Stable Matching in Large Economies*

Yeon-Koo Che, Jinwoo Kim, Fuhito Kojima†

April 24, 2015

Abstract

Complementarities of preferences have been known to jeopardize the stability of two-sided matching, and yet they are a pervasive feature of many markets. We revisit the stability issue with such preferences in a large market. Workers have preferences over firms while firms have preferences over distributions of workers and may exhibit complementarity. We demonstrate that if each firm's choice changes continuously as the set of available workers changes, then there exists a stable matching even with complementarity. Building on this result, we show that there exists an approximately stable matching in any large finite economy. We extend our framework to accommodate indifferences in firms' preferences, construct a stable mechanism that is strategy-proof and equitable for workers perceived as indifferent by firms, and apply the analysis to probabilistic and time-share matching models with a finite number of firms and workers.

JEL Classification Numbers: C70, D47, D61, D63.

Keywords: two-sided matching, stability, complementarity, strategy-proofness, large economy

---

*We are grateful to Nikhil Agarwal, Nick Arnosti, Eduardo Azevedo, Péter Biró, Aaron Bodoh-Creed, Pradeep Dubey, Piotr Dworczak, Tadashi Hashimoto, John William Hatfield, Johannes Hörner, Yuichiro Kamada, Michihiro Kandori, Scott Kominers, Ehud Lehrer, Jacob Leshno, Bobak Pakzad-Hurson, Jinjae Park, Parag Pathak, Marcin Peski, Larry Samuelson, Ilya Segal, Rajiv Sethi, Bob Wilson, and seminar participants at Kyoto, Paris, Seoul, Stanford, Tokyo, ASSA Meetings 2015, NBER Market Design Workshop, INFORMS 2014, International Conference on Game Theory at Stony Brook 2014, Monash Market Design Workshop, SAET 2014, and Midwest Economic Theory Conference 2014 for helpful comments.

†Che: Department of Economics, Columbia University (email: yeonkooche@gmail.com); Kim: Department of Economics, Seoul National University (email: jikim72@gmail.com); Kojima: Department of Economics, Stanford University (email: fuhitokojima1979@gmail.com). We acknowledge the financial support from the National Research Foundation through its Global Research Network Grant (NRF-2013S1A2A2035408). Kojima gratefully acknowledges financial support from the Sloan Foundation.

1 Introduction

Since the celebrated work by Gale and Shapley (1962), matching theory has taken a center stage in market design and economic theory more broadly. In particular, its successful application in medical matching and school choice has fundamentally changed how these markets are organized. A key desideratum in the design of such matching markets is "stability"—that the mechanism admits no incentives for its participants to "block" (i.e., side-contract around) the suggested matching. Stability is crucial for long-term sustainability of a market; unstable matching would be undermined by the parties side-contracting around it either during or after a market.[1] When one side of the market is under centralized control, as with school choice, blocking by a pair of agents on both sides is less of a concern; but even in this case, stability is desirable from a fairness standpoint, as it would eliminate justified envy—envy that cannot be explained away by the preferences of the agents on the other side. In the school choice application, if schools' preferences rest on the test score or other priority that a student feels entitled to, eliminating justified envy appears to be a necessary requirement.

Unfortunately, a stable matching exists only under restricted conditions. It is well known that existence of a stable matching is not generally guaranteed unless the preferences of participants, say firms, are substitutable.[2] In other words, complementarity can lead to nonexistence of a stable matching.

This is a serious limitation on the applicability of centralized matching mechanisms, since complementarities of preferences are a pervasive feature of many matching markets. Firms often seek to hire workers with complementary skills. For instance, in professional athletic leagues, teams demand athletes that complement one another in skills as well as in the positions they play. Some public schools in New York City seek diversity of their student bodies in their skill levels. US colleges tend to exhibit a desire to assemble a class that is complementary and diverse in terms of their aptitudes, life backgrounds, and demographics.

Unless we can get a handle on complementarities, we would not know how to organize

[1]Table 1 in Roth (2002) shows that unstable matching algorithms tend to die out while stable ones survive the test of time.

[2]Substitutability here means that a firm's demand for a worker never grows with more workers being available. More precisely, if a firm does not wish to hire a worker from a set of workers, then it never prefers to hire that worker from a bigger (in the sense of set inclusion) set of workers. Existence of a stable matching under substitutable preferences is established by Kelso and Crawford (1982), Roth (1985), and Hatfield and Milgrom (2005), while substitutability was shown to be the maximal domain for existence by Sönmez and Ünver (2010), , Hatfield and Kojima (2008), and Hatfield and Kominers (2014).

such markets, and the applicability of centralized matching will remain severely limited. The limitation is particularly pertinent for many decentralized markets that may potentially benefit from centralization. College admissions and graduate admissions are obvious examples. Decentralized matching leaves much to be desired in terms of efficiencies and fairness, and of the yield management burden put on the institutions (see Che and Koh (2013)). Despite the potential benefit from centralizing these markets, the exact benefit as well as the method of centralized matching remains unclear, given the instability that may arise from complementary preferences of the participants.

