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Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints

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Abstract

We show that all-pay auctions dominate first-price sealed-bid auctions when bidders face budget constraints. This ranking is explained by the fact that budget constraints bind less frequently in the all-pay auctions, which leads to more aggressive bidding in that format.

Keywords: Budget constraints; First-price sealed-bid auctions; All-pay auctions

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1. Introduction

Sealed-bid auctions are used to sell U.S. Treasury bills and bonds, as well as timber harvesting and mineral extraction rights, to mention just a few examples. Another form of auction is the all-pay auction. In an all-pay auction, the highest bidder wins and all participants forfeit their bids. Implicit forms of all-pay auctions are prevalent. Contests in which prizes are awarded on the basis of contestants' effort, such as job-promotion competitions, R&D competitions, and political campaigns, are forms of all-pay auctions. Political lobbying can also be seen as an all-pay auction.

In this paper we show that all-pay auctions generate higher expected revenue than first-price sealed-bid auctions when buyers face budget constraints. There are many settings in which buyers may face budget constraints.[1] A buyer's liquid wealth could simply be low relative to

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[1] There is substantial empirical evidence consistent with the existence of financial constraints, especially among small firms. For example, an extensive literature documents the impact of financial constraints on investment (see Fazzari et al., 1988a,b, or Holtz-Eakin et al., 1994a,b). Government policy also reflects concern over the existence of financial constraints. For example, the U.S. Federal Communications Commission recently set aside a large number of licenses in its Personal Communications Services auctions for bidding by small firms (see Cramton, 1994).

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the stream of benefits that the object will generate. If the buyer has limited access to capital markets, then he is budget constrained. (We focus on buyers whose liquid assets constitute an absolute constraint on their bids, although this is not crucial.) Similarly, the buyer could be a firm with a low cash flow. The budget constraint could also stem from the need to control a purchasing agent who only internalizes the benefits of acquisitions. Finally, in a job-promotion competition or other tournament setting, a convex cost of effort has the same effect as a budget constraint.

When selling an object, a seller has the option of holding an explicit all-pay auction in place of a first-price sealed-bid auction. In other settings, this same choice is available. For example, if the sponsor of a job-promotion competition is an employer who values aggregate effort, then she will prefer to use a criterion based on recent performance (all-pay) rather than one based on proposed effort (first-price) when the cost of effort is highly convex.

2. The model

A seller has an indivisible object that she values at zero. There are N risk-neutral buyers. Buyer i has wealth $w_i \in [\underline{w}, \bar{w}]$, which is private information. One can interpret w_i as the buyer's liquid wealth, or his budget. Wealth is independently and identically distributed across buyers, with a cumulative distribution function $F(\cdot)$ and density $f(\cdot)$. Each buyer values the object at v, and this valuation is publicly known. This common-value specification allows us to focus on the impact of budget constraints.

Buyers have limited access to capital and, consequently, they cannot pay more than their wealth. A strategy of bidding more than one's wealth is infeasible in the all-pay auction since buyers must pay up front. Bidding more than one's wealth in the first-price auction can be ruled out by not giving the object to a buyer who reneges on his bid and by imposing a small penalty on him.[2]

We assume that the seller does not impose a binding reserve price or entry fee. The dominance of the all-pay auction generalizes to the case where the seller optimally chooses a reserve price or entry fee (see Che and Gale, 1994).

3. Equilibria in the two formats

3.1. First-price sealed-bid auctions

All buyers are unconstrained if $v \leq \underline{w}$, since each is able to pay v. Bertrand competition generates revenue equal to v in equilibrium. Now suppose that $v > \underline{w}$. Let

$$U^{\mathrm{f}}(w) \equiv \max_{\underline{w} \leq b \leq w} (v - b)F(b)^{N-1}.$$

[2] This point has practical significance. In a recent auction of TV licenses in Australia, one bidder made several bids and reneged on each one until reaching the highest rival bid, which had become public knowledge by then (*Business Week*, 14 March 1994, p. 48).

