Efficient Assignment with Interdependent Values*

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March 26, 2015

Abstract: We study the "house allocation" problem in which n agents are assigned n objects, one for each agent, when the agents have interdependent values. We show that there exists no mechanism that is Pareto efficient and ex-post incentive compatible, and the only mechanism that is ex-post group incentive compatible is constant across states. By contrast, we demonstrate that a Pareto efficient and Bayesian incentive compatible mechanism exists in the two agent house-allocation problem, given sufficient congruence of prefer-

*We thank So Yoon Ahn, Yuchen Liu, Kelly Rader, Keeyoung Rhee, and especially Fanqi Shi for excellent research assistance. We also thank the Editor, the Associate Editor, and two anonymous referees, Oguz Afacan, Eduardo Azevedo, Dirk Bergemann, Wing-Sum Cheung, Jon Eguia, Daniel Fragiadakis, Tadashi Hashimoto, Navin Kartik, Konrad Mierendorff, Benny Moldovanu, Bruno Strulovici, and seminar participants at Seoul National University, Stanford University, MITRE Auctions and Matching Workshop at University of Michigan, the 2012 NBER Market Design Workshop, and SING8 at Budapest for comments. The authors acknowledge the financial support from the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology through its grant of Global Research Network (2013S1A2A2035408). Kim gratefully acknowledges the financial support from the Institute of Economic Research of Seoul National University. Kojima gratefully acknowledges the support of a Sloan Research Fellowship. This work was completed while Jinwoo Kim and Fuhito Kojima were visiting Yonsei and Columbia University, whose support and hospitality are gratefully acknowledged.

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ences and the standard single crossing property. By contrast, we show that (approximate) Pareto efficiency can be achieved once we relax the incentive compatibility requirements to approximate ex-post incentive compatibility or Bayesian incentive compatibility.

1 Introduction

Many real life allocation problems involve assigning indivisible objects to individuals without using monetary transfers. Examples include university housing allocation, office assignment, and student placement in public schools. A typical goal in such a problem is to assign the objects efficiently while eliciting true preferences of the participants. The literature on matching and market design has made considerable advances on this problem under the assumption that agents have private values, namely that participants know the values of the objects being assigned. Given the private-value assumption, studies in this literature have identified a number of mechanisms that implement a Pareto efficient allocation in a **strategy-proof** fashion, making it a dominant strategy for each participant to reveal his preferences truthfully.[1] These and related studies have helped the redesign of school choice programs in cities such as Boston and New York City.[2]

In many resource allocation problems, however, the information about the objects being allocated is dispersed among participants, and each agent is often unable to assess their values based solely on his or her limited information. School choice is a case in point. Students and parents participating in a school choice program typically have some information but find it insufficient to form clear preferences on different schools. They consult school websites, information booths, fairs and campus tours. But they also seek advice from others through word-of-mouth, online social networks, and guidebooks, and often get swayed by the information they receive from these sources.[3]

[1]Examples of mechanisms with these features include serial dictatorships (Svensson, 1999; Abdulkadiroğlu and Sönmez, 1998), top trading cycles mechanisms (Abdulkadiroğlu and Sönmez, 2003), hierarchical exchanges (Pápai, 2000), and trading cycles mechanisms (Pycia and Ünver, 2009). In fact, these mechanisms are also group strategy-proof, so no group of agents can benefit by joint manipulations.

[2]See Abdulkadiroğlu, Pathak and Roth (2005), and Abdulkadiroğlu et al. (2005) who helped design student placement mechanisms in New York City and Boston.

[3]For instance, high school applicants "constantly talk about which colleges each high school sends its

The scenario described here departs starkly from the private-value setting portrayed by most existing studies in matching and market design.[4] Instead, dispersed information and the relevance of local information and others' personal experiences make parents' preferences interdependent.[5] That is, a parent's information affects the preferences of other parents. Interdependence of preferences is also present in other allocation problems, such as student housing assignments, course allocation, after-school program assignment, and others. How should one design allocation mechanisms in such an environment? How does preference interdependence affect the performance of allocation mechanisms?

One important consideration in designing allocation mechanisms is the robustness of incentives. It has long been recognized in the mechanism design literature (Wilson, 1987; Bergemann and Morris, 2005) and reinforced by recent market design experiences (Abdulkadiroğlu, Pathak and Roth, 2005; Abdulkadiroğlu et al., 2005) that robust incentives, such as strategy-proofness in the private-value setting, ensure that participants not be harmed by reporting their preferences truthfully, irrespective of their beliefs about other players. Unfortunately, preference interdependence makes strategy-proofness virtually impossible

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graduates to, where there might be more interesting students, how long the subway ride would be." (see "Even an Expert's Resolve Is Tested by the City's High School Admissions Process," *New York Times*, December 8, 2008). The importance of the information (or lack thereof) about schools also appears to be behind the immense popularity of websites such as GreatSchools.org, RateMyProfessors.com, and Insideschools.org. The first two websites enjoy more than 800,000 and 13,000,000 ratings and reviews on schools and college professors by students and their parents, respectively. There several influential guide books, such as *New York City's Best Public Schools* series, written by Clara Hemphill, which is "regarded as the bible for navigating school choices," according to the aforementioned article.

[4]In private-values models, parents have clear preferences about their choices but are concerned about how to "play" the application game. For many parents, a more difficult problem is to determine what school is good for their child. To see how differently a parent in this latter scenario would behave relative to the one in the former scenario, suppose that in a Boston mechanism, a parent receives a word-of-mouth information suggesting that many other parents view a given school as desirable. According to the viewpoint from the existing theory (first scenario), the parent will more likely respond to that information by avoiding ranking that school at the top of her list. But the parent in the latter scenario may more likely rank it at the top, realizing that the school is actually good.

[5]The term "interdependence" refers to informational externalities, namely, that one's value of an object depends on the private information held by others. Importantly, it does not include allocative externalities — namely, that one's preference depends on the other agents' assignments, as would be the case with peer effects.

to attain. A natural notion in this setting is **ex-post incentive compatibility**, which requires that truth-telling form mutual best responses for every signal profile. Ex-post incentive compatibility makes it safe for participants to report signals truthfully as long as others do so as well, by making truth-telling a best response irrespective of the information possessed by other agents or beliefs about it.[6]

Our main finding is that such robustness comes with a high price. We show that there exists no mechanism that is Pareto efficient and ex-post incentive compatible whenever non-trivial preference interdependence exists (and the preference domain is sufficiently rich). Further, if we require the mechanism to be ex-post "group" incentive compatible—namely, that there be no group of agents who can benefit from joint manipulations—, we find that only mechanisms that prescribe a constant outcome across states satisfy this property. These negative results hold even when the value interdependence is arbitrarily small so the preferences are almost private, which stands in stark contrast to efficiency obtained in "pure" private value models.

Our inefficiency results are fairly general, but they rest crucially on the ex-post incentive compatibility notion. If incentive compatibility is required to hold only approximately or in the Bayesian sense (as opposed to the ex-post sense), then positive results are obtained. Specifically, we show that as the preference interdependence vanishes (so the preferences become almost private), a suitable extension of serial dictatorship is approximately group-strategy-proof and produces an approximately efficient assignment. Next, with two agents and two objects, there exists a Bayesian incentive compatible mechanism that attains Pareto efficiency, provided that a standard single crossing property holds and agents' preferences are sufficiently congruent. Further, for a general setting with n agents and n objects, the standard serial dictatorship mechanism implements an approximately efficient outcome in a perfect Bayesian equilibrium, if agents' preferences have a small degree of preference interdependence. Our analysis suggests the importance of not insisting upon exact ex-post incentive compatibility if interdependence of preferences exists. This is in a sharp contrast to private-values setting, in which various studies in recent matching and market design literature have emphasized the importance of strategy-proofness (see

[6]Given its appeal, the concept of ex-post incentive compatibility is used extensively in mechanism design. See Bergemann and Välimäki (2002), Crémer and McLean (1985), Esö and Maskin (2002), Krishna (2003), and Perry and Reny (2002) for instance.

Abdulkadiroğlu, Pathak and Roth (2009) for instance).

2 Related Literature

Our findings intersect with several strands of existing research. First, the central theme of our paper agrees with Jehiel and Moldovanu (2001) and Jehiel et al. (2006) who investigate the difficulties associated with interdependent values under the transferable utility setup. Specifically, the former paper establishes generic impossibility of implementing the efficient allocation in Bayesian equilibrium; and the latter proves the generic impossibility of implementing an allocation that varies nontrivially with states in ex-post equilibrium. In the public decision setting, Li, Rosen and Suen (2001) also establish impossibility of Bayesian implementing an efficient outcome without transfers. While our results reinforce and complement these papers, there are several important distinctions.

First, the inefficiency result of Jehiel and Moldovanu (2001) (established in Bayesian implementation) may at first glance appear to imply ours (established in ex-post implementation, which is a stronger requirement), but the efficiency requirements are different between the two models. Specifically, they employ utilitarian efficiency as the welfare criterion, whereas we focus on Pareto efficiency.[7] The latter is much weaker, and there are often many Pareto efficient allocations for a given signal profile. Therefore, there are many Pareto efficient mechanisms. Hence, to show the impossibility of efficiency, one must show that *all* such mechanisms violate ex-post incentive compatibility. Second, unlike the public decision problem Jehiel et al. (2006) consider, our constancy result is derived in the private-object setting. In the private-object setting, each agent is indifferent across a number of allocations as long as her own assignment is identical. As shown by Bikhchandani (2006), this fact can be exploited to provide non-constant mechanisms in the private-object setting (with monetary transfers available).[8] Finally, the results of both Jehiel and Moldovanu (2001)

[7]The reason for the difference is the environments that these two papers focus on. Jehiel and Moldovanu (2001) consider a transferable utility environment in which Pareto efficiency boils down to utilitarian efficiency. Utilitarian efficiency is not implied by Pareto efficiency, however, in our non-transferable utility environment.

[8]By contrast, the impossibility of efficiency under Bayesian incentive compatibility continues to hold in the private object setting (see Example 14 of Jehiel and Moldovanu (2006)). We thank Benny Moldovanu

and Jehiel et al. (2006) require that agents have multi-dimensional signals. In contrast, our main results, Theorem 1 and 2, hold with both single-dimensional and multi-dimensional signals. Due to these distinctions, our impossibility results are not implied by these papers, but rather extend their insights to a non-transferable utility environment.

As will be seen, the Bayesian impossibility result by Li, Rosen and Suen (2001) arises from the type of public decision problems they consider as well as from their restriction on deterministic mechanisms. Within this framework, they show that any Bayesian-implementable mechanism must be partitional with respect to agents' signals, and that the partitional structure is incompatible with Pareto efficiency. In our private-good environment with no restriction on the mechanism, the partitional structure does not preclude Pareto efficiency. In Appendix A, we provide a (random) mechanism that has the partitional structure but nonetheless Bayesian-implements the Pareto-efficient allocation. As we shall discuss in Remark 6, this possibility result applies to a type of public decision problem labeled "negative externality" by Bergemann and Morris (2008). This difference further highlights the role played by ex-post implementation in our impossibility result.

Our model is also related to Chakraborty, Citanna and Ostrovsky (2010) and Chakraborty, Citanna and Ostrovsky (2014), who study preference interdependence in a matching context. Their setup deals with two-sided matching in which agents on one side are matched with agents on the other side, whereas agents are assigned objects in our setup. This difference matters both in terms of the problems studied and the main thrust of the analysis. For instance, the primary concern in their paper is stability of matching between the two sides, whereas our primary focus is on the efficiency of allocations.[9]

Finally, the current study is part of a growing research field of matching and market design. Gale and Shapley (1962) formalized the two-sided matching problem, and Roth (1984) stimulated early applications of matching theory to economic problems. In particular, market design for student placement due to Balinski and Sönmez (1999) and Abdulkadiroğlu and Sönmez (2003) has been extensively studied in recent years. As mentioned above, the main difference of the current paper from this line of studies is our attention to interde-

for informing us of this result.

[9]While Chakraborty, Citanna and Ostrovsky (2014) also consider efficiency, two-sidedness of matching makes their notion quite distinct from ours. They assume the agents on one side have common preferences of the agents on the other side, so every non-wasteful (full) matching is Pareto efficient.

pendent values. The field is too large to summarize here. Instead, we refer to surveys of the literature by Roth and Sotomayor (1990), Roth (2002), Sönmez and Ünver (2011), and Pathak (2011).

3 Illustrative Example

We illustrate the main insight for our impossibility results in a simple setup in which two agents, 1 and 2, are assigned two objects, a and b, one for each agent. There is no money in this economy. Let $v_o^i(s^1, s^2) > 0$ denote agent $i = 1, 2$'s value of receiving object $o = a, b$, when agent $j = 1, 2$ has signal $s^j \in [0, 1]$. Without loss, consider agent i's net utility gain $u^i(s) := v_a^i(s) - v_b^i(s)$ from receiving a instead of b, when the signal profile is $s = (s^1, s^2)$. The function u^i for each agent $i = 1, 2$ is increasing in both signals and satisfies the single crossing property:

$$\frac{\partial u^i(s)}{\partial s^i} > \frac{\partial u^{-i}(s)}{\partial s^i}, \forall s \in [0, 1]^2, \tag{1}$$

that is, one's signal affects his own value more than the other's.

Let $S_{o_1 o_2}$ denote the set of signal profiles such that agent 1 prefers object $o_1 \in \{a, b\}$ and agent 2 prefers object $o_2 \in \{a, b\}$, strictly for at least one agent.[10] A **Pareto efficient** allocation must assign a to 1 and b to 2 when the signal profile is in S_{ab} (because 1 likes a more than b, and 2 likes b more than a in S_{ab}), and likewise must assign a to 2 and b to 1 when the signal profile is in S_{ba}. Assume that both of these sets are nonempty. These sets are depicted as two shaded areas, respectively, in Figure 1. (In this figure, agent i's indifference curve depicts the locus of signal profiles that make her indifferent between the two objects; i.e., the set $\{s \in [0, 1]^2 | u^i(s) = 0\}$.) Note that Pareto efficiency does not uniquely determine the assignment when both agents prefer a to b (i.e., when the signal profile is in S_{aa}) or when both agents prefer b to a (i.e., when the signal profile is in S_{bb}), or when both of them are indifferent.[11]

We first show that there exists no mechanism that is Pareto efficient and ex-post incen-

[10] For instance, $S_{ab} := \{s \in [0, 1]^2 \mid u^1(s) > 0, u^2(s) \leq 0 \text{ or } u^1(s) \geq 0, u^2(s) < 0\}$.

[11] Hence, there are infinitely many Pareto efficient allocations at such a signal profile. By contrast, the utilitarian efficient allocation is uniquely pinned down (e.g., it is utilitarian efficient to assign agents 1 and 2 objects a and b, respectively, below the dashed curve, and the other way around above the curve).

