Optimal Pricing and Replenishment in a
Single-Product Inventory System

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Abstract

We study an inventory system that supplies price-sensitive demand modeled by Brownian motion, focusing on the optimal pricing and inventory replenishment decisions under both long-run average and discounted objectives. Optimal solutions are derived, and related to or contrasted against previously known results. In addition, we bring out the interplay between the pricing and the replenishment decisions, and the way they react to demand uncertainty. We show that the joint optimization of both decisions may result in significant profit improvement over the traditional way of making the decisions separately or sequentially. We also show that multiple price changes will only result in a limited profit improvement over the optimal single price.

Keywords: Joint pricing-replenishment decision, price sensitive demand, Brownian model.

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1 Introduction

We study a single-product continuous-review inventory model with price-sensitive demand. The cumulative demand process is modeled by a Brownian motion with a drift rate that is a function of the price. Replenishment is instantaneous, and demands are satisfied immediately upon arrival. Consequently, the replenishment follows a simple order-up-to policy, with the order-up-to level denoted by $S$.

We allow the pricing decisions to be dynamically adjusted over time. Specifically, we divide $S$ into $N$ equal segments, with $N$ a given integer. For each segment, there is a price. All $N$ prices are optimally determined, jointly with the replenishment level $S$, so as to maximize the expected long-run average or discounted profit.

There has been a substantial and growing literature on the joint pricing and inventory control. We refer the reader to two recent survey papers, Yano and Gilbert (2003), and Elmaghraby and Keskinocak (2003). Our review below will focus on papers that relate closely to our study.

Whitin (1955), Porteus (1985a), Rajan et al. (1992), among others, study demands that are deterministic functions of prices. Whitin (1955) connects pricing and inventory control in the EOQ (economic order quantity) framework, and Porteus (1985a) provides an explicit solution for the linear demand instance. Rajan et al. (1992) investigates continuous pricing for perishable products for which demands may diminish as products age.

Other works study stochastic demand models. Li (1988) considers a make-to-order production system with price-sensitive demand. Both production and demand are modeled by Poisson processes with controllable intensities. The control of demand intensity is through pricing. A barrier policy is shown to be optimal: when the inventory level reaches an upper barrier, the production stops; when the inventory level drops to zero, the demand stops (or the demand is lost). It is also shown that the optimal price is a non-increasing function of the inventory level.

Federgruen and Heching (1999) examine a model in which the firm periodically reviews inventory and decides both the replenishment quantity and the price to charge over the period. The replenishment cost is linear, without a fixed setup cost. The prices are changed only at the beginning of each period (as opposed to the continuous pricing scheme in Rajan et al. 1992). It is shown that a base-stock list-price policy is optimal for both average and discounted objectives. Earlier related works include Zabel (1972) and Thowsen (1975).

Thomas (1974), Polatoglu and Sahin (2000), Chen and Simchi-Levi (2004a,b), Feng and Chen (2004), and Chen, Ray and Song (2003) extend the above model to include a replenishment setup cost. In the full-backlog setting, it is first conjectured in Thomas (1974), and then proved in Chen
and Simchi-Levi (2004a), that the \((s, S, p)\) policy is optimal for additive demand (a deterministic demand function plus a random noise) in a finite horizon. Chen and Simchi-Levi (2004b) further proves the optimality of the stationary \((s, S, p)\) policy for general demand in an infinite horizon. Feng and Chen (2004) proves the optimality of \((s, S, p)\) policy under more general demand functions, but restricting the prices to a finite set. Assuming lost sales, Polatoglu and Sahin (2000) obtains rather involved optimal policies under a general demand model and provides restrictive conditions under which the \((s, S, p)\) policy is optimal. Chen, Ray and Song (2003) proves that \((s, S, p)\) policy is optimal under additive demand and lost sales.

Continuous-review models are studied in Feng and Chen (2003), and Chen and Simchi-Levi (2003). Feng and Chen (2003) models the demand as a price-sensitive Poisson process. Pricing and replenishment decisions are made upon finishing serving each demand, but the prices are restricted to a given finite set. An \((s, S, p)\) policy is proved optimal, with the optimal prices depending on the inventory level in a rather structured manner. Chen and Simchi-Levi (2003) generalizes this model by considering compound renewal demand process with both the inter-arrival times and the size of the demand depending on the price. It is shown that the \((s, S, p)\) policy is still optimal.

Our work differs from the above papers in two aspects. First, we model the demand process by Brownian motion, with a drift term being a function of the price. (In a more general model, we allow the diffusion term to depend on price as well.) The Brownian model, with its continuous path, is appropriate for modeling fast-moving items. It is a natural model when demand forecast involves Gaussian noises. Furthermore, it allows us to bring out explicitly the impact of demand variability in the optimal pricing and replenishment decisions, whereas results along this line are quite limited in the existing literature. Second, most of the papers cited above focus on establishing the optimality of the order-up-to replenishment policy. In contrast, our focus is on how, given the order-up-to policy, the optimal order quantity and the optimal prices vary and interact with other model parameters, particularly the demand variability. Indeed, by assuming zero lead time (which is appropriate when lead time is insignificant relative to the length of the replenishment cycle, and consistent with practices such as just-in-time or express delivery services), it is immediate that the order-up-to replenishment policy is optimal.

The Brownian demand model in a continuous-time setting is consistent with and is a natural extension of modeling the demand by a normal random variable in a discrete-time setting. The latter has been widely used both in the inventory literature and in practice. For instance, Zipkin (2000) presents a normal approximation (to a Poisson demand) for an inventory model with constant lead-time under a base-stock policy and demonstrates that the approximation works well.
For most of the paper, we shall focus on the Brownian demand model in (1) below, which treats the demand variability (the diffusion coefficient) as independent of the price. There are good reasons for studying such a model. First, the noise part in the demand model is usually due to factors other than price, such as forecast error and inventory accounting error: the former mainly depends on the information available, whereas the latter relies heavily on the prevailing technology. (For instance, the new radio frequency identification (RFID) technology has demonstrated great advantages in improving inventory accounting.) Many discrete inventory and pricing papers (Federgruen and Heching (1999), Chen and Simchi-Levi (2004a,b), Feng and Chen (2004), Chen, Ray and Song (2003)) have considered a similar demand model, i.e., a deterministic price-dependent part plus a random price-independent noise. The Brownian demand model in (1) is an analogous model in the continuous-time setting.

On the other hand, we show in Section 4.3 that most of our results can be extended to a Brownian demand model with a price-dependent diffusion coefficient.

We conclude this introductory section with a summary of some of the key findings and new insights from our study. These include:

- Demand variability incurs an additional inventory holding cost.
- The traditional way of separating the pricing and replenishment decisions could result in significant profit loss, as compared with the joint decision.
- In the single price case, as demand variability increases, the optimal price decreases and the optimal replenishment level increases.
- In the multiple price setting, optimal prices are increasing when inventories deplete in a cycle.
- Multiple price changes will only result in a limited profit improvement over a single price (when both are optimally determined). The relative improvement, however, becomes more significant in applications where the profit margin is low.

The rest of the paper is organized as follows. In Section 2, we present a formal description of our model. In Section 3, we study the optimal pricing and replenishment decisions under the long-run average objective. We start from making these decisions separately, so as to highlight the comparisons against prior studies and known results, and then present the joint optimization model and demonstrate the profit improvement. We conclude the paper pointing out several possible extensions in Section 4.
2 Model Description and Preliminary Results

We consider a continuous-review inventory model with a price-sensitive demand. The objective is to determine the inventory replenishment and pricing decisions that strike a balance between the sales revenue and the cost for holding and replenishing inventory over time, so as to maximize the expected long-run average or discounted profit. The specifics of the demand model, the cost parameters and the control policy are elaborated in the following subsections.

2.1 The Demand Model

The subject of our study is a single-product inventory system supplying a price-sensitive demand stream. The cumulative demand up to time \( t \) is denoted as \( D(t) \), and modeled by a diffusion process:

\[
D(t) = \int_0^t \lambda(p_u) du + \sigma B(t), \quad t \geq 0,
\]

where \( p_t \) is the price charged at time \( t \); \( \lambda(p_t) \) is the demand rate at time \( t \), which is a decreasing function of \( p_t \); \( B(t) \) denotes the standard Brownian motion; and \( \sigma \) is a positive constant measuring the variability of the demand (or the error of demand forecast). (The extension to price-dependent demand variability will be considered in Section 4.3.)

Let \( P \) and \( L \) denote, respectively, the domain and the range of the demand rate function \( \lambda(\cdot) \). Both are assumed to be intervals of \( \mathbb{R}_+ \) (the set of nonnegative real numbers).

**Assumption 1 (on demand rate)** The demand rate \( \lambda(p) \) and its inverse \( p(\lambda) \) are both strictly decreasing and twice continuously differentiable in the interior of \( P \) and \( L \), respectively. The revenue rate \( r(\lambda) = p(\lambda) \lambda \) is strictly concave in \( \lambda \).

Many commonly-used demand functions satisfy the above assumption, including the following examples, where the parameters \( \alpha, \beta \) and \( \delta \) are all positive:

- The linear demand function \( \lambda = \alpha - \beta p, \ p \in [0, \alpha/\beta] \): \( r(\lambda) = \frac{\alpha}{\beta} \lambda - \frac{1}{\beta} \lambda^2 \) is strictly concave;
- The exponential demand function \( \lambda = \alpha e^{-\beta p}, \ p \geq 0 \): \( r(\lambda) = -\frac{1}{\beta} \lambda \log(\lambda/\alpha) \) is strictly concave;
- The power demand function \( \lambda = \beta p^{-\delta}, \ p \geq 0 \): \( r(\lambda) = \lambda^{-\frac{1}{\delta} + 1} \beta^\frac{1}{\delta} \) is strictly concave if \( \delta \geq 1 \).
2.2 Cost Parameters

Let $h$ be the cost for holding one unit of inventory for one unit of time. Let $c(S)$ be the cost to replenish $S$ units of inventory.

**Assumption 2 (on replenishment cost)** The replenishment cost function $c(S)$ is twice continuously differentiable and increasing in $S$ for $S \in (0, \infty)$. The average cost $a(S) = c(S)/S$ is strictly convex in $S$, and $a(S) \to \infty$, as $S \to 0$.

Consider a special case: $c(S) = K + cS^{\delta}$, $S > 0$, where $K, c, \delta > 0$. When $\delta = 1$, this is the most commonly used linear function with a setup cost $K$. The average cost function, $a(S) = \frac{K}{S} + cS^{\delta-1}$ is convex if $\delta \in (0, 1] \cup [2, \infty]$. If $\delta \in (1, 2)$, $a(S)$ is convex in the region where $a'(S) \leq 0$. To see this, note that

$$a'(S) = \frac{1}{S^2}(-K + c(\delta - 1)S^{\delta}),$$

and when $\delta \in (1, 2)$ and $a'(S) \leq 0$, we have

$$a''(S) = -\frac{2-\delta}{S^3}\left(-\frac{2}{2-\delta}K + c(\delta - 1)S^{\delta}\right) \geq -\frac{a'(S)(2-\delta)}{S} \geq 0.$$

2.3 Pricing and Replenishment Policies

Assume replenishment is instantaneous, i.e., zero leadtime. We further assume that all orders (demands) will be supplied immediately upon arrival; i.e., no back-order is allowed, or there is an infinite back-order cost penalty. (Our results extend readily to the back-order case; refer to Section 4.2.)

The replenishment follows a continuous-review order-up-to policy. Specifically, whenever the inventory level drops to zero, it is brought up to $S$ instantaneously via a replenishment, where $S$ is a decision variable. We shall refer to the time between two consecutive replenishment epochs as a cycle.

We adopt the following dynamic pricing strategy. Let $N \geq 1$ be a given integer, and let $S = S_0 > S_1 > \cdots > S_{N-1} > S_N = 0$. Immediately after a replenishment at the beginning of a cycle, price $p_1$ is charged until the inventory drops to $S_1$; price $p_2$ is then charged until the inventory drops to $S_2$; ...; and finally when the inventory level drops to $S_{N-1}$, price $p_N$ is charged until the inventory drops to $S_N = 0$, when another cycle begins. The same pricing strategy applies to all cycles. For simplicity, we set $S_n = S(N - n)/N$. That is, we divide the full inventory of $S$ units into $N$ equal segments, and price each segment with a different price as the inventory is depleted by demand. In general, the segments are not necessarily equal, but our experiments show that
the optimal segments (optimized jointly with prices and replenishment level) are quite even and the corresponding profit is almost the same as using equal segments. Furthermore, in practical situations when the prices are within a discrete set, our algorithm based on equal segments can effectively tell how many prices to use, when to change prices and what prices to change to. (See Example 12 for elaboration.)

In summary, the decision variables are: \((S, \mathbf{p})\), where \(S \in \mathbb{R}_+\), and \(\mathbf{p} = (p_1, \ldots, p_N) \in \mathcal{P}^N\). Within a cycle, we shall refer to the time when the price \(p_n\) is applied as period \(n\).

## 2.4 The Inventory Process

Without loss of generality, suppose at time zero the inventory is filled up to \(S\). We focus on the first cycle which ends at the time when inventory reaches zero. Let \(T_0 = 0\), and let \(T_n\) be the first time when inventory drops to \(S_n\):

\[
T_n := \inf \{t \geq 0 : D(t) = nS/N\}, \quad n = 1, 2, \ldots, N.
\]

The length of period \(n\) is therefore \(\tau_n := T_n - T_{n-1}\). Since \(T_n\)'s are stopping times, by the strong Markov property of Brownian motion, \(\tau_n\) is just the time during which \(S/N\) units of demand has occurred under the price \(p_n\). That is,

\[
\tau_n \overset{\text{dist.}}{=} \inf \{t \geq 0 : \lambda(p_n)t + \sigma B(t) = S/N\}.
\]

(2)

Let \(X(t)\) denote the inventory-level at \(t\). We have,

\[
X(t) = S - D(t), \quad t \in [0, T_N).
\]

Since our replenishment-pricing policy is stationary, \(X(t)\) is a regenerative process with the replenishment epochs being its regenerative points.

We conclude this section with a lemma, which gives the first two moments and the generating function of the stopping time \(\tau_n\). (The proof is in the appendix.)