This paper takes a step toward accommodating complementarities and other forms of general preferences. The general impossibility means, however, that the notion of stability needs to be weakened in some way. Our approach is to consider a large market. Specifically, we consider a market which consists of a large number of workers/students on one side and a finite number of firms/colleges with large capacities on the other, and ask whether stability can be achieved in an "asymptotic" sense—i.e., whether participants' incentives for blocking disappears as the number of workers and firms' capacities grow large. Large markets we envision approximate college admissions and labor markets. Our stability notion also preserves the motivation behind the original notion of stability: as long as the incentive for blocking is sufficiently weak, the instability and fairness concerns will not be so serious as to jeopardize the mechanism.

We first consider a continuum model in which there are a finite number of firms and a continuum of workers. Each worker desires to match with at most one firm. Firms have preferences over groups of workers, and importantly, their preferences may exhibit complementarities. A matching is a distribution of workers across firms. The model generalizes Azevedo and Leshno (2011) who assume responsive preferences (a special case of substitutable preferences) for the firms.

Our main result is that there exists a stable matching if firms' preferences exhibit continuity—that is, the set of workers chosen by each firm varies continuously as the set of workers available to that firm changes. This result is quite general since continuity is satisfied by a rich class of preferences including those exhibiting complementarities.[3] The existence of a stable matching follows from two results: (i) a stable matching can be characterized as a fixed point of a suitably defined mapping over a functional space, and (ii) such a fixed point exists given the continuity assumption. The construction of our fixed point mapping differs from the existing matching literature such as Adachi (2000), Hatfield

[3]For instance, it allows for Leontief-type preferences with respect to alternative types of workers, desiring to hire all types in equal size (or density).

and Milgrom (2005), and Echenique and Oviedo (2006), among others. The existence of a fixed point is established by using the Kakutani-Fan-Glicksberg fixed point theorem—a generalization of Kakutani's fixed point theorem to functional spaces—which appears to be new to the matching literature.

Building on our analysis of the continuum model, we show that there is a sense in which it serves as a legitimate approximation of large finite economies. More specifically we demonstrate that, for any large finite economy that is sufficiently close to our continuum economy (in terms of the distribution of worker types and firms' preferences), there exists an approximately stable matching in the sense that the incentives for blocking is arbitrarily small.

Although the basic model assumes that firm preferences are strict, our framework can be extended to allow for indifferences in the firms' preferences. Accommodating indifferences is particularly important in the school choice context, in which preferences are given by coarse priorities that put many students in the same priority class. Firms' indifferences raise a further issue about fairness, for the standard notion of stability does not ensure that the workers who are perceived by firms as equivalent are treated equitably. Hence, it may be important, particularly in the school choice context, to strengthen the notion of stability to eliminate discrimination among such workers (in addition to eliminating justified envy). How such workers are treated also raises an incentive issue, for inequitable treatment among workers perceived by firms as equivalent may lead to untruthful reporting by workers. To accommodate indifferences, we represent a firm's preference as a choice correspondence (as opposed to a function). We then extend both the fixed point characterization (via a correspondence defined on a functional space) and the proof of the existence result. We finally show that there exists a matching mechanism that satisfies both the stronger notion of stability ("strong stability," as defined by Kesten and Ünver (2014)) and strategy-proofness for workers.

Further, our general model with indifferences has a natural application to fractional/time share matching models. These models study how schools/firms and students/workers can share time or match probabilistically in a stable manner in a finite economy (see Sotomayor (1999), Alkan and Gale (2003), and Kesten and Ünver (2014), among others). Our continuum model lends itself to studying such a probabilistic/time share environment; we can simply interpret types in different subsets within the type space as probabilistic/time units belonging to alternative (finite) workers. Our novel contribution is to allow for more general preferences for firms, including complementarities as well as indifferences. As mentioned, accommodating indifferences is important in school choice design, and complementarities are also relevant since some schools (as those in NYC) seek diversity in their student bodies.[4]

Relationship with the Literature

The present paper is connected with several strands of literature. Most importantly, it is related to the growing literature on matching and market design. Since the seminal contributions by Gale and Shapley (1962) and Roth (1984), stability has been recognized as the most compelling solution concept in matching markets.[5] As argued and demonstrated by Sönmez and Ünver (2010), Hatfield and Milgrom (2005), Hatfield and Kojima (2008), and Hatfield and Kominers (2014) in various situations, the substitutability condition is necessary and sufficient for guaranteeing the existence of a stable matching when the number of agents is finite. Our paper contributes to this line of studies by showing that substitutability is not needed for the existence of a stable matching once there is a continuum of agents on one side of the market and, moreover, there exists an approximately stable matching in large finite markets.

Our study was inspired by a recent research on matching with a continuum of agents by Azevedo and Leshno (2011).[6] As in our paper, they assume that there are a finite number of firms and a continuum of workers and, among other things, show the existence and uniqueness of a stable matching in that setting. The crucial difference relative to the current work is that they assume firms have responsive preferences (which is a special case of substitutability). One of our contributions is that, while almost universally assumed in the literature, restrictions on preferences such as responsiveness or even substitutability are unnecessary for guaranteeing the existence of a stable matching in the continuum markets. Also, one of the uniqueness results by Azevedo and Leshno (2011) is obtained as a special case of our uniqueness result under substitutable, not necessarily responsive, preferences.

An independent study by Azevedo and Hatfield (2012) also analyzes matching with a

[4]As mentioned in Conclusion, we investigate further issues. First, we study a setting with substitutable preferences, provide a condition for uniqueness of a stable matching, and apply it to the setup that generalizes Azevedo and Leshno (2011) by allowing for affirmative action constraints. Second, we generalize our basic model to the model of matching with contracts.