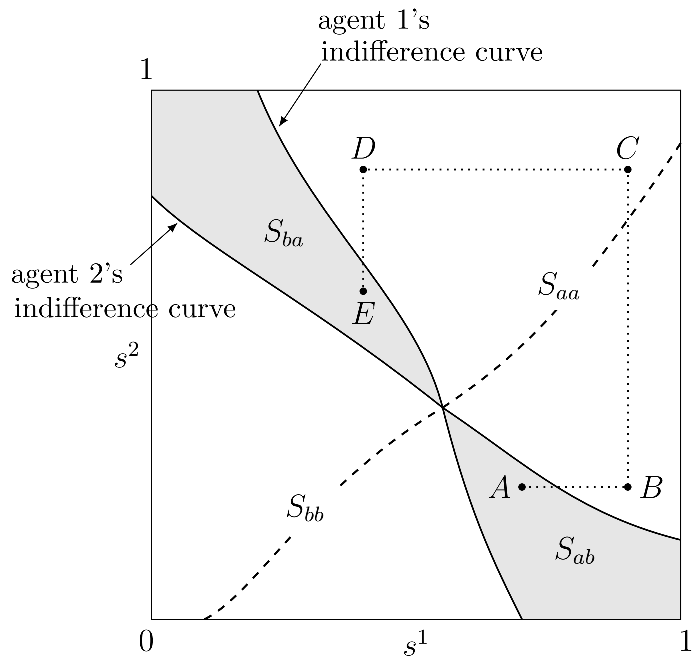

Figure 1: Impossibility of Ex-Post Incentive Compatibility

tive compatible.[12] To see this, suppose otherwise. Then, by the revelation principle, there is a direct mechanism that is ex-post incentive compatible and Pareto efficient. Then, at state $A = (s^1_A, s^2_A) \in S_{ab}$, agent 1 must receive a and agent 2 must receive b, and reporting the signal truthfully is a mutual best response. Now consider state $B = (s^1_B, s^2_A)$. Note that B differs from A only in agent 1's signal, and further that agent 1's (ordinal) preference remains unchanged. These two facts mean that, for the mechanism to be ex-post incentive compatible, agent 1 must receive a at B; or else, agent 1 has incentives to report s^1_A instead at state B and receive a for sure. Hence, the assignment remains unchanged between states A and B. Now consider state $C = (s^1_B, s^2_C)$. State C differs from state B only in agent 2's signal, and that agent's preference is the same between B and C. This means that the allocation must be the same at these two states. To see this, suppose for contradiction that agent 2 receives a with positive probability at C (and b with the remaining probability, as required by Pareto efficiency). Then agent 2 has incentives to misreport

[12] As will be seen, the notion of ex-post incentive compatibility must be defined more precisely for the ordinal preference/non-transferable utility environment. A few alternative concepts will be considered in the paper, but the distinction in the notions does not matter here since there are only two objects.

her signal at state B, reporting s_C^2 instead. The same logic implies that the allocation at state $D = (s_D^1, s_C^2)$ must be exactly the same as the allocation at C. And similarly, the allocations at $E = (s_D^1, s_E^2)$ must be the same as the allocation at D. Recalling the series of equivalences, we conclude that the allocation at E must be the same as the one at state A — that is, agent 1 receives a and agent 2 receives b.[13] But this allocation is not Pareto efficient since $E \in S_{ba}$, showing that there exists no mechanism that is Pareto efficient and ex-post incentive compatible.

In fact, the above argument implies much more than merely the impossibility of ex-post incentive compatibility and Pareto efficiency. We can show that any ex-post incentive compatible mechanism (that assigns both objects) must prescribe a constant allocation across all states! To see this, take two states arbitrarily, say s and $\hat{s} \neq s$. As above, one can construct a connected path of states, $s_0 \to s_1 \to .... \to s_m$, such that $s_0 = s, s_m = \hat{s}$, any adjacent states s_j and s_{j+1} have the same signal for one agent and different signals for the other, and the latter agent's ordinal preferences are the same and strict for both states. Ex-post incentive compatibility then implies that the allocation is unchanged between any adjacent states. Therefore, we conclude that the allocation is identical between any pair of states s and $\hat{s}$.

Several remarks are worth making. First, the latter result—that the ex-post incentive compatibility means that only a constant allocation can be implemented—is reminiscent of Jehiel et al. (2006), who arrive at the same conclusion *under the transferable utility* setup. Despite the resemblance, however, the current result is not implied by theirs. One reason is that their result requires that agents have multi-dimensional signals while ours does not. In fact, the absence of monetary transfers is needed for our result. If monetary transfers were available in our example, a Pareto efficient and ex-post incentive compatible mechanism exists, despite the fact that Pareto efficiency would entail a stronger allocative requirement in the presence of monetary transfer.

This can be seen as follows. Note first that, given transferable utilities, Pareto effi-

[13] Our argument may be reminiscent of studies of local incentive compatibilities by Carroll (2012) and Sato (2013). They consider sufficient conditions under which local incentive constraints are equivalent to incentive constraints. Our analysis similarly considers a series of (small) signal changes to obtain a conclusion about global behavior of a mechanism. However, these models have private values while ours has interdependent values. Moreover, the questions they are analyzing are distinct from ours.

ciency implies **utilitarian efficiency**, which requires that agents 1 and 2 receive a and b, respectively, if $u^1(s) > u^2(s)$ (which corresponds to the region below the dashed curve in Figure 1) and b and a, respectively, if $u^1(s) < u^2(s)$ (the region above the dashed curve in Figure 1). To see how this outcome can be implemented in an ex-post incentive compatible mechanism, let

$$\sigma^i(s^j) := \sup\{s^i \in [0,1] | u^j(s^i, s^j) \geq u^i(s^i, s^j)\}$$

for $i, j = 1, 2, i \neq j$, if the set is nonempty, or else let $\sigma^i(s^j) := 0$. Suppose that the mechanism designer collects reports $(s^1, s^2) \in [0,1]^2$ from the agents and assigns the objects in a utilitarian-efficient manner, while charging agent i a tariff $p^i(s^j) := u^i(\sigma^i(s^j), s^j)$ whenever she receives a. It then follows that, given the associated tariffs, agent i prefers to receive a to b at any signal profile in which she receives a and that she prefers to receive b to a at any signal profile in which she receives b, so truthful reporting is ex-post incentive compatible.[14]

Second, the particular assumptions made above — convexity of $S_{o_1 o_2}$, the single crossing property, and the assumption that there are only two agents and two objects — are not needed for the impossibility of implementing the efficient allocation above. Section 4 will establish inefficiency in a general setting in which these assumptions are relaxed. By contrast, the second impossibility result above (i.e. the impossibility of non-constant mechanisms) does not generalize straightforwardly beyond the two-agents two-objects case. To see this, suppose that there are three agents, 1, 2, and 3, and three objects, a, b, and c. A mechanism that always assigns object c to agent 3, but assigns a and b between agents 1 and 2 in a way that varies only with agent 3's signal, is clearly ex-post incentive compatible.

[14] The detailed argument for incentive compatibility is similar to that in Maskin (1992) and as follows. By the single crossing property and the definitions of σ^1 and σ^2, if $u^i(s) - u^j(s) \geq 0$, then $s^i \geq \sigma^i(s^j)$ and $s^j \leq \sigma^j(s^i)$, and if $u^i(s) - u^j(s) = 0$, then $s^i = \sigma^i(s^j)$ and $s^j = \sigma^j(s^i)$. Suppose first $u^1(s) - u^2(s) \geq 0$. Then, $s^1 \geq \sigma^1(s^2)$, so

$$u^1(s^1, s^2) - p^1(s^2) = u^1(s^1, s^2) - u^1(\sigma^1(s^2), s^2) \geq 0.$$

Hence, she (weakly) prefers a to b, so she will have incentives to report her signal truthfully. Suppose $u^1(s) - u^2(s) \leq 0$. Then $s^1 \leq \sigma^1(s^2)$, so

$$u^1(s^1, s^2) - p^1(s^2) = u^1(s^1, s^2) - u^1(\sigma^1(s^2), s^2) \leq 0.$$

Hence, the agent again has incentives to report her signal truthfully. The argument is symmetric for agent 2.

This example points to another difference of the current model from Jehiel et al. (2006). Unlike their public decision problem, agents are indifferent across some allocations in our setting; for instance, agent 3 is indifferent on how a and b are allocated between 1 and 2. This indifference can be exploited to implement a non-constant allocations in ex-post incentive compatible mechanisms. In Subsection 4.3, we provide a generalization of the second impossibility result by strengthening the incentive requirement to ex-post group incentive compatibility.

Third, the impossibility results without monetary transfers rest crucially on ex-post incentive compatibility. Interdependence of preferences does not preclude efficiency if one relaxes the equilibrium notion to Bayesian Nash equilibrium. Perhaps surprisingly, it is possible to implement a Pareto efficient allocation, even without transfers, via a Bayesian incentive compatible mechanism, when the agents' preferences are sufficiently congruent. We will present this Bayesian possibility result in Section 5.

4 Ex-Post Incentive Compatible Mechanisms

4.1 Setup

Suppose there are n agents and n objects. Let N and O denote the set of agents and objects, respectively. A (pure) **assignment** is a one-to-one mapping μ from N to O, where $\mu^i = a$ means that agent i is assigned object a under assignment μ. Let $\mathcal{M}$ be the set of all assignments. A **random assignment** is a probability distribution over pure assignments. A random assignment $P \in \Delta(\mathcal{M})$ assigns agent $i \in N$ to object $a \in O$ with probability

$$P_a^i = \sum_{\mu \in \mathcal{M}} P(\mu) \mathbf{1}_{\{\mu^i = a\}},$$

where $\mathbf{1}_{\{\mu^i=a\}}$ is the indicator function (whose value is one if $\mu^i = a$ and zero otherwise).[15]

Each agent i receives a private signal $s^i \in S^i$. We denote a profile of signals by $s = (s^1, ..., s^n) \in S \equiv \prod_{i \in N} S^i$. We assume that S^i is a convex subset of $\mathbb{R}^{m^i}$ for some positive integer m^i for each $i \in N$. Agent i has value $v_a^i(s)$ for object a at signal profile s, which is differentiable (thus continuous in particular) in s. Let $\pi : O \to \{1, ..., n\}$ be a function that

[15]Given set X, we denote by $\Delta(X)$ the set of probability distributions over X.

represents ordinal preferences of an individual: An agent with preference π prefers a to b if $\pi_a < \pi_b$ and is indifferent between them if $\pi_a = \pi_b$. For each agent i and signal profile s, agent i's value function v^i induces an associated preference relation $\pi^i(s)$ where $\pi^i_a(s)$ denotes the ranking of object a in preference relation $\pi^i(s)$ induced by the value function $v^i(s) := (v^i_a(s))_{a\in O}$: Formally, $\pi^i_a(s) < \pi^i_b(s)$ if and only if $v^i_a(s) > v^i_b(s)$. A preference relation π^i is said to be **strict** if $\pi^i_a \neq \pi^i_b$ for any pair of objects $a \neq b$. A preference profile $\pi = (\pi^i)_{i\in N}$ is said to be strict if π^i is strict for every $i \in N$.

An assignment μ is **Pareto efficient** at preference profile π if there exists no assignment $\hat{\mu}$ such that $\pi^i_{\hat{\mu}^i} \leq \pi^i_{\mu^i}$ for all $i \in N$, with strict inequality for at least one $i \in N$. A **mechanism** is a mapping $\varphi : S \to \Delta(\mathcal{M})$ from a vector $s \in S$ of signals to a random assignment. A mechanism φ is **Pareto efficient** if, for all $s \in S$, every (pure) assignment in the support of $\varphi(s)$ is Pareto efficient at $\pi(s) = (\pi^i(s))_{i\in N}$.

We now introduce incentive compatibility concepts we shall use. To begin, we say that a random assignment P **first-order stochastically dominates** another random assignment $\hat{P}$ for i at preference π^i if

$$\sum_{b\in O:\pi^i_b\leq\pi^i_a} P^i_b \geq \sum_{b\in O:\pi^i_b\leq\pi^i_a} \hat{P}^i_b,$$

for all $a \in O$. If all these inequalities hold and at least one of them holds strictly, then we say that P **strictly first order stochastically dominates** $\hat{P}$ at π^i. We then say that a mechanism φ is **weakly ex-post incentive compatible** if there exist no agent i, signal profile $s = (s^i, s^{-i}) \in S$, and signal $\bar{s}^i$ for i such that $\varphi(\bar{s}^i, s^{-i})$ strictly first-order stochastically dominates $\varphi(s^i, s^{-i})$ at $\pi^i(s)$. A mechanism is **ex-post incentive compatible** if, for every agent i and signal profile $s = (s^i, s^{-i})$, $\varphi(s^i, s^{-i})$ first-order stochastically dominates $\varphi(\bar{s}^i, s^{-i})$ for all $\bar{s}^i$ at $\pi^i(s)$. Unlike weak ex-post incentive compatibility, ex-post incentive compatibility even eliminates the possibility that an agent's random assignments under true and misreported preferences are incomparable with respect to first-order stochastic dominance. Clearly, ex-post incentive compatibility is a stronger requirement than weak ex-post incentive compatibility.

Remark 1. As is clear, our notions of incentive compatibility are ordinal.[16] The ordinal concept has the additional benefit of being robust to the specific assumptions about agents'

[16] Bogomolnaia and Moulin (2001) define ordinal incentive compatibility concepts in private-values environments. Our concepts reduce to theirs under private values.

attitudes toward risk or uncertainty. One could alternatively define ex-post incentive compatibility based on expected utilities. This alternative concept is weaker than our notion of ex-post incentive compatibility but stronger than weak ex-post incentive compatibility. Note that all these concepts coincide if one restricts attention to deterministic mechanisms as is often done in mechanism design, for instance Jehiel et al. (2006).

4.2 Inefficiency in Weak Ex-Post Implementation

We now present our first impossibility result. To do so, we introduce a few assumptions on the signal space. (In Appendix B.1, we demonstrate that these assumptions are important for establishing our impossibility result.) The first assumption is central to our study: it formalizes the requirement that there be at least some interdependence in agents' valuations.

Assumption 1 (Interdependence)**.** *For any* $i, j \in N$*,* $a, b \in O$ *such that* $a \neq b$*, and* $s \in S$ *such that* $v_a^i(s) = v_b^i(s)$*, there exists* $z^j \in \mathbb{R}^{m^j}$ *with* $\|z^j\| = 1$ *such that* $\nabla_{z^j} v_a^i(s) \neq \nabla_{z^j} v_b^i(s)$*.*[17]

This assumption requires that agent j's signal influences agent i's relative preferences between any pair of objects, at least when agent i is indifferent between these two objects. This condition captures the notion of interdependence. It is worth noting that the condition does not require the value interdependence to be large. As will be seen from Example 1, the condition may hold even with very little interdependence (i.e., almost private values).[18]

The next assumption means that the signal space is sufficiently rich. To state the condition, fix any pair of agents i and j, two objects a and b, and signal profile $s^{-ij} \in S^{-ij}$. Then, for $k, k' \in \{a, b\}$, we define $S_{kk'}^{ij}(s^{-ij}) \subset S^i \times S^j$ to be the (open) set of i and j's signal profiles for which (i) agent i ranks k strictly above $o \in \{a, b\} \setminus \{k\}$, and o strictly above any $k'' \notin \{a, b\}$, (ii) agent j ranks k' strictly above $o \in \{a, b\} \setminus \{k'\}$, and o strictly

[17] Here, $\nabla_{z^j} v_o^i(s), o = a, b$, denotes the directional derivative of the function v_o^i along a given vector $z^j \in \mathbb{R}^{m^j}$ with Euclidean norm $\|z^j\| = 1$ at a given signal profile $s \in S$. To be concrete, $\nabla_{z^j} f(s) := \lim_{h \to 0} \dfrac{f(s^j + hz^j, s^{-j}) - f(s^j, s^{-j})}{h}$ for any $z^j \in \mathbb{R}^{m^j}$ with $\|z^j\| = 1$ such that $s^j + hz^j \in S^j$ for sufficiently small $h > 0$.