**Lemma 1** For the stopping time \(\tau_n\) in (2), we have

\[
\begin{align*}
E[\tau_n] &= \frac{S}{N\lambda_n}, \\
E[\tau_n^2] &= \frac{\sigma^2}{\lambda_n^2}E[\tau_n] + E^2[\tau_n] = \frac{\sigma^2 S}{N\lambda_n^2} + \frac{S^2}{N^2\lambda_n^2}, \\
E[e^{-\gamma \tau_n}] &= \exp \left[ -\frac{\sqrt{\lambda_n^2 + 2\sigma^2\gamma} - \lambda_n S}{\sigma^2} \right],
\end{align*}
\]

where \(\lambda_n = \lambda(p_n) > 0\), and \(\gamma > 0\) is a parameter.
3 Long-Run Average Objective

To optimize the long-run average profit, thanks to the regenerative structure of the inventory process, it suffices for us to focus on the first cycle. Recall that period $n$ refers to the period in which the price $p_n$ applies, and the inventory drops from $S_{n-1} = \frac{(N-n+1)S}{N}$ to $S_n = \frac{(N-n)S}{N}$. We first derive the total inventory over this period. Applying integration by parts, and recognizing $dX(t) = -dD(t)$, $X(T_n-1) = S_n-1$, $X(T_n) = S_n$, we have

$$\int_{T_{n-1}}^{T_n} X(t)dt = T_nS_n - T_{n-1}S_{n-1} - \int_{T_{n-1}}^{T_n} tdX(t)$$

$$= T_nS_n - T_{n-1}S_{n-1} - \int_{T_{n-1}}^{T_n} T_{n-1}dX(t) + \int_{T_{n-1}}^{T_n} (t - T_{n-1})dD(t)$$

$$= \tau_nS_n + \int_{T_{n-1}}^{T_n} (t - T_{n-1})[\lambda(p_n)dt + \sigma dB(t)].$$

A simple change of variable yields

$$\int_{T_{n-1}}^{T_n} (t - T_{n-1})\lambda(p_n)dt = \int_{0}^{\tau_n} u\lambda(p_n)du = \frac{1}{2}\lambda(p_n)\tau_n^2,$$

whereas

$$E\left[\int_{T_{n-1}}^{T_n} (t - T_{n-1})dB(t)\right] = E\left[\int_{T_{n-1}}^{T_n} tdB(t)\right] - E\left[T_{n-1}\right]E\left[B(T_n) - B(T_{n-1})\right] = 0,$$

follows from the martingale property of $B(t)$ and the optional stopping theorem.

Let $v_n(S, p_n)$ denote the expected profit (sales revenue minus inventory holding cost) during period $n$. Then, making use of the above derivation, along with Lemma 1, we have

$$v_n(S, p_n) = \frac{p_nS}{N} - E\left[\int_{T_{n-1}}^{T_n} hX(t)dt\right]$$

$$= \frac{p_nS}{N} - hE[\tau_n]S_n - \frac{1}{2}h\lambda(p_n)E[\tau_n^2]$$

$$= \frac{p_nS}{N} - hS^2(N - n + \frac{1}{2}) - \frac{h\sigma^2S}{2N\lambda(p_n)},$$

(3)

Note in the above expression, the first term is the sales revenue from period $n$, the second term is the inventory holding cost attributed to the deterministic part of the demand (i.e., the drift part of the Brownian motion), and the last term is the additional holding cost incurred by demand uncertainty.
For ease of analysis, below we shall often use \(\{\mu_n = \lambda(p_n)^{-1}, n = 1, \ldots, N\}\) as decision variables and denote \(\mu = (\mu_1, \ldots, \mu_N) \in \mathcal{M}^N\) and \(\mathcal{M} = \{\lambda(p)^{-1} : p \in \mathcal{P}\}\). Then, the long-run average objective can be written as follows:

\[
V(S, \mu) = \frac{\sum_{n=1}^{N} v_n(S, p_n) - c(S)}{S N \sum_{n=1}^{N} \mu_n} = \frac{\sum_{n=1}^{N} \left[p(\frac{1}{\mu_n}) - \frac{hS}{N} (N - n + \frac{1}{2})\mu_n - \frac{h\sigma^2}{2} \mu_n^2 - a(S)\right]}{\sum_{n=1}^{N} \mu_n}.
\]

(4)

The additional holding cost rate due to demand uncertainty is represented by \(\frac{h\sigma^2 \sum_{n=1}^{N} \mu_n^2}{2 \sum_{n=1}^{N} \mu_n}\).

For the special case when \(N = 1\), the price and the demand rate are both constants in a cycle (and hence constant throughout the horizon). The objective function (4) takes a simpler form. For comparison with some classical work, we use \(\lambda\) as the decision variable and denote the long-run average profit under single-price policy as \(V(S, \lambda)\). Then,

\[
V(S, \lambda) = \frac{v_1(S, \lambda) - c(S)}{S/\lambda} = r(\lambda) - \frac{hS}{2} - \lambda a(S) - \frac{h\sigma^2}{2\lambda}.
\]

(5)

The classical EOQ model only consists of the second and third terms in (5), which are the average inventory cost if the demand is deterministic with a constant rate. Whitin (1955) and Porteus (1985a) considered the price-sensitive EOQ model, which involves the first three terms in (5). Our model gives rise to an additional holding cost due to demand variability.

### 3.1 Optimal Replenishment with Fixed Prices

In this case, the set of prices \(p\), or equivalently, \(\mu\) is given, and the firm’s problem is

\[
\max_{S > 0} V(S, \mu).
\]

Under Assumption 1, \(V(S, \mu)\) is strictly concave in \(S\). Note that \(V(S, \mu) \to -\infty\) as \(S \to \infty\) or \(S \to 0\) (the latter is due to Assumption 2 that \(a(S) \to \infty\) as \(S \to 0\)). Thus, the unique optimal replenishment level is determined by the first-order condition:

\[
S^* = a^{-1} \left(- \frac{hN}{N^2} \sum_{n=1}^{N} (N - n + \frac{1}{2})\mu_n\right),
\]

(6)

where \(a^{-1}\) is well-defined since \(a(\cdot)\) is strictly convex and \(a'(\cdot)\) is strictly increasing under Assumption 2. In practice, the average cost as a function of quantity is usually first decreasing (due to economy of scale) and then increasing (due to capacity or other technological restrictions). However, at the optimal replenishment level, we have \(a'(S^*) < 0\) for any fixed prices. This observation helps to reduce the search space when the replenishment level is optimized jointly with pricing decisions (see Section 3.3).
The way that the demand variability and the holding cost impact the optimal replenishment level can be readily derived from (6).

**Proposition 1** With prices fixed,

(i) the optimal replenishment level $S^*$ is independent of $\sigma$, and decreasing in $h$ and in $p_n$ (for any $n$);

(ii) the optimal profit is decreasing in $\sigma$ and $h$.

**Proof.** Part (i) is obvious from (6). Part (ii) is also obvious since the objective in (4) is decreasing in $\sigma$ and $h$ in any case. ■

The interpretation of this proposition is quite intuitive, except that the optimal replenishment level is independent of $\sigma$. The latter results from the fact that the demand variability impacts on the average profit function through the additional holding cost term, $h\sigma^2/(2\lambda)$, which does not affect the replenishment decision. This will not be the case under a discounted objective, as will be shown in Section 4.1.

**Example 1** Consider the linear replenishment cost: $c(S) = K + cS$, where $K, c > 0$. The first-order condition in (6) leads to the familiar EOQ formula:

$$S^* = \sqrt{\frac{2K\lambda_a}{h}}, \quad \text{where} \quad \lambda_a = \left(\sum_{n=1}^{N} \frac{2N - 2n + 1}{N^2} \mu_n\right)^{-1}. \quad (7)$$

In the standard EOQ model, the demand rate is taken as the average demand per time unit, which in our setting becomes

$$\bar{\lambda} = \left(\frac{1}{N} \sum_{n=1}^{N} \mu_n\right)^{-1}. \quad (8)$$

Suppose that the fixed prices satisfy $p_1 \leq p_2 \leq \cdots \leq p_N$ (or $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$), i.e., a lower price is charged when the inventory level is higher (we will see this price pattern in Proposition 3). Higher weights are given to smaller $\mu_n$’s in (7), while $\mu_n$’s are equally weighted in (8). Hence, $\bar{\lambda} \leq \lambda_a$. This implies that the standard EOQ with a time-average demand rate may lead to a replenishment level lower than the optimally desirable. ■

**Example 2** Let $c(S) = 100 + 5S$, $\lambda(p) = 50 - p$, $h = 1$ and $N = 1$. In Figure 1(a), the EOQ quantity is shown as a function of the single fixed price, $S^* = 10\sqrt{100 - 2p}$. Figure 1(b) illustrates the optimal profit corresponding to various levels of demand variability. The peak of each $V^*$ curve is reached when both the price and the replenishment level are jointly optimized. When the price deviates far away from the optimum, the profit falls dramatically. ■
3.2 Optimal Pricing with a Fixed Replenishment Level

3.2.1 The Single-Price Problem

Here $N = 1$. We want to optimize the single price, or equivalently, the demand rate, as follows:

$$\max_{\lambda \in \mathcal{L}} V(S, \lambda) = r(\lambda) - \frac{hS}{2} - \lambda a(S) - \frac{h\sigma^2}{2\lambda}.$$ 

Under Assumptions 1 and 2, $V(S, \lambda)$ is strictly concave in $\lambda$. Assuming an interior solution (which is rather innocuous, since the last two terms in the objective forbid $\lambda$ to be extreme), the unique optimal $\lambda$ follows from the first-order condition:

$$r'(\lambda) + \frac{h\sigma^2}{2\lambda^2} = a(S). \quad (9)$$

We have the following proposition.

**Proposition 2** Suppose a single price is optimized while the replenishment level is held fixed.

(i) The optimal price $p^*$ is decreasing in $\sigma$ and $h$.

(ii) The optimal price $p^*$ is increasing in $a(\cdot)$; if $a(S)$ is decreasing in $S$, then $p^*$ is decreasing in $S$.

(iii) The optimal profit is decreasing in $\sigma$ and $h$.

**Proof.** For (i) and (ii), note that $\frac{\partial^2 V}{\partial \lambda \partial \sigma} = \frac{h\sigma}{\lambda^2} > 0$, $\frac{\partial^2 V}{\partial \lambda \partial h} = \frac{\sigma^2}{2\lambda^2} > 0$, and $\frac{\partial^2 V}{\partial \lambda \partial a(S)} = -1 < 0$. That is, the objective function $V$ is supermodular in $(\lambda, \sigma)$, supermodular in $(\lambda, h)$ and submodular in $(\lambda, a(S))$. Hence, (i) and (ii) follow from standard results for supermodular/submodular functions (Topkis 1978). Part (iii) is obvious. 

**Example 3** If $\lambda(p) = \beta p^{-1}$, $c(S) = K + cS$, where $\beta, K, c > 0$, then $r(\lambda) = \beta$, and the first-order condition (9) becomes

$$\frac{h\sigma^2}{2\lambda^2} = \frac{K}{S} + c,$$

leading to the optimal price

$$p^* = \frac{\beta}{\lambda^*} = \frac{\beta}{\sigma} \sqrt{\frac{2(K + c)}{h}},$$

which is clearly decreasing in $\sigma$, $h$ and $S$. 

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Example 4 Under the same setting as Example 2, Figure 2 plots the optimal price and the corresponding profit against various levels of replenishment and demand variability. The results depict the qualitative trends in Proposition 3.2. The peak of each $V^*$ curve is reached when the price and inventory are jointly optimized. In contrast to Figure 1(b), the profit here appears less sensitive to the replenishment level, as long as the pricing decision is optimized. ■

Figure 2: Optimal price with a fixed replenishment level

3.2.2 The $N$-Price Problem

In this case, the decision variables are the set of prices $p$, or equivalently, $\mu$. Specifically, the problem is

$$
\max_{\mu \in \mathcal{N}} V(S, \mu) = \frac{\sum_{n=1}^{N} \left[ p\left(\frac{1}{\mu_n}\right) - \frac{hS}{N}(N - n + \frac{1}{2})\mu_n - \frac{ha^2}{2} \mu_n^2 - a(S) \right]}{\sum_{n=1}^{N} \mu_n}.
$$

(10)

The strict concavity of the revenue function $r(\lambda)$ (Assumption 1) implies that the function $p\left(\frac{1}{\mu}\right)$ is strictly concave in $\mu$. This is because $r''(\lambda) = 2p'(\lambda) + \lambda p''(\lambda)$ and $d^2 p\left(\frac{1}{\mu}\right)/d\mu^2 = \frac{1}{\mu^3} (2p'(1/\mu) + \frac{1}{\mu} p''(1/\mu))$ have the same sign. Thus, the numerator of the objective is strictly concave in $\mu$. The ratio of a concave function over a positive linear function is known to be pseudo-concave (Mangasarian 1970). Hence, $V(S, \mu)$ is pseudo-concave in $\mu$. Furthermore, it is strictly pseudo-concave in the sense that

$$
\nabla_\mu V(S, \mu_1) (\mu_2 - \mu_1) \leq 0 \quad \Rightarrow \quad V(S, \mu_2) < V(S, \mu_1).
$$

Hence, the optimal $\mu$ must be unique. (This follows from a straightforward extension of Mangasarian (1970) from pseudo-concavity to strict pseudo-concavity.) Suppose the optimality is achieved at an interior point. Then the optimal $\mu$ is uniquely determined by the first-order condition:

$$
\frac{\partial \tilde{v}_n(S, \mu_n)}{\partial \mu_n} = \frac{\sum_{k=1}^{N} \tilde{v}_k(S, \mu_k) - c(S)}{\sum_{n=1}^{N} \mu_k}, \quad n = 1, \ldots, N,
$$

(11)

where $\tilde{v}_n(S, \mu_n) = v_n(S, p\left(\frac{1}{\mu_n}\right))$. Note that the right side of the above does not depend on $n$. This implies that the optimal pricing must be such that the marginal profit is the same across all periods (assuming an interior optimum).

The following proposition describes the basic optimal pricing pattern.

**Proposition 3** For any replenishment quantity $S$, the optimal prices are increasing over the periods, i.e., $p_1^* \leq p_2^* \leq \cdots \leq p_N^*$.  

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Proof. Since \( p_n = p(\frac{1}{\mu_n}) \) is increasing in \( \mu_n \), it suffices to show that the optimal \( \mu_n^* \) is increasing in \( n \). But this follows immediately from (10), since given any \( (\mu_1, \ldots, \mu_N) \), rearranging these variables (but not changing their values) in increasing order will maximize the term \( \sum_{n=1}^{N} n\mu_n \) in the numerator, while all other terms remain unchanged. 

In other words, within each cycle, the optimal prices are increasing over the periods. Intuitively, when the inventory is higher at the beginning of a cycle, we charge a lower price to induce higher demand so as to reduce the inventory holding cost.

The next proposition is about the impact of the parameters, \( \sigma, h \) and \( S \) on the optimal prices. (The proof is in the appendix.)

**Proposition 4** Assuming interior optimum,

(i) The highest price \( p_N^* \) is decreasing in \( \sigma \).
(ii) The lowest price \( p_1^* \) is decreasing in \( h \).
(iii) If \( a'(S) < 0 \), then \( p_1^* \) is decreasing in \( S \).
(iv) The optimal profit is decreasing in \( \sigma \) and \( h \).

Part (i)-(iii) of the above proposition rely on the interior optimum assumption, but we find through numerical tests that these monotonicity results hold even without the interior optimum assumption. This proposition corresponds to Proposition 2 for the single price case. The next example provides comparison plots for these two propositions and also provides an illustration for Proposition 3.

**Example 5** (a) Let \( p(\lambda) = 50 - 10\lambda, c(S) = 500 + 5S, S = 100, h = 2 \), and let \( \sigma \) vary from 0 to 20. For \( N = 4 \), the optimal prices are plotted in Figure 3(a). The highest price is decreasing in \( \sigma \) while the others are not. There also exist other instances in which all the prices are decreasing in \( \sigma \). The demand variability incurs additional holding cost \( \frac{h\sigma^2}{2} \sum \mu_n^2 \). When \( \sigma \) increases, in order to balance out the increase of this cost, intuitively we need to decrease \( \mu \) and at the same time decrease the spread of \( \mu \) (i.e., \( \mu_N - \mu_1 \)). The composite effect is mixed except for \( \mu_N^* \).

(b) Let \( p(\lambda) = 50 - 10\lambda, c(S) = 500 + 5S, S = 100, \sigma = 1 \), and let \( h \) vary from 0 to 3. The optimal prices are shown in Figure 3(b). The lowest price is decreasing in \( h \). The spread of the prices increases significantly with the increase in \( h \).