[5]See Roth (1991) and Kagel and Roth (2000) for empirical and experimental evidence on the importance of stability in labor markets, and Abdulkadiroğlu and Sönmez (2003) for the interpretation of stability as a fairness concept in school choice.

[6]Also related, although formally different, are various recent studies on large matching markets, such as Roth and Peranson (1999), Immorlica and Mahdian (2005), Kojima and Pathak (2009), Kojima and Manea (2010), Manea (2009), Che and Kojima (2010), Lee (2012), Liu and Pycia (2013), Che and Tercieux (2015*b*), Che and Tercieux (2015*a*), Ashlagi, Kanoria and Leshno (2014), Kojima, Pathak and Roth (2013), and Hatfield, Kojima and Narita (2014*b*).

continuum of agents.[7] Like the current paper, their study finds that a stable matching exists even when not all agents have substitutable preferences. There are a number of notable differences between their study and ours, however. First, they consider a large number (more precisely, continuum) of firms each employing a finite number of workers, so they consider a continuum of agents on both sides of the market. By contrast, we consider a finite number of firms each employing a large number (continuum) of workers. These two models provide complementary approaches for studying large markets. In the school choice context, for example, in many school districts, there are usually a small number of schools each admitting hundreds of students, which fits well with our modeling approach. But in a large school district such as New York City, the number of schools is also large, so their model may offer a reasonably good approximation. Second, Azevedo and Hatfield (2012) assume that there is a finite number of firm and worker types. This enables them to use Brouwer's fixed point theorem to show the existence of a stable matching. By contrast, we put no restriction on the number of workers' types, allowing for both finite and infinite numbers of types. This generality in type spaces requires a topological fixed point theorem from functional analysis. This type of mathematics has never been applied to discrete two-sided matching literature to our knowledge, and we view the introduction of these tools to the matching literature as one of our methodological contributions. Our model also has an advantage of subsuming the previous work by Azevedo and Leshno (2011) as well as many others mentioned above, which assume a continuum of worker types. Finally, they also consider many-to-many matchings, although our applications to time-share and probabilistic matching models allow for many-to-many matching.

Our methodological contribution is also related to another recent advance in matching theory based on the monotone method. In the one-to-one matching context, Adachi (2000) defines a certain operator whose fixed points are equivalent to stable matchings. His work has been generalized in many directions by such papers as Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Ostrovsky (2008), and Hatfield and Kominers (2014). We also define an operator whose fixed points are equivalent to stable matchings. A crucial difference is, however, that these previous studies impose restrictions on preferences (e.g., responsiveness or substitutability) so that the operator is monotone, which enables one to apply Tarski's fixed point theorem to show existence of stable matchings. By contrast, we do not impose responsiveness or substitutability restrictions and instead rely on the continuum of workers, along with continuity of firms' preferences, to

[7] Although less related, our study also has some analogy with Azevedo, Weyl and White (2013). They show the existence of competitive equilibrium in an exchange economy with continuum agents and indivisible objects.

guarantee continuity of the operator (in an appropriately chosen topology). That approach allows us to use (a generalization of) Kakutani fixed point theorem, a more familiar tool in traditional economic theory such as the existence proofs of general equilibrium and the Nash equilibrium in mixed strategies.

The current paper is also related to the literature on matching with couples. Like a firm in our model, a couple can be seen as a single agent with complementary preferences over contracts. Roth (1984) and unpublished work by Sotomayor show that there does not necessarily exist a stable matching if there exists a couple. Klaus and Klijn (2005) provide a condition to guarantee the existence of stable matchings. A more recent work by Kojima, Pathak and Roth (2013) presents conditions under which the probability that a stable matching exists even in the presence of couples converges to one as the market becomes infinitely large, and similar conditions have been further analyzed by Ashlagi, Braverman and Hassidim (2014). Pycia (2012) and Echenique and Yenmez (2007) study many-to-one matching with complementarity as well as peer effect. Our paper is different from these studies in various respects, but it complements these papers by formalizing a sense in which finding a stable matching becomes easier in a large market even in the presence of complementarities.

The remainder of this paper is organized as follows. Section 2 provides an example that illustrates the main contribution of our paper. Section 3 describes a matching model in the continuum economy. Section 4 establishes the existence of a stable matching under general, continuous preferences. In Section 5, we use this existence result to show that an approximately stable matching exists in any large finite economy. In Section 6, we extend our analysis to the case where firms may have multi-valued choice mappings (that is, choice correspondences), and apply it to study fairness and incentive compatibility, as well as time share/probabilistic matching models. Section 7 concludes.

2 Illustrative Example

Before proceeding, we illustrate the main contribution of our paper using an example. We first illustrate how complementary preferences may lead to non-existence of a stable matching when there are a finite number of agents. To this end, suppose that there are two firms f_1 and f_2 and two workers θ and θ'. The agents have the following preferences:

$$\theta : f_1 \succ f_2; \qquad\qquad f_1 : \{\theta, \theta'\} \succ \emptyset;$$
$$\theta' : f_2 \succ f_1; \qquad\qquad f_2 : \{\theta\} \succ \{\theta'\} \succ \emptyset.$$

That is, worker θ prefers f_1 to f_2, and worker θ' prefers f_2 to f_1; firm f_1 prefers employing both workers to employing no one, which the firm in turn prefers to employing only one of them; and firm f_2 prefers worker θ to θ', which it in turn prefers to employing neither. Firm f_1 has a "complementary" preference, and this creates instability. To see this, recall stability requires that there be no blocking coalition. Due to f_1's complementary preference, it must employ either both workers or neither in any stable matching. The former case is unstable since worker θ' prefers firm f_2 to firm f_1, and f_2 prefers θ' to being unmatched, thus they can block the matching. The latter is also unstable since, in such a case, f_2 will only hire θ, leaving θ' unemployed; and this outcome will be blocked by f_1 forming a coalition with θ and θ', benefiting all members of the coalition.

