9

Coordinated Replenishment and Rework with Multiple Unreliable Supply Sources

Having started in Chapter 3 from the customer end and gone through several different configurations of production facilities in Chapters 4 through 8, we have now reached the other end of the supply chain. Our focus here is a set of suppliers, which have different grades of quality, and collectively form the sources of supply to a production-inventory system. The system, in turn, fills in a single stream of customer demands. In addition to placing orders of different size with the difference sources, the system can choose to inspect a certain number of units received from each source and repair any identified defectives. We first study this optimal inspection decision, assuming the order quantities are given. This is formulated as an integer optimization problem, which minimizes an objective function that takes into account the inspection and repair costs, as well as the penalty for unmet demand, and the salvage value of any surplus units. As the objective function is non-linear and non-separable, such an integer optimization problem is difficult to solve in general. However, due to certain special properties of the objective function — properties that appear to be strengthenings of supermodularity and convexity (Proposition 9.1), the optimal solution to the inspection problem has a special structure: it is only necessary to inspect those units supplied from sources that fall in the middle range of the quality spectrum, i.e., we can forgo the inspection of units from the best and the worst sources. Consequently, the optimal solution is easily identified through a greedy algorithm.

When the replenishment decision is incorporated into the model, the above problem structure essentially remains intact. And the optimal solution to the order quantities can also be greedily generated, and inspection is optimally applied only to (at most) a single source identified by the greedy algorithm. Furthermore, in the case of linear cost functions (in terms of quality), it is optimal to place orders from two sources only, i.e., dual sourcing is optimal.

These single-period results can be extended to an infinite horizon, with a long-run average cost objective. The optimal policy is an order-up-to policy, with the order-up-to level derived from solving a single-period problem.

When supply imperfection takes the form of a reduced quantity ("yield loss"), the model is easily adapted to generate the optimal replenishment decisions. In fact, the quality control mechanism can be translated into a provision of paying an additional premium to guarantee the delivery quantity.

Below, we start with model description and formulation in §9.1, focusing first on the optimal inspection problem, assuming the replenishment decisions have already been made. Key properties of the objective function are established in §9.2, and the optimal inspection problem is solved in §9.3 via a greedy algorithm. Solutions to the optimal replenishment quantities, taking into account the inspection decision, are studied in §9.4. Extensions to the infinite horizon are presented in §9.5. Adaptation of the model to the setting of yield loss is presented in §9.6.

9.1 The Inspection/Rework Model

There are k sources of supply, indexed by $i = 1, ..., k$; and we shall also refer to the products supplied from source i as type i below. Each unit from source i has a defective rate of θ_i — it is defective with probability θ_i, and non-defective with probability $1 - \theta_i$, independent of all other units. All the defective rates are given constants, with $0 < \theta_1 < \theta_2 < \cdots < \theta_k \leq 1$. In addition, denote $\theta_0 \equiv 0$, signifying a non-defective unit. (As noted in Chapter 3, this simple, binomial defect model may not be appropriate in some applications, and a better and more general model is to make θ_i a random variable too. This, however, will result in an added layer of sequential decisions for inspection, which we do not treat here.)

We start with assuming the order (batch) size from source i, N_i, as given, for all $i = 1, ..., k$; later in §9.4, the order sizes will be treated as decision variables. There is a random demand, denoted D, with a known distribution. The demand can be supplied by products from all k sources, along with some type of warranty or service contract. Suppose the expected warranty/service costs associated with a defective unit and a non-defective unit are, respectively, C_d and C_g; and denote $\Delta := C_d - C_g$. Naturally, assume $\Delta \geq 0$. Defective units can be identified through inspection. Each identified defective unit will be repaired and converted into a non-defective unit. (Hence, the defective rate of any inspected unit is $\theta_0 = 0$.) The unit inspection and repair costs are C_I and C_R, respectively. To ensure that there is enough incentive to repair all identified defective units, assume $C_R \leq \Delta$.

For any surplus unit after demand is satisfied, there is a salvage value, which is a decreasing function of the defective rate, denoted $s(\theta)$. Hence, the salvage value for a surplus unit of batch i is $s(\theta_i)$, if the unit is not inspected; whereas any surplus unit that is inspected (and repaired if necessary) has a salvage value of $s(0)$.

We further assume that supplying demand from a type with a lower index is less costly:

$$C_d\theta_{i-1} + C_g(1-\theta_{i-1}) - s(\theta_i) \leq C_d\theta_i + C_g(1-\theta_i) - s(\theta_{i-1}),$$

or,

$$\Delta(\theta_i - \theta_{i-1}) \geq s(\theta_{i-1}) - s(\theta_i), \qquad \text{for } i = 1, 2, \cdots, k. \tag{9.1}$$

The above guarantees that any demand will always be supplied by the best available unit in terms of quality: starting from the inspected units, followed by the (uninspected) units in batch 1, and then batch 2, and so forth.

Let $(n_1, \cdots, n_k)$ denote the "state" variable, in which n_i units of batch i have been inspected, $i = 1, ..., k$, with any identified defectives repaired. For convenience, denote

$$n_{i,j} := \sum_{\ell=i}^{j} n_\ell \qquad \text{and} \qquad N_{i,j} := \sum_{\ell=i}^{j} N_\ell \qquad \text{for } 1 \leq i \leq j \leq k;$$

and denote $n_{i,j} = N_{i,j} = 0$ if $i > j$. Let $W(D, n_1, \cdots, n_k)$ denote the warranty cost minus salvage value, given that the demand is D and that inspection is terminated in state $(n_1, \cdots, n_k)$. Then,

$$\begin{aligned} & W(D, n_1, \cdots, n_k) \\ = \; & \min\{n_{1,k}, D\}C_g \\ & + \sum_{i=1}^{k} \min\{N_i - n_i, (D - n_{i,k} - N_{1,i-1})^+\}(C_g + \theta_i\Delta) \\ & -(n_{1,k} - D)^+ s(0) \\ & - \sum_{i=1}^{k} [N_i - n_i - (D - n_{i,k} - N_{1,i-1})^+]^+ s(\theta_i). \end{aligned} \tag{9.2}$$

Note that the first two terms on the right hand side above correspond to the warranty costs for inspected and uninspected units that are used to supply demand, while the other two terms correspond to the salvage value of the surplus units that are inspected and uninspected.

Let $\Pi(n_1, n_2, \cdots, n_k)$ denote the expected total cost — including inspection and repair costs, as well as the warranty cost minus salvage value, if we stop inspection in state $(n_1, \cdots, n_k)$. Then,

$$\begin{aligned} & \Pi(n_1, n_2, \cdots, n_k) \\ = \;& C_I n_{1,k} + C_R \sum_{i=1}^{k} \theta_i n_i + \mathsf{E}[W(D, n_1, n_2, \cdots, n_k)], \end{aligned} \tag{9.3}$$

for $n_i \le N_i$, $i = 1, \cdots, k$. We want to find the best solution $(n_1^*, n_2^*, \cdots, n_k^*)$, the number of units inspected for each batch, so as to minimize the expected total cost Π.

9.2 Properties of the Cost Function

The properties of the cost function Π in the proposition below will play a central role in identifying the structure of the optimal solution.

Proposition 9.1 For $i = 2, ..., k$,

$$\begin{aligned} & \Pi(n_1, \cdots, n_{i-1}+1, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1}, n_i+1, \cdots, n_k) \\ = \;& -C_R(\theta_i - \theta_{i-1}) + s(\theta_{i-1}) - s(\theta_i) \\ & +[\Delta(\theta_i - \theta_{i-1}) - (s(\theta_{i-1}) - s(\theta_i))] \\ & \quad \cdot\{[\mathsf{E}(D - n_{i,k} - N_{1,i-1})^+ - \mathsf{E}(D - n_{i,k} - N_{1,i-1} - 1)^+]\}, \end{aligned} \tag{9.4}$$

which is decreasing in $n_{i,k}$, and in particular, decreasing in n_j for all $j \ge i$; furthermore,

$$\begin{aligned} & \Pi(n_1, \cdots, n_{i-1}, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1}, n_i+1, \cdots, n_k) \\ = \;& -(C_I + C_R\theta_i) \\ & + \sum_{j=1}^{i} \Big\{ [s(\theta_{j-1}) - s(\theta_j)] + [\Delta(\theta_j - \theta_{j-1}) - (s(\theta_{j-1}) - s(\theta_j))] \\ & \quad \cdot [\mathsf{E}(D - n_{j,k} - N_{1,j-1})^+ - \mathsf{E}(D - n_{j,k} - N_{1,j-1} - 1)^+] \Big\}, \end{aligned} \tag{9.5}$$

which is decreasing in $n_{j,k}$, for all $j = 1, ..., k$, and hence decreasing in n_j, for all j.

Proof. Making use of the relation

$$\min\{a, b\} = a - (a-b)^+ = a - (b-a)^-,$$

and noticing that $(a^+ - b)^+ = (a-b)^+$ for $b \ge 0$, we rewrite the terms on the right hand side of (9.2) as follows:

$$\min\{n_{1,k}, D\} C_g$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}\min\{N_i-n_i,(D-n_{i,k}-N_{1,i-1})^+\}(C_g+\theta_i\Delta)\\
=\;& [D-(D-n_{1,k})^+]\,C_g\\
&+\sum_{i=1}^{k}\{(D-n_{i,k}-N_{1,i-1})^+\\
&\qquad-[(D-n_{i,k}-N_{1,i-1})^+-(N_i-n_i)]^+\}(C_g+\theta_i\Delta)\\
=\;& [D-(D-n_{1,k})^+]C_g\\
&+\sum_{i=1}^{k}[(D-n_{i,k}-N_{1,i-1})^+-(D-n_{i+1,k}-N_{1,i})^+](C_g+\theta_i\Delta)\\
=\;& [D-(D-N_{1,k})^+]C_g\\
&+\sum_{i=1}^{k}(D-n_{i,k}-N_{1,i-1})^+(\theta_i-\theta_{i-1})\Delta-(D-N_{1,k})^+\theta_k\Delta
\end{aligned}
$$

and

$$
\begin{aligned}
&-\sum_{i=1}^{k}[N_i-n_i-(D-n_{i,k}-N_{1,i-1})^+]^+s(\theta_i)\\
=\;&\sum_{i=1}^{k}\{\min[N_i-n_i,(D-n_{i,k}-N_{1,i-1})^+]-(N_i-n_i)\}s(\theta_i)\\
=\;&\sum_{i=1}^{k}\{(D-n_{i,k}-N_{1,i-1})^+-[(D-n_{i,k}-N_{1,i-1})^+\\
&\qquad-(N_i-n_i)]^+-(N_i-n_i)\}s(\theta_i)\\
=\;&\sum_{i=1}^{k}\{(D-n_{i,k}-N_{1,i-1})^--(D-n_{i+1,k}-N_{1,i})^-\}s(\theta_i)\\
=\;&\sum_{i=1}^{k}(D-n_{i,k}-N_{1,i-1})^-[s(\theta_i)-s(\theta_{i-1})]\\
&\qquad+(D-n_{1,k})^-s(\theta_0)-(D-N_{1,k})^-s(\theta_k).
\end{aligned}
$$