(c) Let \( p(\lambda) = 50 - \lambda, c(S) = 100 + S, h = 0.5, \sigma = 3 \), and let \( S \) vary from 0 to 400. The optimal prices are shown in Figure 3(c). Note that \( a'(S) < 0 \) for all \( S \). The lowest price is decreasing in \( S \). The spread of the prices increases in \( S \).
Figure 3: Optimal $N$ prices and the optimal single price

All the above examples show $p_1^* < \cdots < p_N^*$. For comparison, the optimal price $p^*$ for the single-price problem is also plotted in Figure 3.

Finally we derive an upperbound to quantify the potential benefit of multiple price changes over a single price. To provide intuition of what enhances and what limits the profit improvement, we note that the holding cost attributed to the deterministic part of the demand is the only motive for varying prices over the periods. In the numerator of (10), the additional holding cost term $-\frac{1}{2}h\sigma^2\mu_n^2$ and the revenue term $\sum p(\frac{1}{\mu_n})$ are, on the contrary, suggesting not varying prices, since they are both concave in $\mu$. These two terms limit the extend to which the optimal prices vary, and thus limit the potential profit improvement. The following lemma and the proposition formalize this intuition.

**Lemma 2** For fixed $S$, let $\mu^*$ be the optimal $N$ prices. Then for $1 \leq m < n \leq N$,

$$\mu_n^* - \mu_m^* \leq \frac{S(n-m)}{N\left(\sigma^2 + \frac{B}{n}\right)},$$

where $B = \inf\left\{ - \frac{d^2 p(\frac{1}{\mu})}{d\mu^2} : \mu \in [\mu_1^*, \mu_N^*] \right\}$.

This lemma indicates that the optimal dynamic pricing may not involve marked price changes under some situations. Note that $\sigma^2$ and $B$ measure the concavity of $-\frac{1}{2}h\sigma^2\mu_n^2$ and $\sum p(\frac{1}{\mu_n})$, respectively. Consistent with our previous intuition, increasing either $\sigma^2$ or $B$ reduces the differences between the optimal prices.

**Proposition 5** For fixed $S$, let $\mu^*$ be the optimal $N$ prices, and let $V_N^*$ be the optimal average profit defined in (10). Then,

$$V_N^* - V_1^* \leq \frac{hS^2\left(1 - N^{-2}\right)}{12\bar{\mu}\left(\sigma^2 + \frac{B}{N}\right)},$$

where $\bar{\mu} = \frac{1}{N}\sum_{n=1}^{N} \mu_n^*$ and $B = \inf\left\{ - \frac{d^2 p(\frac{1}{\mu})}{d\mu^2} : \mu \in [\mu_1^*, \mu_N^*] \right\}$.

**Proof.** We consider a feasible single-price policy that charges a price $p(\bar{\mu}^{-1})$, and compare this with the optimal $N$-price policy. We show the latter has a lower average revenue, a higher additional holding cost due to demand variability, and the same average ordering cost. Hence, for the optimal $N$-price policy to yield a higher profit, it must have a lower average holding cost attributed to the
deterministic part of the demand. By the same reasoning, the difference between these two average holding-cost terms corresponding to the two policies constitute an upperbound on the improvement from the single-price policy to the $N$-price policy.

Specifically, the average profit of charging a single price $p(\bar{\mu}^{-1})$ is

$$V_1 = p(\frac{1}{\bar{\mu}}) - \frac{h_S}{2} \bar{\mu} - \frac{h_2}{2} \bar{\mu}^2 - a(S).$$

We have

$$V_N^* - V_1^* \leq V_N^* - V_1 = \sum_{n=1}^N \left[ p(\frac{1}{\mu_n^*}) - p(\frac{1}{\bar{\mu}}) \right] - \frac{h_S}{N} \sum_{n=1}^N \left[ (N - n + \frac{1}{2})\mu_n^* - \frac{N}{2} \bar{\mu} \right] - \frac{h_2 S}{2} \sum_{n=1}^N \mu_n^* - \frac{\bar{\mu}}{2}.$$ 

The first term in the numerator, $\sum_{n=1}^N \left[ p(\frac{1}{\mu_n^*}) - p(\frac{1}{\bar{\mu}}) \right]$ is $\leq 0$, since $p(\frac{1}{\mu})$ is concave in $\mu$; and the last term in the numerator $\sum_{n=1}^N \mu_n^* - \bar{\mu} \geq 0$, which is essentially the Cauchy-Schwartz inequality. Thus,

$$V_N^* - V_1^* \leq \frac{h_S}{N} \sum_{n=1}^N \left[ (N - n + \frac{1}{2})\mu_n^* - \frac{N}{2} \bar{\mu} \right]$$

where in the last line, the series ends with $\mu_{N/2+1}^* - \mu_{N/2}^*$ if $N$ is even, and ends with $2(\mu_{(N+3)/2}^* - \mu_{(N-1)/2}^*)$ if $N$ is odd.

Applying Lemma 2 and the identity:

$$(N - 1)^2 + (N - 3)^2 + \cdots + (N + 1 - 2 \left\lfloor \frac{N}{2} \right\rfloor)^2 = \frac{(N-1)N(N+1)}{6},$$

we have

$$V_N^* - V_1^* \leq \frac{h S^2}{\mu (\sigma^2 + \frac{\mu}{N})} \left( \frac{(N - 1)^2 + (N - 3)^2 + \cdots + (N + 1 - 2 \left\lfloor \frac{N}{2} \right\rfloor)^2}{2N^3} \right)$$

$$= \frac{h S^2}{\mu (\sigma^2 + \frac{\mu}{N})} \left( \frac{(N - 1)N(N+1)}{12N^3} \right)$$

$$= \frac{h S^2 (1 - N^{-2})}{12\bar{\mu} (\sigma^2 + \frac{\mu}{N})}.$$
The bound in the above proposition indicates that the \( N \)-price policy cannot improve much over the single-price policy when the replenishment quantity \( S \) and the holding cost rate \( h \) are low, and the demand variability \( \sigma^2 \) and the concavity of the revenue function \( B \) are high. These parameters affect the above bound in the same way as they do to the price increment bound in Lemma 2.

Also note that this bound does not rely on the interior optimum assumption. It will be used in the next section to derive a bound on improvement for the joint pricing and replenishment strategy.

### 3.3 Joint Pricing-Replenishment Optimization

#### 3.3.1 One Price

In this case, the firm’s problem is

\[
\max_{S > 0, \lambda \in \mathcal{L}} \quad V(S, \lambda) = r(\lambda) - \frac{hS}{2} - \lambda a(S) - \frac{h\sigma^2}{2\lambda}.
\]

The first-order conditions are given by (6) and (9). The difficulty is, \( V(S, \lambda) \) may not be concave in \((S, \lambda)\), and it may even have multiple local maxima (see Example 6 below). However, under linear demand and linear replenishment cost, a solution satisfying the first-order conditions and yielding positive profit must be the global optimum, as demonstrated in the following proposition and example. See appendix for the proof.

**Proposition 6** Suppose \( \lambda(p) = \alpha - \beta p \), \( c(S) = K + cS \), where \( \alpha, \beta, K, c > 0 \). If \( S^* > 0 \) and \( \lambda^* \in (0, \alpha) \) satisfy the first-order conditions in (6) and (9), and \( V(S^*, \lambda^*) > 0 \), then \( (S^*, \lambda^*) \) solves problem (12).

**Example 6** Consider linear demand \( \lambda(p) = 50 - p \), linear replenishment cost \( c(S) = 500 + 2S \), \( \sigma = 0.2 \) and \( h = 1 \). The optimal \( S \) for any fixed \( \lambda \) is given by \( S(\lambda) = 10\sqrt{10\lambda} \). Substituting this EOQ into the first-order condition (9) gives

\[
50 - 2\lambda + \frac{1}{50\lambda^2} = \frac{5\sqrt{10}}{\sqrt{\lambda}} + 2.
\]

However, simply solving the above equation yields three stationary points: \( \lambda_1 = 0.01619 \), \( \lambda_2 = 0.1010 \), and \( \lambda_3 = 22.327 \). Using the second-order condition, it can be verified that \( \lambda_1 \) and \( \lambda_3 \) are local maxima, while \( \lambda_2 \) is a local minimum. Furthermore, \( V(S(\lambda_1), \lambda_1) = -4.482 \) and \( V(S(\lambda_3), \lambda_3) = 423.8 \). Hence \( \lambda^* = \lambda_3 = 22.327 \) and \( S^* = 149.42 \).  ■
The monotonicity of the joint optimum is explored in the following proposition (compared with Proposition 1 and 2 where a single decision is optimized).

**Proposition 7** If there is a unique optimal price and replenishment level, then

(i) the optimal price is decreasing in $\sigma$;

(ii) the optimal replenishment level is increasing in $\sigma$ and decreasing in $h$;

(iii) the optimal profit is decreasing in $\sigma$ and $h$.

**Remark:** When the optimal solutions are not unique, the results in the above proposition continue to hold. In lieu of increasing and decreasing, the relevant properties are *ascending* and *descending*, respectively. Refer to Topkis (1978).

Intuitively, the higher the unit holding cost $h$, the less inventory should be held; and the less order quantity means the higher average cost and thus the higher price. But this intuition is correct only when the demand is deterministic ($\sigma = 0$). To see this, note that when $\sigma = 0$, the optimal $\lambda$ is given by $\max \ r(\lambda) - \lambda a(S^*(h))$, i.e., $\lambda^*$ depends on $h$ only through $S^*$. But we know $S^*$ is increasing in $h$, while $r(\lambda) - \lambda a(S^*)$ is supermodular in $(S^*, \lambda)$. Hence, the price is increasing in $h$. However, when $\sigma > 0$, a lower price may offset the additional holding cost due to demand variability; the composite effect is mixed, as shown in the following example.

**Example 7** Let $\lambda(p) = \alpha e^{-p}$ with $\alpha > 0$. Let $c(S) = S(K - \log S)$ for $0 \leq S \leq e^{K-1}$, where $K > 0$ is given. Note that $r(\lambda) = \lambda \log(\alpha/\lambda)$ is strictly concave for $\lambda \in (0, \alpha]$, $c(S)$ is strictly increasing for $S \in [0, e^{K-1}]$ and $a(S) = K - \log S$ is strictly convex and approaches to infinity as $S \to 0$, so Assumptions 1 and 2 are satisfied.

In this case, the first-order conditions (6) and (9) become

$$\log \alpha - \log \lambda - 1 = K - \log S - \frac{h\sigma^2}{2\lambda^2} \quad \text{and} \quad -\frac{1}{S} = -\frac{h}{2\lambda},$$

which determine the optimal solution:

$$\lambda^* = \frac{\sigma}{2} \sqrt{\frac{2h}{K + 1 + \log(h/2\alpha)}} \quad \text{and} \quad S^* = \frac{2\lambda^*}{h}.$$

(The above is indeed a global optimal solution when $\lambda^* < \alpha$, and $S^* \leq e^{K-1}$; the latter is satisfied if we choose $K$ large enough.) Now, the partial derivative,

$$\frac{\partial(\lambda^*)^2}{\partial h} = \frac{\sigma^2 (K + \log(h/2\alpha))}{2(K + 1 + \log(h/2\alpha))^2},$$

varies from negative to positive, as $h$ increases from zero. Thus, $\lambda^*$ first decreases and then increases in $h$, or equivalently, $p^*$ first increases and then decreases in $h$. ■
Finally, we compare the joint optimization here with the sequential optimization scheme that is usually followed in practice: the marketing/sales department first makes the pricing decision, and then the purchase department decides the replenishment quantity based on demand projection as a consequence of the pricing decision. For instance, the marketing/sales department solves the problem:

\[ \lambda^* = \arg \max_\lambda \{ r(\lambda) \}, \]

and sets \( p^* = p(\lambda^*) \). Then, the purchase department takes \( \lambda^* \) as given and solves the problem:

\[ S^* = \arg \min_S \{ \lambda^* a(S) + hS/2 \}. \]

Clearly, the sequential decision procedure does not take demand variability into account. The optimal price and inventory level found by the sequential scheme certainly satisfy the first-order condition in (6). But the first-order condition in (9) holds only when the demand variability happens to be \( \hat{\sigma} = \lambda^* \sqrt{2a(S^*)/h} \).

**Proposition 8** Comparing to the joint optimization,

(i) if \( \sigma < \hat{\sigma} \), then the sequential decision underprices and overstocks;

(ii) if \( \sigma > \hat{\sigma} \), then the sequential decision overprices and understocks.

The proof is a straightforward application of the monotonicity result in Proposition 7, and will be omitted. In general, the sequential optimization is sub-optimal. If the demand variability is far away from \( \hat{\sigma} \), the sequential decision can lead to significant profit loss, as illustrated in the following examples.

**Example 8** Let \( \lambda(p) = 20 - p \), \( c(S) = 100 + 5S \) and \( h = 1 \). Suppose the marketing/sales department sets price at \( p^* = 10 \) in order to maximize the revenue rate. Then, the purchase department minimizes the operating cost using the EOQ model: \( S^* = 20\sqrt{5} \approx 44.7 \). The threshold \( \hat{\sigma} = \lambda^* \sqrt{2a(S^*)/h} \approx 38 \). In Figure 4 we compare the sequential decision with the joint decision. The parameter range is chosen such that the best decision will be able to achieve a positive profit. Substantial profit loss is observed: when \( \sigma = 0 \), the sequential decision both underprices and overstocks by 28%, resulting 73% profit loss compared to the joint decision; when \( \sigma = 10 \), the sequential decision underprices by 25% and overstocks by 22%, making almost no profit. ■
3.3.2 N Prices

The problem is

$$\max_{S > 0, \mu \in \mathcal{M}_N} V(S, \mu) = \sum_{n=1}^{N} \left[ p(\frac{1}{\mu_n}) - \frac{h}{N} (N - n + \frac{1}{2}) \mu_n - \frac{h\sigma^2}{2} \mu_n^2 - a(S) \right] \sum_{n=1}^{N} \mu_n. \tag{13}$$

As in the one-price case, the objective function $V(S, \mu)$ may not be jointly concave. Under the interior optimum assumption, the optimal solution must satisfy the first-order conditions in (6) and (11).

The result in Proposition 3 (which holds for any replenishment level) continues to hold here, i.e., $p_1^* \leq p_2^* \leq \cdots \leq p_N^*$. However, the monotonicity in Proposition 4 and Proposition 7 need not hold here, as evident from the following example.