Can stability be restored if the market becomes large? As long as the market remains finite, the answer is no. To see this, consider a scaled-up version of the above model: there are q workers of type θ and q workers of type θ', and they have the same preferences as above. Firm f_2 prefers type-θ workers to type-θ' workers, and wishes to hire in that order but at most q workers in total. Firm f_1 has a complementary preference for hiring exactly identical numbers of type-θ and type-θ' workers (with no capacity limit). Formally, if x and x' are the numbers of available workers of types θ and θ', respectively, then firm f_1 would choose $\min\{x, x'\}$ workers of each type.

As long as q is odd (including the original economy with $q = 1$), there exists no stable matching.[8] To see this, first note that if firm f_1 hires more than $q/2$ workers of each type, then firm f_2 has a vacant position, so f_2 can block with a type-θ' worker who prefers f_2 to f_1. If f_1 hires fewer than $q/2$ workers of each type, then some workers will remain unmatched (since f_2 hires at most q workers). If a type-θ worker is unmatched, then f_2 will form a blocking coalition with that worker. If a type-θ' worker is unmatched, then firm f_1 will form a blocking coalition by adding that worker along with a θ worker (possibly matched with f_2).

Consequently, "exact" stability is not guaranteed even in a large market. Nevertheless, one may hope to achieve approximate stability. This is indeed the case with the above example; the "magnitude" of instability diminishes as the economy grows large. To see this, let q be odd and consider a matching in which f_1 hires $\frac{q+1}{2}$ workers of each type while f_2 hires $\frac{q-1}{2}$ workers of each type. This matching is unstable because f_2 has one vacant position it wants to fill and a type-θ' worker who is matched to f_1 prefers f_2. However, note that this is the only possible block of this matching, and it involves only one worker. As

[8] Here we sketch the argument, which is in Section S.1 of the Supplementary Notes in fuller form. When q is even, a matching in which each firm hires $\frac{q}{2}$ of each type of workers is stable.

the economy grows large, if the additional single worker becomes insignificant for firm f_2 relative to its size—and this is what the continuity of a firm's preference captures—, then the payoff consequence of forming such a block must also become insignificant, suggesting that the instability problem becomes insignificant as well.

This can be seen most clearly in the limit of the above economy. Suppose there is a unit mass of workers, half of whom are of type θ and the other half are of type θ'. Their preferences are the same as before. And suppose firm f_1 wishes to maximize $\min\{x, x'\}$, where x and x' are the measures of type-θ and type-θ' workers, respectively. Firm f_2 can hire at most measure $\frac{1}{2}$ of workers, and prefers to fill as much of this quota as possible with type-θ workers and fill the remaining quota with type-θ' workers. In this economy, there is a (unique) stable matching in which each firm hires exactly one half of workers of each type. To see this, note that any blocking coalition involving firm f_1 requires taking away a positive, and identical, measure of type-θ' and type-θ workers from firm f_2, which is impossible since type-θ' workers will object to it. Also, any blocking coalition involving firm f_2 requires taking away a positive measure type-θ workers away from firm f_1 and replacing the same measure of type-θ' workers in its workforce, which is impossible since type-θ workers will object to it. Our analysis below will show that the continuity of firms' preferences, to be defined more clearly, is responsible for guaranteeing existence of a stable matching in the continuum economy and approximate stability in the large finite economies in this example.

3 Model of a Continuum Economy

Agents and their measures. There exist a finite set $F = \{f_1, \ldots, f_n\}$ of firms and a unit mass of workers. Let $\o$ be the null firm, representing the workers' option of not being matched with any firm, and define $\tilde{F} := F \cup \{\o\}$. The workers are identified with types $\theta \in \Theta$, where Θ is a compact metric space. Let Σ denote a Borel σ-algebra of space Θ. Let $\overline{\mathcal{X}}$ be the set of all nonnegative measures such that for any $X \in \overline{\mathcal{X}}$, $X(\Theta) \leq 1$. Assume that the entire population of workers is distributed according to a finite, nonnegative (Borel) measure $G \in \overline{\mathcal{X}}$ on (Θ, Σ). That is, for any $E \in \Sigma$, $G(E)$ is the measure of workers belonging to E. Assume $G(\Theta) = 1$ for normalization. For illustration, the limit economy of the example in the previous section is a continuum economy with $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\{\theta\}) = G(\{\theta'\}) = 1/2$.[9] In the sequel, we shall use this as our leading example for

[9]From now on, given any measure X, we will write $X(\theta)$ to denote a measure of singleton set $\{\theta\}$ to simplify notation.

the purpose of illustrating various concepts we develop.