Hence, (9.2) becomes:

$$
\begin{aligned}
&W(D,n_1,\cdots,n_k)\\
=\;&[D-(D-N_{1,k})^+]C_g-(D-N_{1,k})^+\theta_k\Delta\\
&+\sum_{i=1}^{k}(D-n_{i,k}-N_{1,i-1})^+(\theta_i-\theta_{i-1})\Delta
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{i=1}^{k}(D - n_{i,k} - N_{1,i-1})^{-}[s(\theta_{i-1}) - s(\theta_i)] - (D - N_{1,k})^{-}s(\theta_k) \\
= \quad & C_g D - (C_g + \Delta\theta_k + s(\theta_k))(D - N_{1,k})^{+} + (D - N_{1,k})s(\theta_k) \\
& +\sum_{i=1}^{k}(D - n_{i,k} - N_{1,i-1})^{+}[(\theta_i - \theta_{i-1})\Delta - (s(\theta_{i-1}) - s(\theta_i))] \\
& +\sum_{i=1}^{k}(D - n_{i,k} - N_{1,i-1})(s(\theta_{i-1}) - s(\theta_i)). \qquad (9.6)
\end{aligned}
$$

(Recall $\theta_0 = 0$.) Making use of the above expression, we can derive

$$
\begin{aligned}
& W(D, n_1, \cdots, n_{i-1} + 1, n_i, \cdots, n_k) \\
& \quad -W(D, n_1, \cdots, n_{i-1}, n_i + 1, \cdots, n_k) \\
= \quad & s(\theta_{i-1}) - s(\theta_i) + [\Delta(\theta_i - \theta_{i-1}) - (s(\theta_{i-1}) - s(\theta_i))] \\
& \quad \cdot[(D - n_{i,k} - N_{1,i-1})^{+} - (D - n_{i,k} - N_{1,i-1} - 1)^{+}], \qquad (9.7)
\end{aligned}
$$

noticing that the two W's above only differ at the i-th term (in the summation). By the condition in (9.1) and the fact that $(D - x)^{+}$ is a convex function of x, (9.7) is decreasing in $n_{i,k}$.

The proof is then completed by incorporating the above results, (9.6) and (9.7) in particular, into (9.3). □

Remark 9.2 It is worthwhile to point out several facts that relate to the properties summarized in the above proposition. The property in (9.5) clearly implies the convexity of $\Pi(n_1, \cdots, n_k)$ in each component. To appreciate the property in (9.4), note that the difference is independent of the value of n_{i-1}, in particular, we have

$$
\begin{aligned}
& \Pi(n_1, \cdots, n_{i-1} + 1, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1}, n_i + 1, \cdots, n_k) \\
= \quad & \Pi(n_1, \cdots, n_{i-1} + 2, n_i, \cdots, n_k) \\
& \quad -\Pi(n_1, \cdots, n_{i-1} + 1, n_i + 1, \cdots, n_k),
\end{aligned}
$$

which can be rewritten as follows:

$$
\begin{aligned}
& [\Pi(n_1, \cdots, n_{i-1}, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1}, n_i + 1, \cdots, n_k)] \\
& -[\Pi(n_1, \cdots, n_{i-1}, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1} + 1, n_i, \cdots, n_k)] \\
= \quad & [\Pi(n_1, \cdots, n_{i-1} + 1, n_i, \cdots, n_k) \\
& \quad -\Pi(n_1, \cdots, n_{i-1} + 1, n_i + 1, \cdots, n_k)] \\
& -[\Pi(n_1, \cdots, n_{i-1} + 1, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1} + 2, n_i, \cdots, n_k)].
\end{aligned}
$$

The difference in the second bracket of the left hand side dominates its counterpart on the right hand side, due to convexity. Therefore, we must have

$$\begin{aligned} & \Pi(n_1, \cdots, n_{i-1}, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1}, n_i + 1, \cdots, n_k) \\ \geq\ & \Pi(n_1, \cdots, n_{i-1} + 1, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1} + 1, n_i + 1, \cdots, n_k), \end{aligned}$$

which is nothing but *supermodularity* (refer to §2.3). In other words, the properties of Π as revealed in Proposition 9.1 imply supermodularity and componentwise convexity. It can be verified, however, that these properties in general will *not* guarantee the optimality of the greedy algorithm below. Of course, these properties are weaker than (implied by) those in Proposition 9.1. Furthermore, the particular *form* of the difference expressions in (9.4, 9.5) also plays an important role, as will become evident below.

From (9.4), we can write

$$\begin{aligned} & g_i(n_{i,k}) \\ :=\ & \Pi(n_1, \cdots, n_{i-1} + 1, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1}, n_i + 1, \cdots, n_k), \end{aligned}$$

and define

$$K_i := \min \{n_{i,k} \leq N_{i,k} : \ g_i(n_{i,k}) < 0\}.$$

By regulation, define $K_i := N_{i,k}$ if no $n_{i,k} \leq N_{i,k}$ satisfies $g_i(n_{i,k}) < 0$. Clearly, K_i is the smallest value for $n_{i,k}$ so that it becomes more desirable to inspect one more unit from batch $i-1$ than to inspect one more unit from batch i in any state $(n_1, n_2, \cdots, n_{i-1}, n_i, \cdots, n_k)$.

Lemma 9.3 Suppose $n_{i-1} < N_{i-1}$ and $n_i < N_i$. Then, it is more desirable in state $(n_1, \cdots, n_{i-1}, n_i, \cdots, n_k)$ to inspect one more unit from batch $i-1$ than to inspect *any* more units from batch i if and only if $n_{i,k} \geq K_i$.

Proof. From the definition of K_i, we know in state $(n_1, \cdots, n_{i-1}, n_i, \cdots, n_k)$, it is more desirable to inspect one more unit from batch $i-1$ than to inspect one more unit from batch i if and only if $n_{i,k} \geq K_i$. But this also holds if we inspect more than one unit from batch i, since this will only increase the value of $n_{i,k}$. □

Corollary 9.4 Suppose $n_\ell = N_\ell$, for $\ell = j, j+1, \cdots, i-1$, $1 \leq j < i \leq k$. That is, all the units in batch j through batch $i-1$ have been inspected. And $n_{j-1} < N_{j-1}$, $n_i < N_i$. Then Lemma 9.3 holds with batch $j-1$ replacing batch $i-1$. That is, it is more desirable to inspect one more unit from batch $j-1$ than to inspect *any* more units from batch i if and only if $n_{i,k} \geq K_i$.

Proof. We can merge batches $j-1$ through $i-1$ into a single, non-defective batch and remove them from further consideration. This will not affect the values of $n_{i,k}$ or K_i. The desired conclusion then follows from Lemma 9.3. □

Remark 9.5 Two points are worth mentioning here:

(i) From (9.5), we know that as more units are inspected in batch i, the cost reduction diminishes. Furthermore, as more units are inspected in batch i, $n_{i,k}$ increases, which, in turn, increases the desirability of switching to inspecting batch $i-1$, following Lemma 9.3. Once $n_{i,k}$ reaches K_i, switching to $i-1$ becomes more desirable, and this remains so even when more units from batch $i-1$ are inspected, since $n_{i,k}$ is independent of n_{i-1}.

(ii) On the other hand, switching inspection to batch $i-1$ might never be desirable. For instance, this can happen when the cost reduction in (9.5) becomes negative but we still have $n_{i,k} < K_i$. In this case, it becomes more desirable to simply stop inspection, rather than inspecting any more unit from either batch i or batch $i-1$.

9.3 Optimal Solution to the Inspection Problem

Below we assume that the salvage value $s(\theta)$ is convex in θ, as well as decreasing in θ as assumed earlier. Consequently, $\frac{s(\theta_{i-1})-s(\theta_i)}{\theta_i-\theta_{i-1}}$ is decreasing in i.

From the definition of $g_i(n_{i,k})$, making use of (9.4), we know that

$$g_i(n_{i,k}) < 0$$

is equivalent to

$$\begin{aligned} & C_R - \frac{s(\theta_{i-1})-s(\theta_i)}{\theta_i-\theta_{i-1}} \\ > \; & [\Delta - \frac{s(\theta_{i-1})-s(\theta_i)}{\theta_i-\theta_{i-1}}]\mathsf{E}[(D-y)^+ - (D-y-1)^+], \end{aligned} \tag{9.8}$$

with $y := n_{i,k} + N_{1,i-1}$. From (9.1), we know the first factor on the right hand side above is non-negative. To start with, suppose it is positive. (Later it will become evident that the same argument below applies to the case when this factor is zero.) Then, the inequality in (9.8) is equivalent to the following:

$$\frac{C_R - \frac{s(\theta_{i-1})-s(\theta_i)}{\theta_i-\theta_{i-1}}}{\Delta - \frac{s(\theta_{i-1})-s(\theta_i)}{\theta_i-\theta_{i-1}}} > \mathsf{E}[(D-y)^+ - (D-y-1)^+]. \tag{9.9}$$

A direct verification shows that when $s(\theta)$ is a decreasing and convex function, the left hand side of (9.9) is increasing in i, taking into account that $\frac{s(\theta_{i-1})-s(\theta_i)}{\theta_i-\theta_{i-1}}$ is decreasing in i, and $C_R \leq \Delta$. On the other hand, as i increases, the right hand side of (9.9) decreases (again, due to the convexity of $(D-x)^+$). In fact, since the right hand side decreases as y increases, its decrease can be carried out in a more detailed manner: say, from the state $(n_1, ..., n_i, ..., n_k)$, we can first increase n_i to N_i, and then n_{i+1} to N_{i+1}, and so forth.