[18] In Figure 1, the interdependence is satisfied as long as agent 1's indifference curve has no vertical segment and agent 2's indifference curve has no horizontal segment.

above any $k'' \notin \{a, b\}$, (iii) all other agents rank both a and b strictly below any $k'' \notin \{a, b\}$ (we suppress the dependence of $S^{ij}_{kk'}(s^{-ij})$ on the set of objects $\{a, b\}$ to simplify notation).

Assumption 2 (Rich Domain)**.** *There exist $i, j \in N$, $a, b \in O$, and $s^{-ij} \in S^{-ij}$ such that $S^{ij}_{kk'}(s^{-ij})$ is non-empty for all $k, k' \in \{a, b\}$.*

As suggested by the name, the Rich Domain assumption postulates that the signal structure is rich enough to generate various preference profiles. Specifically, it means that one should be able to find two agents, a signal profile for all other agents, and two objects a and b such that the two prefer a and b to the other objects, the other agents find them the two least preferred, and the two agents find either a or b to be the most preferred depending on their signals. Notice a similarity between this assumption and the Universal Domain assumption—which is often used in the private value models—in that the latter means all preference types are possible. In fact, as shown in Appendix B.3, Rich Domain is implied by Universal Domain if preferences are private.

In order to state the next assumption, fix the two agents i and j, two objects a and b, and signal profile $s^{-ij} \in S^{-ij}$ as before. For $k \in \{a, b\}$, let $S^{ij}_{k\bullet}(s^{-ij}) \subset int(S^i \times S^j)$ denote the set of i and j's interior signal profiles[19] for which (i) and (iii) in the above definition of $S^{ij}_{kk'}(s^{-ij})$ hold but the property (ii) is relaxed to: (ii') agent j ranks a and b strictly above any $k'' \notin \{a, b\}$. That is, $S^{ij}_{k\bullet}(s^{-ij})$ only differs from $S^{ij}_{kk'}(s^{-ij})$ in that agent j's ranking between a and b is unspecified in the former set. Similarly, $S^{ij}_{\bullet k'}(s^{-ij})$ denotes the set of i and j's interior signal profiles for which (ii) and (iii) in the above definition of $S^{ij}_{kk'}(s^{-ij})$ hold but (i) is replaced by a weaker property: (i') agent i ranks a and b strictly above any $k'' \notin \{a, b\}$.

Assumption 3 (Connectedness)**.** *For some $i, j \in N$, $a, b \in O$, and $s^{-ij} \in S^{-ij}$ satisfying the Rich Domain assumption (Assumption 2), and for some $k \in \{a, b\}$, both $S^{ij}_{k\bullet}(s^{-ij})$ and $S^{ij}_{\bullet k}(s^{-ij})$ are connected.*[20]

In our context of Euclidean spaces, connectedness of the open set $S^{ij}_{k\bullet}(s^{-ij})$ means that any two points in that set can be linked by a path contained in that set. Roughly, it means

[19] By taking the interior of $S^i \times S^j$, we are ruling out points on the boundary of $S^i \times S^j$, which will give us some room to perturb signal profiles when needed.

[20] Note that a set is **connected** if it cannot be partitioned into two sets that are open in the relative topology.

that the ordinal preferences vary stably with the changes in signals of agent i and j when others' signals remain fixed. This condition is relatively mild and in particular weaker than the assumption that the set $S^{ij}_{k\cdot}(s^{-ij})$ is convex.

We are now ready to present our first impossibility theorem (all proofs are in the Appendix A).

Theorem 1. *Suppose $n \geq 2$. If Interdependence, Rich Domain, and Connectedness hold, then there exists no mechanism φ that is both Pareto efficient and weakly ex-post incentive compatible.*

The key assumption used for the theorem is Rich Domain. Clearly, this assumption is easier to satisfy when each agent's signal is multidimensional, but multidimensionality of individual signals is not needed for the assumption. In fact, the Rich Domain assumption is satisfied even in fairly natural models with single dimensional signals. This point is illustrated in the following example, which one can see as a natural extension of the two-agent example described in the earlier Section 3.[21]

Example 1 (Canonical one-dimensional signal model). Assume that each agent i has signal $s^i \in [0,1]$. The set of objects is given by $O = \{o_1, ..., o_n\}$. Given signal profile $s \in [0,1]^n$, agent i's utility from object o_k is given by $v^i_{o_k}(s) = \alpha_k w^i(s) + \beta_k$, where $\alpha_k, \beta_k > 0$ and $w^i(s) := \gamma s^i + (1-\gamma)\frac{\sum_{j\neq i} s^j}{n-1}$ with $\gamma \in (\frac{1}{2}, 1)$. Assume that $\beta_{k-1} - \beta_k = \delta$ for some $\delta > 0$ and $\Delta_k := \alpha_k - \alpha_{k-1}$ is positive and strictly decreasing in k for $k = 2, ..., n$. We assume in addition that $\Delta_n\gamma > \delta > \Delta_2(1-\gamma)$.[22] The first (resp. second) inequality, combined with the previous assumptions, implies that if s^i is sufficiently close to 1 (resp. 0), then agent i prefers o_n the most (resp. least) and o_{n-1} the second most (resp. second least) irrespective of the others' signals.[23] This is illustrated in Figure 2 below for the case of 3 agents and

[21] While the utility functions are linear in the example, the linearity assumption is made only for convenience. Our assumptions of Interdependence, Rich Domain, and Connectedness can be seen to hold with nonlinear utility functions which possess the same qualitative features as the linear utility functions described here.

[22] Since $\Delta_2 > \Delta_n$, this assumption requires γ to be sufficiently large. One could think of this as a strengthening of the single crossing property. As long as this assumption is satisfied, we can allow for asymmetric value functions with α^k, β^k, and γ differing across agents.

[23] To see this, we can obtain $v^i_{o_k}(s) - v^i_{o_{k-1}}(s) = \Delta_k(\gamma s^i + (1-\gamma)\frac{\sum_{j\neq i} s^j}{n-1}) - \delta$ and note that this expression is positive (resp. negative) for all k and s^{-i} if $\Delta_n\gamma > \delta$ and $s^i \simeq 1$ (resp. $\Delta_2(1-\gamma) < \delta$ and $s^i \simeq 0$).

3 objects. As can be seen in the figure, for $s^1, s^2 \simeq 1$ and $s^3 \simeq 0$, agents 1 and 2 prefer $o_3 - o_2 - o_1$ in that order, whereas agent 3 prefers $o_1 - o_2 - o_3$ in that order.

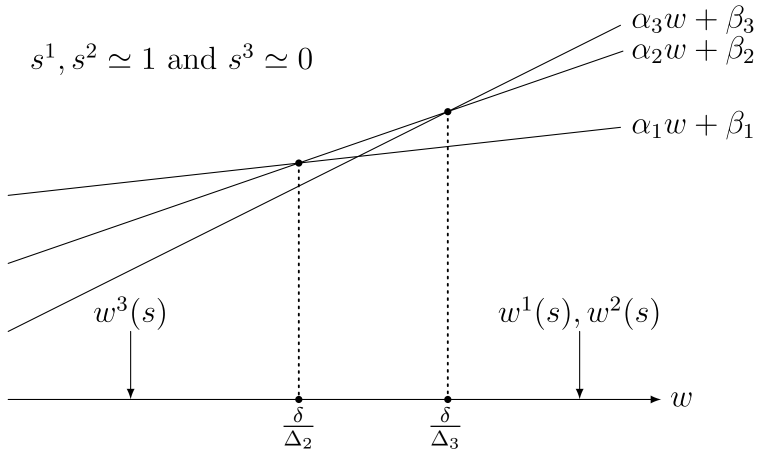

Figure 2: Illustration of Rich Domain Assumption with $n = 3$

Let $a = o_n$ and $b = o_{n-1}$, and fix each s^k, $k \neq i, j$, to be sufficiently close to zero so that a and b are the two least preferred objects for agent $k \neq i, j$, irrespective of i and j's signals. Now, one can find a signal profile $(\hat{s}^i, \hat{s}^j) \in (0, 1)^2$ for agents i and j such that

$$w^i(\hat{s}^i, \hat{s}^j, s^{-ij}) = w^j(\hat{s}^i, \hat{s}^j, s^{-ij}) = \frac{\delta}{\Delta_n},$$

which means that given the signal profile $\hat{s} = (\hat{s}^i, \hat{s}^j, s^{-ij})$, both agents i and j are indifferent between a and b.[24] It is easy to check that this condition also implies that both i and j prefer a and b to all other objects. Then, the Rich Domain assumption is satisfied with small $\varepsilon > 0$ chosen so that $(\hat{s}^i+\varepsilon, \hat{s}^j+\varepsilon) \in S^{ij}_{aa}(s^{-ij})$, $(\hat{s}^i+\varepsilon, \hat{s}^j-\varepsilon) \in S^{ij}_{ab}(s^{-ij})$, $(\hat{s}^i-\varepsilon, \hat{s}^j+\varepsilon) \in S^{ij}_{ba}(s^{-ij})$, and $(\hat{s}^i - \varepsilon, \hat{s}^j - \varepsilon) \in S^{ij}_{bb}(s^{-ij})$.

Further, the Connectedness assumption is satisfied since the linearity of $v^i_{o_k}$ means that $S^{ij}_{k\bullet}(s^{-ij})$ and $S^{ij}_{\bullet k}(s^{-ij})$, each of which is a set defined by finitely many linear inequalities, are convex. Finally, to see that the Interdependence (Assumption 1) condition holds, suppose that agent i is indifferent between objects o_k and o_ℓ where $k > \ell$. It then follows that

[24] To see that such $\hat{s}^i$ and $\hat{s}^j$ exist, first take $s^i = s^j$, which implies $w^i(s^i, s^j, s^{-ij}) = w^j(s^i, s^j, s^{-ij})$ by definition of $w^i(\cdot)$ and $w^j(\cdot)$. Then note that these are smaller than δ/Δ_n for $s^i = s^j = 0$, while the reverse inequality holds for $s^i = s^j = 1$. Since utility functions are continuous, by the mean value theorem there exists a value $\hat{s}^i = \hat{s}^j \in (0, 1)$ of the signals such that the desired equality holds.

$(\alpha_k - \alpha_\ell)w^i(s) = \beta_\ell - \beta_k$. It is easy to see that this tie is broken by a slight change in any agent's signal since $\frac{\partial w^i(s)}{\partial s^j} > 0, \forall i, j$. In particular, the required interdependence $(1-\gamma) > 0$ can be arbitrarily small, in which case the agents' preferences become almost private. This implies that our inefficiency result holds even when the preferences are almost private. By contrast, with private values (i.e. $\gamma = 1$), the efficient assignment is dominant-strategy (and hence ex-post) implementable. This means that there is a discontinuity between private and interdependent values setups.

Remark 2. As stated in Remark 1, our (ordinal) notion of weak ex-post incentive compatibility is weaker than the cardinal notion of ex-post incentive compatibility based on expected utilities. Hence, our inefficiency result continues to hold when one employs the latter concept of incentive compatibility.

4.3 Limits of Ex-Post Group Incentive Compatibility

In this section, we consider joint manipulations by multiple agents, and a mechanism that is robust against such manipulations in the ex-post sense. Formally, we say that a mechanism φ is **manipulable by group** $N' \subset N$ **at** $s \in S$ if there exists a signal profile $\hat{s}^{N'} \in \prod_{i \in N'} S^i$ such that, for all $i \in N'$, $\varphi(s^{N'}, s^{-N'})$ does not strictly first-order stochastically dominate $\varphi(\hat{s}^{N'}, s^{-N'})$ at $\pi^i(s)$, and $\varphi^i(s^{N'}, s^{-N'}) \neq \varphi^i(\hat{s}^{N'}, s^{-N'})$ for at least one $i \in N'$. A mechanism φ is said to be **ex-post group incentive compatible** if it is not manipulable by any group $N' \subset N$ at any $s \in S$. This concept is a strengthening of ex-post incentive compatibility, requiring that the mechanism eliminates profitable misreporting of signals not only by an individual agent, but also by a group of agents. We will show that this strengthening of incentive compatibility leads to an even stronger impossibility result, namely that only a constant allocation can be implemented.

To begin, we say that a mechanism φ is **constant** if $\varphi(s) = \varphi(\hat{s})$ for all $s, \hat{s} \in S$ such that $\pi(s)$ and $\pi(\hat{s})$ are strict preference profiles. Our result is that any ex-post group incentive compatible mechanism must be constant. To obtain this result, we shall invoke again the Interdependence assumption (Assumption 1), and two variants of Assumptions 2 and 3:[25]

[25] It is straightforward, if tedious, to verify that there is no logical relationship between the Rich Domain and Rich Domain* assumptions or between the Connectedness and Connectedness* assumptions. See

Assumption 4 (Rich Domain*). *For any preference profile π, there exists a signal profile $s \in int(S)$ such that $\pi(s) = \pi$.*

The Rich Domain* assumption requires that, given any preference profile, there is an interior signal that induces it.[26]

Assumption 5 (Connectedness*). *For any strict preference profile π, the set $S_\pi := \{s \in S | \pi(s) = \pi\}$ is connected.*

Theorem 2. *Suppose $n \geq 2$. If Interdependence, Rich Domain*, and Connectedness* hold, then any ex-post group incentive compatible mechanism φ is constant.*

To see why group incentive compatibility is needed for this result, recall the example in Section 3 with three agents, 1, 2, and 3, and three objects, a, b, and c. Consider a mechanism that always assigns object c to agent 3, but assigns a and b between agents 1 and 2 in a way that varies only with the signal of agent 3. Such a mechanism is ex-post incentive compatible, but it is not ex-post group incentive compatible since either 1 or 2 will stand to gain from a joint manipulation with agent 3.[27]

While ex-post group incentive compatibility is a strong requirement, the constancy result is not expected from the traditional private value model. Observe that when the values are private, our notion of ex-post group incentive compatibility reduces to group strategy-proofness (see Pápai (2000) and Pycia and Ünver (2009) for instance). This latter requirement is met by a large class of mechanisms that attain efficiency in the private value setting (Pycia and Ünver, 2009). In this regard, the constancy result of Theorem 2 is striking, particularly since it holds even when the preferences are "almost" private.

To obtain the intuition of the proof, let s and $\hat{s}$ be signal profiles at which preferences of all agents are strict. We construct a connected path of states, $s_0 \to s_1 \to .... \to s_m$, such that $s_0 = s, s_m = \hat{s}$, preferences of all agents are strict at each of these states, and

Appendix B.2.

[26]In Appendix B.3, we establish the equivalence between Rich Domain* and Universal Domain in the private value case.

[27]To see why this is the case, first note that by the Rich Domain* condition, there always exists a signal profile $s = (s^1, s^2, s^3)$ such that both agents 1 and 2 prefer a to b. By definition of the mechanism, there exists signal $\hat{s}^3$ such that $\varphi(s^1, s^2, s^3) \neq \varphi(s^1, s^2, \hat{s}^3)$. Since agent 3 receives c with certainty at any signal profile, $\varphi^i_a(s^1, s^2, s^3) < \varphi^i_a(s^1, s^2, \hat{s}^3)$ for an agent $i \in \{1, 2\}$. Thus, agent 3 can report $\hat{s}^3$ to benefit agent i at s, so φ is manipulable by $\{i, 3\}$ at s.