$U^{\mathrm{f}}(w)$ is the highest expected surplus that a buyer with wealth w could receive if all other buyers bid their wealth. (Note that $U^{\mathrm{f}}(\cdot)$ is weakly increasing.) We now show that $U^{\mathrm{f}}(w)$ is the expected surplus that accrues to a buyer with wealth w in equilibrium. The proof is in the appendix.

Lemma 1. A buyer with wealth w receives an expected surplus of $U^{\mathrm{f}}(w)$ in equilibrium in the first-price auction. In particular, it is equilibrium behavior for such a buyer to bid $v - U^{\mathrm{f}}(w)/F(w)^{N-1}$.

The first-price auction extracts revenue from at most one buyer. In the presence of budget constraints, the winning bidder often receives a substantial net surplus. His presence clearly lowers the winner's bid if his own constraint binds. In addition, unconstrained bidders shade their bids below their valuation, knowing that other bidders may be constrained. Thus, their presence lowers the winner's bid if he is unconstrained, too.

3.2. All-pay auctions

In an all-pay auction, buyers submit non-negative bids simultaneously. The highest bidder wins, but *all* active bidders pay their bids. If a buyer submits a bid b and wins, his surplus is $v - b$. If he loses, his surplus is $-b$.

For each w, let

$$U^{\mathrm{a}}(w) \equiv \max_{0 \leq b \leq w} vF(b)^{N-1} - b.$$

$U^{\mathrm{a}}(w)$ equals the expected surplus that accrues to a bidder with wealth w, if all others bid their wealth. The following lemma, the proof of which is omitted, is proven in a manner similar to Lemma 1.

Lemma 2. A buyer with wealth w receives an expected surplus of $U^{\mathrm{a}}(w)$ in equilibrium in the all-pay auction. In particular, it is equilibrium behavior to bid $vF(w)^{N-1} - U^{\mathrm{a}}(w)$.[3]

If $v \leq \underline{w}$, then $U^{\mathrm{a}}(w) = 0$ for all w. By Lemma 2, all buyers then receive zero expected surplus. Since all surplus is extracted, and the object sells with probability one, the seller's expected revenue is v. The expected revenue may equal v even when $v > \underline{w}$. In particular, if $F(w) \leq (w/v)^{1/(N-1)}$ for all w, then there is an equilibrium in which a buyer with wealth w bids $vF(w)^{N-1}$. This generates the same distribution of bids as the symmetric (mixed-strategy) equilibrium without budget constraints (Baye et al., 1993b). We have purified that equilibrium strategy here by using wealth as the randomizing device.

[3] This bid function has the same form as the bid function for the first-price auction in Lemma 1. One obvious difference arises because a buyer pays his bid with probability one in the all-pay auction. The expected surplus that accrues to a buyer with wealth w can also differ across the two auctions.

4. Revenue comparison

Both mechanisms yield expected revenue of v if $v \leq \underline{w}$, since all buyers are unconstrained. For the remainder of this section we focus on $v > \underline{w}$. We demonstrate that the all-pay auction strictly dominates the first-price auction by showing that every buyer's expected surplus is lower in the all-pay auction.

Proposition 1. The all-pay auction yields a strictly higher seller's expected revenue if $v > \underline{w}$.

Proof. Consider a first-price auction. By Lemma 1, the expected surplus for a buyer with wealth w is to

$$\max_{\underline{w} \leq b \leq w} (v - b)F(b)^{N-1}.$$

We now consider an all-pay auction. By Lemma 2, a buyer with wealth w receives expected surplus equal to

$$\max_{\underline{w} \leq b \leq w} vF(b)^{N-1} - b,$$

if this exceeds zero. Otherwise, he receives zero expected surplus. (Note that in the definition of $U^{a}(\cdot)$, a maximum never occurs in $(0, \underline{w})$ because $F(b) = 0$ in that interval.)