Therefore, starting from the zero state, $(0, ..., 0)$, we increase each component n_i, following the order $i = 1, ..., k$, from 0 to N_i. Let i^* be the first i index and $\hat{n}_{i^*}$ be the smallest corresponding component value such that $y^* = \hat{n}_{i^*} + N_{1,i^*-1}$ is the smallest y that satisfies (9.9). For the time being, suppose such a y^* does exist. This has the following implications:

(a) After $\hat{n}_{i^*}$ units from batch i^* are inspected, it becomes more desirable to switch to inspecting units from batch i^*-1. In other words, $K_{i^*} = y^* - N_{1,i^*-1} = \hat{n}_{i^*}$.

(b) We can then inspect each batch $i = i^* - 1, ..., 1$ in decreasing order of i; and there is no need to switch to batch $i-1$ until all units of batch i have been inspected. This is because in this range ($i < i^*$) the left hand side of (9.9) is dominated by the right hand side (i.e., the inequality is satisfied in the reverse direction), which implies, via (9.4) and (9.5), that the cost reduction in (9.5) dominates the same cost reduction when the index i is changed to $i-1$. Hence, in this case, $K_i = y^* - N_{1,i-1}$, for $i = 1, ..., i^* - 1$.

(c) No unit should be inspected from any batch $i^* + 1, ..., k$, until all the batches $1, ..., i^*$ have been inspected, since for $i > i^*$, (9.9) is always satisfied. In other words, $K_i = 0$, for $i = i^* + 1, ..., k$.

(d) In view of Remark 9.5 (ii), however, in cases (a) and (b), as we increase the number of inspected units in each batch, we still need to make sure that the cost reduction in (9.5) is positive. Once the cost reduction becomes non-positive, inspection should be terminated for all batches.

(e) On the other hand, if the cost reduction in (9.5) stays positive, and all units in the batches $i = 1, ..., i^* - 1$ have been inspected, then we need to return to batch i^*. Note that switching to $i^* - 1$ in (a) was due to the positive cost reduction in (9.4). Now, although this cost reduction remains positive, switching to a lower indexed batch becomes out of question, since all units in those batches have been inspected. Hence, inspecting more units from batch i^* is warranted, as long as the cost reduction in (9.5) is positive. In fact, we need to consider the batches $i > i^*$ as well, in increasing order of i, for the same reason as in (b), since here the cost reduction in (9.5) dominates the same cost reduction when the index i is changed to $i+1$.

There are cases in which the inequality in (9.8) just cannot be satisfied. For instance, if the left hand side of (9.8) is negative, then the inequality cannot hold, since both factors on the right hand side are non-negative. Another case is when y^* (as defined above) does not exist: even when i is increased to k and y is increased to its upper limit $N_{1,k}$, the left hand side of (9.9) is still dominated by the right hand side. Both of these two instances imply that inspecting batch i is always more preferable than inspecting batch $i-1$ for any i, or, $K_i = N_{i,k}$; and hence, the optimal solution is to inspect the batches in decreasing order of i, and to stop inspection whenever the cost reduction in (9.5) becomes non-positive. A third case is when the salvage value $s(\theta)$ is a linear function, and

$$\Delta = \frac{s(\theta_{i-1}) - s(\theta_i)}{\theta_i - \theta_{i-1}} \leq C_R \tag{9.10}$$

for all i. This, along with the assumption that $C_R \leq \Delta$, implies that the inequality in (9.10) holds as an equality, and hence both sides of (9.8) are zero. In fact, in this case the right hand side of (9.5) becomes: $-C_I - C_R\theta_0 \leq 0$. This leads to the optimality of the trivial, "do-nothing" solution, $(0, ..., 0)$. Correspondingly, $K_i = 0$ for all $i = 1, ..., k$.

We now return to the case when the first factor on the right hand side of (9.8) is zero for some i. Since this factor is non-negative and increasing in i, let $\bar{i}$ be the largest i for which it stays at zero. Then, the left hand side of (9.8) is non-positive for all $i \leq \bar{i}$, since $C_R \leq \Delta$. That is, (9.8) is not satisfied for all $i \leq \bar{i}$. For $i > \bar{i}$, on the other hand, the factor in question becomes positive, and we can repeat the earlier argument based on (9.9). In particular, we know $i^* > \bar{i}$ in this case.

To summarize the above discussion, we have:

Theorem 9.6 Suppose the salvage value $s(\theta)$ is a convex and decreasing function. Let $i^* \leq k$ be the smallest batch index and let $\hat{n}_{i^*} \leq N_{i^*}$ be the smallest corresponding component value, such that $y^* = \hat{n}_{i^*} + N_{1,i^*-1}$ is the smallest y that satisfies (9.9).

(i) When such a y^* exists, the optimal solution is either

$$(0, ..., 0, n'_h, N_{h+1}, ..., N_{i^*-1}, n'_{i^*}, 0, ..., 0),$$

or

$$(N_1, ..., N_{j^*-1}, n'_{j^*}, 0, ..., 0),$$

where,

- n'_{i^*} is equal to either $\hat{n}_{i^*}$, or the smallest n_i (with $i = i^*$) value that results in a non-positive cost reduction in (9.5), if this value is less than $\hat{n}_{i^*}$;
- $1 \leq h \leq i^*$, $n'_h \leq N_h$, and n'_h is the smallest n_i (with $i = i^*$) value that results in a non-positive cost reduction in (9.5);

- $j^* \geq i^*$, and $n'_{j^*} \leq N_{j^*}$ is the smallest n_i (with $i = i^*$) value that results in a non-positive cost reduction in (9.5); or, if no such value exists, then $n'_{j^*} = N_k$ with $j^* = k$.

(ii) If y^* as defined above does not exist, or if the left hand side of (9.8) is negative for all i, then the optimal solution is obtained as follows: inspect the batches in decreasing order of i, starting from $i = k$, and stop inspection whenever the cost reduction in (9.5) becomes non-positive. In this case, the optimal solution is

$$(0, ..., 0, n'_{j^*}, N_{j^*+1}, ..., N_k),$$

with $1 \leq j^* \leq k$.

(iii) If the cost data satisfy the relation in (9.10), then the optimal solution is $(0, ..., 0)$, i.e., inspect no unit at all.

Proof. (i) It is clear from the construction of both solutions that a decrease of any of the positive components will result in a sacrifice of some positive cost reduction. This includes reducing some positive component while increasing another component (by the same amount) — a procedure that we shall refer to as "shifting" below. So it suffices to argue that none of the components can be increased either; and we only need to examine those components that have not reached the given batch sizes.

Consider the first solution. Clearly, we cannot increase n'_{i^*} without resulting in a cost increase (via (9.4,9.5)). Suppose $n_j > 0$ for $j = i^* + 1$. Then, clearly, $y = n_{j,k} + N_{1,j-1} > y^*$ satisfies (9.9) (with $i = j$). Therefore, the overall cost will decrease, following Lemma 9.3 and Corollary 9.4, if we shift one unit from batch j to batch i^* (if $n'_{i^*} < N_{i^*}$), or to batch $i \leq h$ (if $n'_{i^*} = N_{i^*}$). A similar argument applies if $n_j > 0$ for $j > i^* + 1$, through repeatedly shifting units from batch j to batches with lower indices.

In the second solution, increasing any component $j \geq j^*$ will further decrease the right hands side of (9.5), resulting in a non-positive cost reduction — beyond what results in the case of $\hat{n}_{j^*}$.

(ii) In this case, same as in (i), decreasing any positive component of the optimal solution, including shifting it to some other component, will result in a cost increase. On the other hand, since the optimal solution is reached when the cost reduction in (9.5) ceases to be positive, increasing any component of the optimal solution will result in a non-positive cost reduction, through increasing the $n_{j,k} + N_{1,j-1}$ value in (9.5).

(iii) In this case, any solution that has a positive component cannot be optimal, since reducing the positive component by one unit will result in a cost reduction of $C_I + C_R\theta_0$, via the discussion following (9.10). □

Remark 9.7 The K_i values in the three cases of the above theorem are:

	Data and Optimal Solutions									
Type i	1	2	3	4	5	6	7	8	9	10
Batch Size N_i	20	20	20	20	20	20	20	20	20	20
defective rate θ_i	.03	.08	.12	.15	.17	.20	.22	.25	.30	.40
$(n_1^*, \cdots, n_{10}^*)$ (N)	0	0	17	20	20	20	20	20	0	0
$(n_1^*, \cdots, n_{10}^*)$ (U)	0	0	12	20	20	20	20	20	3	0
$(n_1^*, \cdots, n_{10}^*)$ (P)	0	4	20	20	20	20	20	17	0	0

TABLE 9.1. Optimal Inspection Policy

(i) $K_i = y^* - N_{1,i-1}$, for $i = 1, ..., i^*$ (in particular, $K_{i^*} = y^* - N_{1,i^*-1} = \hat{n}_{i^*}$); and $K_j = 0$, for $j = i^* + 1, ..., k$.

(ii) $K_i = N_{i,k}$, for $i = 1, ..., k$.

(iii) $K_i = 0$, for $i = 1, ..., k$.

Note that in all three cases, K_i is decreasing in i. As evident from the discussions above, this turns out to be the key to the threshold structure of the optimal solution in Theorem 9.6.

Example 9.8 Consider the following inspection problem. Suppose we have a total of ten types of products, with batch sizes and defective rates listed in Table 9.1 below. The cost data are:

$$C_d = 7.0, \quad C_g = 1.0, \quad C_I = 0.2, \quad C_R = 2.9;$$

and the salvage value is a convex function, $s(\theta) = 1.5 - 2.5\theta + 3\theta^2$. Consider three types of demand distributions:

(a) normal (N) with mean 150 and standard deviation 20,

(b) uniform (U) over the interval $(115.36, 184.64)$,

(c) Poisson (P) with mean 150.

Note that all three distributions have the same mean; and in addition, the normal and the uniform distributions have the same standard deviation. The optimal solutions, $(n_1^*, \cdots, n_{10}^*)$, under the three demand distributions are listed in Table 9.1. The corresponding optimal objective values are listed in Table 9.2, in comparison with those under full (100%) inspection and zero inspection.