**Example 9** Let $p(\lambda) = 10 - 10^{-3}\lambda + \lambda^{-1}$, $c(S) = 50 + S^2$ and $h = 0.2$. (Note the term $\lambda^{-1}$ in $p(\lambda)$ only adds a constant to the objective function to ensure that the average profit is positive.) Consider $N = 2$. The optimization problem is:

$$\max_{\mu_1, \mu_2, S} \frac{p(\frac{1}{\mu_1}) + p(\frac{1}{\mu_2}) - \frac{hS}{4} (3\mu_1 + \mu_2) - \frac{h\sigma^2}{2} (\mu_1^2 + \mu_2^2) - 2a(S)}{\mu_1 + \mu_2}.$$

The optimal solutions are plotted in Figure 5. Two observations emerge from the figure. First, there exist jumps in the optimal policy, due to multiple local maxima. For example, for $\sigma = 0.243$, there are at least two local maximizers: $(S = 4.917, \mu_1 = 0.401, \mu_2 = 41.51)$ and $(S = 4.464, \mu_1 = 5.637, \mu_2 = 43.44)$. The first yields an objective value $V = 0.2639$, which is slightly higher than the second one ($V = 0.2638$). For $\sigma = 0.244$, the two local maximizers are slightly different: $(S = 4.918, \mu_1 = 0.448, \mu_2 = 41.33)$ and $(S = 4.425, \mu_1 = 6.248, \mu_2 = 43.41)$. However, the objective value corresponds to the second one ($V = 0.261908$) is slightly better than the first ($V = 0.261903$). Hence, the optimal policy exhibits discontinuity when $\sigma$ varies from 0.243 to 0.244. (These numerical results are accurate to all the decimal places used. Furthermore, these phenomena can also be verified analytically.) Our second observation is that when the optimal solutions are continuous in $\sigma$, there exists a range in which both $\mu_1$ and $\mu_2$ are increasing in $\sigma$, while $S$ is decreasing in $\sigma$. This is in sharp contrast with the results in the single-price case.  

Next, we develop a bound on the profit improvement as the number of prices ($N$) increases.
**Proposition 9** Let \((S^*_N, \mu^*)\) be the optimal joint pricing-replenishment decision, and \(V^*_N\) be the corresponding optimal profit in (13). Then,

\[
V^*_N - V^*_1 = V^*_N(S^*_N) - V^*_1(S^*_1) \leq V^*_N(S^*_N) - V^*_1(S^*_1) \leq \frac{hS^*_N^2(1 - N^{-2})}{12\mu^*}\left(\sigma^2 + \frac{B_N}{N}\right),
\]

where \(\mu^* = \frac{1}{N}\sum_{n=1}^{N} \mu^*_n\) and \(B_N = \inf \left\{ -\frac{d^2p(\frac{1}{\mu^*})}{d\mu^2} : \mu \in [\mu^*_1, \mu^*_N] \right\}\).

**Proof.** Let \(V^*_N(S)\) denote the optimal profit when \(S\) is given (i.e., pricing decision only). If \(S\) happens to be fixed at \(S^*_N\), then the profit is \(V^*_N\), i.e., \(V^*_N = V^*_N(S^*_N)\). Applying Proposition 5, we have the desired bound immediately:

\[
V^*_N - V^*_1 = V^*_N(S^*_N) - V^*_1(S^*_1) \leq \frac{hS^*_N^2(1 - N^{-2})}{12\mu^*}\left(\sigma^2 + \frac{B_N}{N}\right).
\]

The shortfall of the above bound is that it involves the solution to the \(N\)-price problem. Heuristically, we can use the solution to the joint single-price and replenishment problem, denoted by \((S^*_1, \mu^*)\), in the upperbound. Specifically, replace \(S^*_N\) by \(S^*_1\), replace \(\mu^*\) by \(\mu^*\), replace \(B_N\) by \(B_1 = -\frac{d^2p(\frac{1}{\mu^*})}{d\mu^2}\), and replace \(1 - N^{-2}\) by \(3/4\) to tighten the bound. That is, the bound in Proposition 9 can be evaluated heuristically as follows:

\[
\frac{hS^*_1^2}{16\mu^*}\left(\sigma^2 + \frac{B_1}{N}\right).
\]

**Example 10** Let \(c(S) = K + cS\) and \(p(\lambda) = a - b\lambda\), where \(a, b, c, K\) are all positive parameters. The optimization problem in (13) becomes

\[
\max_{S > 0, \mu \in [b/a, \infty)^N} V(S, \mu) = \frac{\sum_{n=1}^{N} \left[ a - \frac{b}{\mu_n} - \frac{hS}{N}(N - n + \frac{1}{2})\mu_n - \frac{h^2}{2}\mu_n^2 - \frac{K}{S} - c \right]}{\sum_{n=1}^{N} \mu_n}.
\]

Applying a change of variables, \(\mu = b\mu^*\) and \(S = K\tilde{S}\), we can rewrite the above problem as follows:

\[
\max_{\tilde{S} > 0, \tilde{\mu} \in [1/a, \infty)^N} V(\tilde{S}, \tilde{\mu}) = \frac{\sum_{n=1}^{N} \left[ a - c - \frac{1}{\tilde{\mu}_n} - \frac{K}{N}(N - n + \frac{1}{2})\tilde{\mu}_n - \frac{b^2}{2}\tilde{\mu}_n^2 - \frac{1}{\tilde{S}} \right]}{b\sum_{n=1}^{N} \tilde{\mu}_n}.
\]

Clearly, the above expression indicates that there are four degrees of freedom in terms of independent parameters: \((N, a - c, Kbh, h\sigma^2)\). Specifically, the four degrees of freedom are determined by \(N\), either \(a\) or \(c\), and two from \((K, h, b, \sigma)\). In the numerical studies reported here and below, we choose to vary \((N, c, h, \sigma)\) while fixing \((K, a, b)\).
Figure 6: Effect of $N$, the number of price changes.

Figure 6 shows the optimal replenishment levels (Figure 6(a)), prices (Figure 6(b)) and profit values (Figure 6(c)) corresponding to different values of $N$, the number of price changes. We see that as $N$ increases the optimal prices are inter-leaved (e.g., the optimal single price is sandwiched between the two-price solutions, which, in turn, are each sandwiched between a neighboring pair of the three-price solutions). (This inter-leaving property seems to persist in all examples we have studied.) Figure 6(c) shows that $N$ has a decreasing marginal effect on profit. Using two prices already achieves most of the potential profit improvement, and beyond $N = 8$, the marginal improvement is essentially nil. ■

Example 11 Here we continue with the last example, but focus on comparing the optimal profits under $N = 1$ and $N = 8$. We consider the parameter values $c \in (0, 30], h \in (0, 50], \sigma \in (0, 50]$, while fixing the others at $K = 100, a = 50, b = 1$.

The results reported in Figure 7 are for $c = 1$. Similar results are observed for $c = 5, 10, 20, 30$.

Figure 7: Multiple prices vs. a single price

Figure 7(a) shows the optimal profit corresponding to a single price. The profit is clearly decreasing in $h$ and $\sigma$ (Proposition 7). Furthermore, the negative effect of $\sigma$ is larger when $h$ is larger and vice versa, suggesting that the profit function is submodular in $(h, \sigma)$. Intuitively, since the effect of $\sigma$ shows up in the profit function through the additional holding cost, the higher the $h$ (resp. $\sigma$) value, the more sensitive is the profit to $\sigma$ (resp. $h$).

Figure 7(b) shows the absolute improvement in profit when using 8 prices ($V^*_8 - V^*_1$). The improvement is increasing in $h$ and decreasing in $\sigma$. Intuitively, as the inventory holding cost increases, the right trade-off among revenue, holding cost and replenishment cost becomes more important, thus more pricing options over time is more beneficial. However, pricing becomes less effective as demand variability ($\sigma$) increases.

Note that when the profit under a single price $V^*_1$ approaches zero, the absolute improvement ($V^*_8 - V^*_1$) does not diminish. In particular, while $V^*_1$ decreases in $h$, the improvement increases in $h$. Indeed, the relative improvement ($= \frac{V^*_8 - V^*_1}{V^*_1}$) is increasing in $h$, and approaching infinity when $V^*_1$ is close to zero, as demonstrated in Figure 7(c) and (d).

Figure 8: Bounds on the profit improvement
To conclude this example, we show in Figure 8 the profit improvement in comparison with the theoretical bound in Proposition 9 and heuristic bound in (14). The heuristic bound can be seen in this example as giving a good estimate of the maximum potential improvement. 

### 3.3.3 The Algorithm: Fractional Programming

To solve the fractional optimization problem in (13), we can instead solve the following:

\[
\max_{S>0, \mu \in \mathcal{M}} V_\eta := \sum_{n=1}^{N} \left[ p\left( \frac{1}{\mu_n} \right) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{h\sigma^2}{2}\mu_n^2 - a(S) \right] - \eta \sum_{n=1}^{N} \mu_n, \tag{16}
\]

where \(\eta\) is a parameter. When the optimal objective value of the problem in (16) is zero, the corresponding solution is the optimal solution to the original problem in (13), and the corresponding \(\eta\) is the optimal value of the original problem. To see the equivalence, write the optimal value in (13) as \(V^* = A^*/B^*\), where \(A\) and \(B\) denote the numerator and denominator in (13), respectively. When \(\eta = V^*\), the optimal value of (16) is zero. On the other hand, suppose there exists an \(\eta\) which yields a zero objective value in (16), specifically, \(A^* - \eta B^* = 0\), then for any feasible \(A, B\), we have \(A - \eta B \leq 0\), which implies \(A/B \leq \eta = A^*/B^*\).

The algorithm described below takes advantage of the separability of the objective function with respect to \(\mu\) when \(S\) and \(\eta\) are given. Specifically, the sub-problems are:

\[
\max_{\mu_n \in \mathcal{M}} g_n(\mu_n) := p\left( \frac{1}{\mu_n} \right) - \frac{hS(N-n+\frac{1}{2})}{N}\mu_n - \frac{h\sigma^2}{2}\mu_n^2 - a(S) - \eta\mu_n, \quad n = 1, \ldots, N. \tag{17}
\]

Under Assumption 1, the objective in (17) is concave in \(\mu_n\).

#### Algorithm for solving (13)

1. Initialize \(\eta = \eta_0\), \(S = S_0\), and \(\mu_n = \mu_0\) for \(n = 1, \ldots, N\).
   (For instance, \(\eta_0 = 0\); \(S_0 = S^*\) and \(\mu_0 = \mu^*\), with \((S^*, \mu^*)\) being the optimal solution to the single-price problem.)
   Set \(\epsilon\) at a small positive value (according to the required precision).

2. Solve the \(N\) single-dimensional concave function maximization problems in (17).

3. Update \(S\) following a specified stepsize or use equation (6).
   If the difference between the new and the old \(S\) values is less than \(\epsilon\), go to step 4; otherwise, return to step 2.

4. Let \(V_\eta\) be the sum of the objective values of the \(N\) sub-problems.
   If \(|V_\eta| \leq \epsilon\), optimal solution found; stop.
   If \(V_\eta > \epsilon\), increase \(\eta\); if \(V_\eta < -\epsilon\), decrease \(\eta\); return to step 2.
In step 2, there are ways to accelerate the search procedure. First, we can take the advantage of the monotonicity of $\mu^*_n$ in $n$ following Proposition 3. Second, we have $g_n(\mu^*_n) \leq g_{n+1}(\mu^*_n) \leq g_{n+1}(\mu^*_{n+1})$, for $n = 1, \ldots, N - 1$. Hence, if $g_1(\mu^*_1) > \varepsilon/N$, then $V_\eta > \varepsilon$, and we can bypass the rest of the algorithm, increase $\eta$ directly and proceed to the next loop. Third, we can make use of the monotonicity of $\mu^*_n$ in $S$ according to Proposition 4 (iii). This is particularly useful when the stepsize for updating $S$ in step 3 is small.

In step 3, if we are in the early stage of the algorithm, i.e., when $V_\eta$ is still substantially away from zero, then the updating of $S$ needs to cover a wide range so as not to miss the true optimum. This can be done by using large stepsize first and then reduce them gradually. When we are at the late stage of fine-tuning $V_\eta$, we can update $S$ to nearby values.

In step 3, we can use (6) as an updating scheme. This is in fact the coordinate assent method: we alternate between optimizing $\mu$ for fixed $S$ and optimizing $S$ for fixed $\mu$. This procedure is guaranteed to converge to a local maximum, but not necessary the global maximum (see Luenberger (1984)). So it is useful once the algorithm enters the region containing the true optimum without other local maxima.

It can be verified that the optimal objective value in (16) is strictly decreasing in $\eta$, so there is a unique zero-crossing point at which $V_\eta$ is zero. Thus, in step 4, we can update $\eta$ following a standard line search algorithm, such as the bi-section or the golden ratio.

The above algorithm with a slight modification appears more suitable for applications. In practice, it is often relevant to ask how many prices to use, when to change prices and what prices to change to. In addition, the prices and the replenishment level may be restricted to some given (discrete) sets of values. It is readily verified that the equivalence between (13) and (16) still holds under discrete sets. To solve the discrete version of (16), we simply modify step 2 of the algorithm such that $N$ single-dimensional concave maximization problems are solved, and modify step 3 such that the updating scheme for $S$ is restricted to the discrete set. Using a large $N$, the solution can effectively answer the questions posed above, as evident from the following example.

**Example 12** Let $\lambda(p) = 50 - p$, $c(S) = 100 + S$, $h = 1$, $\sigma = 10$. Suppose the prices have to be integer valued, and $S$ has to be a multiple of 5. Using the algorithm adapted as above, and choosing $N = 140$, we find the optimal policy is to order 70 units for each cycle, charge a price of $25 until the inventory level drops to 67 units, charge $26 until inventory drops to 19, and charge $27 until the inventory runs out. The policy uses only three prices despite the large $N$ value. In other words, the equal partitioning of $[0, S]$ (into 140 segments) does not prevent the algorithm from finding the optimal partitioning (of three uneven segments).
4 Extensions

We have demonstrated the effectiveness of the Brownian motion model for price-sensitive demand, in particular in making optimal pricing and replenishment decisions, quantifying the profit improvement, and bringing out the impact of demand variability. The Brownian motion model also enables us to work out explicit analytical solutions, which facilitate making connections to and comparisons against, wherever applicable, previously known results in both deterministic and stochastic settings.

In the rest of this paper, we elaborate on several possible extensions of our model and results developed above.

4.1 Discounted Objective: An Example

Many results in Section 3 continue to hold if the long-run average objective is replaced by a discounted objective. While we relegate the technical details to an online companion of this paper, we present below a simple example to highlight the contrast between the two cases.

Let $\gamma > 0$ be the discounted rate. Under a single price policy, the discounted profit starting from zero inventory takes the form (equality (33) in the online companion):

$$
V_{\gamma}(S, \lambda) = \frac{r(\lambda)}{\gamma} - \frac{S(h/\gamma + a(S))}{1 - e^{-b(\lambda)S}} + \frac{h \lambda}{\gamma^2},
$$

where $b(\lambda) = 2\gamma/(\lambda^2 + 2\sigma^2 + \lambda)$. Unlike the average objective case, the additional cost due to demand variability, defined as the difference between the value of $V_{\gamma}(S, \lambda)$ with a positive $\sigma$ and that with $\sigma$ set at zero, now depends on $S$.

We fix the price and only optimize the replenishment level, that is

$$
\max_{S>0} V_{\gamma}(S, \lambda).
$$

The first-order condition is

$$
[h/\gamma + a(S) + Sa'(S)](e^{bS} - 1) = bS[h/\gamma + a(S)].
$$

(18)

Recall, under the average objective and fixed price, the demand variability has no effect on the optimal replenishment level. In contrast, the demand variability will raise the replenishment level under the discounted objective. (See the proof in the online companion.) The following example illustrates this effect.
Suppose $c(S) = K + cS$, where $K, c > 0$. When $\gamma$ is small, $b$ is also small and the first-order condition (18) can be written as
\[(h/\gamma + c)(bS + \frac{1}{2}b^2S^2 + o(b^2)) = bS(h/\gamma + K/S + c).
\]
Then,
\[S^*_\gamma \approx \sqrt{\frac{2K}{b(h + c\gamma)}}.
\]
Approximating $b$ by Taylor series: $b \approx \frac{\gamma}{\lambda} - \frac{\sigma^2\gamma^2}{2\lambda^3}$, we have
\[S^*_\gamma \approx \sqrt{\frac{2K(\lambda + \frac{\sigma^2\gamma^2}{2\lambda})}{h + c\gamma}}.
\]
Porteus (1985b) pointed out that the EOQ solution under a discounted objective can be approximated by $\sqrt{\frac{2K\lambda}{h + c\gamma}}$. Our result generalizes this result by incorporating the effect of demand variability.