Any subset of the population, or **subpopulation**, is represented by a nonnegative measure X on (Θ, Σ) such that $X(E) \leq G(E)$ for all $E \in \Sigma$. Let $\mathcal{X} \subset \overline{\mathcal{X}}$ denote the set of all subpopulations. We further say that a nonnegative measure $\tilde{X} \in \mathcal{X}$ is a **subpopulation of** $X \in \mathcal{X}$, denoted $\tilde{X} \sqsubset X$, if $\tilde{X}(E) \leq X(E)$ for all $E \in \Sigma$. We use $\mathcal{X}_X$ to denote the set of all subpopulations of X.

Given the partial order $\sqsubset$, for any $X, Y \in \mathcal{X}$, we define $X \vee Y$ (join) and $X \wedge Y$ (meet) to be the supremum and infimum of X and X', respectively.[10] That $X \vee Y$ and $X \wedge Y$ are well-defined, i.e. they are also measures belonging to $\mathcal{X}$, follows from the next lemma, whose proof is in Section S.2.1 of the Supplementary Notes.

Lemma 1. *The partially ordered set* $(\mathcal{X}, \sqsubset)$ *is a complete lattice.*

The join and meet of X and Y in $\mathcal{X}$ can be illustrated via a couple of examples. In our leading example with two types of workers, given $X := (x, x'), Y := (y, y')$, we have $X \vee Y = (\max\{x, y\}, \max\{x', y'\})$, and $X \wedge Y = (\min\{x, y\}, \min\{x', y'\})$. Consider next a continuum economy with types $\Theta = [0, 1]$ and suppose the measure G admits a bounded density g for all $\theta \in [0, 1]$. In this case, it easily follows that for $X, Y \sqsubset G$, their densities x and x' are well defined,[11] and $Z := (X \vee Y)$ and $Z' := (X \wedge Y)$ admit densities z and z' defined by $z(\theta) = \max\{x(\theta), y(\theta)\}$ and $z'(\theta) = \min\{x(\theta), y(\theta)\}$ for all θ, respectively. As usual, for any two measures $X, Y \in \mathcal{X}$, $X + Y$ and $X - Y$ denote their sum and difference, respectively.[12]

Consider the space of all (signed) measures (of bounded variation) on (Θ, Σ). We endow this space with a weak-$*$ topology and its subspace $\mathcal{X}$ with the relative topology. Given a sequence of measures (X_k) and a measure X on (Θ, Σ), we write $X_k \xrightarrow{w^*} X$ to indicate that (X_k) converges to X as $k \to \infty$ under weak-$*$ topology, and simply say that (X_k) **weakly converges** to X.[13]

[10]That is, $X \vee Y$ for instance is the smallest measure of which both X and Y are subpopulations. One can show that for all $E \in \Sigma$,

$$(X \vee Y)(E) = \sup_{D \in \Sigma} X(E \cap D) + Y(E \cap D^c),$$

which is a special case of Lemma 1.

[11]Since $|X([0, \theta']) - X([0, \theta])| \leq |G([0, \theta']) - G([0, \theta])| \leq N|\theta' - \theta|$, where $N := \sup_s g(s)$, so $X([0, \theta))$ is Lipschitz continuous, and its density is well defined.

[12]In case $X + Y \not\sqsubset G$ (resp., $0 \not\sqsubset X - Y$), we abuse notation to let $X + Y$ (resp., $X - Y$) denote the meet of $X + Y$ and G (resp., the join of of $X - Y$ and 0).

[13] We use the term "weak convergence" since it is common in statistics and mathematics, though weak-$*$ convergence is a more appropriate term from the perspective of functional analysis. As is well known,

Agents' preferences. We now describe agents' preferences. Each worker is assumed to have a strict preference over $\tilde{F}$. Let $\mathcal{P}$ denote the (finite) set of all possible worker preferences, and let $P \in \mathcal{P}$ denote its generic element (i.e., a particular worker preference). We write $f \succ_P f'$ to indicate that f is strictly preferred to f' according to P. For each $P \in \mathcal{P}$, let $\Theta_P \subset \Theta$ denote the set of all worker types whose preference is given by P, and assume that Θ_P is measurable and $G(\partial\Theta_P) = 0$, where $\partial\Theta_P$ denotes the boundary of Θ_P.[14] Since all worker types have strict preferences, Θ can be partitioned into the sets in $\mathcal{P}_\Theta \equiv \{\Theta_P : P \in \mathcal{P}\}$.

We next describe firms' preferences. We do so indirectly by defining a firm f's **choice function**, $C_f : \mathcal{X} \to \mathcal{X}$, where $C_f(X)$ is a subpopulation of X for any $X \in \mathcal{X}$ and satisfies the following **revealed preference** property: for any $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, if $C_f(X) \sqsubset X'$, then $C_f(X') = C_f(X)$.[15] Note we are assuming that the firm's demand is unique given any set of available workers. In Section 6.1, we consider a generalization of the model in which the firm's choice is not unique. Let $R_f : \mathcal{X} \to \mathcal{X}$ be a **rejection function** defined by $R_f(X) := X - C_f(X)$. By convention, we let $C_\o(X) = X, \forall X \in \mathcal{X}$, meaning that $R_\o(X)(E) = 0$ for all $X \in \mathcal{X}$ and $E \in \Sigma$. In our leading example, the choice functions of firms f_1 and f_2 are given respectively by $C_{f_1}(x_1, x_1') = (\min\{x_1, x_1'\}, \min\{x_1, x_1'\})$ and $C_{f_2}(x_2, x_2') = (\min\{x_2, \frac{1}{2}\}, \min\{\frac{1}{2} - x_2, x_2'\})$, when x_i of type θ-workers and x_i' of type-θ' workers are available to firm f_i, $i = 1, 2$.