- any adjacent states s_k and s_{k+1} differ in the signal of only one agent, say j_k, and
- the ordinal preferences remain unchanged between s_k and s_{k+1} for all agents except for at most one agent, say i_k, who is different from j_k.

Between two adjacent states s_k and s_{k+1}, the assignment for agent j_k (whose signal varies across those states) cannot change due to ex-post incentive compatibility (since her ordinal preferences are strict and remain unchanged per our construction). Ex-post group incentive compatibility then implies that the assignments for every other agent whose preferences do not change should remain unchanged as well, because otherwise the agent whose assignment changes can profitably manipulate jointly with j_k. It then follows that the assignment for i_k (whose strict preferences vary across the states as described above) must also remain unchanged, since the assignments for all other agents remain unchanged (recall that there exists at most only one agent, i_k, whose preferences vary, by our construction). Thus the entire assignments remain unchanged between s_k and s_{k+1} for each k, and hence between s and $\hat{s}$, which implies the result. The detailed proof is in Appendix A.

Assumptions 1, 4, and 5 enable us to construct a connected path of states with the above desired properties. However, the result can also be obtained even in other cases without these assumptions if such a path can be constructed, as our proof method can be applied to such cases. To see this, recall the canonical one-dimensional signal model in Example 1. The preference specification of this example does not admit a full set of ordinal preferences, so it does not satisfy Rich Domain*.[28] Yet, it can be shown that there exists a connected path required for the proof of the theorem.

Proposition 1. *Any ex-post group incentive compatible mechanism is constant in the Canonical one-dimensional signal model in Example 1.*

Remark 3. The "non-wastefulness" feature of the house allocation model — that all objects are assigned — plays a role in the constancy result of Theorem 2. Without this assumption, it is ex-post group incentive compatible, for instance, to assign agent 1 his most preferred object (which varies across signals) and assign all other agents "no" objects; the agents would then have no incentive to lie about signals individually or jointly. In this sense,

[28] This can be seen easily for the case with $n = 3$. Letting $a := o_3, b := o_2, c := o_1$, the specification admits preference orderings: abc, bac, bca, cba, but it does not admit acb or cab.

Theorem 2 can be rephrased as establishing "constancy" among non-wasteful mechanisms. As can be inferred from the example, though, there is a sense in which the extent of implementable "variation" in allocation is limited even in a wasteful mechanism. If one restricts attention to a deterministic mechanism (one that implements a deterministic allocation for each profile of signals), then only one agent's allocation can change between any two signal profiles.[29]

Remark 4. Ex-post group incentive compatibility is a strong requirement. However, a closer look at the proof reveals that the full force of this condition is *not* needed for the result. More specifically, the only requirement we need is that *no individual or pair of agents* can benefit from misreporting their preferences. In other words, precluding manipulations by groups of arbitrary sizes is not needed. To see this point, simply observe that the proof, as outlined above, applies the condition for only individuals and pairs.

5 The Role of Ex-Post Incentive Compatibility

We show here that the impossibility results of the previous sections depend crucially on our implementation notion, ex-post incentive compatibility. To this end, we introduce two weaker notions of implementation, *approximate ex-post incentive compatibility* and *Bayesian incentive compatibility.* While we do not aim to provide a general analysis using the alternative notions, we demonstrate that in some restricted domains, positive results can be obtained under the relaxed notions.

5.1 Approximate Ex-Post Incentive Compatibility

In this section, we relax the solution concept by allowing incentive constraints to hold only approximately. Under this relaxed notion, we show that non-constant allocation—even approximately efficient allocation—can be implemented if the value interdependence becomes sufficiently small. Recall that our impossibility results in the previous section hold

[29]Roughly speaking, if allocations vary for two agents (call them "inside" agents), even only as a function of the signals of the other agents (call them "outside" agents), there is a scope for the outside agents to jointly manipulate with one of the inside agents to improve his assignment, when both inside agents prefer the same object.

even with vanishingly small value interdependence, which suggests a discontinuity between the private and interdependent value cases since the efficiency can be achieved in the former case. The positive results in this section demonstrate that the apparently counterintuitive discontinuity disappears once the requirement of *exact* ex-post incentive compatibility is relaxed, even if it is still required to hold approximately.

Let us say that the preference of agent i is **ϵ-interdependent** if $|v_a^i(s) - v_a^i(\tilde{s})| \leq \epsilon$ for every $a \in O$ and $s, \tilde{s} \in S$ such that $s^i = \tilde{s}^i$.

We say that a (deterministic) mechanism φ is **ϵ-strategy-proof** if

$$v^i_{\varphi(s^i,\tilde{s}^{-i})}(s) + \epsilon \geq v^i_{\varphi(\tilde{s}^i,\tilde{s}^{-i})}(s)$$

for every $i \in N$, $s^i, \tilde{s}^i \in S^i$, and $s^{-i}, \tilde{s}^{-i} \in S^{-i}$. That is, no agent should be able to misreport his type and make himself better off by more than ϵ against some strategy profile of the other agents. The notion of approximate incentive compatibility can be extended to the group-strategy-proofness: A mechanism φ is **ϵ-group-strategy-proof** if, for any $N' \subset N$ and $s = (s^{N'}, s^{-N'})$, there are no strategy profiles $\tilde{s}^{N'}$ and $\tilde{s}^{-N'}$ such that

$$v^i_{\varphi(\tilde{s}^{N'},\tilde{s}^{-N'})}(s) > v^i_{\varphi(s^{N'},\tilde{s}^{-N'})}(s) + \epsilon, \forall i \in N'.$$

That is, no group of agents should be able to jointly misreport their types and make all of them better off by more than ϵ against some strategy profile of the other agents. Note that ϵ-group-strategy proofness implies ϵ-strategy-proofness.

Let us define the following mechanism and refer to it as the **serial dictatorship**:

1. Let there be an exogenous order over the agents. Without loss of generality, let the order be $1, 2, \ldots, n$. Let there also be an exogenously fixed signal profile $\bar{s}$;

2. Let each agent i report her signal s^i;

3. Iterate the following rounds through $i = 1, \ldots, n$[30]:

   Round i: Agent i obtains the object a^i such that

$$v^i_{a^i}(s^i, \bar{s}^{-i}) = \begin{cases} \max_{b \in O} v^1_b(s^1, \bar{s}^{-1}) & \text{for } i = 1 \\ \max_{b \in O \setminus \{a^1,\ldots,a^{i-1}\}} v^i_b(s^i, \bar{s}^{-i}) & \text{for } i \geq 2. \end{cases} \quad (2)$$

[30] If there are multiple objects satisfying the criterion, then pick one object according to an exogenously specified order. The same remark applies to all subsequent rounds.

This mechanism is the same as the standard serial dictatorship (as defined in the private value setting), except for two differences: First, instead of reporting their own preferences, agents report signals (which may affect other agents' preferences). Second, assignment for each agent i is based on his reported signal s^i and *fixed* signals $\bar{s}^{-i}$ of others (as opposed to their reported signals s^{-i}).[31]

Proposition 2. *Suppose that the payoff function of every agent is ϵ-interdependent. Then any serial dictatorship is 2ϵ-group-strategy-proof and thus 2ϵ-strategy-proof.*

The above result contrasts with the impossibility result in Theorem 2: Observe that the serial dictatorship mechanism need not be constant.[32] Furthermore, we show below that the allocation from the serial dictatorship is Pareto efficient for a large set of signal profiles, which contrasts with the impossibility result in Theorem 1. To this end, we assume **preferences are strict almost everywhere**: That is, $\nu(\overline{S}) = 0$, where ν is the probability measure on the set of signal profiles S, and

$$\overline{S} := \{s \in S \mid v^i_a(s) = v^i_b(s) \text{ for some } i \in N \text{ and some } a \neq b\},$$

is the set of signal profiles at which indifference occurs. Note that this is analogous to the strict preference assumption often made in the private values setup.

Proposition 3. *Suppose that the preference of every agent is strict almost everywhere and ϵ-interdependent. Then, the allocation resulting from the serial dictatorship is Pareto efficient with probability that goes to one as $\epsilon \to 0$.*

Remark 5. The result in Proposition 3 does not rule out the possibility that under the serial dictatorship, very inefficient outcomes arise for signal profiles with a small measure. One may ask if the serial dictatorship also satisfies an alternative notion of approximate efficiency that rules out such a possibility. To answer this, call a mechanism φ **ϵ-efficient** if for any $s \in S$, there does not exist another assignment $\mu \in \mathcal{M}$ such that, for all i such that $\mu^i \neq \varphi^i(s^i, s^{-i})$,

$$v^i_\mu(s^i, s^{-i}) > v^i_{\varphi(s^i,s^{-i})}(s^i, s^{-i}) + \epsilon.$$

[31]This latter feature plays an important role for proving Proposition 2.

[32]This will hold as long as each agent's ranking over the objects varies nontrivially with his own signal.

That is, an ϵ-efficient mechanism achieves approximate Pareto efficiency at *all* signal profiles. The following result shows that the serial dictatorship satisfies this alternative efficiency notion as well.

Proposition 4. *Suppose that the preference of every agent is ϵ-interdependent. Then, the serial dictatorship is 2ϵ-efficient.*

5.2 Bayesian Incentive Compatibility

In this section, we relax the incentive requirement to the Bayesian incentive compatibility. A mechanism φ is **Bayesian incentive compatible** if truth-telling is a Bayesian Nash equilibrium of mechanism φ — namely, for each agent, reporting his true signal maximizes the expected utility, given that the other also reports his true signal. We first consider the 2×2 setup and provide a condition under which Bayesian incentive compatibility allows for Pareto efficiency. We then consider the general $n \times n$ case with ϵ-interdependent preference to show that any (perfect) Bayesian equilibrium of the standard serial dictatorship yields an approximately efficient allocation.

Suppose that there are two objects a and b to be assigned between agents 1 and 2. Each agent i's signal s^i is assumed to be drawn independently of s^j from the interval $[0, 1]$ via cdf F^i. As in Section 3, let $u^i(s) = v_a^i(s) - v_b^i(s)$ and assume $u^i(\cdot)$ to be increasing in both signals and satisfy the single crossing property (1).

We assume that there is a signal profile $(\bar{s}^1, \bar{s}^2)$ at which both agents are indifferent between a and b, i.e., $u^1(\bar{s}^1, \bar{s}^2) = u^2(\bar{s}^1, \bar{s}^2) = 0$. We call them **threshold types**. Note that $(\bar{s}^1, \bar{s}^2)$ corresponds to the intersection of the two indifferent curves given in Figure 1.[33] We say the **threshold types' preferences are congruent in expectation** if either

$$\int_0^1 u^1(\bar{s}^1, s^2) dF^2(s^2) \geq 0 \geq \int_0^1 u^2(s^1, \bar{s}^2) dF^1(s^1) \tag{3}$$

[33] Due to the single crossing condition, there can be at most one intersection. When there is no intersection, one of the two areas S_{ab} and S_{ba} in Figure 1 disappears so there is only one area where Pareto efficiency pins down an allocation. Then, a mechanism that implements that allocation irrespective of signal profiles achieves Pareto efficiency.

or

$$\int_0^1 u^1(\bar{s}^1, s^2) dF^2(s^2) \leq 0 \leq \int_0^1 u^2(s^1, \bar{s}^2) dF^1(s^1). \tag{4}$$

In words, the threshold types' preferences are congruent in expectation if their preferred objects, as evaluated by their expected payoffs, are distinct.

Proposition 5. *Suppose $n = 2$. If the threshold types' preferences are congruent in expectation, then there exists a Bayesian incentive compatible mechanism that is Pareto efficient.*[34]

The next result shows that the congruence condition is not only sufficient but also necessary for efficiency when the mechanism is required to be **ex-post monotonic**: that is, for each $i = 1, 2$, $\varphi_a^i(\cdot, s^j)$ is non-decreasing (in s^i) for all s^j.

Proposition 6. *Suppose $n = 2$. There exists a Bayesian incentive compatible mechanism that is Pareto efficient and ex-post monotonic if and only if the threshold types' preferences are congruent in expectation.*

The above results, together with Theorems 1 and 2, suggest that the two incentive requirements, Bayesian and ex-post incentive compatibilities, entail dramatic differences in what can be implemented at least for the two agent case. While efficient allocations can be implemented by a Bayesian incentive compatible mechanism, only a constant allocation can be implemented if one insists upon the (exact) ex-post incentive compatibility. The difference remains relevant even in an "almost private value" model. For instance, consider a two-agent model in which $u^i(s) = \gamma s^i + (1-\gamma)s^{-i} - 0.5$, for $\gamma \in (1/2, 1)$ and s^i is drawn uniformly from $[0, 1]$. Recall our impossibility results hold under exact ex-post implementation even for $\gamma \approx 1$, exhibiting discontinuity at $\gamma = 1$. By contrast, the congruence assumption is satisfied regardless of $\gamma \in (1/2, 1)$, so an efficient assignment is Bayesian implementable, and there is no discontinuity at $\gamma = 1$.

[34] Proposition 5 can be generalized to the case in which signals are (weakly) positively correlated. To be concrete, the same result can be proven with the assumption that, for each i and $j \neq i$, the conditional cdf $F^i(s^i|s^j)$ is nonincreasing in s^j, i.e. $F^i(\cdot|s^j)$ first-order stochastically dominates $F^i(\cdot|\hat{s}^j)$ for $\hat{s}^j < s^j$. The sufficient conditions in the statement are unchanged, except for replacing each cumulative distribution function $F^i(s^i)$ in the inequalities to a conditional cdf at the threshold type, $F^i(s^i|\bar{s}^j)$.

The positive conclusion for Bayesian implementation can be generalized to the $n \times n$ setup even without the congruence condition, as long as preferences exhibit small interdependence and efficiency is relaxed to approximate efficiency. To establish this result, we consider the **standard serial dictatorship** wherein agents choose objects one at a time according to an exogenous serial order; at the time of choosing, each agent is informed of his own signal and the objects that remain at that time. We employ perfect Bayesian equilibrium, so agents form beliefs about signals of others based on the observed histories and the equilibrium strategies, and they make choices optimally given their beliefs.[35]

Proposition 7. *Suppose that the preference of every agent is strict almost everywhere and ϵ-interdependent. Then, in any perfect Bayesian equilibrium of the standard serial dictatorship, the allocation is Pareto efficient with probability that goes to one as $\epsilon \to 0$.*

Remark 6. While our model is framed in a private good setting, our possibility result also applies to a public decision setting. To see this clearly, consider a binary public decision model in which two agents 1 and 2 decide between two alternatives A and B. The payoff of agent $i = 1, 2$ from A is (normalized to be) zero, and the payoff of i from B is $\nu^i(s^1, s^2)$. Bergemann and Morris (2008) classify the public decision problem as **positive (informational) externality** if both ν^1 and ν^2 are increasing in (s^1, s^2), and **negative (informational) externality** if ν^1 and $-\nu^2$ are increasing in (s^1, s^2). In words, positive externality means that a favorable change in signals for one agent is also favorable for the other, and negative externality means that a favorable change in signals for one agent is unfavorable for the other agent. Li, Rosen and Suen (2001) focus on the positive externality public decision problem and find impossibility of Bayesian-implementing a Pareto efficient outcome via a deterministic and monotonic mechanism. While positive externality arises in many contexts, the negative externality is relevant for some important contexts.[36] One can easily see that the negative externality public decision problem is isomorphic to our

[35]As is clear from the proof, the conclusion of Proposition 7 holds more generally, even when agents have arbitrary beliefs. In particular, the result is robust to the information the mechanism reveals to each agent about the choices of other agents.