### 4.2 Full Backlogging

Our model can be extended to allow full backlogging, in which case the replenishment takes the form of a $(-s, S)$ policy with $s, S > 0$: whenever the inventory level (on-hand inventory net the backlogs) reaches $-s$, an order of size $S + s$ is issued to bring the inventory level back to $S$. Assume zero leadtime as before, and let $b$ be cost of backlogging one unit of demand for one unit of time.

The pricing policy is modified accordingly: equally divide $S$ and $s$ into $N$ and $M$ (positive integers) segments, respectively, such that
\[S = S_0 > S_1 > \cdots > S_N = 0 > S_{N+1} > \cdots > S_{N+M} = -s,
\]
and apply price $p_n$ until the inventory level falls to $S_n$, $n = 1, \ldots, N + M$.

Most of the results we have derived for the no-backlog case will continue to hold, with suitable modifications. For instance, the optimal pricing will satisfy the monotonicity:
\[p_1^* \leq p_2^* \leq \cdots \leq p_N^*, \quad p_{N+1}^* \geq p_{N+2}^* \geq \cdots \geq p_{N+M}^*.
\]

As before, the optimal prices increase as the on-hand inventory is depleted, but decrease as the backlog increases. Intuitively, by inducing higher demand when the backlog increases, we shorten the time of reaching $-s$ and reduce the backlog cost.

The bound on the profit improvement of dynamic pricing over a single price is (assuming $M = N$):
\[V_N^* - V_1^* \leq \frac{(1 - N^{-2})}{12} \left[ \frac{hS^2}{\mu_h \left( \sigma^2 + \frac{B}{b} \right)} + \frac{bs^2}{\mu_b \left( \sigma^2 + \frac{B}{b} \right)} \right],
\]
where \( \bar{\mu}_h = \frac{1}{N} \sum_{n=1}^{N} \mu_n^* \), \( \bar{\mu}_b = \frac{1}{N} \sum_{n=N+1}^{2N} \mu_n^* \), and \( B \) denotes, as before, the lower bound for \( -\frac{d^2 p(\frac{1}{\mu})}{d\mu^2} \) over the spread of \( \mu^* \).

### 4.3 Price-Sensitive Demand Variability

Here we extend the Brownian demand model in (1) by allowing the diffusion coefficient to depend on the demand rate, which varies with the price. Specifically,

\[
D(t) = \int_0^t \lambda(p_u) du + \int_0^t \sigma(\lambda(p_u)) dB(u), \quad t \geq 0, \tag{19}
\]

where the diffusion coefficient \( \sigma(\lambda) \) is a function of the demand rate.

With the general model in (19), under the same pricing and replenishment policies, the long-run average profit functions (4) and (5) become

\[
V(S, \mu) = \sum_{n=1}^{N} \left[ \frac{p(\frac{1}{\mu_n}) - \frac{hS}{N}(N-n+\frac{1}{2})\mu_n - \frac{1}{2}h\sigma(\mu_n)^2\mu_n^2 - a(S)}{\sum_{n=1}^{N} \mu_n} \right], \tag{20}
\]

\[
V(S, \lambda) = r(\lambda) - \frac{hS}{2} - \lambda a(S) - \frac{h\sigma(\lambda)^2}{2\lambda}, \tag{21}
\]

respectively. Obviously, the specific functional form of \( \sigma(\lambda) \) will affect the properties of the optimal decisions and profit. Let \( C_v(\lambda) = \sigma(\lambda)/\sqrt{\lambda} \) denote the coefficient of variation. Intuitively, \( C_v(\lambda) \) provides a measure for the relative forecast error. In most applications, low volume demands are known to be more difficult to forecast; hence, it is reasonable to assume that the relative error is convex decreasing in the demand rate. Indeed, assuming \( C_v(\lambda) \) to be convex in \( \lambda \), we can show that many results considered in the previous sections extend readily to the general model of price-dependent variability in (19). We summarize these results with necessary modifications below.

(The proofs of some key results are included in the online companion.)

First, \( C_v(\lambda) \) being convex in \( \lambda \) implies that \( \sigma(\frac{1}{\mu})^2\mu^2 \) is convex in \( \mu \). Then, \( V(S, \lambda) \) is concave in \( \lambda \), and \( V(S, \mu) \) is pseudo-concave in \( \mu \). The first-order conditions are thus sufficient to determine the optimal pricing decisions. In particular, we can show that all the key results in Propositions 3, 5 and 9 prevail. Specifically, the conclusion in Proposition 3 that for any fixed \( S \) the optimal prices are increasing over the periods still holds here (with the same line of the original proof).

The results on the upperbounds in Proposition 5 and 9 require some modification, with \( \sigma^2 \) in the original upperbound being replaced by \( G = \inf \left\{ \frac{d^2(\frac{1}{2}\sigma(\frac{1}{\mu})^2\mu^2)}{d\mu^2} : \mu \in [\mu_1^*, \mu_N^*] \right\} \). Hence, the bound in Proposition 9 becomes

\[
V_N^* - V_1^* \leq \frac{hS_N^2 (1 - N^{-2})}{12\bar{\mu} \left( G + \frac{B_N}{h} \right)}.
\]
Note that if $\sigma(\lambda)$ is a constant, then $G = \sigma^2$ and we recover the original bound. The derivation of this bound relies on the convexity assumption of $\sigma(\lambda)^2/\lambda$ and a modification of Lemma 2: for $1 \leq m < n \leq N$, $\mu_n^* - \mu_m^* \leq \frac{s(n-m)}{N(G+\frac{B}{\lambda})}$. All these results are proven the online companion.

5 Appendix

Proof of Lemma 1. (This lemma follows from well-known results regarding optional stopping applied to Brownian motion; refer to, e.g., Karlin and Taylor (1975), and Ross (1996). The proof here is included for completeness.) Omit the subscript $n$, and write $T := \tau_n$, $x := S/N$. We have

$$\lambda T + \sigma B(T) = x. \quad (22)$$

This, combined with $E[B(T)] = 0$, which follows from applying optional stopping to $B(t)$, leads to $E[T] = x/\lambda$.

To derive the second moment of $T$, we have

$$\sigma^2[B^2(T) - T] = (x - \lambda T)^2 - \sigma^2 T.$$ Applying optional stopping again, to the martingale $B^2(t) - t$, we have

$$E[(x - \lambda T)^2] = \sigma^2 E[T],$$

which can be expressed as

$$\lambda^2 \text{Var}(T) = \sigma^2 E[T].$$ This establishes

$$E[T^2] = \frac{\sigma^2}{\lambda^2} E[T] + E^2[T] = \frac{\sigma^2 x}{\lambda^3} + \frac{x^2}{\lambda^2}.$$ Now consider $E[e^{-\gamma T}]$, where $\gamma > 0$ is a constant (discount rate). Write

$$-\gamma T = aB(T) - \frac{1}{2}a^2T - bx, \quad (23)$$

where $a$ and $b$ are parameters to be determined. Then, applying optional stopping to the exponential martingale, $e^{aB(t)-\frac{1}{2}a^2t}$, we have

$$E[e^{-\gamma T}] = e^{-bx} E[e^{aB(T)-\frac{1}{2}a^2T}] = e^{-bx}.$$ The power $b$ can be derived as follows. Substituting (22) into the right hand side of (23), we have

$$-\gamma T = aB(T) - \frac{1}{2}a^2T - b\sigma B(T) - b\lambda T.$$
Equating the coefficients of $T$ and $B(T)$ on both sides yields:

\[ a = b\sigma, \quad \frac{1}{2}a^2 + b\lambda = \gamma, \]

leading to

\[ \sigma a^2 + 2\lambda a - 2\gamma \sigma = 0. \]

Hence,

\[ a = \sqrt{\lambda^2 + 2\sigma^2\gamma} - \lambda, \]

and

\[ \mathbb{E}[e^{-\gamma T}] = e^{-bx} = \exp\left[-\frac{\sqrt{\lambda^2 + 2\sigma^2\gamma} - \lambda}{\sigma^2}x\right]. \]

Proof of Lemma 2. First, assuming interior optimum. The first-order conditions in (11) hold, which imply that

\[ \frac{d\tilde{v}_m(\mu_m)}{d\mu_m} = \frac{d\tilde{v}_n(\mu_n)}{d\mu_n}, \]

or equivalently,

\[ \frac{dp}{d\mu}(\frac{1}{\mu_m}) - \frac{dp}{d\mu}(\frac{1}{\mu_n}) + \frac{b}{N}(m - n) + h\sigma^2(\mu_n^* - \mu_m^*) = 0. \tag{24} \]

By Proposition 3, $\mu_n^* \geq \mu_m^*$ for $n > m$. Next, applying the mean-value theorem and using the definition of $B$, we have

\[ \frac{dp}{d\mu}(\frac{1}{\mu_m}) \leq \frac{dp}{d\mu}(\frac{1}{\mu_n}) + \frac{bS}{N}(m - n) + h\sigma^2(\mu_n^* - \mu_m^*) \geq B(\mu_n^* - \mu_m^*), \tag{25} \]

where $\zeta \in [\mu_m^*, \mu_n^*]$. Combining (24) and (25), we have

\[ \frac{bS}{N}(m - n) + (h\sigma^2 + B)(\mu_n^* - \mu_m^*) \leq 0, \]

or

\[ \mu_n^* - \mu_m^* \leq \frac{S(n - m)}{N(\sigma^2 + \frac{B}{h})}. \]

Next, we show the above result holds even when the optimum is on the boundary. If $\mu_m^* = \mu_n^*$, then the desired inequality obviously holds. If $\mu_m^* < \mu_n^*$, consider the following alternative pricing
policy: \((\mu^*_1, \ldots, \mu^*_m + \delta, \ldots, \mu^*_n - \delta, \ldots, \mu^*_N)\), where \(0 < \delta < \frac{\mu^*_n - \mu^*_m}{2}\). Since this alternative cannot be better than the optimum, we have

\[
p\left(\frac{1}{\mu^*_n + \delta}\right) + p\left(\frac{1}{\mu^*_n - \delta}\right) - \frac{hS}{N} \left[ (N - m + \frac{1}{2})(\mu^*_m + \delta) + (N - n + \frac{1}{2})(\mu^*_n - \delta) \right] - \frac{h\sigma^2}{2} [(\mu^*_m + \delta)^2 + (\mu^*_n - \delta)^2] \\
\leq p\left(\frac{1}{\mu^*_n}\right) + p\left(\frac{1}{\mu^*_n}\right) - \frac{hS}{N} \left[ (N - m + \frac{1}{2})\mu^*_m + (N - n + \frac{1}{2})\mu^*_n \right] - \frac{h\sigma^2}{2} [\mu^*_m^2 + \mu^*_n^2].
\]

Taylor’s expansion with straightforward algebra simplifies the inequality to

\[
\frac{dp}{d\mu} \left(\frac{1}{\mu^*_n + \zeta_1 \delta}\right) - \frac{dp}{d\mu} \left(\frac{1}{\mu^*_n - \zeta_2 \delta}\right) + \frac{hS}{N} \delta (m - n) + h\sigma^2 \delta (\mu^*_n - \mu^*_m - \delta) \leq 0,
\]

where \(\zeta_1, \zeta_2 \in [0, 1]\). Dividing both sides by \(\delta\), and then letting \(\delta \to 0\), we have

\[
\frac{dp}{d\mu} \left(\frac{1}{\mu^*_n}\right) - \frac{dp}{d\mu} \left(\frac{1}{\mu^*_n}\right) + \frac{hS}{N} (m - n) + h\sigma^2 (\mu^*_n - \mu^*_m) \leq 0.
\]

Combining (25) and (26) yields the desired inequality. \(\blacksquare\)

**Proof of Proposition 4.** We shall write \(\mu^*(\sigma), \mu^*(h)\) and \(\mu^*(S)\) to emphasize the dependence of the maximizer of (10) on these parameters. First we prove the following lemma.

**Lemma 3** Assuming interior optimum,

(i) \(\mu^*(\sigma)\) is differentiable and \(\frac{d\mu^*_k(\sigma)}{d\sigma} < 0\) if and only if

\[
\sum_{k=1}^{N} \mu_k^*(\sigma)^2 - 2\mu_n^*(\sigma) \sum_{k=1}^{N} \mu_k^*(\sigma) < 0; \tag{27}
\]

(ii) \(\mu^*(h)\) is differentiable and \(\frac{d\mu^*_k(h)}{dh} < 0\) if and only if

\[
\frac{S}{N} \sum_{k=1}^{N} (n - k)\mu_k^* + \sigma^2 \sum_{k=1}^{N} \left(\frac{\mu_k^*}{2} - \mu_n^*\mu_k^*\right) < 0; \tag{28}
\]

(iii) \(\mu^*(S)\) is differentiable and \(\frac{d\mu^*_k(S)}{dS} < 0\) if and only if

\[
\frac{h}{N^2} \sum_{k=1}^{N} (n - k)\mu_k^*(S) < -a'(S). \tag{29}
\]

**Proof of Lemma 3.** We only prove part (i); the proof of part (ii) and (iii) is completely analogous. Consider an equivalent problem to the fractional problem (10):

\[
\max_{\mu} \sum_{n=1}^{N} \left[ p\left(\frac{1}{\mu_n}\right) - \frac{hS(N - n + \frac{1}{2})}{N} \mu_n - \frac{h\sigma^2}{2} \mu_n^2 - a(S) \right] - \eta(\sigma) \sum_{n=1}^{N} \mu_n,
\]

29
where \( \eta(\sigma) \) is the optimal value of (10) when the demand variability is \( \sigma \) and all the other parameters are fixed. To solve the above problem, we can maximize each \( \mu_n \) separately:

\[
\max_{\mu_n} \quad g_n(\mu_n, \sigma) = p\left( \frac{1}{\mu_n} \right) - \frac{hS(N - n + \frac{1}{2})}{N} \mu_n - \frac{h\sigma^2}{2} \mu_n^2 - \eta(\sigma) \mu_n.
\]

Since \( \mu^*(\sigma) \) is in the interior, applying the envelope theorem, we have

\[
\frac{d\eta}{d\sigma} = -\frac{h\sigma}{\eta(\sigma)} \sum_{k=1}^{N} (\mu_k^* - \mu^*)^2.
\]

To establish the desired monotonicity, it suffices to establish the submodularity of \( g_n(\mu_n, \sigma) \):

\[
\frac{\partial^2 g_n(\mu_n^*, \sigma)}{\partial \mu_n \partial \sigma} = -2h\sigma \mu_n^* - \frac{\partial \eta}{\partial \sigma} = -2h\sigma \mu_n^* + \frac{h\sigma}{\sum_{k=1}^{N} \mu_k^*} \sum_{k=1}^{N} \mu_k^*^2.
\]

If \( \sum_{k=1}^{N} \mu_k^*^2 - 2\mu_n^* \sum_{k=1}^{N} \mu_k^* < 0 \), then the objective \( g_n(\mu_n, \sigma) \) is submodular in \((\mu_n, \sigma)\) in a neighborhood of \((\mu_n^*, \sigma)\). Thus, \( \mu_n^* \) is (locally) decreasing in \( \sigma \). If the converse is true, \( \mu_n^* \) is (locally) increasing in \( \sigma \).