The expected total surplus is v in the two auctions. The buyers' expected surplus is weakly higher in the first-price auction because $(v - b)F(b)^{N-1}vF(b)^{N-1} - b$ for $\underline{w} > b > \underline{w}$, so the maximum over $[\underline{w}, w]$ is strictly higher in the first-price auction, for all $w \in (\underline{w}, \bar{w})$. Thus, the seller receives strictly more expected revenue by holding an all-pay auction. □

In a first-price auction, the highest bidder receives the object, and he alone makes a payment to the seller, whereas all bidders pay their bids in an all-pay auction. A given bidder submits a smaller bid in the all-pay auction so, roughly speaking, budget constraints bind less frequently in that format.[4] Since the two formats generate the same expected revenue in the absence of constraints, the all-pay auction dominates when constraints are present.

5. Concluding remarks

We have shown that all-pay auctions dominate first-price auctions when buyers face budget constraints. Thus, in job-promotion competitions or other tournament settings, a sponsor who values aggregate effort wil prefer a criterion based on past performance to one based on proposed effort, when the cost of effort is highly convex.[5] The revenue ranking is also

[4] In first-price auctions, buyers bid for the right to obtain the good with probability one, whereas buyers bid for the right to obtain the good with probability less than one in all-pay auctions. All-pay auctions therefore divide the good into smaller 'probability' units.

[5] Research foundations often make awards on the basis of the amount and quality of research already undertaken, rather than on the strength of the proposal. Similarly, many R&D tournaments require the building of a prototype, in addition to the proposal.

interesting because political lobbying is usually seen as a second-best outcome. The argument is that politicians and officials of government agencies use this form of rent extraction because they are not permitted to sell political prizes explicitly (see Baye et al., 1993a, for example). Since the all-pay auction generates higher expected revenue than the first-price auction, a politician who maximizes expected revenue may prefer to receive revenue via lobbying.[6]

The revenue ranking here has some parallels in auctions without budget constraints. In particular, the same ranking has been found by Amann and Leininger (1994) and Krishna and Morgan (1994), with risk-neutral bidders and affiliated values. With independent private values, however, the two formats are revenue-equivalent (see Myerson, 1981, Riley and Samuelson, 1981). The ranking here also differs from that found with risk-averse bidders (see Maskin and Riley, 1984, for a general treatment of risk-averse bidders). To see this, consider the limit as bidders become infinitely risk averse. Then, revenue approaches zero in the all-pay auction, whereas it approaches the highest valuation in the first-price auction.

The preferences for all-pay auctions here is robust to changes in the model. For example, the all-pay auction still dominates the standard first-price auction if borrowing is possible at a fixed interest rate. Likewise, the result is unchanged if buyers can differ in both valuations and budgets.

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Appendix: Proof of Lemma 1

Let $U(\cdot)$ denote the expected surplus in a candidate equilibrium. If $v \leq \underline{w}$, then $U(w) = 0$ for all w, since Bertrand competition leads to a price of v. It remains to complete the proof for $v > \underline{w}$. A bid b wins with probability $F(b)^{N-1}$ or more, since the other buyers cannot bid more than their wealth. The bid gives an expected surplus of at least $(v-b)F(b)^{N-1}$, so

$$U(w) \geq U^{\mathrm{f}}(w) = \max_{\underline{w} \leq b \leq w} (v-b)F(b)^{N-1} .$$

Now suppose that there exists w' such that $U(w') > U^{\mathrm{f}}(w')$. We show that this provides a contradiction.

Let $z \equiv \inf\{w \mid U(w) = U(w')\}$ denote the infimum of the wealths for which the equilibrium expected surplus equals $U(w')$. We assume, for now, that $U(z) = U(w')$. (In principle,

[6] An obvious question is: How easily can lobbying funds be converted to personal use? A recently amended law allowed members of the U.S. Congress to convert campaign contributions to personal use (2 U.S.C. Section 439a). An alternative method of conveying funds is to direct business to a politician's law firm, for example.