[36]For instance, the decision could be an action against global warming, and the agents are different regions, some negatively affected (southern regions subject to extreme climates) and others positively affected (northern regions which will see productivity increase or opening of new shipping routes), and the signals are the information indicating the severity of the global warning; or it could be a decision for a free trade agreement whose success will affect some parties positively (e.g., consumers) and others

private good model, by simply relabeling $u^1(s^1, s^2) = \nu^1(s^1, s^2)$ and $u^2(s^1, s^2) = -\nu^2(s^1, s^2)$ and maintaining the same assumptions. Then, Theorem 5 implies a possibility result for this type of public decision, as long as one does not insist on a deterministic mechanism.

6 Conclusion

We have studied the implication of providing robust incentives in the matching of indivisible objects to agents with interdependent preferences. In contrast to the private value setting where efficiency can be achieved via a strategy-proof mechanism, we have found that a similar robustness requirement in interdependent preferences—namely ex-post incentive compatibility—severely restricts implementable outcomes. Pareto efficiency is shown to be impossible to achieve via any ex-post incentive compatible mechanism, and only a constant allocation can be attained if one also requires ex-post group incentive compatibility.

Our impossibility results differ from many other negative findings in the matching literature both in terms of its robustness and the methodology. First, the impossibility is robust to the degree of preference interdependence; even a vanishing amount of interdependence is sufficient for our results. Second, our methodology differs from a typical argument that "embeds" a negative small-market example in a given (possibly larger) market. In particular, Theorem 2 establishes that assignments are *constant* across *all* generic signals, which cannot be reduced to an analysis on a small subset of the market. Even Theorem 1, which does employ an embedding technique, required a special care in our interdependence setting because every agent's signal affects preferences of everyone else.

Meanwhile, relaxing the incentive requirement raises some hope, as we have shown that it is possible to attain (approximate) Pareto efficiency through approximately strategy-proof or Bayesian mechanisms if agents have almost private preferences or there are only two agents with congruent preferences. While this opens up the possibility of implementing the efficient allocation, we leave it for future research to extend the positive results to more

negatively (e.g., domestic industries), and the signals are the competitiveness of the exporting industry of its trading partner country; or the decision could be a national health plan such as Obamacare whose successful implementation may be favored by one party (e.g., democrats and the currently uninsured) and disfavored by the other (e.g., republicans and the currently insured), and the signals could be health needs by the currently uninsured.

general setup.

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Appendix A: Proofs

In the proof of Theorem 1, we will invoke the following mathematical result.

Lemma 1 (Proposition 2.10, Stewart and Tall (1983))**.** *Suppose X is an open and connected subset of a (multi-dimensional) Euclidean space. Then X is "step-connected" in the following sense: For any $x, x' \in X$, there exists a sequence $x_0, x_1, \ldots, x_m$ in X such that*

1. *For each $k = 0, \ldots, m-1$, there exists $i(k) \in \{1, \ldots, n\}$ such that $x_k^{-i(k)} = x_{k+1}^{-i(k)}$;*[37]

2. *$\overline{x_k x_{k+1}} \subset X, \forall k = 0, \ldots, m-1$ with $x_0 = x$ and $x_m = x'$, where $\overline{x_k x_{k+1}}$ denotes the line segment connecting x_k and x_{k+1} (including the end points).*

The set (or path) $\bigcup_{k=0}^{m-1} \overline{x_k x_{k+1}}$ in the above Lemma will be referred to as a **step-wise path between** x **and** x'.

Proof of Theorem 1. Assume for contradiction that φ is Pareto efficient and weakly ex-post incentive compatible. Take any $i, j \in N$, $a, b \in O$, and $s^{-ij} \in S^{-ij}$ that satisfy the Rich Domain and Connectedness assumptions. Suppose without loss that the Connectedness assumption is satisfied with $k = a$. Note that $S^{ij}_{a\cdot}(s^{-ij})$ and $S^{ij}_{\cdot a}(s^{-ij})$ are step-connected since they are connected and open. We first observe that there exists a signal profile $\hat{s}^{ij} = (\hat{s}^i, \hat{s}^j) \in S^{ij}_{\cdot a}(s^{-ij})$ such that given $\hat{s} := (\hat{s}^{ij}, s^{-ij})$, for any $c \neq a, b$,

$$\begin{aligned} v^i_a(\hat{s}) = v^i_b(\hat{s}) > v^i_c(\hat{s}),\ v^j_a(\hat{s}) > v^j_b(\hat{s}) > v^j_c(\hat{s}),\ \textit{and} \\ v^k_c(\hat{s}) > \max\{v^k_a(\hat{s}), v^k_b(\hat{s})\}, \forall k \neq i, j. \end{aligned} \tag{5}$$

To see this, use the Rich Domain assumption to choose any $r^{ij} = (r^i, r^j) \in S^{ij}_{aa}(s^{-ij})$ and $t^{ij} = (t^i, t^j) \in S^{ij}_{ba}(s^{-ij})$. Then, by the Connectedness assumption, there must be some continuous path between r^{ij} and t^{ij} that is contained in $S^{ij}_{\cdot a}(s^{-ij})$. Given the continuity of that path and value functions, we must have some signal profile $\hat{s}^{ij} \in S^{ij}_{\cdot a}(s^{-ij})$ such that $v^i_a(\hat{s}^{ij}, s^{-ij}) = v^i_b(\hat{s}^{ij}, s^{-ij})$ by intermediate value theorem. Clearly, $\hat{s} = (\hat{s}^{ij}, s^{-ij})$ satisfies the desired preference relationship.

By the Interdependence assumption and relation (5), there exists $z^j \in \mathbb{R}^{m^j}$ such that $(\hat{s}^i, \hat{s}^j - z^j) \in S^{ij}_{ba}(s^{-ij})$ and $(\hat{s}^i, \hat{s}^j + z^j) \in S^{ij}_{aa}(s^{-ij})$. Since φ is Pareto efficient, $\varphi^j(\hat{s}^i, \hat{s}^j -$

[37]Note that $x^{-i} = (x^j)_{j \neq i}$ denotes the components of vector x except for its i'th components.

$z^j, s^{-ij}) = a$.[38] Since j strictly prefers a most at both $\pi^j(\hat{s}^i, \hat{s}^j - z^j, s^{-ij})$ and $\pi^j(\hat{s}^i, \hat{s}^j + z^j, s^{-ij})$ and φ is weakly ex-post incentive compatible, we must have $\varphi^j(\hat{s}^i, \hat{s}^j + z^j, s^{-ij}) = a$. Again since φ is Pareto efficient, $\varphi^i(\hat{s}^i, \hat{s}^j + z^j, s^{-ij}) = b$. Therefore we conclude that there exists a signal profile $\tilde{s}^{ij} = (\tilde{s}^i, \tilde{s}^j) \in S^{ij}_{aa}(s^{-ij})$ such that $\varphi^i(\tilde{s}^{ij}, s^{-ij}) = b$ and $\varphi^j(\tilde{s}^{ij}, s^{-ij}) = a$ (simply define $\tilde{s}^i = \hat{s}^i$, $\tilde{s}^j = \hat{s}^j + z^j$).

Consider now any profile $\check{s}^{ij} \in S^{ij}_{ab}(s^{-ij}) \subset S^{ij}_{a\bullet}(s^{-ij})$. By Lemma 1, the connectedness of $S^{ij}_{a\bullet}(s^{-ij})$, and the fact that $\tilde{s}^{ij} \in S^{ij}_{aa}(s^{-ij}) \subset S^{ij}_{a\bullet}(s^{-ij})$, one can find a step-wise path $\bigcup_{k=0}^{m-1} \overline{s^{ij}_k s^{ij}_{k+1}} \subset S^{ij}_{a\bullet}(s^{-ij})$ between $\check{s}^{ij} = s^{ij}_0$ and $\tilde{s}^{ij} = s^{ij}_m$. Since this path as well as value functions are continuous, one can find some signal profile $\bar{s}^{ij} = (\bar{s}^i, \bar{s}^j) \in \overline{s^{ij}_\ell s^{ij}_{\ell+1}}$ for some ℓ such that (1) $v^j_a(\bar{s}^{ij}, s^{-ij}) = v^j_b(\bar{s}^{ij}, s^{-ij})$ and (2) $s^{ij} \in S^{ij}_{aa}(s^{-ij})$ for all profiles $s^{ij} \neq \bar{s}^{ij}$ on the (sub)step-wise path between $\bar{s}^{ij}$ and $\tilde{s}^{ij}$. (That is, $\bar{s}^{ij}$ is the last point on the step-wise path going from $\check{s}^{ij}$ to $\tilde{s}^{ij}$ at which agent j is indifferent between a and b.) One can also choose ℓ so that $\bar{s}^{ij} \neq s^{ij}_{\ell+1}$. We then prove the following claim:

Claim 1. *$\varphi^i(s^{ij}_k, s^{-ij}) = b$ and $\varphi^j(s^{ij}_k, s^{-ij}) = a$ for all $k = \ell + 1, \cdots, m$.*

Proof. Note first that the following is true:

1. $\varphi^i(s^{ij}_m, s^{-ij}) = \varphi^i(\tilde{s}^{ij}, s^{-ij}) = b$ and $\varphi^j(s^{ij}_m, s^{-ij}) = \varphi^j(\tilde{s}^{ij}, s^{-ij}) = a$,
2. For each $k = \ell + 1, \ldots, m - 1$, either $s^i_k = s^i_{k+1}$ or $s^j_k = s^j_{k+1}$,[39] and
3. $(s^i_k, s^j_k) \in S^{ij}_{aa}(s^{-ij})$ for $k = \ell + 1, \ldots, m$.

For each $k = \ell + 1, \ldots, m - 1$, by items 2 and 3 above and ex-post weak incentive compatibility of φ, $\varphi^i(s^i_k, s^j_k, s^{-ij}) = \varphi^i(s^i_{k+1}, s^j_{k+1}, s^{-ij})$ if $s^i_k \neq s^i_{k+1}$ and $\varphi^j(s^i_k, s^j_k, s^{-ij}) = \varphi^j(s^i_{k+1}, s^j_{k+1}, s^{-ij})$ if $s^j_k \neq s^j_{k+1}$. In either case, this and the Pareto efficiency of φ imply $\varphi^h(s^i_k, s^j_k, s^{-ij}) = \varphi^h(s^i_{k+1}, s^j_{k+1}, s^{-ij})$ for $h = i, j$, which, combined with item 1 above, gives us the desired result. □

Now, to establish the desired contradiction, we consider two cases: (i) $\bar{s}^j = s^j_{\ell+1}$; (ii) $\bar{s}^i = s^i_{\ell+1}$.

[38] We write $P^j = a$ for a degenerate random assignment such that $P^j_a = 1$.

[39] This is because, by Lemma 1, we can take the sequence in such a way that s_k and s_{k+1} differ only in one dimension, and hence in one agent's signal.

Case (i): Let $\Delta s^i := s^i_{\ell+1} - \bar{s}^i$. Then, for all $\varepsilon \in (0,1]$, $(\bar{s}^i + \varepsilon\Delta s^i, \bar{s}^j) = (\bar{s}^i + \varepsilon\Delta s^i, s^j_{\ell+1}) \in \overline{\bar{s}^{ij} s^{ij}_{\ell+1}} \subset S^{ij}_{aa}(s^{-ij})$. With small enough $\varepsilon > 0$, it also holds that $(\bar{s}^i - \varepsilon\Delta s^i, s^j_{\ell+1}) \in S^{ij}_{ab}(s^{-ij})$. Note that $\varphi^i(s^{ij}_{\ell+1}, s^{-ij}) = b$ by the above Claim 1, and observe that since agent i likes a most at both $\pi^i(\bar{s}^i + \varepsilon\Delta s^i, s^j_{\ell+1}, s^{-ij})$ and $\pi^i(s^i_{\ell+1}, s^j_{\ell+1}, s^{-ij})$, the weak ex-post incentive compatibility and Pareto efficiency of φ imply $\varphi^i(\bar{s}^i + \varepsilon\Delta s^i, s^j_{\ell+1}, s^{-ij}) = \varphi^i(s^i_{\ell+1}, s^j_{\ell+1}, s^{-ij}) = b$. Similarly, the weak ex-post incentive compatibility for agent i also implies $\varphi^i(\bar{s}^i - \varepsilon\Delta s^i, s^j_{\ell+1}, s^{-ij}) = b$, which means the inefficiency arises since $(\bar{s}^i - \varepsilon\Delta s^i, s^j_{\ell+1}) \in S^{ij}_{ab}(s^{-ij})$.

Case (ii): By the Interdependence assumption and the fact in (1) above that $v^j_a(\bar{s}^{ij}, s^{-ij}) = v^j_b(\bar{s}^{ij}, s^{-ij})$, one can find $z^i \in \mathbb{R}^{m^i}$ such that $(\bar{s}^i + \varepsilon z^i, \bar{s}^j) \in S^{ij}_{aa}(s^{-ij})$ and $(\bar{s}^i - \varepsilon z^i, \bar{s}^j) \in S^{ij}_{ab}(s^{-ij})$ for all small enough $\varepsilon > 0$. The Pareto efficiency then implies that $\varphi^i(\bar{s}^i - \varepsilon z^i, \bar{s}^j, s^{-ij}) = a$, which, in turn, implies $\varphi^i(\bar{s}^i + \varepsilon z^i, \bar{s}^j, s^{-ij}) = a$ due to the weak ex-post incentive compatibility for agent i with $\bar{s}^i + \varepsilon z^i$. Thus, by the Pareto efficiency,

$$\varphi^j(\bar{s}^i + \varepsilon z^i, \bar{s}^j, s^{-ij}) = b. \tag{6}$$

With small enough ε, we also have $(\bar{s}^i + \varepsilon z^i, s^j_{\ell+1}) \in S^{ij}_{aa}(s^{-ij})$ since $(\bar{s}^i, s^j_{\ell+1}) = (s^i_{\ell+1}, s^j_{\ell+1}) \in S^{ij}_{aa}(s^{-ij})$. Then, since agent i likes a most at both $\pi^i(\bar{s}^i + \varepsilon z^i, s^j_{\ell+1}, s^{-ij})$ and $\pi^i(s^i_{\ell+1}, s^j_{\ell+1}, s^{-ij})$, the weak ex-post incentive compatibility and Pareto efficiency of φ imply $\varphi^i(\bar{s}^i + \varepsilon z^i, s^j_{\ell+1}, s^{-ij}) = \varphi^i(s^i_{\ell+1}, s^j_{\ell+1}, s^{-ij}) = b$, which, in turn, implies $\varphi^j(\bar{s}^i + \varepsilon z^i, s^j_{\ell+1}, s^{-ij}) = a$ by the Pareto efficiency of φ. Given this and (6), agent j with $\bar{s}^j$ would prefer (mis)reporting $s^j_{\ell+1}$ to obtain a rather than b when others' type profile is $(\bar{s}^i + \varepsilon z^i, s^{-ij})$. $\square$

Proof of Theorem 2. Consider two signal profiles $s, \hat{s} \in S$ such that $\pi(s)$ and $\pi(\hat{s})$ are strict. We will assume $s, \hat{s} \in int(S)$ and show $\varphi(s) = \varphi(\hat{s})$. Later we will extend our argument to signals on the boundary.

