac{dp}{d\mu}(\tfrac{1}{\mu_m^*+\zeta_1\delta})\delta - \tfrac{dp}{d\mu}(\tfrac{1}{\mu_n^*-\zeta_2\delta})\delta + \tfrac{hS}{N}\delta(m-n) + h\delta\Big(\tfrac{df}{d\mu}(\mu_n^*-\zeta_3\delta) - \tfrac{df}{d\mu}(\mu_m^*+\zeta_4\delta)\Big) \;\; \leq \;\; 0,$$

where $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0,1]$. Dividing both sides by δ, and then letting $\delta \to 0$, we have

$$\tfrac{dp}{d\mu}(\tfrac{1}{\mu_m^*}) - \tfrac{dp}{d\mu}(\tfrac{1}{\mu_n^*}) + \tfrac{hS}{N}(m-n) + h\Big(\tfrac{df}{d\mu}(\mu_n^*) - \tfrac{df}{d\mu}(\mu_m^*)\Big) \;\; \leq \;\; 0. \tag{22}$$

Applying the mean-value theorem and using the definition of B and G, we have

$$\tfrac{dp}{d\mu}(\tfrac{1}{\mu_m^*}) - \tfrac{dp}{d\mu}(\tfrac{1}{\mu_n^*}) \;=\; -\tfrac{d^2p}{d\mu^2}(\tfrac{1}{\zeta_5})(\mu_n^* - \mu_m^*) \;\geq\; B(\mu_n^* - \mu_m^*), \tag{23}$$

$$\tfrac{df}{d\mu}(\mu_n^*) - \tfrac{df}{d\mu}(\mu_m^*) \;=\; \tfrac{d^2f}{d\mu^2}(\zeta_6)(\mu_n^* - \mu_m^*) \;\geq\; G(\mu_n^* - \mu_m^*) \tag{24}$$

where $\zeta_5, \zeta_6 \in [\mu_m^*, \mu_n^*]$. Combining (22)-(24) yields the desired inequality. ∎

Proof of Proposition 2. We shall write $\boldsymbol{\mu}^*(\sigma)$, $\boldsymbol{\mu}^*(h)$ and $\boldsymbol{\mu}^*(S)$ to emphasize the dependence of the maximizer of (9) on these parameters. First we prove the following lemma.

Lemma 3 *(i) If $\sigma(\lambda) = \sigma$, then $\mu_n^*(\sigma)$ is decreasing in σ if and only if*

$$\sum_{k=1}^{N} \mu_k^{*2} - 2\mu_n^* \sum_{k=1}^{N} \mu_k^* \quad < \quad 0; \tag{25}$$

and $\mu_n^*(h)$ is decreasing in h if and only if

$$\frac{S}{N} \sum_{k=1}^{N} (n-k)\mu_k^* + \sigma^2 \sum_{k=1}^{N} \left(\frac{\mu_k^{*2}}{2} - \mu_n^* \mu_k^* \right) \quad < \quad 0; \tag{26}$$

(ii) If $\sigma(\lambda) = \sigma\lambda$, then $\mu_n^*(\sigma)$ is increasing in σ for all $n = 1, \ldots, N$, and $\mu_n^*(h)$ is decreasing in h if and only if

$$\frac{S}{N^2} \sum_{k=1}^{N} (n-k)\mu_k^* \quad < \quad -\frac{\sigma^2}{2}; \tag{27}$$

(iii) If $\sigma(\lambda) = \sigma\sqrt{\lambda}$, then $\boldsymbol{\mu}^*(\sigma)$ is independent of σ, and $\mu_n^*(h)$ is decreasing in h if and only if

$$\frac{S}{N^2} \sum_{k=1}^{N} (n-k)\mu_k^* \quad < \quad 0; \tag{28}$$

(iv) For general $\sigma(\lambda)$ that satisfies Assumption 2, $\mu_n^*(S)$ is decreasing in S if and only if

$$\frac{h}{N^2} \sum_{k=1}^{N} (n-k)\mu_k^* \quad < \quad -a'(S). \tag{29}$$

Proof. (i) Suppose $\sigma(\lambda) = \sigma$. Consider an equivalent problem to the fractional problem (9):

$$\max_{\boldsymbol{\mu} \in \mathcal{M}^N} \quad \sum_{n=1}^{N} \left[p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{h\sigma^2}{2}\mu_n^2 - a(S) \right] - \eta(\sigma) \sum_{n=1}^{N} \mu_n,$$

where $\eta(\sigma)$ is the optimal value of (9) when the demand variability is σ and all the other parameters are fixed. (This equivalence is discussed in Section 3.3.3.) To solve the above problem, we can maximize each μ_n separately:

$$\max_{\mu_n \in \mathcal{M}} \quad g_n(\mu_n, \sigma) \; := \; p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{h\sigma^2}{2}\mu_n^2 - \eta(\sigma)\mu_n. \tag{30}$$