9.4 Optimal Replenishment Quantities

Here we extend the earlier model to include the batch sizes, N_i, $i = 1, ..., k$, as decision variables. Specifically, we want to decide the order quantity

	Objective Value		
	Optimal Inspection	Full Inspection	Zero Inspection
Normal	248.5	276.0	282.7
Uniform	245.1	273.1	276.5
Poisson	244.2	275.4	280.4

TABLE 9.2. Objective Values under Different Inspection Policies

of each product type, taking into account that these products will be inspected, following the optimal rule discussed in the earlier sections, and then used to supply demand.

In addition to the cost data in §9.1, there is a penalty cost, C_P, for each unit of shortage (unfilled demand). Assume

$$C_P - s(\theta_k) \geq C_g + \Delta\theta_k, \tag{9.11}$$

which implies

$$C_P - C_g \geq \Delta\theta_k + s(\theta_k) \geq \Delta\theta_i + s(\theta_i),$$

for all $i \leq k$. This simply guarantees that any demand will be supplied if there is a product available. There is also a purchasing cost, $c(\theta_i)$, for each unit of product i, $i = 1, ..., k$. We assume that the purchasing cost net the salvage value, $c(\theta) - s(\theta)$, is a decreasing and convex function of the defective rate θ. Note that this implies the decreasing convexity of $c(\theta)$, since $s(\theta)$ is a decreasing and convex function, as assumed earlier.

Furthermore, we assume that type 1 products are of perfect quality: $\theta_1 = 0$ (in contrast with $\theta_0 = 0$ in the previous sections). This way, we can address the tradeoff between purchasing perfect units at a higher cost and purchasing lower quality units but spending more on inspection and repair. Note that ordering products with defective rates $\theta = 0$ and $\theta > 0$ corresponds, respectively, to the "selective purchase" and the "blind purchase" in [55].

Let $v(N_1, \cdots, N_k)$ denote the optimal cost function considered in last section, given the batch sizes, N_i units of product i, for $i = 1, 2, \cdots, k$. Let $V(N_1, \cdots, N_k)$ denote the new objective (cost) function here, i.e.,

$$\begin{aligned} & V(N_1, \cdots, N_k) \\ := & \sum_{i=1}^{k} N_i c(\theta_i) + v(N_1, \cdots, N_k) + C_P \mathsf{E}(D - N_{1,k})^+. \end{aligned} \tag{9.12}$$

We want to find $(N_1, \cdots, N_k)$ so as to minimize the V function above. We need the following lemma.

Lemma 9.9 Let

$$\ell^* = arg \min_{2 \leq i \leq k} \{i : c(\theta_i) + C_R \theta_i\}. \tag{9.13}$$

Suppose $c(0) \leq c(\theta_{l^*}) + C_I + C_R\theta_{\ell^*}$. Then, it is optimal not to inspect any unit. Hence, the cost function in (9.12) is reduced to:

$$\begin{aligned}
& V(N_1, \cdots, N_k) \\
= \; & \sum_{i=1}^{k} N_i c(\theta_i) + \Pi(0, \cdots, 0; N_1, \cdots, N_k) \\
& + C_P \mathsf{E}(D - N_{1,k})^+, \qquad (9.14)
\end{aligned}$$

where $\Pi(0, \cdots, 0; \cdot) = \Pi(0, \cdots, 0)$ in (9.3). Furthermore,

$$\begin{aligned}
& V(N_1, \cdots, N_i, \cdots, N_k) - V(N_1, \cdots, N_i + 1, \cdots, N_k) \\
= \; & -c(\theta_i) + s(\theta_i) + [C_P - C_g - \Delta\theta_k - s(\theta_k)] \\
& \cdot[\mathsf{E}(D - N_{1,k})^+ - \mathsf{E}(D - N_{1,k} - 1)^+] \\
& + \sum_{j=i+1}^{k} [\Delta(\theta_j - \theta_{j-1}) - (s(\theta_{j-1}) - s(\theta_j))] \\
& \cdot[\mathsf{E}(D - N_{1,j-1})^+ - \mathsf{E}(D - N_{1,j-1} - 1)^+], \qquad (9.15)
\end{aligned}$$

which is decreasing in N_j, for any $j = 1, ..., k$; and

$$\begin{aligned}
& V(N_1, \cdots, N_{i-1} + 1, N_i, \cdots, N_k) - \\
& \quad V(N_1, \cdots, N_{i-1}, N_i + 1, \cdots, N_k) \\
= \; & c(\theta_{i-1}) - c(\theta_i) - [s(\theta_{i-1}) - s(\theta_i)] \\
& -[\Delta(\theta_i - \theta_{i-1}) - (s(\theta_{i-1}) - s(\theta_i))] \\
& \quad \cdot[\mathsf{E}(D - N_{1,i-1})^+ - \mathsf{E}(D - N_{1,i-1} - 1)^+], \qquad (9.16)
\end{aligned}$$

which is increasing in $N_{1,i-1}$.

Proof. First, it is easy to see that ℓ^* is the only product type (among types 2 through k) which we may consider for inspection. (Type 1 does not need inspection any way.) This is because, instead of ordering N_i and N_{ℓ^*} units, for any $i \neq 1, \ell^*$, and inspecting n_i and n_{ℓ^*} units, respectively, we can order $N_i - n_i$ and $N_{\ell^*} + n_i$ units, and then inspect 0 and $n_{\ell^*} + n_i$ units for type i and type ℓ^*, respectively. This way, the overall cost reduction (in V) is:

$$n_i \left\{ [c(\theta_i) - c(\theta_{\ell^*})] + C_R(\theta_i - \theta_{l^*}) \right\},$$

which is non-negative, following the definition of ℓ^* in (9.13).

Hence, we can obtain a perfect unit, at an expected cost of $c(\theta_{\ell^*}) + C_I + C_R\theta_{\ell^*}$, by ordering and inspecting one unit of type ℓ^*. Alternatively, we can also get a perfect unit, at a cost $c(0)$, by ordering one unit of product 1, since $\theta_1 = 0$. This alternative is certainly preferred, when

$$c(0) \leq c(\theta_{l^*}) + C_I + C_R\theta_{l^*}.$$

In this case we will never inspect any unit. Because should we choose to inspect any unit, in types 2 through k, we would prefer not to order the unit, but instead, replace it by a type ℓ^* unit and then inspect the latter. But then, we would further prefer to replace the type ℓ^* unit by a type 1 unit and forgo inspection.

To establish the properties for (9.15) and (9.16), letting

$$(n_1, ..., n_k) = (0, ..., 0)$$

in (9.3) and (9.6), we have

$$\begin{aligned}
& \Pi(0, ..., 0; N_1, ..., N_k) \\
= \quad & C_g \mathsf{E}[D - (D - N_{1,k})^+] - \Delta\theta_k \mathsf{E}(D - N_{1,k})^+ - s(\theta_k)\mathsf{E}(D - N_{1,k})^- \\
& + \sum_{j=1}^{k} \{\Delta(\theta_j - \theta_{j-1})\mathsf{E}(D - N_{1,j-1})^+ \\
& \quad -(s(\theta_{j-1}) - s(\theta_j))\mathsf{E}(D - N_{1,j-1})^-\}. \qquad (9.17)
\end{aligned}$$

Substituting the above into (9.14), we have

$$\begin{aligned}
& V(N_1, \cdots, N_i, \cdots, N_k) - V(N_1, \cdots, N_i + 1, \cdots, N_k) \\
= \quad & -c(\theta_i) + (C_P - C_g)[\mathsf{E}(D - N_{1,k})^+ - \mathsf{E}(D - N_{1,k} - 1)^+] \\
& + \sum_{j=i+1}^{k} \{ \ \Delta(\theta_j - \theta_{j-1})[\mathsf{E}(D - N_{1,j-1})^+ - \mathsf{E}(D - N_{1,j-1} - 1)^+] \\
& \qquad -(s(\theta_{j-1}) - s(\theta_j))[\mathsf{E}(D - N_{1,j-1})^- - \mathsf{E}(D - N_{1,j-1} - 1)^-] \ \} \\
& -\Delta\theta_k[\mathsf{E}(D - N_{1,k})^+ - \mathsf{E}(D - N_{1,k} - 1)^+] \\
& -s(\theta_k)[\mathsf{E}(D - N_{1,k})^- - \mathsf{E}(D - N_{1,k} - 1)^-].
\end{aligned}$$

Collecting terms and making use of the identity $x^+ - x^- = x$, we can simplify the above to (9.15). The decreasing property follows from (9.1), (9.11) and the convexity of $(D - x)^+$. Next, since

$$\begin{aligned}
& V(N_1, \cdots, N_{i-1} + 1, N_i, \cdots, N_k) - V(N_1, \cdots, N_{i-1}, N_i + 1, \cdots, N_k) \\
= \quad & [V(N_1, \cdots, N_i, \cdots, N_k) - V(N_1, \cdots, N_i + 1, \cdots, N_k)] \\
& -[V(N_1, \cdots, N_{i-1}, \cdots, N_k) - V(N_1, \cdots, N_{i-1} + 1, \cdots, N_k)],
\end{aligned}$$

Applying (9.15) twice yields (9.16). The increasing property follows from the convexity of $(D - x)^+$ and (9.1). □

Based on (9.16), we can write

$$\begin{aligned}
& G_i(N_{1,i}) \\
:= \quad & V(N_1, \cdots, N_i + 1, N_{i+1}, \cdots, N_k) - V(N_1, \cdots, N_i, N_{i+1} + 1, \cdots, N_k).
\end{aligned}$$

Define

$$M_i := \min\{N_{1,i} : G_i(N_{1,i}) > 0\} \qquad (9.18)$$

for $i = 1, \cdots, k-1$; and $M_0 := 0$. Note that $G(N_{1,i}) > 0$ is equivalent to

$$\frac{[c(\theta_i) - s(\theta_i)] - [c(\theta_{i+1}) - s(\theta_{i+1})]}{\theta_{i+1} - \theta_i} \tag{9.19}$$

$$> [\Delta - \frac{s(\theta_i) - s(\theta_{i+1})}{\theta_{i+1} - \theta_i}][\mathsf{E}(D - N_{1,i})^+ - \mathsf{E}(D - N_{1,i} - 1)^+], \tag{9.20}$$

which means that it becomes more desirable to order one more unit of type $i+1$ than to order one more unit of type i.