**Proof of Proposition 4.**

(i) By Lemma 3, the inequality in (27) always holds for \( n = N \), as \( \mu_N^* \) is the largest component of \( \mu \) following Proposition 3. Hence \( \mu_N^* \) is decreasing in \( \sigma \).

(ii) Denote the left side of (28) by \( L_n \). We will show that \( L_1 + L_N < 0 \) and \( L_1 < L_N \), which then implies \( L_1 < 0 \).

\[
L_1 + L_N = \frac{S}{N} \sum_{k=1}^{N} (N + 1 - 2k)\mu_k^* + \sigma^2 \sum_{k=1}^{N} \left[ \mu_k^*^2 - (\mu_1^* + \mu_N^*)\mu_k^* \right] = \frac{S}{N} \left[ (N - 1)(\mu_1^* - \mu_N^*) + (N - 3)(\mu_2^* - \mu_{N-1}^*) + \ldots \right] + \sigma^2 \sum_{k=1}^{N} \left[ \mu_k^*(\mu_k^* - \mu_N^*) - \mu_k^* \mu_k^* \right] < 0,
\]

where in the second equality, the series ends with \( \mu_{N/2}^* - \mu_{N/2+1}^* \) if \( N \) is even, and ends with \( 2(\mu_{(N-1)/2}^* - \mu_{(N+3)/2}^*) \) if \( N \) is odd; the last inequality follows from \( \mu_n^* \) increasing in \( n \). Next, note that

\[
L_1 - L_N = \left[ \frac{S(1 - N)}{N} + \sigma^2(\mu_1^* - \mu_N^*) \right] \sum_{k=1}^{N} \mu_k^* \leq 0,
\]
where the last inequality follows from Lemma 2. Hence \( L_1 < 0 \), and \( p_1^*(h) \) is decreasing in \( h \).

(iii) By Lemma 3, if \( a'(S) < 0 \) and \( n = 1 \), the inequality in (29) always holds. Hence \( \mu^*_1 \) is decreasing in \( S \).

(iv) This part is obvious. ■

**Proof of Proposition 6.** From Example 1, the optimal replenishment level for each \( \lambda \) is 
\[
S^*(\lambda) = \sqrt{2KL}/h.
\]
We replace \( S \) in the objective by \( S^*(\lambda) \), and then find the optimal \( \lambda \). Let 
\[
W(\lambda) := V(S^*(\lambda), \lambda) = (\alpha - \lambda)\lambda/\beta - c\lambda - \sqrt{2Kh\lambda} - \frac{h\sigma^2}{2\lambda}.
\]

We prove that \( \lambda^* \) satisfying \( W'(\lambda^*) = 0 \) and \( W(\lambda^*) > 0 \) is the global maximizer.

Let \( f(\lambda) := (\alpha - \lambda)\lambda/\beta - c\lambda - \sqrt{2Kh\lambda} \). It is easy to see that \( f(0) = 0 \), \( f'(0) = -\infty \) and \( f''(\lambda) = -\frac{2}{\beta} + \sqrt{\frac{hK}{8\lambda^3}} \). Let \( \hat{\lambda} \) be the inflection point, i.e., \( f''(\hat{\lambda}) = 0 \). Then, \( f(\lambda) \) starts decreasing from zero, is convex in \( [0, \hat{\lambda}] \) and concave in \( [\hat{\lambda}, \infty) \).

Suppose \( f'(\hat{\lambda}) \leq 0 \), then by convexity/concavity, \( f'(\lambda) \leq f'(\hat{\lambda}) \leq 0 \) for all \( \lambda \geq 0 \). Therefore, \( W(\lambda) < f(\lambda) \leq f(0) = 0 \) for all \( \lambda \geq 0 \), so there exists no maximizer with positive profit in this case. Hence, we must have \( f'(\hat{\lambda}) > 0 \).

Next, let \( \lambda_1 \) be the local minimizer for \( f(\lambda) \) in the convex part of the function. Obviously, \( \lambda_1 < \hat{\lambda} \). First, \( \lambda^* \notin [0, \lambda_1] \) because in this region \( W(\lambda) < f(\lambda) \leq 0 \). Second, \( \lambda^* \notin [\lambda_1, \hat{\lambda}] \) because in this region \( W'(\lambda) = f'(\lambda) + \frac{h\sigma^2}{2\lambda^2} > 0 \). Hence, \( \lambda^* \in [\hat{\lambda}, \alpha] \). (\( \hat{\lambda} < \alpha \) in order for \( \lambda^* \) to exist.) Since \( W(\lambda) = f(\lambda) - \frac{h\sigma^2}{2\lambda^2} \) is strictly concave in \( [\hat{\lambda}, \alpha] \), \( \lambda^* \) is the unique maximizer for \( W(\lambda) \) in \( [\hat{\lambda}, \alpha] \). It is the global maximizer since all other local maximizers (if any) must be within \( [0, \lambda_1] \), the region where \( W(\lambda) < 0 \). ■

**Proof of Proposition 7.** (i) The joint optimization problem can be solved sequentially. We first optimize \( S \) for each fixed \( \lambda > 0 \) (same as Section 3.1). From Proposition 1, the optimal \( S \) is invariant to \( \sigma \), and we denote it by \( S^*(\lambda) \). Then, the optimal \( \lambda \) can be found by

\[
\max_{\lambda \in \mathcal{L}} \bar{V}(\lambda, \sigma) := r(\lambda) - \frac{hS^*(\lambda)}{2} - \lambda a(S^*(\lambda)) - \frac{h\sigma^2}{2\lambda}.
\]

Now, \( \bar{V}(\lambda, \sigma) \) is supermodular in \( (\lambda, \sigma) \) since

\[
\frac{\partial^2 \bar{V}}{\partial \lambda \partial \sigma} = \frac{h\sigma}{\lambda^2} \geq 0,
\]

and therefore \( \lambda^*(\sigma) \) is increasing in \( \sigma \), or equivalently, the optimal price is decreasing in \( \sigma \).
(ii) That $S^*$ is increasing in $\sigma$ follows immediately from (i) and Proposition 1 (i). To examine the effect of $h$ on $S^*$, we first optimize $\lambda$ for each fixed $S > 0$ (same as Section 3.2), and denote the maximizer by $\lambda^*(S, h)$. Then, the optimal $S$ is determined by

$$\max_{S > 0} \tilde{V}(S, h) := r(\lambda^*(S, h)) - \frac{hS}{2} - \lambda^*(S, h)a(S) - \frac{h\sigma^2}{2\lambda^*(S, h)}.$$ 

Let $\lambda^*_S$, $\lambda^*_h$ and $\lambda^*_{Sh}$ denote the partial derivatives. We have

$$\frac{\partial^2 \tilde{V}}{\partial S \partial h} = r''\lambda^*_S\lambda^*_h + r'\lambda^*_S - \frac{1}{2} - a\lambda^*_S - a'\lambda^*_h + \frac{\sigma^2}{2\lambda^*2} \lambda^*_S - \frac{h\sigma^2}{\lambda^*3} \lambda^*_S \lambda^*_h + \frac{h\sigma^2}{2\lambda^*2} \lambda^*_S.$$ 

From (9), we have

$$r' - a + \frac{h\sigma^2}{2\lambda^*2} = 0, \quad \lambda^*_S = \frac{a'}{r'' - \frac{h\sigma^2}{\lambda^*3}}, \quad \lambda^*_h = -\frac{\sigma^2}{2\lambda^*2r'' - \frac{h\sigma^2}{\lambda^*3}}.$$ 

Then, the first and the last terms in (30) are zero. We have

$$\frac{\partial^2 \tilde{V}}{\partial S \partial h} = -\frac{1}{2} - a'\lambda^*_h = -\frac{1}{2} + \frac{a'\sigma^2}{\lambda^*2r'' - \frac{h\sigma^2}{\lambda^*3}}.$$ 

Now we show that the above is less than zero if evaluated at $S^*$. This is because $S^*$ satisfies (6), which implies $a'(S^*) = -\frac{h}{2\lambda^*}$, and

$$\frac{\partial^2 \tilde{V}}{\partial S \partial h} = -\frac{1}{2} + \frac{h\sigma^2}{\lambda^*2r'' + \frac{h\sigma^2}{\lambda^*3}} < -\frac{1}{2} + \frac{h\sigma^2}{\lambda^*2r'' + \frac{h\sigma^2}{\lambda^*3}} < -\frac{1}{4}.$$ 

Hence, $\tilde{V}(S, h)$ is submodular in $(S, h)$ in the neighborhood of the optima, and therefore, $S^*(h)$ is decreasing in $h$.

(iii) This part is obvious. 

References


Figure 1. Optimal replenishment with a fixed price
Figure 2. Optimal price with a fixed replenishment level
Figure 3. Optimal $N$ prices and the optimal single price
Figure 4. Sequential vs. joint pricing and replenishment decisions
Figure 5. Impact of demand variability on joint optimal solutions
Figure 6. Effect of $N$, the number of price changes.
Figure 7. Multiple prices vs. a single price

\[ \frac{V_8^* - V_1^*}{V_1^*} (\%) \]

\[ \sigma \]

\[ h \]

\[ \% \]
Figure 8. Bounds on the profit improvement
A Discounted Objective

In this section, we consider our model under a discounted objective. The demand model, cost parameters and policy specifications are described in Section 2 of the paper. Parallel to the analysis in Section 3, we first derive the profit functions, and then consider optimal replenishment and pricing decisions, with the other decision variable fixed or optimized jointly. We will focus on the contrasts rather than the similarities between these two classes of models. So as to present the results with minimal distraction, all proofs are relegated to Section B.

Recall that period \( n \) refers to the period in which the price \( p_n \) applies and the inventory drops from \( S_{n-1} = \frac{(N-n+1)S}{N} \) to \( S_n = \frac{(N-n)S}{N} \). Let \( \gamma > 0 \) be the discount rate. Let \( v_{n,\gamma}(\lambda_n) \) denote the expected profit over period \( n \) discounted to the beginning of the period. We have

\[
v_{n,\gamma}(\lambda_n) = E \left[ \int_0^{\tau_n} e^{-\gamma t} \left( p(\lambda_n)dD(t + T_{n-1}) - hX(t + T_{n-1})dt \right) \right]
\]

\[
= E \left[ \int_0^{\tau_n} e^{-\gamma t} \left( p(\lambda_n) + \frac{h}{\gamma} \right)dD(t + T_{n-1}) + \frac{he^{-\gamma \tau_n}S_n}{\gamma} - \frac{hs_{n-1}}{\gamma} \right]
\]

\[
= \left( p(\lambda_n) + \frac{h}{\gamma} \lambda_n \right) \frac{1}{\gamma} \left( 1 - E[e^{-\gamma \tau_n}] \right) + \frac{hE[e^{-\gamma \tau_n}]S_n}{\gamma} - \frac{hs_{n-1}}{\gamma}
\]

\[
= f(\lambda_n) + \frac{hS(1 - \Theta(\lambda_n))}{N\gamma}
\]

(31)

where,

\[
f(\lambda) = \left( p(\lambda) + \frac{h}{\gamma} \lambda \right) \left( 1 - \Theta(\lambda) \right) + \frac{h\Theta(\lambda)S}{\gamma} - \frac{hS(N + 1)}{N\gamma},
\]

\[
\Theta(\lambda) = E[e^{-\gamma \tau_n(\lambda)}] = e^{-b(\lambda)S/N},
\]

\[
b(\lambda) = \frac{\sqrt{\lambda^2 + 2\sigma^2} - \lambda}{\sigma^2} = \frac{2\gamma}{\sqrt{\lambda^2 + 2\sigma^2} + \lambda}.
\]

Here Lemma 1 is applied to derive the expression for \( \Theta(\lambda) \). Assuming that the optimal \( \lambda_n \) is finite, then \( \Theta(\lambda_n) < 1 \). Let \( \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathcal{L}^N \), and let \( V_\gamma(S, \lambda) \) denote the discounted profit starting from zero inventory under the policy \((S, \lambda)\). Then,

\[
V_\gamma(S, \lambda) = \frac{v_\gamma(S, \lambda)}{1 - \Theta_\gamma(S, \lambda)},
\]

(32)

where,

\[
v_\gamma(S, \lambda) = v_{1,\gamma}(\lambda_1) + \Theta(\lambda_1)v_{2,\gamma}(\lambda_2) + \cdots + \Theta(\lambda_1)\cdots\Theta(\lambda_{N-1})v_{N,\gamma}(\lambda_N) - c(S),
\]

\[
\Theta_\gamma(S, \lambda) = \Theta(\lambda_1)\cdots\Theta(\lambda_N) < 1.
\]
Note in the above expressions \( v_n, \gamma \) and \( \Theta \) depend on \( S \) as well, but we have suppressed \( S \) to simplify notation. For the same reason, we shall omit \( S \) in \( v_\gamma(S, \lambda) \) and \( \Theta_\gamma(S, \lambda) \) when \( S \) is not a decision variable.

For a single price, (32) becomes

\[
V_\gamma(S, \lambda) = \frac{r(\lambda)}{\gamma} - \frac{S(h/\gamma + a(S))}{1 - e^{-b(\lambda)S}} + \frac{h\lambda}{\gamma^2}. \tag{33}
\]

**Optimal Replenishment**

The problem is

\[
\max_{S > 0} V_\gamma(S, \lambda).
\]

The first-order condition is

\[
\left[\frac{h}{\gamma} + a(S) + Sa'(S)\right](e^{bs} - 1) = bS\left[\frac{h}{\gamma} + a(S)\right]. \tag{34}
\]

**Proposition A.1** \( V_\gamma(S, \lambda) \) is strictly concave in \( S \) if \( a'(S) < 0 \) and \( c''(S) \geq 0 \).

The condition \( a'(S) < 0 \) holds if \( S \) satisfies (34). Hence, the first-order condition in (34) is sufficient for optimality if \( c''(S) \geq 0 \). We have also identified examples of multiple stationary points when \( c''(S) < 0 \). (This example is \( c(S) = 2 + 10S - S^2 \) for \( S \in [0, 5] \), \( \lambda(p) = 10 - p \), \( \sigma = 0.5 \), \( h = 1 \) and \( \gamma = 0.3 \). Two stationary points can be found.)

**Proposition A.2** With price fixed,

(i) the optimal replenishment level \( S_\gamma^* \) is increasing in \( \sigma \), decreasing in \( h \) and \( p \);

(ii) the optimal profit is decreasing in \( \sigma \) and \( h \).

When there are multiple optima, in lieu of increasing and decreasing in part (i), the relevant properties are ascending and descending, respectively.

Recall, under the average objective and fixed price, the demand variability has no effect on the optimal replenishment level. In contrast, the demand variability will raise the replenishment level under the discounted objective. (See the example provided in Section 4.1 of the paper.) Other monotonicity properties in Proposition 1 continue to hold here.

**The Single-Price Problem**

The problem is

\[
\max_{\lambda \in \mathcal{L}} V_\gamma(S, \lambda)
\]
The first-order condition is

\[ r'(\lambda) + \frac{hS + \gamma c(S)}{(1 - e^{-b(\lambda)S})^2} Se^{-b(\lambda)S} b'(\lambda) + \frac{h}{\gamma} = 0. \]  

(35)

**Proposition A.3** \( V_\gamma(S, \lambda) \) is strictly concave in \( \lambda \).

Hence the optimal single price is uniquely determined by the first-order condition.