Consider a sequence of strict preference profiles $\pi_0, \pi_1, \ldots, \pi_m$ and a sequence of non-strict preference profiles $\tilde{\pi}_0, \tilde{\pi}_1, \ldots, \tilde{\pi}_{m-1}$[40] such that

1. $\pi_0 = \pi(s)$, $\pi_m = \pi(\hat{s})$,

2. For each $k = 0, \ldots, m-1$, there exists $i_k \in N$ such that $\pi_k^{-i_k} = \tilde{\pi}_k^{-i_k} = \pi_{k+1}^{-i_k}$ and $\pi_k^{i_k}$, $\tilde{\pi}_k^{i_k}$, and $\pi_{k+1}^{i_k}$ are "adjacent" with one another where $\pi_k^{i_k} \neq \pi_{k+1}^{i_k}$ are strict while $\tilde{\pi}_k^{i_k}$ is non-strict. That is, there exist $a_k, b_k \in O$ such that $a_k \neq b_k$ whose rankings are (equal

[40] With abuse of notation, the subscript here does not specify objects (as in π_a and π_b) as before.

or) next to each other at $\pi_k^{i_k}$, $\tilde{\pi}_k^{i_k}$, and $\pi_{k+1}^{i_k}$ such that $\pi_k^{i_k}$, $\tilde{\pi}_k^{i_k}$, and $\pi_{k+1}^{i_k}$ are different only in the ranking between a_k and b_k and $\pi_{k,a_k}^{i_k} < \pi_{k,b_k}^{i_k}, \tilde{\pi}_{k,a_k}^{i_k} = \tilde{\pi}_{k,b_k}^{i_k}, \pi_{k+1,a_k}^{i_k} > \pi_{k+1,b_k}^{i_k}$.[41]

By the Rich Domain* assumption, there exists a sequence of signal profiles $s_0, s_1, \ldots, s_{m-1} \in int(S)$ such that $\pi(s_k) = \tilde{\pi}_k$ for each k. Take a sequence of agents $j_0, j_1, \ldots, j_{m-1}$ such that $j_k \neq i_k$ for each k (such agents exist because $|N| \geq 2$). By the Interdependence assumption and the fact that $s_0, s_1, \ldots, s_{m-1} \in int(S)$, there exist signals $(\tilde{s}_{k-}^{j_k}, \tilde{s}_{k+}^{j_k})_{k=0}^{m-1}$ such that $\pi(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) = \pi_k$ and $\pi(\tilde{s}_{k+}^{j_k}, s_k^{-j_k}) = \pi_{k+1}$.[42]

Claim 2. *For each $k = 0, 1, \ldots, m-1$,*

$$\varphi(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) = \varphi(\tilde{s}_{k+}^{j_k}, s_k^{-j_k}).$$

Proof. First note, by construction, that $\pi^{j_k}(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) = \pi^{j_k}(\tilde{s}_{k+}^{j_k}, s_k^{-j_k}) = \pi_k^{j_k}$ and this preference is strict. These facts as well as (group) ex-post incentive compatibility of φ imply that

$$\sum_{b:\pi_{k,b}^{j_k} \leq \pi_{k,a}^{j_k}} \varphi_b^{j_k}(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) = \sum_{b:\pi_{k,b}^{j_k} \leq \pi_{k,a}^{j_k}} \varphi_b^{j_k}(\tilde{s}_{k+}^{j_k}, s_k^{-j_k}),$$

for each $a \in O$ (otherwise, group ex-post incentive compatibility of φ is violated at either $(\tilde{s}_{k-}^{j_k}, s_k^{-j_k})$ or $(\tilde{s}_{k+}^{j_k}, s_k^{-j_k})$ by a singleton "group" j_k). These equalities imply

$$\varphi^{j_k}(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) = \varphi^{j_k}(\tilde{s}_{k+}^{j_k}, s_k^{-j_k}). \tag{7}$$

To show $\varphi(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) = \varphi(\tilde{s}_{k+}^{j_k}, s_k^{-j_k})$ suppose, for contradiction, that

$$\varphi^{j}(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) \neq \varphi^{j}(\tilde{s}_{k+}^{j_k}, s_k^{-j_k}), \tag{8}$$

for some $j \in N$. By equality (7), $j \neq j_k$. If inequality (8) holds for i_k, then, by equality (7) and the assumption that each of the n objects are assigned to exactly one of the n agents, there is another agent $j \neq i_k$ for whom inequality (8) holds. Thus we can assume $j \neq i_k, j_k$ without loss of generality. Since $\pi^{j}(\tilde{s}_{k-}^{j_k}, s_k^{-j_k}) = \pi^{j}(\tilde{s}_{k+}^{j_k}, s_k^{-j_k}) = \pi_k^{j}$ and this preference is

[41] For any agent i, index k, and object a, $\pi_{k,a}^{i}$ denotes the ranking of object a at preference π_k^{i}.

[42] The argument is similar to the one for constructing $\tilde{s}^{i}$ in the proof of Theorem 1.

strict for any such j by assumption, inequality (8) implies that there exists an object $a \in O$ such that

$$\sum_{b:\pi^j_{k,b} \leq \pi^j_{k,a}} \varphi^j_b(\tilde{s}^{j_k}_{k-}, s^{-j_k}_k) > \sum_{b:\pi^j_{k,b} \leq \pi^j_{k,a}} \varphi^j_b(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k), \text{ or}$$
$$\sum_{b:\pi^j_{k,b} \leq \pi^j_{k,a}} \varphi^j_b(\tilde{s}^{j_k}_{k-}, s^{-j_k}_k) < \sum_{b:\pi^j_{k,b} \leq \pi^j_{k,a}} \varphi^j_b(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k).$$

In the former case, φ is manipulable by $N' = \{j_k, j\}$ at $(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k)$ since $\varphi(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k)$ does not strictly first-order stochastically dominate $\varphi(\tilde{s}^{j_k}_{k-}, s^{-j_k}_k)$ for j_k, j and $\varphi^j(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k) \neq \varphi^j(\tilde{s}^{j_k}_{k-}, s^{-j_k}_k)$. In the latter case, φ is manipulable by $N' = \{j_k, j\}$ at $(\tilde{s}^{j_k}_{k-}, s^{-j_k}_k)$ since $\varphi(\tilde{s}^{j_k}_{k-}, s^{-j_k}_k)$ does not strictly first-order stochastically dominate $\varphi(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k)$ for j_k, j and $\varphi^j(\tilde{s}^{j_k}_{k-}, s^{-j_k}_k) \neq \varphi^j(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k)$. Therefore, φ is not ex-post group incentive compatible. □

Claim 3. *For each k,*

$$\varphi(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k) = \varphi(\tilde{s}^{j_{k+1}}_{(k+1)-}, s^{-j_{k+1}}_{k+1}).$$

Proof. First note that by construction of the signals,

$$\pi(\tilde{s}^{j_k}_{k+}, s^{-j_k}_k) = \pi(\tilde{s}^{j_{k+1}}_{(k+1)-}, s^{-j_{k+1}}_{k+1}) = \pi_{k+1}.$$

Also observe that $S_{\pi_{k+1}}$ is open because π_{k+1} is a strict preference profile and utility functions are continuous in signal profiles, and connected by the Connectedness* assumption. Thus, by Lemma 1, there exists a sequence $s(0), s(1), \ldots, s(\bar{l}) \in S_{\pi_{k+1}}$ and $i(0), i(1), \ldots, i(\bar{l}-1) \in N$ such that

1. $s(0) = (\tilde{s}^{j_k}_{k+}, s^{-j_k}_k), s(\bar{l}) = (\tilde{s}^{j_{k+1}}_{(k+1)-}, s^{-j_{k+1}}_{k+1})$,
2. $s(l)^{-i(l)} = s(l+1)^{-i(l)}$ for each $l \in \{0, 1, \ldots, \bar{l}-1\}$.

For each l, since $\pi(s(l)) = \pi_{k+1}$ is a strict preference profile and φ satisfies (group) ex-post incentive compatibility, an argument analogous to what lead to relation (7) in the proof of Claim 2 implies

$$\varphi^{i(l)}(s(l)) = \varphi^{i(l)}(s(l+1)). \tag{9}$$

Since φ satisfies group ex-post incentive compatibility and π_{k+1} is a strict preference profile, this implies $\varphi(s(l)) = \varphi(s(l+1))$ for each l by an argument similar to the last part of the proof of Claim 2. Thus $\varphi(s(0)) = \varphi(s(\bar{l}))$, completing the proof. □

To complete the proof of the Theorem, observe that Claims 2 and 3 imply that

$$\varphi(\tilde{s}_{0-}^{j_0}, s_0^{-j_0}) = \varphi(\tilde{s}_{(m-1)+}^{j_{m-1}}, s_{m-1}^{-j_{m-1}}). \tag{10}$$

By arguments identical to the proof of Claim 3,

$$\varphi(s) = \varphi(s_{0-}^{j_0}, s_0^{-j_0}), \tag{11}$$

$$\varphi(\hat{s}) = \varphi(s_{(m-1)+}^{j_{m-1}}, s_{m-1}^{-j_{m-1}}). \tag{12}$$

Relations (10)-(12) imply $\varphi(s) = \varphi(\hat{s})$.

Consider now a signal profile s on the boundary, i.e. $s \in S \backslash int(S)$, that is associated with strict preference $\pi(s)$. Choose any i for whom $s^i \in S^i \backslash int(S^i)$.

Claim 4. *There exists a signal profile $\tilde{s}$ such that $\tilde{s}^i \in int(S^i)$, $\tilde{s}^{-i} = s^{-i}$, and $\pi(s) = \pi(\tilde{s})$.*

Proof. Let $\hat{s}$ be a signal such that $\hat{s}^i \in int(S^i)$ and $\hat{s}^{-i} = s^{-i}$. Because S^i is a convex set, the agents' utility functions are continuous, and $\pi(s)$ is a strict preference profile, there exists $\lambda \in (0, 1)$ such that $\tilde{s} := \lambda\hat{s} + (1-\lambda)s$ is in S and $\pi(\tilde{s}) = \pi(s)$. Note that $\tilde{s}^{-i} = s^{-i}$ by definition of $\tilde{s}$. To show that $\tilde{s}^i \in int(S^i)$, first note that there exists $\varepsilon > 0$ such that, for any $\bar{s}^i \in \mathbb{R}^{m^i}$, $\|\bar{s}^i - \hat{s}^i\| < \varepsilon \Rightarrow \bar{s}^i \in S^i$ because $\hat{s}^i \in int(S^i)$ by assumption. This fact and convexity of S^i imply that there exists $\varepsilon' > 0$ such that, for any $\bar{s}^i \in \mathbb{R}^{m^i}$, $\|\bar{s}^i - \tilde{s}^i\| < \varepsilon' \Rightarrow \bar{s}^i \in S^i$.[43] This means that $\tilde{s}^i \in int(S^i)$, completing the proof. $\square$

Let $\tilde{s}$ be a signal profile $\tilde{s}$ such that $\tilde{s}^i \in int(S^i)$, $\tilde{s}^{-i} = s^{-i}$, and $\pi(s) = \pi(\tilde{s})$: such a signal $\tilde{s}$ exists by Claim 4. Then, a proof similar to that in Claim 1 above can be used to show $\varphi(s) = \varphi(\tilde{s})$. Repeating this argument for each i whose signal s^i is on the boundary, we can establish that $\varphi(s) = \varphi(\tilde{s}) = \cdots = \varphi(\hat{s})$ for some $\hat{s} \in int(S)$, which completes the proof. $\square$

Proof of Proposition 1. First we write $v_{o_k}^i(s) = \alpha_k w^i(s) + \beta_k$, where $w^i(s) := \gamma s^i + (1-\gamma)\frac{\sum_{j \neq i} s^j}{n-1}$. Fix any $s \in S_\pi$ and $\hat{s} \in S_{\hat{\pi}}$. We show that there exists a step-wise path of the desired form between the two points. We show these in two steps.

[43]The claim holds for $\varepsilon' = \lambda\varepsilon$ for instance. To see this, let $\bar{s}^i$ be a point in $\mathbb{R}^{m^i}$ such that $\|\bar{s}^i - \tilde{s}^i\| < \varepsilon'$. Let $\check{s}^i \in \mathbb{R}^{m^i}$ be defined as $\check{s}^i := \frac{\bar{s}^i - (1-\lambda)s^i}{\lambda}$. Because $\hat{s}^i = \frac{\tilde{s}^i - (1-\lambda)s^i}{\lambda}$ by definition of $\tilde{s}^i$, it follows that $\|\check{s}^i - \hat{s}^i\| = \|\bar{s}^i - \tilde{s}^i\|/\lambda < \frac{\varepsilon'}{\lambda} = \varepsilon$, implying that $\check{s}^i \in S^i$. Because $\bar{s}^i$ can be expressed as a convex combination $\bar{s}^i = \lambda\check{s}^i + (1-\lambda)s^i$, this and convexity of S^i imply $\bar{s}^i \in S^i$.

Claim 5. *Fix any $s \in S_\pi$ and $\hat{s} \in S_{\hat{\pi}}$ for strict preference profiles π and $\hat{\pi}$. There exists a continuous path $\sigma : [0,1] \to S$ with $\sigma(0) = s$ and $\sigma(1) = \hat{s}$ such that $\sigma(t) \in S_{\pi'}$ for a strict preference profile π' for all $t \in [0,1]$, except possibly for finite values, $\{t_1, ..., t_K\}$, and for each value $t \in \{t_1, ..., t_K\}$, $\sigma(t) \in S_{\pi''}$ for π'' that is strict for all agents except for one. (In words, there exists a continuous path connecting s and $\hat{s}$ that crosses an indifference curve of at most one agent at a time and only finitely many times.)*

Proof. For each agent i, there exists only a finite number of w^i's between $w^i(s)$ and $w^i(\hat{s})$ such that $\alpha_k w^i + \beta_k = \alpha_l w^i + \beta_l$ for some $k \neq l$. This means that for any open neighborhood W containing $w(\hat{s}) := (w^1(\hat{s}), ..., w^n(\hat{s}))$, the set

$$W' := \left\{ w \in W \middle| \exists i, j \in N, i \neq j, \exists k, l, k', l' \in O, k \neq l, k' \neq l', \text{ and } \exists t \in [0,1] \text{ such that } (\alpha_k - \alpha_l)(tw^i + (1-t)w^i(s)) = \beta_l - \beta_k \text{ and } (\alpha_{k'} - \alpha_{l'})(tw^j + (1-t)w^j(s)) = \beta_{l'} - \beta_{k'} \right\}$$

is a lower-dimensional subset of W.[44] In particular, $W \setminus W'$ is nonempty.