To start with, consider $i = 1$. The right hand side of (9.19) is decreasing in $N_{1,1} \equiv N_1$. (Note that the first factor on the right hand side is non-negative, following (9.1).) Hence, when N_1 is large enough to satisfy the inequality, i.e., $N_1 = M_1$ following (9.18), we should stop ordering any more units of type 1, and switch to ordering type 2 units. Next, consider $i = 2$. Since $c(\theta) - s(\theta)$ is decreasing and convex in θ as assumed earlier, the left hand side of (9.19) is decreasing in i. And, since $s(\cdot)$ is decreasing and convex, the first factor on the right hand side of (9.19) is increasing in i. Hence, the smallest $N_{1,2}$ that satisfies the inequality, i.e., M_2 as denoted in (9.18), will be no less than M_1; and the order size for type 2 units is up to $M_2 - M_1$: after that limit is reached we should switch to ordering type 3 units, and so forth. In general, when type i has been ordered to its maximum, $M_i - M_{i-1}$, we should switch to ordering from type $i+1$.

On the other hand, before $N_{1,i}$ reaches M_i, (9.19) holds in the reverse direction, which means the right hand side of (9.16) is non-positive. This, in turn, implies

$$\begin{aligned} & [V(N_1, \cdots, N_{i-1}, N_i, \cdots, N_k) - V(N_1, \cdots, N_{i-1} + 1, N_i, \cdots, N_k)] \\ \geq\ & [V(N_1, \cdots, N_{i-1}, N_i, \cdots, N_k) - V(N_1, \cdots, N_{i-1}, N_i + 1, \cdots, N_k)]. \end{aligned}$$

That is, until $N_{1,i}$ reaches M_i, or until type i has been ordered to its maximum, $M_i - M_{i-1}$, there is no need to switch to ordering type $i+1$.

From (9.15), we know that as more units of product i is ordered, the cost reduction decreases. Hence, in ordering each additional unit of type i, even before reaching the limit $M_i - M_{i-1}$, we need to make sure that the cost reduction in (9.15) is positive. Should this cost reduction become non-positive before the limit $M_i - M_{i-1}$ is reached, we should stop ordering altogether — not just for type i, but for all types $j > i$. Hence, (9.15) plays the same role as (9.5) in the earlier model. If $N_i = M_i - M_{i-1}$ for $i = 1, 2, \cdots, k-1$, then N_k is determined by (9.15) with $i = k$, i.e., it is the smallest order quantity of product k so that (9.15) becomes non-positive.

Finally, a special case is of particular interest: When both $c(\theta)$ and $s(\theta)$ are linear functions, from the above discussion, it is easy to see from (9.19) that $M_1 = M_2 = \cdots = M_{k-1}$ in this case. Consequently, product $2, 3, \cdots, k-1$ will not be ordered, and N_1 and (possibly) N_k are the only non-zero components in the optimal solution. In other words, it is optimal to use only two supply sources, 1 and k. To summarize, we have

Theorem 9.10 Suppose $c(0) \leq c(\theta_{\ell^*}) + C_I + C_R\theta_{l^*}$. Then the optimal solution to the order quantities is obtained as follows:

- Order the units in increasing order of the type index i, starting from $i = 1$.
- Every time a unit of type i is ordered, check whether the cost reduction in (9.15) stays positive; if not, stop ordering any more unit from any type.
- As long as the cost reduction in (9.15) stays positive, keep ordering type i units, until $N_i = M_i - M_{i-1}$, then switch to ordering type $i+1$ units, for $i = 1, 2, \cdots, k-1$. Here M_i follows the specification in (9.18);
- If $N_i = M_i - M_{i-1}$ for $i = 1, 2, \cdots, k-1$, then keep ordering type k units until (9.15), with $i = k$, becomes non-positive.

Furthermore, following Lemma 9.9, in this case it is optimal not to inspect any unit from any type. When $c(\theta)$ and $s(\theta)$ are linear functions, $N_i = 0$ for $i = 2, ..., k-1$. That is, it is optimal to use only two supply sources, 1 and k.

Proof. Following the specification in the Theorem, we can write the optimal solution as $(N_1^*, ..., N_{i^*-1}^*, N_{i^*}^*, 0, ..., 0)$, where $N_i^* \leq M_i - M_{i-1}$ for $i \leq i^*$, and $1 \leq i^* \leq k$ is the smallest i value in (9.15) that makes its right hand side non-positive. (Recall, from the analysis above, this right hand side is decreasing in i.) Similar to the argument in the proof of Theorem 9.6, it is clear that decreasing any of the positive components amounts will forgo some positive cost reduction. On the other hand, increasing the value of any component $i \leq i^* - 1$ is not as good as increasing that of $i+1$, since the right hand side of (9.16) is positive; and increasing the value of any component $i \geq i^*$ will result in a cost increase, via (9.15). □

Next, consider the case of $c(0) > c(\theta_{\ell^*}) + C_I + C_R\theta_{\ell^*}$. In this case, it does not pay to order any unit of product 1. Instead we are better off ordering units of type ℓ^*, and converting them into non-defective units via inspection and possible repair. Denote the units so obtained as of type $1'$. Then, we can solve the problem following Theorem 9.10, treating product $1'$ as product 1, with a zero defect rate, and with the purchasing cost $c(\theta_1) = c(0)$ replaced by $c(\theta_{\ell^*}) + C_I + C_R\theta_{\ell^*}$.

Theorem 9.11 Suppose $c(0) > c(\theta_{l^*}) + C_I + C_R\theta_{l^*}$. Then, order zero units of type 1. Instead, replace type 1 by a type $1'$, which has zero defective rate, and a unit purchasing cost of $c(\theta_{\ell^*}) + C_I + C_R\theta_{\ell^*}$, with ℓ^* following (9.13). Follow Theorem 9.10 to derive the order quantities. Type $1'$ units

	Data and Optimal Solutions											Obj.Val.
Type i	$1'$	2	3	4	5	6	7	8	9	10	11	
Defective Rate θ_i	.00	.03	.08	.12	.15	.17	.20	.22	.25	.30	.40	
$(N^*_{1'}, N^*_2, \cdots, N^*_{11})$ (N)	133	14	5	4	2	3	2	3	4	0	0	643.5
$(N^*_{1'}, N^*_2, \cdots, N^*_{11})$ (U)	128	17	7	5	3	3	3	3	4	0	0	638.3
$(N^*_{1'}, N^*_2, \cdots, N^*_{11})$ (P)	139	9	3	2	2	1	2	1	3	0	0	626.9

TABLE 9.3. Optimal Order Quantities

are then ordered from type ℓ^*, with all units inspected (and repaired if necessary). Inspect no unit from any other types. When $c(\theta)$ and $s(\theta)$ are linear functions, it is optimal to only order from two supply sources, $1'$ and k.

Remark 9.12 The dual sourcing result in Theorems 9.10 and 9.11 indicates that ordering from a single source is, in general, suboptimal, even with linear cost and linear salvage value. The intuitive reason is this: As we assume $c(\theta_i) - s(\theta_i)$ to be decreasing in i, units with a better quality (naturally) also has a higher *net* cost (i.e., cost minus salvage value). Hence, it does not pay to order a better quality product but unable to use it to supply demand. Since demand is random, there is always a possibility that some units ordered will be left over as surplus inventory. Hence, it is more desirable, following the optimality of dual sourcing, to order some units for possible backup from a second, low-quality/low-cost source.

Finally, suppose there is *no* perfect type like product 1 to start with. This clearly corresponds to the case in Theorem 9.11. That is, we can always pay a unit purchasing cost of $c(\theta_{\ell^*}) + C_I + C_R\theta_{\ell^*}$ to obtain a perfect unit, by ordering from type ℓ^* along with inspection and possible repair. This way, we have effectively created a perfect type. Another way to view this case is to set $c(0) = \infty$, signifying the unavailability of a perfect type. Then, naturally Theorem 9.11 applies.

Example 9.13 Consider the problem in Example 9.8. In addition to the data given there, we have the penalty cost, $C_P = 10.0$, and the ordering cost, $c(\theta) = 3.0 - 5\theta + 6\theta^2$ for $\theta > 0$. There is also a perfect type, indexed as $i = 1$, with ordering cost $c(0) = 3.1$, and salvage value $s(0) = 3.1/2$. Accordingly, here we re-index the original ten product types as $2, 3, \cdots, 11$. We can identify $\ell^* = 6$, and $c(0) > c(\theta_{l^*}) + C_I + C_R\theta_{l^*}$. Hence, Theorem 9.11 applies, and no perfect unit should be ordered. For the three types of demand distributions in Example 9.8, the results are summarized in Table 9.3.

Note that here type $1'$ units are obtained by ordering type ℓ^*. For instance, when demand follows the normal distribution, we should order 136 units of type 6, among them 133 units are inspected and repaired (if necessary).

	Data and Optimal Solutions			Obj.Val.
Type i	$1'$	2	11	
defective rate θ_i	.00	.03	.40	
$(N^*_{1'}, N^*_2, N^*_{11})$ (N)	0	164	5	647.2
$(N^*_{1'}, N^*_2, N^*_{11})$ (U)	0	167	5	641.9
$(N^*_{1'}, N^*_2, N^*_{11})$ (P)	0	158	3	630.2

TABLE 9.4. Two Supply Sources

Next, suppose there are only two types available in the above example, 2 and 11, the best and the worst types. Then, the optimal order quantities and the corresponding objective values are summarized in Table 9.4 below. Type 2 is identified as the ℓ^* type; on the other hand, none of the type 2 units should be inspected, i.e., $N^*_{1'} = 0$. For instance, when demand follows the normal distribution, it is optimal to order 164 units of type 2 and 5 units of type 11, and none of them should be inspected.

9.5 Optimal Replenishment over an Infinite Horizon

We now extend the single-period model of the last section to the case of optimal replenishment over an infinite horizon, with an independent and identically distributed demand sequence, $\{D_t\}$, where D_t denotes the demand quantity in period t, with $t = 0, 1, 2, \ldots$. Any unsatisfied demand is lost, with a penalty of C_P per unit. On the other hand, for any surplus after demand is supplied, in lieu of the salvage value $s(\theta_i)$, here we assume there is a holding cost $h(\theta_i)$ for each surplus unit of product i at the end of each period. Assume that $h(\theta)$, like $s(\theta)$, is a decreasing and convex function of the defective rate. Analogous to assuming that $c(\theta) - s(\theta)$ is a decreasing and convex function, here we assume that $c(\theta) - h(\theta)$ is a decreasing and convex function. This is automatically satisfied if, for instance, when the holding cost is charged as a (fixed) proportion of the purchasing cost.