**Proposition A.4** With the replenishment level fixed at \( S \), let the optimal price be \( p^* \). Then
(i) \( p^* \) is decreasing in \( h \);
(ii) \( p^* \) is decreasing in \( S \) for \( S \) satisfying \( a'(S) \leq 0 \);
(iii) the optimal profit is decreasing in \( \sigma \) and \( h \).

Most results in Proposition 2 of the paper continue to hold here. The only difference is that the optimal price is decreasing in demand variability under the average objective, while here the effect of demand variability is rather unclear. Numerically, we have observed that in most cases the optimal price is still decreasing in \( \sigma \) under the discounted objective, but exceptions do exist.

**The N-Price Problem**

The problem is

\[ \max_{\lambda \in L^N} V_\gamma(S, \lambda). \]  

(36)

This problem has an equivalent dynamic programming formulation. Let \( V_{n, \gamma} \) denote the optimal discounted profit-to-go starting from the beginning of period \( n \). Following the principle of optimality, we have the following recursions:

\[
V_{1, \gamma} = \max_{\lambda \in L} \{-c(S) + v_{1, \gamma}(\lambda) + \Theta(\lambda)V_{2, \gamma}\}, \\
V_{n, \gamma} = \max_{\lambda \in L} \{v_{n, \gamma}(\lambda) + \Theta(\lambda)V_{n+1, \gamma}\}, \quad n = 2, \ldots, N - 1, \\
V_{N, \gamma} = \max_{\lambda \in L} \{v_{N, \gamma}(\lambda) + \Theta(\lambda)V_{1, \gamma}\}. 
\]  

(37)

The following result parallels Proposition 3 for the average objective in the paper.

**Proposition A.5** For any fixed replenishment level \( S \), the optimal prices under the discounted objective are increasing over the periods, i.e., \( p_1^* \leq p_2^* \leq \ldots \leq p_N^* \).

Instead of searching in \( L^N \) for the optimal \( \lambda \), we provide an iterative algorithm with each step solving a one-dimensional problem in \( L \).

A fixed-point algorithm for solving the dynamic program (37)
1. Set an arbitrary initial value $V_0^{\gamma}$, and set $\epsilon > 0$ small.

2. Set $V_1^{\gamma}$ equal to $V_0^{\gamma}$ in the last equation of (37).
   Solve for $V_{N,\gamma}, V_{N-1,\gamma}, \ldots, V_1^{\gamma}$ recursively following (37).

3. If $|V_1^{\gamma} - V_1^{\gamma}| < \epsilon$, stop. Otherwise, set $V_1^{\gamma}$ equal to $V_1^{\gamma}$, and go to Step 2.

To explore the convergence of the algorithm, we substitute the equations for $V_2^{\gamma}, \ldots, V_N^{\gamma}$ in (37) recursively into the first equation and obtain the following:

$$V_1^{\gamma} = \max_{\lambda \in \mathcal{L}^N} \left\{ v_\gamma(\lambda) + \Theta_\gamma(\lambda)V_1^{\gamma} \right\} \equiv \max_{\lambda \in \mathcal{L}^N} \psi(\lambda, V_1^{\gamma}) \equiv \Psi(V_1^{\gamma}).$$

(38)

The mapping $\Psi$ is not necessarily a contraction mapping, yet we are able to prove the convergence of the above algorithm.

**Proposition A.6** Let $\psi$ and $\Psi$ be defined as above, then

(i) the mapping $\Psi(V)$ is increasing and Lipschitz continuous with modulus 1;

(ii) there exists a unique fixed point satisfying $V^* = \Psi(V^*) = \psi(\lambda^*, V^*)$. Furthermore, $\lambda^*$ solves (36) with the maximum discounted profit being $V^*$; and

(iii) The above algorithm for solving (37) converges to the optimal solution from any initial value $V_0^{\gamma}$.

**Joint Pricing-Replenishment**

In general, the pricing-replenishment joint optimal decisions lack the monotonicity properties with respect to the parameters $\sigma$ and $h$ even under the single price.

When $N$ prices are optimized jointly with the replenishment level $S$, we can use the iterative algorithm developed above for each fixed $S$, and then do a line search to find the optimal $S$.

**Convergence to the Average Objective**

The discounted profit function converges to the average profit function in the following sense.

**Proposition A.7** Let $v_{n,\gamma}$ and $V_\gamma$ be defined as (31) and (32), and let $v_n$ and $V$ follow (3) and (4) under the average objective. Then,

$$\lim_{\gamma \to 0} v_{n,\gamma} = v_n \quad \text{and} \quad \lim_{\gamma \to 0} \gamma V_\gamma = V.$$ 

The optimal pricing-replenishment decisions under the discounted objective converge to those under the average objective, notwithstanding the contrasts highlighted above.
Proposition A.8 Let \((S^*_\gamma, p^*_\gamma)\) and \((S^*, p^*)\) denote, respectively, the optimal solution under the discounted and the average objectives. Then, we have

\[
\lim_{\gamma \to 0} S^*_\gamma = S^* \quad \text{and} \quad \lim_{\gamma \to 0} p^*_\gamma = p^*.
\]

B Proofs

Proof of Proposition A.1. We can derive

\[
\frac{\partial^2 V_\gamma}{\partial S^2} = -\frac{(bSe^{-bS} + 2e^{-bS} + bS - 2)e^{-bS}b(\frac{h}{\gamma} + a(S))}{(1 - e^{-bS})^3} + \frac{2bSe^{-bS}a'(S)}{(1 - e^{-bS})^2} - \frac{c''(S)}{1 - e^{-bS}}.
\]

The first term on the right side is non-positive due to the fact that

\[
h(x) := xe^{-x} + 2e^{-x} + x - 2 \ge 0 \quad \text{for all} \quad x \ge 0,
\]

since \(h(0) = h'(0) = 0\) and \(h''(x) = xe^{-x} \ge 0\) for all \(x \ge 0\); the second term is negative since \(a'(S) < 0\); the third term is also non-positive if \(c(S)\) is convex in \(S\). Hence \(V_\gamma(S, \lambda)\) is strictly concave in \(S\) if \(a'(S) < 0\) and \(c''(S) \ge 0\).

Proof of Proposition A.2. (i) To show that \(S^*_\gamma\) is decreasing in \(p\), it suffices to verify that \(V_\gamma(S, \lambda)\) is supermodular in \((S, \lambda)\); i.e., \(\frac{\partial^2 V_\gamma}{\partial S \partial \lambda} \ge 0\). Since the optimal replenishment level must satisfy \(a'(S) < 0\), it suffices to verify the supermodularity in the sub-lattice where \(a'(S) < 0\).

We have

\[
\frac{\partial^2 V_\gamma(S, \lambda)}{\partial S \partial \lambda} = g^2 e^{-bS}b'S\left[\frac{h}{\gamma} + c'(S) - \left(\frac{h}{\gamma} + a(S)\right)(bS(1 + e^{-bS}) - 1)\right].
\]

On the right side of the above, we have \(b'(\lambda) < 0\) and \(\frac{h}{\gamma} + c'(S) < \frac{h}{\gamma} + a(S)\). (The latter is implied by \(a'(S) < 0\).) Thus, \(\frac{\partial^2 V_\gamma}{\partial S \partial \lambda} \ge 0\) can be established if we have

\[
bS\frac{1 + e^{-bS}}{1 - e^{-bS}} - 1 \ge 1.
\]

But this last inequality follows immediately from (39).

The proof of \(S^*\) increasing in \(\sigma\) is completely analogous: we can follow the above to establish \(\frac{\partial^2 V_\gamma}{\partial S \partial \sigma} \ge 0\) in the sub-lattice where \(a'(S) < 0\); the only change is to replace \(b'(\lambda)\) by \(b'(\sigma)\). Note in particular that \(b'(\sigma) < 0\) too.

To show that \(S^*\) is decreasing in \(h\), note that \(V_\gamma\) is submodular in \((S, h)\):

\[
\frac{\partial^2 V_\gamma}{\partial S \partial h} = -\frac{1 - e^{-bS} - bSe^{-bS}}{\gamma(1 - e^{-bS})^2} \le 0,
\]
since $1 - e^{-bS} \geq bSe^{-bS}$.

(ii) This part is obvious. ■

**Proof of Proposition A.3.** It suffices to show that $g(\lambda) := \frac{1}{1 - e^{-b(\lambda)S}}$ is convex in $\lambda$. To this end, we derive

$$g''(\lambda) = g^2e^{-bS}S\left(b'^2S(2ge^{-bS} + 1) - b''\right)$$

$$= \frac{g^2e^{-bS}S}{\lambda^2 + 2\gamma\sigma^2}\left(b'^2S(2ge^{-bS} + 1) - \frac{2\gamma}{\sqrt{\lambda^2 + 2\gamma\sigma^2}}\right),$$

where the last equality follows from $b' = -\frac{b}{\sqrt{\lambda^2 + 2\gamma\sigma^2}}$ and $b'' = \frac{2\gamma}{(\lambda^2 + 2\gamma\sigma^2)^{3/2}}$. We have

$$b'^2S(2ge^{-bS} + 1) - \frac{2\gamma}{\sqrt{\lambda^2 + 2\gamma\sigma^2}} > b'^2S(2ge^{-bS} + 1) - 2b$$

$$= bg(bSe^{-bS} + bS + 2e^{-bS} - 2)$$

$$\geq 0,$$

where the last inequality is again due to (39). Hence, $g(\lambda)$ is convex in $\lambda$, and $V_\gamma(S, \lambda)$ is concave in $\lambda$. ■

**Proof of Proposition A.4.** (i) To prove the monotonicity in $h$, we show that $V_\gamma$ is supermodular in $(\lambda, h)$:

$$\frac{\partial^2 V}{\partial \lambda \partial h} = \frac{S^2e^{-bS}b'}{\gamma(1 - e^{-bS})^2} + \frac{1}{\gamma^2}$$

$$= \frac{1}{b\gamma^2}\left[-\frac{e^{-bS}S^2b'}{(1 - e^{-bS})^2\sqrt{\lambda^2 + 2\sigma^2\gamma}} + b\right]$$

$$\geq \frac{1}{b\gamma^2}\left[-\frac{\gamma}{\sqrt{\lambda^2 + 2\sigma^2\gamma}} + b\right]$$

$$\geq 0,$$

where the first inequality results from the following:

$$\frac{x^2e^{-x}}{(1 - e^{-x})^2} = \frac{x^2}{e^x + e^{-x} - 2} = \sum_{n=1}^{\infty} \frac{x^2}{2x^{2n}/(2n)!} \leq 1,$$

and in the last inequality we used the fact that

$$\frac{\gamma}{\sqrt{\lambda^2 + 2\sigma^2\gamma}} \leq b = \frac{2\gamma}{\sqrt{\lambda^2 + 2\sigma^2\gamma} + \lambda} \leq \frac{\gamma}{\lambda}.$$
(ii) The monotonicity in $S$ relies on the supermodularity of $V_\gamma$ in $(S, \lambda)$, which has been established in the proof of Proposition A.2 (i).

(iii) This part is obvious. 

**Proof of Proposition A.5.** Let the optimal profit-to-go be $V_{n,\gamma}^*$, for $n = 1, \ldots, N$, which satisfy (37). For the ease of exposition, define $V_{N+1,\gamma}^* := V_{1,\gamma}^*$. In view of (37) and (31), we can write

$$\lambda_n^* \in \arg \max_{\lambda \in L} \{ F(n, \lambda) \}, \quad \text{for} \ n = 1, \ldots, N,$$

where

$$F(n, \lambda) = f(\lambda) + \frac{hS(1 - \Theta(\lambda))}{N\gamma} n + \Theta(\lambda)V_{n+1,\gamma}^*.$$

Thus, $\lambda_n^*$ is decreasing in $n$ if we can show that $F(n, \lambda)$ is submodular in $(n, \lambda)$. To this end, notice that

$$F(n, \lambda) - F(n-1, \lambda) = \frac{hS(1 - \Theta(\lambda))}{N\gamma} + \Theta(\lambda)(V_{n+1,\gamma}^* - V_{n,\gamma}^*).$$

Since $\Theta(\lambda)$ is decreasing in $\lambda$, the right side of the last equation is decreasing in $\lambda$ if $V_{n+1,\gamma}^* - V_{n,\gamma}^* \geq \frac{hS}{N\gamma}$. This is indeed the case. Consider starting from the inventory level $\frac{S(N-n+1)}{N}$ but following the optimal policy for the inventory starting at $\frac{S(N-n)}{N}$. This implies that $\frac{S}{N}$ units of inventory will be held for ever. Consequently, the profit associated with this strategy is $V_{n+1,\gamma}^* - \frac{hS}{N\gamma}$, where $\frac{hS}{N\gamma}$ is the discounted cost of holding the extra $\frac{S}{N}$ units. Since this strategy is sub-optimal, we have

$$V_{n+1,\gamma}^* - V_{n,\gamma}^* \geq V_{n+1,\gamma}^* - \left( V_{n+1,\gamma}^* - \frac{hS}{N\gamma} \right) = \frac{hS}{N\gamma}.$$

The submodularity of $F(n, p)$ follows, and hence $\lambda_n^*$ is decreasing in $n$, or equivalently, $p_n^*$ is increasing in $n$. 

**Proof of Proposition A.6.**

(i) For $V > \tilde{V}$, let $\lambda$ and $\tilde{\lambda}$ be the maximizers (not necessarily in the interior) of $\psi(\cdot, V)$ and $\psi(\tilde{\lambda}, \tilde{V})$, respectively (refer to (38)). Then,

$$\Psi(V) - \Psi(\tilde{V}) = \psi(\lambda, V) - \psi(\tilde{\lambda}, \tilde{V}) \geq \psi(\tilde{\lambda}, V) - \psi(\tilde{\lambda}, \tilde{V}) = \Theta(\tilde{\lambda})(V - \tilde{V}) \geq 0,$$

which implies that $\Psi(V)$ is increasing in $V$. Furthermore,

$$\Psi(V) - \Psi(\tilde{V}) = \psi(\lambda, V) - \psi(\tilde{\lambda}, \tilde{V}) \leq \psi(\lambda, V) - \psi(\lambda, \tilde{V}) = \Theta(\lambda)(V - \tilde{V}). \quad (40)$$
Since \( \Theta_\gamma(\lambda) \leq 1 \), the above inequality implies that \( \Psi \) is Lipschitz continuous, with modulus no larger than 1. That is, \( \Psi(V) \) is non-expansive.

(ii) Let \( \lambda^* \) be the optimal solution to (36) with optimal value \( V^* \). We eliminate the uninteresting case where \( \lambda^* = \infty \). Thus, \( \Theta_\gamma(\lambda^*) < 1 \) and \( V^* = \frac{v_\gamma(\lambda^*)}{1-\Theta_\gamma(\lambda^*)} \), or equivalently, \( V^* = \psi(\lambda^*, V^*) \).

On the other hand, \( V^* \geq \frac{v_\gamma(\lambda)}{1-\Theta_\gamma(\lambda)} \), or equivalently, \( V^* \geq \psi(\lambda, V^*) \) for all \( \lambda \in \mathcal{L}^N \). Thus, \( V^* = \max_{\lambda \in \mathcal{L}^N} \psi(\lambda, V^*) = \Psi(V^*) \); and hence, \( V^* \) is a fixed point of \( \Psi \). This proves the existence.