Noticing that the feasible set $\mathcal{M}$ is independent of σ, and applying the envelope theorem, we have

$$\frac{d\eta}{d\sigma} \quad = \quad -\frac{h\sigma \sum_{k=1}^{N} \mu_k^{*2}}{\sum_{k=1}^{N} \mu_k^*}.$$

To establish the desired monotonicity, it suffices to establish the submodularity of $g_n(\mu_n, \sigma)$ (refer to Topkis 1978):

$$\frac{\partial^2 g_n(\mu_n^*, \sigma)}{\partial \mu_n \partial \sigma} \quad = \quad -2h\sigma\mu_n^* - \frac{d\eta}{d\sigma} \quad = \quad \frac{h\sigma}{\sum_{k=1}^{N} \mu_k^*} \left(\sum_{k=1}^{N} \mu_k^{*2} - 2\mu_n^* \sum_{k=1}^{N} \mu_k^* \right),$$

If (25) holds, then the objective $g_n(\mu_n, \sigma)$ is submodular in (μ_n, σ) in a neighborhood of (μ_n^*, σ). Thus, μ_n^* is (locally) decreasing in σ. If the converse is true, μ_n^* is (locally) increasing in σ.

The proofs for the rest of the lemma are completely analogous, so we only list the objective functions as that in (30) and the cross-derivatives.

Part (i) regarding the monotonicity in h:

$$g_n(\mu_n, h) \;:=\; p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{h\sigma^2}{2}\mu_n^2 - \eta(h)\mu_n.$$

$$\begin{aligned}\frac{\partial^2 g_n(\mu_n^*, h)}{\partial \mu_n \partial h} &= -\frac{S(N-n+\frac{1}{2})}{N} - \sigma^2\mu_n^* + \frac{\sum_{k=1}^N \left[\frac{S}{N}(N-k+\frac{1}{2})\mu_k^* + \frac{\sigma^2}{2}\mu_k^{*2}\right]}{\sum_{k=1}^N \mu_k^*} \\ &= \frac{1}{\sum_{k=1}^N \mu_k^*}\left[\frac{S}{N}\sum_{k=1}^N (n-k)\mu_k^* + \sigma^2 \sum_{k=1}^N \Big(\frac{\mu_k^{*2}}{2} - \mu_n^*\mu_k^*\Big)\right],\end{aligned}$$

Part (ii) regarding the monotonicity in σ:

$$g_n(\mu_n, \sigma) \;:=\; p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{h\sigma^2}{2} - \eta(\sigma)\mu_n.$$

$$\frac{\partial^2 g_n(\mu_n^*, \sigma)}{\partial \mu_n \partial \sigma} \;=\; -\frac{\partial \eta}{\partial \sigma} \;=\; \frac{h\sigma N}{\sum_{k=1}^N \mu_k^*} > 0$$

Part (ii) regarding the monotonicity in h:

$$g_n(\mu_n, h) \;:=\; p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{h\sigma^2}{2} - \eta(h)\mu_n.$$

$$\begin{aligned}\frac{\partial^2 g_n(\mu_n^*, h)}{\partial \mu_n \partial h} &= -\frac{S(N-n+\frac{1}{2})}{N} + \frac{\sum_{k=1}^N \left[\frac{S}{N}(N-k+\frac{1}{2})\mu_k^* + \frac{\sigma^2}{2}\right]}{\sum_{k=1}^N \mu_k^*} \\ &= \frac{1}{\sum_{k=1}^N \mu_k^*}\left[\frac{S}{N}\sum_{k=1}^N (n-k)\mu_k^* + \frac{N\sigma^2}{2}\right],\end{aligned}$$

Part (iii) regarding the monotonicity in σ: When $\sigma(\lambda) = \sigma\sqrt{\lambda}$, the profit function (4) becomes

$$V(S, \boldsymbol{\mu}) = \frac{\sum_{n=1}^N \left[p(\frac{1}{\mu_n}) - \frac{hS}{N}(N-n+\frac{1}{2})\mu_n - a(S)\right]}{\sum_{n=1}^N \mu_n} - \frac{1}{2}h\sigma^2.$$

Clearly, $\boldsymbol{\mu}^*$ is invariant to σ.