Observe next that the gradient matrix $\nabla w(s) := [\nabla_{s^j} w^i]$ has $\gamma > 1/2$ on the diagonal entries and $\frac{1-\gamma}{n-1}$ on the off diagonal entries, so has a full rank. Hence, for an open neighborhood $U \subset S_{\hat{\pi}}$ containing $\hat{s}$, there exists an open neighborhood W containing $w(\hat{s}) := (w^1(\hat{s}), ..., w^n(\hat{s}))$ such that for each $w \in W$ there exists $\tilde{s} \in U$ with $w = w(\tilde{s}) = (w^1(\tilde{s}), ..., w^n(\tilde{s}))$. In particular, one can choose $\tilde{s}$ with $w(\tilde{s}) = w$ for some $w \in W \setminus W'$. The path $\tilde{\sigma}(t) = t\tilde{s} + (1-t)s$ then crosses at most one agent's indifference surface at a given t, and there can be only a finite number of such t's in $[0,1]$. By construction, the path $\hat{\sigma}(t) = t\hat{s} + (1-t)\tilde{s}$ stays inside $S_{\hat{\pi}}$ since the latter set is convex (and hence crosses no agent's indifference surface). Finally, a path $\sigma(t) := \tilde{\sigma}(2t)$ for $t \in [0, \frac{1}{2}]$ and $\sigma(t) := \hat{\sigma}(2t-1)$ for $t \in (\frac{1}{2}, 1]$ satisfies the requirement. □

Claim 6. *There exists a step-wise path of the required form connecting s and $\hat{s}$.*

[44] The fact that W' is a lower-dimensional subset of W can be seen because

$$W' = \bigcup_{i \neq j, k \neq l, k' \neq l'} \left\{ w \in W \middle| \frac{(\beta_l - \beta_k) - (\alpha_k - \alpha_l)w^i(s)}{(\alpha_k - \alpha_l)(w^i - w^i(s))} = \frac{(\beta_{l'} - \beta_{k'}) - (\alpha_{k'} - \alpha_{l'})w^j(s)}{(\alpha_{k'} - \alpha_{l'})(w^j - w^j(s))} \in [0,1] \right\}$$

and the latter set is clearly lower-dimensional.

Proof. By Claim 5, there exists a continuous path $\sigma(t)$ with $\sigma(0) = s$ and $\sigma(1) = \hat{s}$ such that σ crosses at most one agent's indifference surface at a given t, only for finitely many t's: $0 < t_1 < ... < t_{K-1} < 1$ for some positive integer K. Let $t_0 \equiv 0$ and $t_K \equiv 1$. Let agent $i_k \in N$ be indifferent over at least a pair of objects at t_k, $0 < k < K$, and let π_k be such that $\sigma(t) \in S_{\pi_k}$ for all $t \in (t_{k-1}, t_k)$. Then, since the specified utility function satisfies the Interdependence assumption, for such k there exists $j_k \neq i_k$ such that $s_{k-} \equiv \sigma(t_k) - \varepsilon e^{j_k} \in S_{\pi_k}$ and $s_{k+} \equiv \sigma(t_k) + \varepsilon e^{j_k} \in S_{\pi_{k+1}}$ for a (positive or negative) real number ε with a sufficiently small absolute value, where e^j is a vector whose component corresponding to j equals one and all other components equal zero. For ε with any sufficiently small absolute value, any signal on the line segment between s_{k-} and s_{k+} gives rise to the same strict preference for all agents, except for agent i_k whose preferences change from $\pi_k^{i_k}$ to $\pi_{k+1}^{i_k}$ as one moves from s_{k-} to s_{k+} along that line segment. Since a set S_{π_k} is an open connected set, by Lemma 1, there exists a step-wise path $\bar{\sigma}_k$ connecting $s_{(k-1)+}$ and s_{k-}, which varies in one agent's signal on each segment and lies within S_{π_k} (where $s_{0+} \equiv s$ and $s_{K-} \equiv \hat{s}$). Piecing together $\bar{\sigma}_k$ with the line segment $\overline{s_{k-}s_{k+}}$, for each $k = 1, ..., K-1$ and finally connecting with $\bar{\sigma}_K$, we construct a step-wise path of the required form. □

The above claims prove Proposition 1. □

Proof of Proposition 2. Let φ be a serial dictatorship with respect to agent order $1, 2, \ldots, n$ and fixed signal profile $\bar{s}$. Let s be the true signal profile. Consider any set $N' \subset N$ of agents who misreport their signals as $\tilde{s}^{N'}$. Let $\tilde{s}^{-N'}$ denote any signal profile reported by the agents outside N'. For each $j \in N$, we denote $a^j = \varphi^j(s^{N'}, \tilde{s}^{-N'})$ and $\tilde{a}^j = \varphi^j(\tilde{s}^{N'}, \tilde{s}^{-N'})$. Letting $i = \min N'$ (i.e., an agent in N' with the earliest order), note that by definition of φ, we must have $a^j = \tilde{a}^j, \forall j < i$. Thus, we must have

$$v^i_{\tilde{a}^i}(s^i, \bar{s}^{-i}) \leq \max_{b \in O \setminus \{a^1, \ldots, a^{i-1}\}} v^i_b(s^i, \bar{s}^{-i}) = v^i_{a^i}(s^i, \bar{s}^{-i}).$$

Because v^i is ϵ-interdependent, this inequality implies

$$v^i_{\tilde{a}^i}(s) - \epsilon \leq v^i_{\tilde{a}^i}(s^i, \bar{s}^{-i}) \leq v^i_{a^i}(s^i, \bar{s}^{-i}) \leq v^i_{a^i}(s) + \epsilon,$$

and thus

$$v^i_{\tilde{a}^i}(s) \leq v^i_{a^i}(s) + 2\epsilon,$$

which means that it is impossible to increase the welfare of agent i, and thus all agents in N', by more than 2ϵ, as is required by 2ϵ-group-strategy-proofness. $\square$

Proof of Proposition 3. Define

$$S_\alpha = \{s \in S \mid \alpha < |v_a^i(s) - v_b^i(s)|, \forall a, b \in O, a \neq b, \forall i \in N\}$$

and note that $\cup_{n=1}^{\infty} S_{1/n} = S_0 = (\overline{S})^c$. Due to the assumption that $\nu(\overline{S}) = 0$ and thus $\nu(S_0) = 1$, we have

$$\lim_{n\to\infty} \nu(S_{1/n}) = \nu\left(\cup_{n=1}^{\infty} S_{1/n}\right) = \nu(S_0) = 1,$$

where the first equality holds since $\{S_{1/n}\}_{n\in\mathbb{N}}$ is an increasing sequence of sets that converge to S_0. Therefore, for any (small) $\epsilon > 0$, one can find $\tilde{n}$ and $\delta(\epsilon) \in [0,1]$ such that $\frac{1}{\tilde{n}+1} \leq 2\epsilon < \frac{1}{\tilde{n}}$ and $\nu(S_{1/\tilde{n}}) = 1 - \delta(\epsilon)$, where $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Now, setting $\tilde{S} = S_{1/\tilde{n}}$, we show that the assignment $\varphi(s)$ at any $s \in \tilde{S}$ is Pareto efficient. It suffices to show that at any Round i, agent i is assigned his most preferred object among the remaining ones according to his (true) preference at $s \in \tilde{S}$. To do so, fix any $s \in \tilde{S}$ and let $O' \subset O$ denote the set of available objects when it is i's turn to choose. Letting a denote agent i's most preferred object from O' according to his true preference at s, we have $v_a^i(s) > v_b^i(s) + \frac{1}{\tilde{n}} > v_b^i(s) + 2\epsilon$ for all $b \in O' \backslash \{a\}$ since $s \in \tilde{S} = S_{1/\tilde{n}}$. Thus, for any such b,

$$v_a^i(s_i, \overline{s}_{-i}) \geq v_a^i(s_i, s_{-i}) - \epsilon > v_b^i(s_i, s_{-i}) + \epsilon \geq v_b^i(s_i, \overline{s}_{-i}),$$

where the weak inequalities hold due to the ϵ-interdependence of i's preference. The above inequality means that according to $\varphi(s)$, agent i is assigned object a, his most preferred object in O' at profile s, as was to be shown. $\square$

Proof of Proposition 4. Let φ be a serial dictatorship with respect to agent order $1, 2, \ldots, n$ and fixed signal profile $\overline{s}$. Fix any alternative (random) assignment P and fix any profile s of signals. Let k be the first agent under the serial dictatorship for whom $P_a^k \neq 1$ where $a := \varphi^k(s)$. Let $O' \subset O$ denote the set of objects that are available for agent k when making his choice under the serial dictatorship. Note that $P_b^i = 0$ for any $b \notin O'$ since $b = \varphi^j(s)$ for some agent $j \neq k$ who chooses before k under the serial dictatorship, and also since $P_b^j = 1$ by definition of k. Note also that for any $b \in O'$, $v_a^k(s^k, \overline{s}^{-k}) \geq v_b^k(s^k, \overline{s}^{-k})$ since both a and b are available at k's turn and he chooses a. Then, by the ϵ-interdependence,

$$v_a^k(s^k, s^{-k}) \geq \sum_{b\in O'} P_b^i v_b^k(s^k, \overline{s}^{-k}) - \epsilon \geq \sum_{b\in O'} P_b^k v_b^k(s^k, \overline{s}^{-k}) - \epsilon \geq v_a^k(s^k, s^{-k}) - 2\epsilon.$$

This shows that there exists at least one agent whose welfare gain from φ is at most 2ϵ. Since this argument works for all assignment $P \neq \varphi$, φ is 2ϵ-efficient. $\square$

Proof of Proposition 5. Given the threshold signal profile $(\overline{s}_1, \overline{s}_2)$, we define a direct mechanism φ as follows: for some $p, p' \in [0,1]$,

$$(\varphi_a^1(s), \varphi_a^2(s)) = \begin{cases} (p, 1-p) & \text{if } s \in [\overline{s}_1, 1] \times [\overline{s}_2, 1] \\ (p', 1-p') & \text{if } s \in [0, \overline{s}_1] \times [0, \overline{s}_2] \\ (0,1) & \text{if } s \in [0, \overline{s}_1) \times (\overline{s}_2, 1] \\ (1,0) & \text{if } s \in (\overline{s}_1, 1] \times [0, \overline{s}_2). \end{cases}$$

Note that $\varphi_b^i(s) = 1 - \varphi_a^i(s)$ for all s and i. Figure 3 illustrates the mechanism φ defined above.

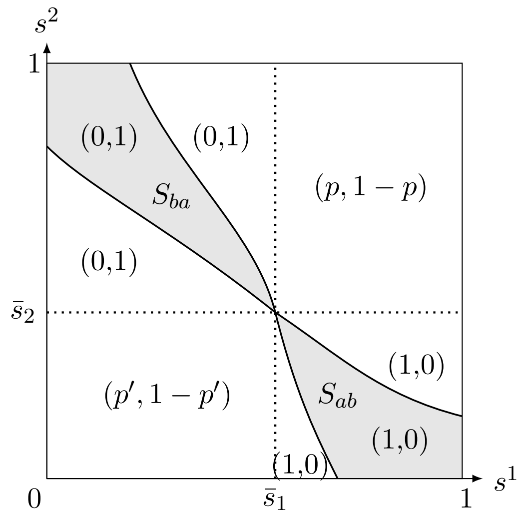

Figure 3: Bayesian Implementation

Clearly, this mechanism is Pareto efficient since agent 1 gets a (resp. b) in the area S_{ab} (resp. S_{ba}). We need to find a pair $p, p' \in [0,1]$ that makes φ Bayesian incentive compatible. To do so, we show that if condition (3) or (4) holds, then there exist $p, p' \in [0,1]$ such that each agent with type $\overline{s}^i$ is indifferent between reporting some $s^i > \overline{s}^i$ and some $\tilde{s}^i < \overline{s}^i$. Given this indifference and the fact that u^i is increasing in s^i, each agent i with $s^i > (<) \overline{s}^i$ would prefer reporting truthfully to reporting any $\tilde{s}^i < (>) \overline{s}^i$ so φ is Bayesian incentive compatible. Now the indifference condition for $\overline{s}^i$ requires: for agent 1,

$$p' \int_0^{\bar{s}^2} u^1(\bar{s}^1, s^2) dF^2(s^2) = \int_0^{\bar{s}^2} u^1(\bar{s}^1, s^2) dF^2(s^2) + p \int_{\bar{s}^2}^1 u^1(\bar{s}^1, s^2) dF^2(s^2)$$

or

$$(1-p') \int_0^{\bar{s}^2} u^1(\bar{s}^1, s^2) dF^2(s^2) + p \int_{\bar{s}^2}^1 u^1(\bar{s}^1, s^2) dF^2(s^2) = 0 \tag{13}$$

and for agent 2,

$$(1-p') \int_0^{\bar{s}^1} u^2(s^1, \bar{s}^2) dF^1(s^1) = \int_0^{\bar{s}^1} u^2(s^1, \bar{s}^2) dF^1(s^1) + (1-p) \int_{\bar{s}^1}^1 u^2(s^1, \bar{s}^2) dF^1(s^1)$$

or

$$p' \int_0^{\bar{s}^1} u^2(s^1, \bar{s}^2) dF^1(s^1) + (1-p) \int_{\bar{s}^1}^1 u^2(s^1, \bar{s}^2) dF^1(s^1) = 0. \tag{14}$$

Define

$$E_-^i := \int_0^{\bar{s}^{-i}} u^i(\bar{s}^i, s^{-i}) dF^{-i}(s^{-i}) \text{ and } E_+^i := \int_{\bar{s}^{-i}}^1 u^i(\bar{s}^i, s^{-i}) dF^{-i}(s^{-i})$$

and note that $E_-^i < 0$ and $E_+^i > 0$. Then, (13) and (14) can be rewritten as

$$(1-p')E_-^1 + pE_+^1 = 0 \tag{15}$$

$$p'E_-^2 + (1-p)E_+^2 = 0. \tag{16}$$

To check the existence of $(p, p') \in [0, 1]$ that solves these equations, in Figure 4 below, we draw solid lines passing through $(p, p') = (0, 1)$ for (15) and dashed lines passing through $(p, p') = (1, 0)$ for (16).

So the linear system (15) and (16) will have a solution if and only if two lines intersect at a point like either A or B. For A to be an intersection, the slope of (15) must be weakly smaller than -1 while the slope of (16) must be weakly greater than -1, i.e.

$$\frac{E_+^1}{E_-^1} \leq -1 \text{ and } \frac{E_+^2}{E_-^2} \geq -1$$

or

$$E_-^1 + E_+^1 \geq 0 \geq E_-^2 + E_+^2,$$

which is equivalent to (3). Similarly, B being an intersection is equivalent to (4). $\square$

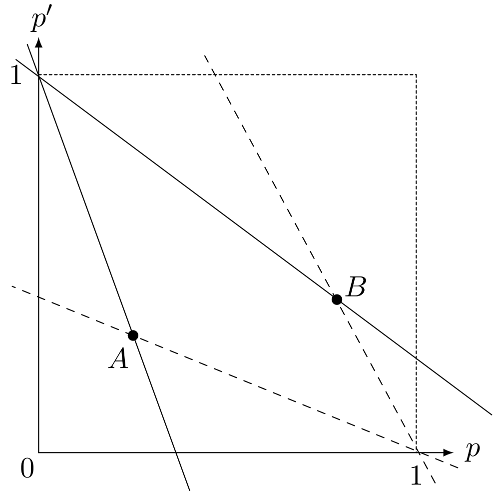

Figure 4: Existence of p and p' that make φ Bayesian incentive compatible

Proof of Proposition 6. The sufficiency of the congruence condition is immediate from the fact that the mechanism φ described in the proof of Proposition 5 is ex-post monotonic. For the necessity, we show that any Pareto efficient and ex-post monotonic mechanism must take the same form almost everywhere as φ in the proof of Proposition 5.

Let $S_{NE} := [\bar{s}^1, 1] \times [\bar{s}^2, 1] \subset [0, 1]^2$, i.e. the northeast square in the right panel of Figure 3. Let S_{NW}, S_{SE}, and S_{SW} be defined similarly. Note that $S_{ab} \subset S_{SE}$ and $S_{ba} \subset S_{NW}$. Consider any Pareto efficient and ex-post monotonic assignment φ. By the Pareto efficiency, we must have $\varphi_a^1(s) = 1 = 1 - \varphi_a^1(\hat{s})$ for all $s \in S_{ab}, \hat{s} \in S_{ba}$.