In sum, a continuum matching model is summarized as a tuple $(G, F, \mathcal{P}_\Theta, C_F)$.

Remark 1. Our model takes firms' choice functions as a primitive, which gives us some flexibility in describing their preferences, in particular preferences over alternatives that are not chosen. This approach is also adopted by other studies in matching theory, which include Alkan and Gale (2003) and Aygün and Sönmez (2013) among others. An alternative, albeit more restrictive, approach would be to assume that each firm is endowed with a complete, continuous preference relation over $\mathcal{X}$. Maximization with such a preference will result in an upper hemicontinuous choice correspondence defined over $\mathcal{X}$.[16] Assuming

$X_k \xrightarrow{w^*} X$ if $\int_\Theta h dX_k \to \int_\Theta h dX$ for all bounded continuous function h. See Theorem 8 in Appendix A to see some implications of this convergence.

[14]Formally, $\partial E := \overline{E} \cap \overline{E^c}$, where $\overline{E}$ and $\overline{E^c}$ are the closures of E and E^c, respectively. This means that if Θ is discrete as in our leading example, then we have $\overline{E} = E$ and $\overline{E^c} = E^c$, so $\overline{E} \cap \overline{E^c} = E \cap E^c = \emptyset$. Hence, the assumption is satisfied.

[15]This property must hold if the choice is made by a firm optimizing with a well-defined preference relation. See for instance Hatfield and Milgrom (2005), Fleiner (2003), and Alkan and Gale (2003) for some implicit or explicit use of the revealed preference property in matching theory literature. Recently, Aygün and Sönmez (2013) have clarified the role of this property in the context of matching with contracts.

[16]This also relies on the fact that the set of alternatives $\mathcal{X}$ is compact, a fact we establish in the proof

a unique optimal choice will then give us a choice function (which is also continuous), although, as will be shown in Section 6.1, our results generalize to the case in which each firm's choice is not unique.

Matchings, and their efficiency and stability requirements. A **matching** is $M = (M_f)_{f\in\tilde{F}}$ such that $M_f \in \mathcal{X}$ for all $f \in \tilde{F}$ and $\sum_{f\in\tilde{F}} M_f = G$. Firms' choice functions can be used to define a partial order on firms' preferences over matchings. For any two matchings, M and M', we say that $M'_f \succeq_f M_f$ (or firm f prefers M'_f to M_f) if $M'_f = C_f(M'_f \vee M_f)$.[17] We also say $M'_f \succ_f M_f$ if $M'_f \succeq_f M_f$ and $M'_f \neq M_f$. The resulting preference (partial) order amounts to taking a minimal stance on the firms' preferences, limiting attention to those revealed via their choices. Given this preference order, we say $M' \succeq_F M$ if $M'_f \succeq_f M_f$ for each $f \in F$.

To discuss workers' welfare, fix any matching M and any firm f. Let

$$D^{\succeq f}(M) := \sum_{P\in\mathcal{P}} \sum_{f'\in\tilde{F}:f'\succeq_P f} M_{f'}(\Theta_P \cap \cdot) \text{ and } D^{\preceq f}(M) := \sum_{P\in\mathcal{P}} \sum_{f'\in\tilde{F}:f'\preceq_P f} M_{f'}(\Theta_P \cap \cdot) \quad (1)$$

denote the measure of workers assigned to firm f or better (according to their preferences) and the measure of workers assigned to firm f or worse (again according to their preferences), respectively, where $M_{f'}(\Theta_P \cap \cdot)$ denotes a measure that takes the value $M_{f'}(\Theta_P \cap E)$ for each $E \in \Sigma$. Starting from M as a default matching, the latter measures the number of workers who are available to firm f for possible rematching. Meanwhile, the former measure is useful for characterizing the workers' overall welfare. For any two matchings M and M', we say that $M' \succeq_\Theta M$ if for each $f \in \tilde{F}$, $D^{\succeq f}(M') \sqsubset D^{\succeq f}(M)$. That is, if, for each firm f, the measure of workers assigned to f or better is larger in one matching than the other, then we can say that the workers' overall welfare is higher in the former matching.

Equipped with these notions, we can define Pareto efficiency and stability.

Definition 1. A matching M is **Pareto efficient** if there exists no matching $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$.

Definition 2. A matching $M = (M_f)_{f\in\tilde{F}}$ is **stable** if

1. For all $P \in \mathcal{P}$, we have $M_f(\Theta_P) = 0$ for any f satisfying $\emptyset \succ_P f$; and for each $f \in F$, $M_f = C_f(M_f)$, and
2. There exist no $f \in F$ and $M'_f \in \mathcal{X}$ such that $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M)$.

of Theorem 4.

[17] This is known as the Blair order in the literature. See Blair (1984).

Condition 1 of this definition, called **individual rationality**, means that each matched worker prefers the matching over being unmatched, and that no firm wishes to unilaterally drop some of its matched workers. Condition 2, called **no blocking**, requires that there be no firm and a set of workers who are not matched together but want to do so. When Condition 2 is violated by f and M'_f, we say that f and M'_f **block** M.