Part (iii) regarding the monotonicity in h:

$$g_n(\mu_n, h) \;:=\; p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \eta(h)\mu_n.$$

$$\frac{\partial^2 g_n(\mu_n^*, h)}{\partial \mu_n \partial h} = -\frac{S(N-n+\frac{1}{2})}{N} + \frac{\sum_{k=1}^N \frac{S}{N}(N-k+\frac{1}{2})\mu_k^*}{\sum_{k=1}^N \mu_k^*}$$
$$= \frac{1}{\sum_{k=1}^N \mu_k^*} \frac{S}{N} \sum_{k=1}^N (n-k)\mu_k^*,$$

Part (iv) regarding the monotonicity in S:

$$g_n(\mu_n, S) := p\Big(\frac{1}{\mu_n}\Big) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{1}{2}h\mu_n\rho\Big(\frac{1}{\mu_n}\Big) - a(S) - \eta(S)\mu_n$$

$$\frac{\partial^2 g_n(\mu_n^*, S)}{\partial \mu_n \partial S} = -\frac{h(N-n+\frac{1}{2})}{N} + \frac{\sum_{k=1}^N \left[\frac{h}{N}(N-k+\frac{1}{2})\mu_k^* + a'(S)\right]}{\sum_{k=1}^N \mu_k^*}$$
$$= \frac{1}{\sum_{k=1}^N \mu_k^*}\left[\frac{h}{N}\sum_{k=1}^N (n-k)\mu_k^* + Na'(S)\right],$$

∎

Proof of Proposition 2.

(i) The inequality in (25) always holds for $n = N$, as μ_N^* is the largest component of $\boldsymbol{\mu}$ following Theorem 1. Hence, by Lemma 3, μ_N^* is decreasing in σ.

To prove the monotonicity in h, denote the left side of (26) by L_n. We will show that $L_1 + L_N < 0$ and $L_1 < L_N$, which then implies $L_1 < 0$.

$$L_1 + L_N = \frac{S}{N}\sum_{k=1}^N (N+1-2k)\mu_k^* + \sigma^2 \sum_{k=1}^N \left[\mu_k^{*2} - (\mu_1^* + \mu_N^*)\mu_k^*\right]$$
$$= \frac{S}{N}\Big[(N-1)(\mu_1^* - \mu_N^*) + (N-3)(\mu_2^* - \mu_{N-1}^*) + \cdots + (\mu^*_{\lfloor \frac{N}{2} \rfloor} - \mu^*_{\lfloor \frac{N+3}{2} \rfloor})\Big]$$
$$+ \sigma^2 \sum_{k=1}^N \Big[\mu_k^*(\mu_k^* - \mu_N^*) - \mu_1^*\mu_k^*\Big]$$
$$< 0,$$

where the last inequality follows from μ_n^* increasing in n. Next, note that

$$L_1 - L_N = \left[\frac{S(1-N)}{N} + \sigma^2(\mu_N^* - \mu_1^*)\right]\sum_{k=1}^N \mu_k^* \le 0,$$

where the last inequality follows from Lemma 2. Hence $L_1 < 0$, and $p_1^*(h)$ is decreasing in h.

(ii) By Lemma 3 (ii), the optimal prices are increasing in σ. By Lemma 3 (ii), if $n = N$, the inequality in (27) cannot holds. Hence μ_N^* is increasing in h.

(iii) By Lemma 3 (iii), the optimal prices are invariant to σ. By Lemma 3 (iii), the inequality in (28) always holds for $n = 1$, and never for $n = N$. Hence μ_1^* is decreasing in h, and μ_N^* is increasing in h.

(iv) By Lemma 3 (iv), if $a'(S) < 0$ and $n = 1$, the inequality in (29) always holds. Hence μ_1^* is decreasing in S. The monotonicity of the optimal average profit is obvious.

∎

Proof of Proposition 3. The joint optimization problem can be solved sequentially. We first optimize S for each fixed $\lambda > 0$ (same as Section 3.1). From Proposition 1, the optimal S is invariant to the demand variability, and we denote it by $S^*(\lambda)$. Then, the optimal λ can be found by

$$\max_{\lambda \in \mathcal{L}} \quad \widetilde{V}(\lambda, \sigma, h) \;:=\; r(\lambda) - \frac{hS^*(\lambda)}{2} - \lambda a(S^*(\lambda)) - \frac{h\rho(\lambda)}{2}.$$

If $\sigma(\lambda) = \sigma$, then $\widetilde{V}$ is supermodular in (λ, σ) since $\partial^2\widetilde{V}/\partial\lambda\partial\sigma = h\sigma/\lambda^2 \geq 0$, and therefore $\lambda^*(\sigma)$ is ascending in σ, or equivalently, the optimal price is descending in σ.