Now consider any signal profile $s \in S_{NW} \cap S_{aa}$. We can find another profile $\hat{s} \in S_{ba}$ with $\hat{s}^1 = s^1$ and $\hat{s}^2 \leq s^2$. The ex-post monotonicity and $\varphi_a^2(\hat{s}) = 1$ imply $\varphi_a^2(s) = 1$, and this implies $\varphi_a^1(s) = 0$. Consider alternatively any profile $s \in S_{NW} \cap S_{bb}$. We can find another profile $\hat{s} \in S_{ba}$ with $\hat{s}^1 \geq s^1$ and $\hat{s}^2 = s^2$. The ex-post monotonicity and $\varphi_a^1(\hat{s}) = 0$ imply $\varphi_a^1(s) = 0$. To sum, we must have $\varphi_a^1(s) = 1 - \varphi_a^2(s) = 0$ for all $s \in S_{NW}$. In a similar fashion, it can be shown that $\varphi_a^1(s) = 1 - \varphi_a^2(s) = 1$ for all $s \in S_{SE}$.

Let us now turn to the region S_{NE}. We need to show that $\varphi_a^1(s)$ is constant almost everywhere in S_{NE}. Consider any two signals $s^1, \hat{s}^1 \in [\bar{s}^1, 1]$ for agent 1 with $\hat{s}^1 > s^1$. Note that by the above argument

$$\varphi_a^1(s^1, s^2) = \varphi_a^1(\hat{s}^1, s^2) = 1, \forall (s^1, s^2), (\hat{s}^1, s^2) \in S_{SE}. \tag{17}$$

Also, by the ex-post monotonicity,

$$\varphi_a^1(s^1, s^2) \le \varphi_a^1(\hat{s}^1, s^2), \forall (s^1, s^2), (\hat{s}^1, s^2) \in S_{NE}. \tag{18}$$

Now the difference in expected payoffs of s^1 when reporting s^1 and when reporting $\hat{s}^1$ can be written as

$$\begin{aligned}
&\int_0^{\bar{s}^2} (\varphi_a^1(s^1, s^2) - \varphi_a^1(\hat{s}^1, s^2))u^1(s)dF^2(s^2) + \int_{\bar{s}^2}^1 (\varphi_a^1(s^1, s^2) - \varphi_a^1(\hat{s}^1, s^2))u^1(s)dF^2(s^2) \\
&= \int_{\bar{s}^2}^1 (\varphi_a^1(s^1, s^2) - \varphi_a^1(\hat{s}^1, s^2))u^1(s)dF^2(s^2),
\end{aligned}$$

where the equality holds since the first integral is equal to zero due to (17). Note that $u^1(s) > 0 \; \forall s \in S_{NE}$. Thus, if (18) holds as a strict inequality for a positive measure of s^2's in S_{NE}, then the payoff difference above would be strictly negative so agent 1 with signal s^1 would be better off reporting $\hat{s}^1$ rather than s^1. So we must have

$$\varphi_a^1(s^1, s^2) = \varphi_a^1(\hat{s}^1, s^2), \text{ for almost all } (s^1, s^2), (\hat{s}^1, s^2) \in S_{NE}. \tag{19}$$

A similar argument can be used to show

$$\varphi_a^2(s^1, s^2) = \varphi_a^2(s^1, \hat{s}^2), \text{ for almost all } (s^1, s^2), (s^1, \hat{s}^2) \in S_{NE}. \tag{20}$$

Combining (19) and (20), we obtain for almost all signal profiles $(s^1, s^2), (\hat{s}^1, \hat{s}^2) \in S_{NE}$,

$$\varphi_a^1(s^1, s^2) = \varphi_a^1(\hat{s}^1, s^2) = 1 - \varphi_a^2(\hat{s}^1, s^2) = 1 - \varphi_a^2(\hat{s}^1, \hat{s}^2) = \varphi_a^1(\hat{s}^1, \hat{s}^2).$$

A symmetric argument can be used to show $\varphi_a^1(s)$ is constant almost everywhere in S_{SW}. □

Proof of Proposition 7. The proof is analogous to that of Proposition 3 and makes use of the set $\tilde{S}$ defined therein. We show that at any Round i, agent i chooses the object among the set O' of available objects that he likes best according to his (true) preference at $s \in \tilde{S}$. Note as before that letting a denote i's most preferred object from O' at $s \in \tilde{S}$, we have $v_a^i(s) > v_b^i(s) + 2\epsilon$ for all $b \in O' \backslash \{a\}$. Let $\tilde{\nu}^{-i} \in \Delta(S^{-i})$ be the belief agent i forms based on his signal and his observation of others' choices. Then, by the ϵ-interdependence,

$$\mathbb{E}_{\tilde{\nu}^{-i}}[v_a^i(s^i, \tilde{s}^{-i})|s^i] \ge v_a^i(s_i, s_{-i}) - \epsilon > v_b^i(s_i, s_{-i}) + \epsilon \ge \mathbb{E}_{\tilde{\nu}^{-i}}[v_b^i(s^i, \tilde{s}^{-i})],$$

where the expectation is taken over $\tilde{s}^{-i}$. This inequality means that given his belief, agent i chooses object a, his most preferred object in O' at profile s. □

Appendix B: Discussion on Assumptions in Section 4

B.1: Importance of Assumptions in Section 4

This section demonstrates that the assumptions made in Section 4 for the general impossibility results are necessary in the sense that the conclusion of Theorem 1 does not hold if any of our Assumptions 1 (Interdependence), 2 (Rich Domain), and 3 (Connectedness) is violated.

First, it is easy to see that interdependence of valuations (Assumption 1) is necessary for our general impossibility. To see this, note that, under private values, it is well known that mechanisms such as serial dictatorship are both Pareto efficient and ex-post incentive compatible.[45] Those mechanisms provide counterexamples to the conclusion of Theorem 1 when Assumption 1 is violated (note that the counterexamples work even if the other assumptions, Assumptions 2 and 3, are satisfied).

Second, it is straightforward to show that the conclusion of Theorem 1 does not necessarily hold if Assumption 2 is violated. To see this point, consider an environment with two agents $1, 2$ and two objects a, b where both agents strictly prefer a to b for every signal profile (note that Assumption 2 is violated while the other two assumptions are satisfied). Then a mechanism that assigns a to 1 and b to 2 for every signal trivially satisfies both Pareto efficiency and ex-post incentive compatibility.

Lastly, consider Assumption 3. To show that the conclusion of Theorem 1 does not necessarily hold if Assumption 3 is violated, consider an environment depicted in Figure 5. In this environment, there are two agents and two objects. Recall that $S_{kk'}$ represents a set of signal profiles where agent 1 prefers k most and 2 prefers k' most. Consider the mechanism that assigns a to 1 and b to 2 in S_{aa} and S_{ab} while assigning b to 1 and a to 2 in S_{bb} and S_{ba}. This mechanism is clearly Pareto efficient. Moreover, it is ex-post incentive compatible since agent 1 is always assigned his preferred object and thus has no incentive to misreport, and agent 2 is assigned his preferred object if $s_1 \in [s_1', s_1'']$ while his report has no impact on assignment if $s_1 \notin [s_1', s_1'']$. Notice here that the example satisfies both Interdependence (Assumption 1) and Rich Domain (Assumption 2), but it fails Connectedness: for instance, the set $S_{\cdot a} = S_{aa} \cup S_{ba}$ is not connected. Therefore, this

[45]Mechanisms satisfying these properties are discussed in Introduction and references therein.

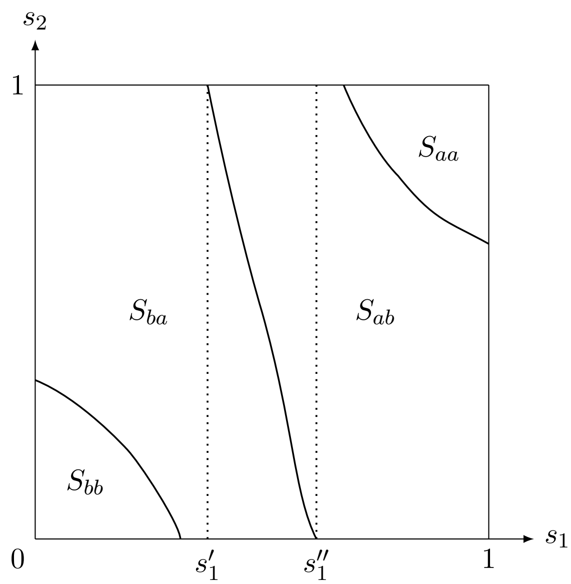

Figure 5: Pareto efficient and Bayesian incentive compatible mechanism without the Connectedness assumption.

example demonstrates that Assumption 3 cannot be dispensed with in Theorem 1.

B.2: Relationships among Different Assumptions

The Rich Domain assumption does not imply the Rich Domain* assumption. As explained in the main text of the paper, the canonical one-dimensional signal model of Example 1 for any $n \geq 3$ is a counterexample.

The Rich Domain* assumption does not imply the Rich Domain assumption. To see this point, consider a model with three agents $1, 2, 3$ and three objects. For each i, her ordinal preference depends only on signal s^{i+1} (where we use the convention that $i + 1 = 1$ for $i = 3$). Moreover, assume that for each i and her ordinal preference π^i, there exists $s^{i+1} \in int(S^{i+1})$ such that the ordinal preference is π^i when the signal of agent $(i + 1)$ is s^{i+1}. Then the Rich Domain* assumption is satisfied by definition. On the other hand, for any $i, j \in N$, and s^{-ij}, the ordinal preference of either i or j is constant across all s^i and s^j by construction, implying that the Rich Domain assumption is violated.

The Connectedness assumption does not imply the Connectedness* assumption. To see this, consider a model with three agents $1, 2, 3$ and three objects a, b, c, with the signal space of each agent being $[0, 1]$. The utility function of agent $i \in \{1, 2\}$ is given by $v^i_a(s) = 0$, $v^i_b(s) = s^i - \frac{1}{2}$, and $v^i_c(s) = -1$ for all s. For agent $j = 3$, let $v^j_a(s) = -1$, $v^j_b(s) = (s^3 - \frac{1}{2})^2 - \frac{1}{8}$, and $v^j_c(s) = 0$ for all s. Then clearly the Connectedness assumption holds with a, b, $i = 1, j = 2$ and any $s^{-ij} = s^3$ in the definition of the condition, while Connectedness* is violated since the subset of the signal space at which agent 3 prefers b to c (and c to a) is not connected.

The Connectedness* assumption does not imply the Connectedness assumption. To see this, consider a model with three agents $1, 2, 3$ and three objects a, b, c, with the signal space of each agent $[0, 1]$. For agent $i \in \{1, 2\}$, $v^i_a(s) = 0$, $v^i_b(s) = (s^i - \frac{1}{2})^2 - \frac{1}{8} + s^3$, and $v^i_c(s) = -1$ for all s. The preference of agent 3 is constant across signals. Then Connectedness* holds, but Connectedness is violated because the Rich Domain assumption can only be satisfied with objects a and b, agents 1 and 2, and $s^3 < 1/8$, in which case, however, the set of agents 1 and 2's signals for which they prefer a to b and b to c is not connected.

B.3: Relation between Rich Domain and Universal Domain Assumptions

Our analysis relies on a pair of rich domain assumptions, Rich Domain (Assumption 2) and Rich Domain* (Assumption 4). More specifically, the Rich Domain assumption is used to obtain the impossibility of ex-post incentive compatible and Pareto efficient mechanism (Theorem 1) while Rich Domain* is used to obtain the constancy result (Theorem 2).

Although these richness conditions may appear unfamiliar, they play similar roles as the more standard "Universal Domain" assumption does in well-known impossibility results in the private value setting. The Universal Domain assumptions are made in most studies, ranging from abstract pubic choice problems with and without strategic agents (Arrow (1951), Gibbard (1973), Satterthwaite (1975)) to more specific problems such as two-sided matching (Roth (1982)) and object allocation (Zhou (1991), Bogomolnaia and Moulin (2001)). In fact, often the Universal Domain assumption is not even explicitly stated, but implicitly assumed.

To make the connection between these concepts precise, let us say that agent i has

private values if $v^i_a(s) = v^i_a(\bar{s})$ for all $a \in O$ and $s, \bar{s} \in S$ such that $s^{-i} = \bar{s}^{-i}$. In the private-values setting, we say that the problem satisfies the **Universal Domain** assumption if no ordinal preferences are precluded from agents' preferences, that is, for all $i \in N$ and ordinal preference π^i, there exists $s^i \in int(S^i)$ such that $\pi^i(s) = \pi^i$, where $s = (s^i, s^{-i})$ and s^{-i} is an arbitrary signal profile for agents other than i (recall that the choice of s^{-i} does not affect agent i's preferences under the private values assumption). Now we are ready to state the relationship between the Universal Domain assumption and our richness assumptions.

Proposition 8. *Suppose that each agent has private values. Then the following relationships hold.*

1. *Universal Domain is equivalent to Rich Domain*.*

2. *Universal Domain implies Rich Domain.*

Proof. **Part 1:** First, we will show that the Universal Domain assumption implies the Rich Domain* assumption. To do so, fix any preference profile $\pi = (\pi^i)_{i \in N}$. By the Universal Domain assumption, for each $i \in N$ there exists $s^i \in int(S^i)$ such that $\pi^i(s^i, \tilde{s}^{-i}) = \pi^i$ for any $\tilde{s}^{-i} \in S^{-i}$. Because agents have private values, this implies that $s := (s^i)_{i \in N} \in int(S)$ and $\pi(s) = \pi$, completing the proof.

Second, we will show that the Rich Domain* assumption implies the Universal Domain assumption. To do so, fix an agent $i \in N$ and her preference π^i. Choose preference profile π^{-i} for all agents other than i arbitrarily, and let $\pi = (\pi^i, \pi^{-i})$. Then, by the Rich Domain* assumption, there exists $s \in int(S)$ such that $\pi(s) = \pi$. Thus we have found that s^i satisfies the requirement for the Universal Domain assumption, as desired.

Part 2: We will show that the Universal Domain assumption implies the Rich Domain assumption. To do so, fix $i, j \in N$, and $a, b \in O$ arbitrarily. By the Universal Domain assumption, for every $i' \in N \setminus \{i, j\}$, there exists $s^{i'}$ such that i' ranks both a and b strictly below any $k'' \notin \{a, b\}$. Fix any such $s^{i'}$ and denote $s^{-ij} = (s^{i'})_{i' \in N \setminus \{i,j\}}$. Again by the Universal Domain assumption, for any $k, k' \in \{a, b\}$, there exist s^i and s^j at which (i) agent i ranks k strictly above $o \in \{a, b\} \setminus \{k\}$, and o strictly above any $k'' \notin \{a, b\}$, and (ii) agent j ranks k' strictly above $o \in \{a, b\} \setminus \{k'\}$, and o strictly above any $k'' \notin \{a, b\}$. Thus the conclusion of the Rich Domain assumption has been obtained. $\square$

This result suggests that our two rich domain assumptions are no stronger than the Universal Domain assumption in the private value setting. In this sense, our two assumptions can be seen as legitimate extensions of the Universal Domain assumption to the interdependent value setting.