$U(z) < U(w')$ is possible. Applying the arguments given below to wealth levels that are above z, but arbitrarily close, gives the same result.) The following relationships must then hold:

$$U(z) = U(w') > U^{\mathrm{f}}(w') \geq (v - z)F(z)^{N-1}. \tag{A1}$$

The first and second hold by assumption, while the third holds by definition. Since $U(z) > (v - z)F(z)^{N-1}$, a buyer with wealth z must bid $b(z) < z$ or else he must win with probability strictly greater than $F(z)^{N-1}$. If $z > \underline{w}$, but $b(z) < z$, then buyers with wealth $w \in [b(z), z)$ would be strictly better off deviating and bidding $b(z)$, since $U(W) < U(z)$ for $w < z$, by assumption. Therefore, $b(z) = z$. If $z = \underline{w}$, then $U(w) > 0$ for all w, which implies that there must be a mass point at the minimum equilibrium bid. (That is, the first-order statistic of the N bids must have a mass point there.) Individual buyers could then obtain a discrete increase in expected surplus by raising their bids infinitesimally above that minimum bid, so $z = \underline{w}$ is not possible.

Since $b(z) = z$, a buyer with wealth z must win with probability greater than $F(z)^{N-1}$ to satisfy (A1). Again, this requires a mass point at z. Thus, we conclude that $U(w) = U^{\mathrm{f}}(w)$ for all w.

We now demonstrate that the postulated bids are equilibrium bids. Let

$$b^*(w) \equiv v - U^{\mathrm{f}}(w)/F(w)^{N-1}.$$

If $U^{\mathrm{f}}(w) = (v - w)F(w)^{N-1}$, then $b^*(w) = w$. Conversely, if $U^{\mathrm{f}}(w) > (v - w)F(w)^{N-1}$, then $b^*(w) < w$. Thus, these bids are feasible.

We now show optimality. Suppose that all other bidders use $b^*(\cdot)$. A buyer with wealth w wins with probability $F(w)^{N-1}$ if he bids $b^*(w)$, because $b^*(\cdot)$ is a strictly increasing function.[7] His expected surplus is then

$$(v - b^*(w))F(w)^{N-1} = U^{\mathrm{f}}(w).$$

The bidder has no incentive to change his bid. To see this, recall that only bids $b \leq w$ are feasible for him. A bid $b \in [\underline{w}, b^*(w)]$ wins with probability $F(b^{*-1}(b))^{N-1}$ and gives expected surplus of $U^{\mathrm{f}}(b^{*-1}(b))$. This surplus is less than or equal to $U^{\mathrm{f}}(w)$, because $U^{\mathrm{f}}(\cdot)$ is increasing, and $b^{-1}(b) \leq w$ in this region. Thus, there is no incentive to deviate and bid lower. If $b^*(w) < w$, then we must also consider $b \in (b^*(w), w]$. Let $w^* \equiv b^{*-1}(w)$, so a buyer with wealth w^* bids w. (Let $w^* \equiv \underline{w}$ if no solution exists.) It is clear that $b^*(z) < z$ for $z \in (w, w^*]$, because $b^*(z) \leq b^*(w) = w$. However, $b^*(z) < z$ if and only if $(v - z)F(z)^{N-1} < U^{\mathrm{f}}(z)$. Since $U^{\mathrm{f}}(\cdot)$ is constant in a neighbourhood of z whenever $(v - z)F(z)^{N-1} < U^{\mathrm{f}}(z)$, we have $U^{\mathrm{f}}(z) = U^{\mathrm{f}}(w)$ for $z \in (w, w^*]$. This is equivalent to $U^{\mathrm{f}}(b^{*-1}(b)) = U^{\mathrm{f}}(w)$ for $b \in (b^*(w), w]$, so there is no incentive to deviate. We conclude that $b^*(\cdot)$ is an equilibrium bid function. □

[7] The interval $[\underline{w}, \bar{w}]$ can be divided into open intervals on which $U^{\mathrm{f}}(\cdot)$ is constant or else it is strictly increasing. When $U^{\mathrm{f}}(\cdot)$ is constant over an interval, the bids are strictly increasing. When $U^{\mathrm{f}}(\cdot)$ is strictly increasing, $b^*(w) = w$, so the bids are again strictly increasing.

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