If $\sigma(\lambda) = \sigma\lambda$, then $\widetilde{V}$ is submodular in (λ, σ) since $\partial^2\widetilde{V}/\partial\lambda\partial\sigma = -h\sigma \leq 0$, and therefore $\lambda^*(\sigma)$ is descending in σ, or equivalently, the optimal price is ascending in σ. From Proposition 1, S^* is increasing in λ, and therefore, $\partial^2\widetilde{V}/\partial\lambda\partial h = -S^{*\prime}(\lambda)/2 - \sigma^2/2 < 0$, implying that $\widetilde{V}$ is submodular in (λ, h) and the optimal price is ascending in h.

If $\sigma(\lambda) = \sigma\sqrt{\lambda}$, then $\rho(\lambda) = \sigma^2$. Thus, σ affects the objective function only by a constant, so will not affect the optimal decisions. The optimal price is ascending in h, due to the submodularity of $\widetilde{V}$: $\partial^2\widetilde{V}/\partial\lambda\partial h = -S^{*\prime}(\lambda)/2 \leq 0$.

The monotonicity of S^* with respect to σ follows immediately from monotonicity of p^* with respect to σ and Proposition 1(i).

To examine the effect of h on S^*, we first optimize λ for each fixed $S > 0$ (same as Section 3.2), and denote the maximizer by $\lambda^*(S, h)$. Then, the optimal S is determined by

$$\max_{S>0} \quad \widetilde{V}(S, h) \;:=\; r(\lambda^*(S, h)) - \frac{hS}{2} - \lambda^*(S, h)a(S) - \frac{h}{2}\rho(\lambda^*(S, h)).$$

Let λ_S^*, λ_h^* and λ_{Sh}^* denote the partial derivatives. We have

$$\begin{aligned} \frac{\partial^2\widetilde{V}}{\partial S\partial h} &= r''\lambda_S^*\lambda_h^* + r'\lambda_{Sh}^* - \frac{1}{2} - a\lambda_{Sh}^* - a'\lambda_h^* - \frac{1}{2}\rho'\lambda_S^* - \frac{h}{2}\rho''\lambda_S^*\lambda_h^* - \frac{h}{2}\rho'\lambda_{Sh}^* \\ &= \lambda_S^*\Big(r''\lambda_h^* - \frac{1}{2}\rho' - \frac{h}{2}\rho''\lambda_h^*\Big) - \frac{1}{2} - a'\lambda_h^* + \Big(r' - a - \frac{h}{2}\rho'\Big)\lambda_{Sh}^* \end{aligned} \tag{31}$$

Assuming interior optimum, $\lambda^*(S, h)$ satisfies $r'(\lambda) - \frac{h}{2}\rho'(\lambda) - a(S) = 0$, and therefore,

$$\lambda_S^* = \frac{a'}{r'' - \frac{h}{2}\rho''}, \qquad \lambda_h^* = \frac{\frac{1}{2}\rho'}{r'' - \frac{h}{2}\rho''},$$

Then, (31) becomes

$$\frac{\partial^2 \widetilde{V}}{\partial S \partial h} = -\frac{1}{2} - a' \lambda_h^*$$

The first-order condition for S^* is $a'(S^*) = -\frac{h}{2\lambda^*}$, and thus,

$$\frac{\partial^2 \widetilde{V}}{\partial S \partial h} = -\frac{1}{2} + \frac{\frac{h}{4\lambda}\rho'}{r'' - \frac{h}{2}\rho''}$$

When $\sigma(\lambda) = \sigma$, $\rho(\lambda) = \sigma^2/\lambda$, we have

$$\frac{\partial^2 \widetilde{V}}{\partial S \partial h} = -\frac{1}{2} + \frac{-\frac{h\sigma^2}{4\lambda^{*3}}}{r'' - \frac{h\sigma^2}{\lambda^{*3}}} \leq -\frac{1}{2} + \frac{\frac{h\sigma^2}{4\lambda^{*3}}}{\frac{h\sigma^2}{\lambda^{*3}}} = -\frac{1}{4}.$$

When $\sigma(\lambda) = \sigma\lambda$, $\rho(\lambda) = \sigma^2\lambda$, we have

$$\frac{\partial^2 \widetilde{V}}{\partial S \partial h} = -\frac{1}{2} + \frac{\frac{h\sigma^2}{4\lambda^*}}{r''} \leq -\frac{1}{2}.$$

When $\sigma(\lambda) = \sigma\sqrt{\lambda}$, $\rho(\lambda) = \sigma^2$, we have

$$\frac{\partial^2 \widetilde{V}}{\partial S \partial h} = -\frac{1}{2}$$

In all cases, $\widetilde{V}(S, h)$ is submodular in (S, h) in the neighborhood of the optima, and therefore, S^* is descending in h. ∎

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