Conversely, suppose \( V^* \) is a fixed point of \( \Psi(V) \) with the corresponding maximizer \( \lambda^* \). Note that \( \lambda^* = \infty \) implies that all inventory can be sold out instantaneously after the replenishment (with the revenue covering exactly the replenishment cost). If we eliminate this uninteresting case, then \( \Theta_\gamma(\lambda^*) < 1 \), and \( V^* = \psi(\lambda^*, V^*) \) or \( V^* = \frac{v_\gamma(\lambda^*)}{1-\Theta_\gamma(\lambda^*)} \). At the same time, \( V^* \geq \psi(\lambda, V^*) \) or \( V^* \geq \frac{v_\gamma(\lambda)}{1-\Theta_\gamma(\lambda)} \) for all \( \lambda \in \mathcal{L}^N \). Hence, \( \lambda^* \) solves the problem in (36), achieving the optimal value \( V^* \).

Now, suppose there exist two fixed points \( V^* > \tilde{V}^* \) corresponding to the maximizers \( \lambda^* \) and \( \tilde{\lambda}^* \), respectively. Then, both \( V^* \) and \( \tilde{V}^* \) are optimal values for the problem in (36), an obvious contradiction (against optimality). This establishes the uniqueness.

(iii) Let \( V^* \) be the unique fixed point. From (i), we have \( \Psi(V) > V \) for \( V < V^* \) and \( \Psi(V) < V \) for \( V > V^* \). Hence, if we start from an initial value \( V_0 < V^* \), the sequence of values generated by the iteration \( V_n = \Psi(V_{n-1}) \) is increasing and bounded from above by \( V^* \). So the sequence converges to a fixed point of the mapping \( \Psi \), which must be \( V^* \). A similar argument holds for an initial value \( V_0 > V^* \).

Proof of Proposition A.7. When \( \gamma \to 0 \), via Taylor expansion, we can derive the following:

\[
b(\lambda) = \frac{\gamma}{\lambda} - \frac{\sigma^2 \gamma^2}{2\lambda^3} + O(\gamma^3),
\]
\[
\Theta(\lambda) = e^{-b(\lambda)S/N} = 1 - \frac{S\gamma}{N\lambda} + \frac{\sigma^2 S \gamma^2}{2N^2 \lambda^3} + \frac{S^2 \gamma^2}{2N^2 \lambda^2} + O(\gamma^3).
\]

Applying (41) and (42), we have

\[
v_{n, \gamma}(\lambda) = \left( p(\lambda) + \frac{h}{\gamma} \right) \left( \frac{S}{N} - \frac{\sigma^2 S \gamma}{2N\lambda^2} - \frac{S^2 \gamma}{2N^2 \lambda} \right) + \frac{hS}{\gamma} - \frac{hS^2}{N\lambda} - \frac{hS(N+1)}{N\gamma} + \frac{hS^2 n}{N^2 \lambda} + o(1)
\]
\[
= \frac{p(\lambda) S}{N} - \frac{hS^2 (N-n+1)}{N^2 \lambda^2} - \frac{hS^2 S}{2N^2 \lambda} + o(1)
\]
\[
= v_n(\lambda) + o(1), \quad \text{as} \ \gamma \to 0.
\]
\[
\gamma V_\gamma(S, \lambda) = \frac{v_\gamma(\lambda)}{\gamma^{-1}(1 - \Theta_\gamma(\lambda))} = \frac{\sum_n v_n(\lambda_n) + o(1)}{\gamma^{-1}(1 - \prod_{n=1}^N (1 - \frac{S\gamma}{N\lambda_n}))} = \frac{\sum_n v_n(\lambda_n) - c(S) + o(1)}{\gamma^{-1}(1 - \sum_n \lambda_n^{-1}) + o(1)} = V(S, \mu) + o(1), \quad \text{as } \gamma \to 0.
\]

**Proof of Proposition A.8.** The first-order condition for \(S^*_\gamma\) in (34) can be written as follows:

\[
(h + \gamma c'(S))\left(S_\gamma \frac{\sigma^2 S^2}{2\lambda^3} - S^2 \frac{\gamma^2}{2\lambda^2}\right) = (1 - \frac{S_\gamma}{\lambda})(\frac{\gamma}{\lambda} - \frac{\sigma^2 \gamma^2}{2\lambda^2})[hS + \gamma c(S)] + o(\gamma^2)
\]

Ignoring higher-order terms, we can simplify the above to the equation in (6), which is the first-order optimality condition for \(S^*_\gamma\).

Using (41) and (42), when \(\gamma \to 0\), the first-order condition in (35) can be approximated as

\[
0 = r'(\lambda) - \frac{h}{\gamma} - a(S) + h \left(\frac{\sigma^2 \gamma^2}{2\lambda^2} - \frac{S\gamma}{\lambda} - \frac{S\gamma}{\lambda} - 3 \frac{\sigma^2 \gamma^2}{2\lambda^2}\right) + h + o(1)
\]

which is exactly the same as the first-order condition in (9) for the average objective.

The convergence of the first-order conditions (34) and (35) to their counterparts in (6) and (9), respectively, implies that the decisions under the discounted objective converge to those under the average objective. ■

**C Price-Sensitive Demand Variability**

In Section 4.3 of the paper, the Brownian model (1) is extended to a more general model (19), which allows a price-dependent diffusion coefficient. Using the same way as that leading to (4) and (5), we can derive the average objective function for the price-sensitive demand model, shown in (20) and (21) in the paper. In this section, we prove the results extending Proposition 3, Lemma 2, Proposition 5 and 9.

The relevant maximization problem is

\[
\max_{\mu \in \mathcal{M}^N} V(S, \mu) = \frac{\sum_{n=1}^N \left[p(\frac{1}{\mu_n}) - \frac{hS}{N}(N - n + \frac{1}{2})\mu_n - \frac{1}{2} h\sigma(\mu_n^{-1})^2 \mu_n^2 - a(S)\right]}{\sum_{n=1}^N \mu_n}.
\]

9
**Proposition C.1** For any replenishment quantity $S$, the optimal prices are increasing over the periods, i.e., $p_1^* \leq p_2^* \leq \cdots \leq p_N^*$.

**Proof.** Same as the original proof for Proposition 3. \(\blacksquare\)

**Assumption 3 (on demand variability)** $C_v(\lambda) = \sigma(\lambda)/\sqrt{\lambda}$ is convex in $\lambda$.

This assumption implies that $C_v^2(\lambda) = \sigma(\lambda)^2/\lambda$ is convex in $\lambda$ and hence that $\sigma(\mu^{-1})^2\mu^2$ is convex in $\mu$. We again note that the holding cost attributed to the deterministic part of the demand is the only motive for varying prices over the periods. In the numerator of (43), the additional holding cost term $-\sum 1/2h \sigma(\mu^{-1})^2\mu^2$ and the revenue term $\sum p(\frac{1}{\mu})$ are, on the contrary, suggesting not varying prices, since they are both concave in $\mu$. These two terms limit the extend to which the optimal prices vary, and thus limit the potential profit improvement.

**Lemma C.1** For fixed $S$, let $\mu^*$ be the optimal $N$ prices. Then for $1 \leq m < n \leq N$,

$$\mu_n^* - \mu_m^* \leq \frac{S(n-m)}{N(G + \frac{B}{h})},$$

where $B = \inf \{ -\frac{d^2(\frac{1}{\mu})^2\mu^2}{d\mu^2} : \mu \in [\mu_1^*, \mu_N^*] \}$ and $G = \inf \{ \frac{d^2(\frac{1}{\mu})^2\mu^2}{d\mu^2} : \mu \in [\mu_1^*, \mu_N^*] \}$.

**Proof.** By Proposition C.1, $\mu_n^* \geq \mu_m^*$ for $n > m$. If $\mu_n^* = \mu_m^*$, then the desired inequality obviously holds. If $\mu_n^* < \mu_m^*$, consider the following alternative pricing policy: $(\mu_1^*, \ldots, \mu_n^* + \delta, \ldots, \mu_m^* - \delta, \ldots, \mu_N^*)$, where $0 < \delta < \frac{\mu_n^* - \mu_m^*}{2}$. Since this alternative cannot be better than the optimum, we have

$$p(\frac{1}{\mu_n^* + \delta}) + p(\frac{1}{\mu_n^* - \delta}) - \frac{hS}{N} \left[ (N - m + \frac{1}{2})(\mu_n^* + \delta) + (N - n + \frac{1}{2})(\mu_n^* - \delta) \right] - h \left[ f(\mu_n^* + \delta) + f(\mu_n^* - \delta) \right] \leq p(\frac{1}{\mu_n^*}) + p(\frac{1}{\mu_n^*}) - \frac{hS}{N} \left[ (N - m + \frac{1}{2})\mu_n^* + (N - n + \frac{1}{2})\mu_n^* \right] - h \left[ f(\mu_n^*) + f(\mu_n^*) \right],$$

where $f(\mu) := \frac{1}{2}\sigma(\frac{1}{\mu})^2\mu^2$. Taylor’s expansion with straightforward algebra simplifies the inequality to

$$\frac{dp}{d\mu}(\frac{1}{\mu_n^* + \delta}) \delta - \frac{dp}{d\mu}(\frac{1}{\mu_n^* - \delta}) \delta + \frac{hS}{N} \delta(m - n) + h\delta \left( \frac{df}{d\mu}(\mu_n^* - \delta) - \frac{df}{d\mu}(\mu_n^* + \delta) \right) \leq 0,$$

where $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in [0, 1]$. Dividing both sides by $\delta$, and then letting $\delta \to 0$, we have

$$\frac{dp}{d\mu}(\frac{1}{\mu_n^*}) - \frac{dp}{d\mu}(\frac{1}{\mu_n^*}) + \frac{hS}{N} (m - n) + h \left( \frac{df}{d\mu}(\mu_n^*) - \frac{df}{d\mu}(\mu_n^*) \right) \leq 0.$$  \hspace{1cm} (44)
Applying the mean-value theorem and using the definition of $B$ and $G$, we have

\[
\frac{dp}{d\mu} \left( \frac{1}{\mu_n^*} \right) - \frac{dp}{d\mu} \left( \frac{1}{\mu_n^*} \right) = -\frac{d^2\mathbb{V}}{d\mu^2} \left( \frac{1}{\mu_n^*} \right) (\mu_n^* - \mu_m^*) \geq B(\mu_n^* - \mu_m^*),
\]

where $\zeta_5, \zeta_6 \in [\mu_n^*, \mu_m^*]$. Combining (44)-(46) yields the desired inequality. \[\blacksquare\]

**Proposition C.2** For fixed $S$, let $\mu^*$ be the optimal $N$ prices, and let $V_N^*$ be the optimal average profit defined in (43). Then,

\[
V_N^* - V_1^* \leq \frac{hS^2 (1 - N^{-2})}{12\bar{\mu} (G + \frac{B}{\bar{\mu}})},
\]

where $\bar{\mu} = \frac{1}{N} \sum_{n=1}^{N} \mu_n^*$ and $B = \inf \{ -\frac{d^2p(\frac{1}{\mu})}{d\mu^2} : \mu \in [\mu_1^*, \mu_N^*] \}$ and $G = \inf \{ \frac{d^2(\frac{1}{2} \sigma(\frac{1}{\mu})^2 \mu^2)}{d\mu^2} : \mu \in [\mu_1^*, \mu_N^*] \}$.

**Proof.** The average profit of charging a single price $p(\bar{\mu}^{-1})$ is

\[
V_1 = \frac{p(\frac{1}{\bar{\mu}}) - \frac{hS}{2} \bar{\mu} - \frac{h}{2} \sigma(\bar{\mu}^{-1})^2 \bar{\mu}^2 - a(S)}{\bar{\mu}}.
\]

We have

\[
V_N^* - V_1^* \leq V_N^* - V_1 = \sum_{n=1}^{N} \left[ p(\frac{1}{\mu_n^*}) - p(\frac{1}{\bar{\mu}}) \right] - \frac{hS}{N} \sum_{n=1}^{N} \left[ (N - n + \frac{1}{2}) \mu_n^* - \frac{N}{2} \bar{\mu} \right] - \frac{h}{N} \sum_{n=1}^{N} \sigma(\frac{1}{\mu_n^*})^2 \mu_n^2 - \sigma(\frac{1}{\bar{\mu}})^2 \bar{\mu}^2.
\]

The first term in the numerator, $\sum_{n=1}^{N} \left[ p(\frac{1}{\mu_n^*}) - p(\frac{1}{\bar{\mu}}) \right]$, is concave in $\mu$; and the last term in the numerator $\frac{1}{N} \sum_{n=1}^{N} \sigma(\frac{1}{\mu_n^*})^2 \mu_n^2 - \sigma(\frac{1}{\bar{\mu}})^2 \bar{\mu}^2 \geq 0$, since $\sigma(\frac{1}{\mu})^2 \mu^2$ is convex in $\mu$. Thus,

\[
V_N^* - V_1^* \leq \frac{hS}{N} \sum_{n=1}^{N} \frac{1}{\mu_n^*} \left[ (N - n + \frac{1}{2}) \mu_n^* - \frac{N}{2} \bar{\mu} \right] - \frac{h}{N} \sum_{n=1}^{N} \sigma(\frac{1}{\mu_n^*})^2 \mu_n^2 - \sigma(\frac{1}{\bar{\mu}})^2 \bar{\mu}^2 \geq 0.
\]
where in the last line, the series ends with $\mu^*_{N/2+1} - \mu^*_{N/2}$ if $N$ is even, and ends with $2(\mu^*_{(N+3)/2} - \mu^*_{(N-1)/2})$ if $N$ is odd.

Applying Lemma C.1 and the identity:

$$(N-1)^2 + (N-3)^2 + \cdots + (N+1-2\left\lfloor \frac{N}{2} \right\rfloor)^2 = \frac{(N-1)N(N+1)}{6},$$

we have

$$V^*_N - V^*_1 \leq \frac{hS^2}{\bar{\mu}(G + \frac{B}{\mu})} \left( \frac{(N-1)^2 + (N-3)^2 + \cdots + (N+1-2\left\lfloor \frac{N}{2} \right\rfloor)^2}{2N^3} \right)$$

$$= \frac{hS^2}{\bar{\mu}(G + \frac{B}{\mu})} \frac{(N-1)N(N+1)}{12N^3}$$

$$= \frac{hS^2 (1-N^{-2})}{12\bar{\mu}(G + \frac{B}{\mu})}.$$

\[ \square \]

**Proposition 10** Let $(S^*_N, \mu^*)$ be the optimal joint pricing-replenishment decision, and $V^*_N$ be the corresponding optimal profit. Then,

$$V^*_N - V^*_1 \leq \frac{hS^2^2 (1-N^{-2})}{12\bar{\mu}(G_N + \frac{B_N}{\mu})},$$

where $\bar{\mu} = \frac{1}{N} \sum_{n=1}^{N} \mu^*_n$, $B_N = \inf \left\{ - \frac{d^2\mu(\frac{1}{2})}{d\mu^2} : \mu \in [\mu^*_1, \mu^*_N] \right\}$, and $G_N = \inf \left\{ \frac{d^2(\frac{1}{2}\sigma(\frac{1}{2})^2\mu^2)}{d\mu^2} : \mu \in [\mu^*_1, \mu^*_N] \right\}$.

**Proof.** Same as the proof for Proposition 9 with $\sigma^2$ replaced by $G_N$. \[ \square \]