Remark 2 (Equivalence to group stability). We say that a matching M is **group stable** if Condition 1 of Definition 2 holds and, in addition,

2'. *There exist no* $F' \subseteq F$ *and* $M'_{F'} \in \mathcal{X}^{|F'|}$ *such that* $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M_f)$ for all $f \in F'$.

This definition is a strengthening of our stability concept, as it requires that the matching be immune to blocks by coalitions potentially involving multiple firms. Such stability concepts with coalitional blocks are analyzed by Sotomayor (1999), Echenique and Oviedo (2006), and Hatfield and Kominers (2014), among others. Clearly any group stable matching is stable, because if Condition 2 of stability is violated by a firm f and M'_f, then Condition 2' of group stability is violated by a singleton set $F' = \{f\}$ and $M'_{\{f\}}$. The converse also holds. To see why, note that if Condition 2' of group stability is violated by $F' \subseteq F$ and $M'_{F'}$, then Condition 2 of stability is violated by any f and M'_f such that $f \in F'$ because $M'_f \succ_f M_f$ and $M'_f \sqsubset D^{\preceq f}(M)$ by assumption.[18]

As in the standard finite market, stability implies Pareto efficiency:

Proposition 1. *If a matching is stable, then it is Pareto efficient.*

Proof. See Section S.2.2 of the Supplementary Notes. ∎

[18]By requiring $M'_f \sqsubset D^{\preceq f}(M_f)$ for all $f \in F'$ in Condition 2', our group stability concept is implicitly assuming that workers who are considering participating in a blocking coalition with $f \in F'$ use the current matching M_{-f} as the reference point. This means that workers are available to firm f as long as they prefer f to their current matching. However, given that a more preferred firm $f' \in F'$ may be making offers to workers in $D^{\preceq f}(M_f)$ as well, the set of workers available to f may be smaller. Such a consideration would result in a weaker notion of group stability. Any such concept, however, will be equivalent to our notion of stability, because in this remark we establish that even the most restrictive notion of group stability, i.e., the concept using $D^{\preceq f}(M_f)$ in Condition 2', is equivalent to stability, while stability is weaker than any group stability concept described above.

4 Existence of a Stable Matching in the Continuum Economy

Finding a stable matching is tantamount to identifying the "opportunities" available to each participant that are compatible with opportunities available to all other participants, given their preferences. The mutually compatible opportunity sets are endogenous and of a fixed point character. For each firm f, its opportunities—or the workers available to the firm—in a stable matching are precisely those workers who have no better choice than f. So, to identify such workers, one must identify the opportunities available to the workers, namely the firms that would like to hire them. To identify these firms in turn requires identifying the workers available to them, thus coming to a full circle.

This logic suggests that a stable matching is associated with a fixed point of a mapping—or more intuitively, a stationary point of a process that repeatedly revises the set of available workers to the firms, based on the preferences of the workers and the firms. Formally, we define a map $T : \mathcal{X}^{n+1} \to \mathcal{X}^{n+1}$ where $T(X) = (T_f(X))_{f \in \tilde{F}}$ for each $X \in \mathcal{X}^{n+1}$. For each $f \in \tilde{F}$, the map $T_f : \mathcal{X}^{n+1} \to \mathcal{X}$ is defined by

$$T_f(X)(E) := \sum_{P: f_-^P = \emptyset} G(\Theta_P \cap E) + \sum_{P: f_-^P \neq \emptyset} R_{f_-^P}(X_{f_-^P})(\Theta_P \cap E), \tag{2}$$

where $f_-^P \in \tilde{F}$, called the **immediate predecessor** of f, is a firm that is ranked immediately above firm f according to P.[19] The map can be interpreted as a tâtonnement process whereby an auctioneer quotes "budgets" of workers that firms can choose from. The auctioneer, just like the classical Walrasian one, revises the budget quotes based on the preferences of the market participants, shrinking the budget for a firm f (i.e., making smaller work force available to it) when more workers are demanded by the firms that they rank ahead of f, and expanding it otherwise. Once the process converges, reaching a fixed point, workers who are "truly" available to firms—in the sense of being compatible with the preferences of other market participants—will have been found.

Alternatively, the mapping can be seen as a process by which firms rationally adjust their beliefs about available workers based on the preferences of the other market participants. The fixed point of the mapping then captures the workers firms can iteratively rationalize as being available to them. To illustrate, fix a firm f. Consider first the worker types $\theta \in \Theta_P$ for which f is at the top of their preference P (i.e., $f_-^P = \emptyset$). Firm f can rationally believe

[19] An immediate predecessor of f is formally defined such that $f_-^P \succ_P f$ and if $f' \succ_P f$ for $f' \in \tilde{F}$, then $f' \succeq_P f_-^P$.

all such workers are available to that firm, which explains the first term of (2). Consider next the worker types $\theta \in \Theta_P$ for which f is the second-best according to P. Firm f can rationally believe that among them only those who would be rejected by their top choice firm are available to it, which explains the second term of (2). Now, consider the worker types for which f is their third-best. Firm f analogously rationalizes as being available to it only those among them who would be rejected by their first-best and second-best firms. But the workers who would be rejected by the first-best are available to the second-best, according to the earlier rationalization. This in turn rationalizes f's belief that the workers available to f are precisely those who are available to but rejected by the second-best firm. In general, for any worker types, the same iterative process of belief rationalization establishes the validity of (2).