Furthermore, we shall assume the following two conditions:

$$C_P + h(\theta_k) \geq C_g + \Delta\theta_k + c(\theta_k), \tag{9.21}$$

and

$$\Delta(\theta_i - \theta_{i-1}) \geq [c(\theta_{i-1}) - c(\theta_i)] - [h(\theta_{i-1}) - h(\theta_i)]. \tag{9.22}$$

Note that (9.21) is analogous to (9.11). It guarantees that for any given type, using it to supply demand is always better than keeping the unit (and hence paying penalty and inventory charges), even if it can be salvaged at purchasing cost. Note that (9.21) is weaker than $C_P \geq C_g + \Delta\theta_k + c(\theta_k)$, which simply gives enough incentive to place orders: the shortage penalty is such that it always pays to order, including the type with the lowest quality. (Otherwise, some types can be pre-eliminated from the model.) And, (9.22) is analogous to (9.1): it ensures that any demand will always be supplied by the best available unit. Specifically, it is equivalent to the following:

$$\begin{aligned} & C_d\theta_{i-1} + C_g(1-\theta_{i-1}) + c(\theta_{i-1}) - h(\theta_{i-1}) \\ \leq \; & C_d\theta_i + C_g(1-\theta_i) + c(\theta_i) - h(\theta_i). \end{aligned}$$

Clearly, Lemma 9.9 applies here as well. Hence, without loss of generality, we shall focus on the case of $c(0) \leq c(\theta_{\ell^*}) + C_I + C_R\theta_{l^*}$, as in Lemma 9.9, since the complementary case can be reduced to this case, as evident from Theorem 9.11.

Let $f(N_1^{(t)}, \cdots, N_k^{(t)}; D_t)$ denote the total cost in period t, excluding the purchasing cost, provided the starting inventory *after* replenishment is $N^{(t)} := (N_1^{(t)}, \cdots, N_k^{(t)})$ and the demand is D_t. Then, following (9.17) but with $-s(\theta)$ replaced by $h(\theta)$ and with the penalty cost added, we have

$$\begin{aligned} & f(N_1^{(t)}, \cdots, N_k^{(t)}; D_t) \\ = \; & C_g D_t + [C_P - C_g - \Delta\theta_k](D_t - N_{1,k}^{(t)})^+ + h(\theta_k)(D_t - N_{1,k}^{(t)})^- \\ & + \sum_{j=1}^{k} [\Delta(\theta_j - \theta_{j-1})(D_t - N_{1,j-1}^{(t)})^+ \\ & \quad + (h(\theta_{j-1}) - h(\theta_j))(D_t - N_{1,j-1}^{(t)})^-] \\ = \; & C_g D_t + [C_P + h(\theta_k) - C_g - \Delta\theta_k](D_t - N_{1,k}^{(t)})^+ - h(\theta_k)(D_t - N_{1,k}^{(t)}) \\ & + \sum_{i=1}^{k} [\Delta(\theta_i - \theta_{i-1}) + (h(\theta_{i-1}) - h(\theta_i))](D_t - N_{1,i-1}^{(t)})^+ \\ & - \sum_{i=1}^{k} [h(\theta_{i-1}) - h(\theta_i)](D_t - N_{1,i-1}^{(t)}). \end{aligned} \tag{9.23}$$

Let $X^{(t)} := (X_1^{(t)}, \cdots, X_k^{(t)})$ denote the inventory level at the beginning of period t, *before* the replenishment. It is equal to the end inventory of period $t-1$, and can be expressed as follows:

$$X_i^{(t)} = [N_i^{(t-1)} - (D_{t-1} - N_{1,i-1}^{(t-1)})^+]^+. \tag{9.24}$$

Given a replenishment policy π, denote

$$N^{\pi(t)} := (N_1^{\pi(t)}, ..., N_k^{\pi(t)}), \quad \text{and} \quad X^{\pi(t)} := (X_1^{\pi(t)}, ..., X_k^{\pi(t)}).$$

Note that for a policy π to be feasible, we must have $N_i^{\pi(t)} \geq X_i^{\pi(t)}$, for all $i = 1, ..., k$, and for all $t = 0, 1, ...$ Let $V_T^{\pi}(x)$ denote the T-period expected cost associated with the policy π, starting from $X^{(0)} = x := (x_1, ..., x_k)$. Then, we can write

$$\begin{aligned} & V_T^\pi(x) \\ = & \sum_{t=0}^{T-1} \mathsf{E}\{\sum_{i=1}^{k} c(\theta_i)[N_i^{\pi(t)} - X_i^{\pi(t)}] + f(N_1^{\pi(t)}, \cdots, N_k^{\pi(t)}; D_t)|X^{(0)} = x\} \\ & -\sum_{i=1}^{k} c(\theta_i)\mathsf{E}[X_i^{\pi(T)}|X^{(0)} = x] \\ = & \sum_{t=0}^{T-1}\sum_{i=1}^{k} c(\theta_i)\mathsf{E}[N_i^{\pi(t)} - X_i^{\pi(t+1)}] - \sum_{i=1}^{k} c(\theta_i)x_i, \end{aligned}$$

with the understanding that $X^{\pi(0)} = x$. Here the last term in (9.25) assumes that any surplus unit at the end of period T can be salvaged at purchasing cost. This term will vanish when we consider the long-run average cost below.

Since $X_i^{\pi(t+1)}$ relates to $N_i^{\pi(t)}$ and D_t following (9.24), the last expression above motivates us to define:

$$\begin{aligned} & F(N; D) \\ := \; & F(N_1, \cdots, N_k; D) \\ := \; & \sum_{i=1}^{k} c(\theta_i)\{N_i - [N_i - (D - N_{1,i-1})^+]^+\} \\ & +f(N_1, \cdots, N_k; D), \end{aligned} \tag{9.25}$$

where D denotes the generic demand per period (i.e., with the same distribution as D_t). Then, we can rewrite $V_T^\pi(x)$ as follows:

$$V_T^\pi(x) = \sum_{t=0}^{T-1} \mathsf{E}[F(N^{\pi(t)}; D_t)|X^{(0)} = x] - \sum_{i=1}^{k} c(\theta_i)x_i.$$

Denote $\bar{V}^\pi(x)$ as the long-run average cost. We have

$$\begin{aligned} \bar{V}^\pi(x) &= \varlimsup_{T\to\infty} \frac{1}{T} V_T^\pi(x) \\ &= \varlimsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathsf{E}[F(N^{\pi(t)}; D_t)|X^{(0)} = x], \end{aligned} \tag{9.26}$$

since $\sum_{i=1}^{k} c(\theta_i)x_i$ is a finite constant.

Lemma 9.14 The F function of (9.25) is a convex function of $(N_1, \cdots, N_k)$, and satisfies the following properties:

$$F(N_1, \cdots, N_i, \cdots, N_k; D) - F(N_1, \cdots, N_i + 1, \cdots, N_k; D)$$

is decreasing in N_j for any $j = 1, \cdots, k$; and

$$F(N_1, \cdots, N_{i-1} + 1, N_i, \cdots, N_k; D) - F(N_1, \cdots, N_{i-1}, N_i + 1, \cdots, N_k; D)$$

is increasing in $N_{1,i-1}$.

Proof. Following the proof of Proposition 9.1, part of the F function in (9.25) can be written as follows:

$$\begin{aligned}
& -\sum_{i=1}^{k} c(\theta_i)[N_i - (D - N_{1,i-1})^+]^+ \\
= & -\sum_{i=1}^{k} (D - N_{1,i-1})^-(c(\theta_{i-1}) - c(\theta_i)) - (D - N_{1,k})^- c(\theta_k) \\
= & -\sum_{i=1}^{k} (D - N_{1,i-1})^+(c(\theta_{i-1}) - c(\theta_i)) \\
& +\sum_{i=1}^{k} (D - N_{1,i-1})(c(\theta_{i-1}) - c(\theta_i)) - (D - N_{1,k})^- c(\theta_k).
\end{aligned}$$

Hence, substituting the above, along with (9.23), into (9.25), we have

$$\begin{aligned}
& F(N_1, \cdots, N_k; D) \\
= & \; C_g D + [C_P + h(\theta_k) - C_g - \Delta\theta_k - c(\theta_k)](D - N_{1,k})^+ \\
& +\sum_{i=1}^{k} [\Delta(\theta_i - \theta_{i-1}) + (h(\theta_{i-1}) - h(\theta_i)) \\
& \quad -(c(\theta_{i-1}) - c(\theta_i))](D - N_{1,i-1})^+ \\
& -\sum_{i=1}^{k} [(h(\theta_{i-1}) - h(\theta_i)) - (c(\theta_{i-1}) - c(\theta_i))](D - N_{1,i-1}) \\
& -(h(\theta_k) - c(\theta_k))(D - N_{1,k}) + \sum_{i=1}^{k} c(\theta_i) N_i.
\end{aligned}$$

Note that on the right hand side, both $(D - N_{1,i-1})^+$ and $(D - N_{1,k})^+$ are convex in $(N_1, ..., N_k)$, and their coefficients are non-negative, following (9.21) and (9.22). The other terms are all linear in $(N_1, ..., N_k)$. Hence, F is convex in $(N_1, \cdots, N_k)$.

The other two properties follow immediately from the close resemblance of $F(N_1, \cdots, N_k; D)$ to the V function in (9.12). □

In view of the above lemma, the minimizer

$$N^* := (N_1^*, \cdots, N_k^*) := \arg\min \mathsf{E} F(N_1, ..., N_k; D) \tag{9.27}$$

is well defined. Also, define

$$\begin{aligned} & G_i^*(N_{1,i}) \\ := \ & \mathsf{E}F(N_1,\cdots,N_{i-1}+1,N_i,\cdots,N_k;D) \\ & -\mathsf{E}F(N_1,\cdots,N_{i-1},N_i+1,\cdots,N_k;D). \end{aligned}$$

Then, analogous to (9.19), $G_i^*(N_{1,i}) > 0$ is equivalent to the following inequality:

$$\begin{aligned} & \frac{h(\theta_i)-h(\theta_{i+1})}{\theta_{i+1}-\theta_i} \\ > \ & [\Delta - \frac{(c(\theta_i)-h(\theta_i))-(c(\theta_{i+1})-h(\theta_{i+1}))}{\theta_{i+1}-\theta_i}] \\ & \cdot[\mathsf{E}(D-N_{1,i})^+ - \mathsf{E}(D-N_{1,i}-1)^+]. \end{aligned} \tag{9.28}$$

Since $h(\theta)$ is decreasing and convex, the left hand side of (9.28) is decreasing in i. Similarly, since $c(\theta)-h(\theta)$ is decreasing and convex, the first factor on the right hand side of (9.28) is increasing in i. Hence, the discussion preceding Theorem 9.10 and the results stated in Theorem 9.10 apply here as well. In particular, the optimal solution in (9.27) can be generated by the greedy algorithm in the last section, with (9.28) replacing (9.19).

Denote the set of vectors,

$$S := \{x : x_i \le N_i^* \text{ for } 1 \le i \le k\ \},$$

with N^* being the minimizer in (9.27). Without loss of generality, we shall assume $\mathsf{E}[D] > 0$ below (otherwise, we have the trivial case of $D \equiv 0$). We are now ready to study the optimal policy that minimizes the long-run average cost objective in (9.26). The theorem below states that the optimal policy is to order up to the level $N^* = (N_i^*)$, unless the inventory (of any type, in any period) already exceeds this level, in which case order nothing.

Theorem 9.15 It is optimal to order up to N^* in period t, whenever $X^{(t)} \in S$; and order nothing, if $X^{(t)} \notin S$.

Proof of Theorem 9.15. From (9.26), we have

$$\begin{aligned} \bar{V}^{\pi}(x) &= \varliminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathsf{E}[F(N^{\pi(t)};D_t)|X^{(0)}=x] \\ &\ge \varliminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} \mathsf{E}F(N^*;D_t) \\ &= \mathsf{E}F(N^*;D). \end{aligned} \tag{9.29}$$

Below, we show $\bar{V}^{\pi^*}(x) = \mathsf{E}F(N^*;D)$ for any initial inventory x, so that π^* is optimal. (The feasibility of π^* is obvious.)

First, note that, if $x = X^{(0)} \in S$, then $X^{\pi^*(t)} \in S$ and hence $N^{\pi^*(t)} = N^*$ for all $t \geq 0$. Therefore,

$$\begin{aligned}
\bar{V}^{\pi^*}(x) &= \varliminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathsf{E}[F(N^{\pi^*(t)}; D_t)|X^{(0)} = x] \\
&= \varliminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathsf{E}F(N^*; D_t) \\
&= \mathsf{E}F(N^*; D).
\end{aligned}$$

Now suppose $x \notin S$. Define

$$T_x := \inf\{0 \leq t < \infty : X^{\pi^*(t)} \in S, X^{(0)} = x\},$$

i.e., T_x is the time until the inventory level drops down into the set S. Note that under the stated policy π^*, nothing will be ordered until T_x. Hence,

$$T_x \leq \hat{T}_x := \min\{T : \sum_{t=0}^{T} D_t \geq \sum_{i=1}^{k} x_i\},$$

and we must have $\mathsf{E}[\hat{T}_x] < \infty$, since $\mathsf{E}[D] > 0$; and hence, $\mathsf{E}[T_x] < \infty$. Furthermore, from (9.23) and (9.25), it is clear that, for each period $t < T_x$, the expected cost $\mathsf{E}[F]$ is bounded:

$$0 \leq \mathsf{E}[F(N^{\pi^*(t)}; D_t)|X^{(0)} = x] \leq B_x,$$

for some constant B_x (which may depend on x). Hence, when $x \notin S$, we have

$$\begin{aligned}
\bar{V}^{\pi^*}(x) &= \varliminf_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \mathsf{E}[F(N^{\pi^*(t)}; D_t)|X^{(0)} = x] \\
&\leq \varliminf_{T\to\infty} \frac{1}{T}\{B_x \mathsf{E}[T_x] + \sum_{t=T_x}^{T-1} \mathsf{E}[F(N^{\pi^*(t)}; D_t)|X^{(0)} = x] \\
&= \varliminf_{T\to\infty} \frac{1}{T} \sum_{t=T_x}^{T-1} \mathsf{E}F(N^*; D_t) \\
&= \mathsf{E}F(N^*; D).
\end{aligned}$$

Combining the above with (9.29), we have $\bar{V}^{\pi^*}(x) = \mathsf{E}F(N^*; D)$. □

Note that when the initial inventory exceeds the desired level of N^*, nothing is ordered, and the inventory level will be brought down to below N^* within a *finite* time, during which the expected one-step cost is bounded. This is guaranteed by the fact that it always pays to use up any unit of inventory to supply demand instead of keeping it, thanks to the assumed

Type i	1	2	3	Obj. Val.
defective rate θ_i	.00	.01	.03	
(N_1^*, N_2^*, N_3^*) (N)	151	3	24	623.0
(N_1^*, N_2^*, N_3^*) (U)	151	3	24	616.9
(N_1^*, N_2^*, N_3^*) (P)	150	2	15	613.3

TABLE 9.5. Infinite horizon; convex $c(\theta)$ and $h(\theta)$

condition in (9.21). Hence, any cost over this finite time will be washed out in the long-run average. This is in contrast with the models in Ignall and Veinott [48] and Veinott [104], where the initial inventory must be restricted to below N^*. Two aspects of those models are different from our model here: (a) multiple demand types, with the possibility of substitution (whereas we only consider a single demand stream); and (b) allowing backlog (we assume lost sales). Hence, in those models it is possible that the initial inventory of some types may be kept forever, while some other types may run into a large amount of backlog if nothing is ordered. This will result in an unbounded one-step cost, and the argument in our proof will not apply.

In summary, to find the optimal replenishment policy in the infinite horizon case amounts to solving a single-period problem, exactly like the one in the last section; in particular, the optimal order-up-to level, N^*, can be derived from the greedy algorithm there. Furthermore, when both $c(\theta)$ and $h(\theta)$ are linear, in the infinite-horizon case we also have the optimality of dual sourcing (from sources 1 and k), just as in the single-period case.

Example 9.16 Continue with the problem in Example 9.13, but with an infinite horizon, and three product types only (with slightly different defective rates). There is a perfect type, indexed as $i = 1$; and the ordering cost, $c(\theta) = 3.0 - 5\theta + 6\theta^2$, applies to all three types. Other data remain the same as before. In addition, assume the holding cost is 20% of the purchasing cost: $h(\theta) = 0.2c(\theta)$. Note that here we have $c(0) < c(\theta_i) + C_I + C_R\theta_i$, for $i = 2, 3$; and consequently, no inspection of any type is performed.

For the three types of demand distributions in Example 9.13, the optimal order-up-to quantities and the corresponding objective values are summarized in Table 9.5.

Table 9.6 repeats the above results, but with both the purchasing and holding costs being linear in θ: $c(\theta) = 3.0 - 5\theta$, and $h(\theta) = 0.2c(\theta)$. Note that $N_2^* = 0$ in all cases, as expected, since dual sourcing is optimal.

9.6 A Random Yield Model with Multiple Sources

We can recast the model studied earlier as a random yield model (refer to, e.g., [2, 37, 99, 56, 68, 111]) as follows. For ease of discussion, we focus on the single-period case. So it would be helpful to relate to the model in §9.4.

Type i	1	2	3	Obj. Val.
defective rate θ_i	.00	.01	.03	
(N_1^*, N_2^*, N_3^*) (N)	150	0	28	623.0
(N_1^*, N_2^*, N_3^*) (U)	149	0	29	616.8
(N_1^*, N_2^*, N_3^*) (P)	150	0	17	613.2

TABLE 9.6. Infinite horizon; linear $c(\theta)$ and $h(\theta)$

Suppose there are k sources of supply, indexed by $i = 1, ..., k$. Each source i has a yield ratio of $1 - \theta_i$. Specifically, a proportion, θ_i, of any quantity ordered from source i may not be delivered; or, in other words, an order of N_i units will result in an expected delivery of $N_i(1 - \theta_i)$ units — the actual "yield" that can be used to supply demand. Assume, as before, the sources are indexed in increasing order of the θ_i values.

Let D denote demand as before. Reinterpret C_g and C_d as the costs for each unit of demand satisfied and unsatisfied. In particular, $-C_g$ is the profit derived from supplying each unit of demand. As before, let $\Delta = C_d - C_g$. Note that with $C_d \geq 0$ and $-C_g \geq 0$, $\Delta \geq 0$ is automatic. Let $s \geq 0$ denote the salvage value for each surplus unit; and we replace condition (9.1) with $\Delta \geq s$. Note that since the sources only differ in their yield ratios, once delivered all units are of equal value in supplying demand, hence, the salvage value is independent of the sources, just like C_g and C_d. Also note that under the new interpretation of C_d, it is necessarily equal to the penalty cost in §9.4, C_P; and hence, the inequality in (9.11) is automatically satisfied. Let $c(\theta_i)$ be the purchasing cost of each unit from source i. For instance, $c(\theta_i) = c(1 - \theta_i)$, where $c > 0$ is the cost rate for each unit *delivered*; hence, $cN_i(1 - \theta_i)$ is the expected purchasing cost for an order quantity of N_i units from source i. As before, assume $c(\theta_i)$ is a decreasing and convex functions.

Our model allows a new feature that is not present in previous random yield models. At a premium — above and beyond the purchasing cost — of $a + b\theta_i$ per unit, the supply (delivery) can be guaranteed. Hence, out of the N_i units ordered from source i, for which we pay a purchasing cost of $N_i c(\theta_i)$, we may choose to guarantee a delivery of n_i units by paying an additional premium of $n_i(a + b\theta_i)$. Clearly, this feature is analogous to upgrading, through inspection and repair, a defective unit to a perfect unit. Hence, the (per unit) premium, $a + b\theta_i$, corresponds to the inspection and repair costs in the earlier model with $a = C_I$ and $b = C_R$. To facilitate comparisons, below we shall continue writing C_I and C_R instead of a and b.

We now illustrate how the model in §9.4 can be adapted to identify the optimal order quantities from a set of k unreliable supply sources, each having a random yield in quantities actually delivered. The replenishment decision is supplemented by the option of paying a premium to secure a guaranteed delivery quantity (which is analogous to inspection in the earlier model); and the objective is to minimize the expected total net cost — purchasing, premium and penalty costs minus profit and salvage value.

We start with the expression in (9.2) for $W(D, n_1, \cdots, n_k)$, reinterpreted here as the penalty cost minus profit and salvage value, given that the demand is D, the replenishment quantities are $(N_1, ..., N_k)$, of which $(n_1, \cdots, n_k)$ are guaranteed by paying premiums. Let $B_i(N_i - n_i)$ denote the number of units from source i that are not guaranteed by premiums but are actually delivered. Below we shall assume that $B_i(N_i - n_i)$ follows a binomial distribution associated with $N_i - n_i$ Bernoulli trials each with a success probability $1 - \theta_i$. This implies that the yield of each unit is independent of all other units. This independence assumption is crucial to the stochastic monotonicity results below.

As before, denote $n_{1,k} = \sum_{i=1}^{k} n_i$; and similarly denote

$$B_{1,k} = \sum_{j=1}^{k} B_j(N_j - n_j),$$

and

$$B_{1,k}^i = \sum_{\substack{j=1 \\ j \neq i}}^{k} B_j(N_j - n_j) + B_i(N_i - n_i - 1).$$

Then we can rewrite the function W in (9.2) as follows:

$$\begin{aligned} & W(D, n_1, \cdots, n_k) \\ = \;& C_g \min\{n_{1,k} + B_{1,k}, D\} + C_d(D - n_{1,k} - B_{1,k})^+ \\ & \quad -s(n_{1,k} + B_{1,k} - D)^+ \\ = \;& C_g D + \Delta(D - n_{1,k} - B_{1,k})^+ - s(n_{1,k} + B_{1,k} - D)^+ \\ = \;& C_g D + (\Delta - s)(D - n_{1,k} - B_{1,k})^+ \\ & +s(D - n_{1,k} - B_{1,k}). \end{aligned} \tag{9.30}$$

The expression for Π in (9.3) remains valid here, with the new interpretation of $C_I + C_R\theta_i$ being the premium to guarantee the delivery of a unit from source i. Therefore, (9.5) becomes:

$$\begin{aligned} & \Pi(n_1, \cdots, n_{i-1}, n_i, \cdots, n_k) - \Pi(n_1, \cdots, n_{i-1}, n_i + 1, \cdots, n_k) \\ = \;& -(C_I + C_R\theta_i) + s\theta_i + (\Delta - s) \\ & \cdot[\mathsf{E}(D - n_{1,k} - B_{1,k})^+ - \mathsf{E}(D - n_{1,k} - B_{1,k}^i - 1)^+]. \end{aligned} \tag{9.31}$$

Note that the second expectation above follows from the fact that when n_i is increased to $n_i + 1$, $n_{1,k}$ becomes $n_{1,k} + 1$ and $B_{1,k}$ becomes $B_{1,k}^i$.

Note the following relation:

$$B_{1,k} = B_{1,k}^i + \delta_i,$$

where δ_i denotes a binary variate that equals 1 with probability $1-\theta_i$. We can modify (9.4) based on (9.31):

$$\begin{aligned}
& \Pi(n_1,\cdots,n_{i-1}+1,n_i,\cdots,n_k)-\Pi(n_1,\cdots,n_{i-1},n_i+1,\cdots,n_k) \\
= \quad & -(C_R-s)(\theta_i-\theta_{i-1})+(\Delta-s) \\
& \cdot[\mathsf{E}(D-n_{1,k}-B_{1,k}^{i-1}-1)^+ - \mathsf{E}(D-n_{1,k}-B_{1,k}^{i}-1)^+] \\
= \quad & -(C_R-s)(\theta_i-\theta_{i-1})+(\Delta-s) \\
& \cdot[\mathsf{E}(D-n_{1,k}-B_{1,k}+\delta_{i-1}-1)^+ - \mathsf{E}(D-n_{1,k}-B_{1,k}^{i}+\delta_i-1)^+] \\
= \quad & -(C_R-s)(\theta_i-\theta_{i-1})+(\Delta-s)(\theta_i-\theta_{i-1}) \\
& \cdot[\mathsf{E}(D-n_{1,k}-B_{1,k})^+ - \mathsf{E}(D-n_{1,k}-B_{1,k}-1)^+]. \qquad (9.32)
\end{aligned}$$

With the same definition of $g_i(n_{i,k})$ as in §9.2, here $g_i(n_{i,k})<0$ is equivalent to

$$C_R-s>(\Delta-s)\mathsf{E}[(D-y)^+-(D-y-1)^+], \qquad (9.33)$$

with $y:=n_{1,k}+B_{1,k}$. Note that the right hand side of (9.33) is decreasing in n_i for any $i=1,...,k$. This fact follows from the standard theory of stochastic monotonicity (Ross [80], Chapter 9), since (a) $(D-x-1)^+-(D-x-2)^+$ is a decreasing function of x, and (b) $n_i+B_j(N_i-n_i)$ is stochastically increasing in n_i. Furthermore, just like the case of (9.8) and (9.9), the right hand side of (9.33) decreases as we increase n_i, in the increasing order of i. (This follows from the fact that $B_{1,k}^i$ is stochastically increasing in i.) Similarly, the right hand side of (9.31) is also decreasing in n_i, for any $i=1,...,k$.

Therefore, with the order quantities, $(N_1,...,N_k)$, given, the decision problem of finding the optimal number of units to guarantee from each source through paying premium has the same structure as the optimal inspection problem in §9.3. In particular, with (9.5) and (9.8) replaced by (9.31) and (9.33), and condition (9.10) replaced by $\Delta=s=C_R$, all the results in Theorem 9.6 are still applicable here. In particular, the optimal solution here should be $(N_1,...,N_{j^*-1},n'_{j^*},0,...,0)$, where $j^*\geq 1$ is the smallest index, and $n'_{j^*}\leq N_{j^*}$ is the smallest n_i (with $i=j^*$) value, such that the cost reduction in (9.31) becomes non-positive; or, if no such value exists, then $j^*=k$ and $n'_{j^*}=N_k$.

Now consider the order sizes as decision variables as well. Lemma 9.9 is still valid here, and hence so is the expression in (9.14) for V, but without the last term, the penalty cost, which is now part of W. Note that when no premium is paid to guarantee any delivery, the first two terms on the right hand side of both (9.31) and (9.32) vanish. Consequently, we have,

$$\begin{aligned}
& V(N_1,\cdots,N_i,\cdots,N_k)-V(N_1,\cdots,N_i+1,\cdots,N_k) \\
= \quad & -c(\theta_i)+(\Delta-s)(\theta_i-\theta_{i-1}) \\
& \cdot[\mathsf{E}(D-B_{1,k})^+ - \mathsf{E}(D-B_{1,k}-1)^+], \qquad (9.34)
\end{aligned}$$

and

$$\begin{aligned}
& V(N_1,\cdots,N_{i-1}+1,N_i,\cdots,N_k)-V(N_1,\cdots,N_{i-1},N_i+1,\cdots,N_k)\\
= \;& c(\theta_{i-1})-c(\theta_i)-(\Delta-s)(\theta_i-\theta_{i-1})\\
& \cdot[\mathsf{E}(D-B_{1,k})^+-\mathsf{E}(D-B_{1,k}-1)^+]. \qquad (9.35)
\end{aligned}$$

Note that in the above expressions, $B_{1,k} := B_{1,k}(N_{1,k})$, which is stochastically increasing in N_i for any $i = 1, ..., k$. Hence, the right hand sides of (9.34) and (9.35) are, respectively, decreasing and increasing in N_i for any $i = 1, ..., k$, just like (9.15) and (9.16). In particular, (9.19) now becomes

$$\begin{aligned}
& \frac{c(\theta_i)-c(\theta_{i+1})}{\theta_{i+1}-\theta_i}\\
> \;& (\Delta-s)[\mathsf{E}(D-B_{1,k}(N_{1,i}))^+-\mathsf{E}(D-B_{1,k}(N_{1,i})-1)^+], \quad (9.36)
\end{aligned}$$

where we have written out explicitly the argument of $B_{1,k}$, in correspondence to the i index on the left hand side. This way, as in §9.4, for each i, we can increase N_i until the above inequality is satisfied. On the other hand, every time i is increased, the left hand side decreases, requiring an increase in the $N_{1,i}$ value to satisfy the inequality in (9.36). This leads to the sequence of M_i values, with $M_i - M_{i-1}$ being the upper limit on the order size from source i. On the other hand, the decreasing property of the right hand side of (9.34) plays the role of a stopping rule, exactly as in §9.4.

Therefore, the solution to the optimal replenishment problem can be obtained following the two theorems in §9.4: We first compare the purchasing cost of the perfectly reliable supply source, $i = 1$, with the supply source ℓ^* that has the lowest combined purchasing cost and premium among all other sources. If source 1 is less expensive, then the optimal order quantities are obtained in increasing order of i, starting from $i = 1$, and following the upper limits specified by the M_i's and the stopping rule signified by the non-positive cost reduction in (9.15); and no premium should be paid to any sources. Otherwise, ignore source 1, replace it by source ℓ^* with premiums paid for all units ordered to guarantee a perfect yield; and then proceed in the same manner as in the previous case. Extensions to the infinite horizon as in §9.5 lead to the optimality of the order-up-to policy.

9.7 Notes

It is quite common for production-inventory systems to have multiple supply sources that have different grades of reliability, in terms of the quantity and quality of orders delivered; refer to, e.g., Anupindi and Akella [2], and Parlar and Wang [68]; also refer to Chen, Yao and Zheng [21], from which most of the materials in this chapter are drawn.

In this kind of setting, it is imperative that replenishment decisions take into account supply uncertainty and related cost implications, in addition to the usual tradeoff between the possibilities of surplus inventory and unmet demand.

Along with replenishment decisions, there are recourse actions that can be taken to offset supply imperfection. For example, certain quality control mechanisms can be applied to the orders received, including rework on any defective units before the orders are supplied to customers. Here, the quality control decision (on inspection and repair) is embedded into the replenishment decision; and typically both decisions have to be made before demand is realized. The existing literature in this area focuses mostly on a single, unreliable supply source; refer to, for instance, Lee [54], Lee and Rosenblatt [55], Peter, Schneider and Tang [72], and Yao and Zheng [112], among others.

Another kind of recourse action to offset yield loss is "substitution," which uses the surplus of higher-grade products (in terms of quality and functionality, for instance) to supply the shortage of demand for lower-grade products. Unlike the inspection-repair mechanism, substitution cannot be carried out until when the demand is realized. On the other hand, like the inspection-repair mechanism, which is tantamount to paying premiums to offset yield loss, substitution incurs extra costs associated with filling demand for a lower-grade product with a higher-grade product. Substitution is the subject of the next chapter.