Remark 3. Our map can be rewritten to mimic Gale and Shapley's deferred acceptance algorithm, where the firms and workers take turns to reject dominated proposals in each round. Specifically, we can write $T = \Psi \circ \Phi$, where, for each profile $X = (X_f)_{f\in\tilde{F}} \in \mathcal{X}^{n+1}$ of workers, the map

$$\Phi_f(X) := G - R_f(X_f)$$

returns the workers that are not rejected by each firm, and for $Y = (Y_f) \in \mathcal{X}^{n+1}$, the subsequent map

$$\Psi_f(Y) := G - \sum_{P\in\mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot),$$

returns workers available for each firm f—more specifically, those that remain after removing the workers who would be accepted by firms they consider better than f. The map written in this way resembles those developed in the context of the finite matching markets (e.g., see Adachi (2000), Hatfield and Milgrom (2005), and Echenique and Oviedo (2006)), but the construction here differs due to dealing with a richer space of worker types. As will be also clear, this new construction is needed for a new method of proof for characterizing the fixed point.

Theorem 1. *A matching M is stable if and only if there is a fixed point $X = T(X)$ such that $M_f = C_f(X_f), \forall f \in \tilde{F}$. Also, any such X and M satisfy $X_f = D^{\preceq f}(M), \forall f \in \tilde{F}$.*

Proof. This result follows as a corollary of Theorem 7. For details, see Appendix A. ∎

Example 1. To illustrate how a stable matching can be found from T mapping, consider our leading example. We can denote candidate measures of available workers by a tuple: $(X_{f_1}, X_{f_2}) = (x_1, x_1'; x_2, x_2') \in [0, \frac{1}{2}]^4$, where $X_{f_1} = (x_1, x_1')$ are the measures of workers of types θ and θ' available to f_1 and $X_{f_2} = (x_2, x_2')$ are the measures of workers of types θ and

θ' available to f_2. Since f_1 is most preferred by θ and f_2 is most preferred by θ', according to our T, all of these workers are available to the respective firms. So, we can without loss set $x_1 = G(\theta) = \frac{1}{2}$ and $x'_2 = G(\theta') = \frac{1}{2}$ and consider $(\frac{1}{2}, x'_1; x_2, \frac{1}{2})$ as our candidate measures. To compute $T(\frac{1}{2}, x'_1; x_2, \frac{1}{2})$, we first consider the choice by each firm given the respective available worker sets. Note $C_{f_1}(\frac{1}{2}, x'_1) = (x'_1, x'_1)$, so $R_{f_1}(\frac{1}{2}, x'_1) = (\frac{1}{2} - x'_1, 0)$. Similarly, $C_{f_2}(x_2, \frac{1}{2}) = (x_2, \frac{1}{2} - x_2)$, so $R_{f_2}(x_2, \frac{1}{2}) = (0, x_2)$. Now applying our formula in (2), we get $T(\frac{1}{2}, x'_1; x_2, \frac{1}{2}) = (\frac{1}{2}, x_2; \frac{1}{2} - x'_1, \frac{1}{2})$. So, $(\frac{1}{2}, x'_1; x_2, \frac{1}{2})$ is a fixed point of T if and only if $(\frac{1}{2}, x'_1; x_2, \frac{1}{2}) = (\frac{1}{2}, x_2; \frac{1}{2} - x'_1, \frac{1}{2})$, or $x'_1 = x_2 = \frac{1}{4}$. The optimal choice from the fixed point then gives a matching

$$M = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{4}\theta + \frac{1}{4}\theta' & \frac{1}{4}\theta + \frac{1}{4}\theta' \end{pmatrix},$$

where the notation here (analogous notation is used throughout) means that each of the firms f_1 and f_2 is matched to mass $\frac{1}{4}$ of worker types θ and θ'. This matching M is stable.

We now introduce a condition on the firms' preferences that ensures existence of a stable matching.

Definition 3. Firm f's preference is **continuous** if, for any sequence $(X_k)_{k\in\mathbb{N}}$ and X in $\mathcal{X}$ such that $X_k \xrightarrow{w^*} X$, it holds that $C_f(X_k) \xrightarrow{w^*} C_f(X)$.

As suggested by the name, continuity of a firm's preferences means that the firm's choice changes continuously with the distribution of workers available to it. Under this assumption, we obtain a general existence result as follows:

Theorem 2. *If each firm's preference is continuous, then there exists a stable matching.*

Proof. This result follows as a corollary of Theorem 4. For details, see Appendix A. ∎

To prove that T admits a fixed point, we first demonstrate that continuity of firms' preferences implies that mapping T is also continuous. We also verify that $\mathcal{X}$ is a compact and convex set. Continuity of T and compactness and convexity of $\mathcal{X}$ allow us to apply the Kakutani-Fan-Glicksberg fixed-point theorem to guarantee that T has a fixed point. Then, the existence of a stable matching follows from Theorem 1, which shows the equivalence between the set of stable matchings and the set of fixed points of T.

Many complementary preferences are compatible with continuous preferences and thus with existence of a stable matching. Recall Example 1. In that example, firm f_1 has a Leontief type preference, for it wishes to hire an equal measure of workers of types θ and θ' (so, in particular, the firm wants to hire type-θ workers only if type-θ' workers are also available, and vice versa). As seen in Example 1, a stable matching exists despite the extreme complementarity. And the reason has to do with the continuity: As the firm's preferences are clearly continuous in that example, the existence of a stable matching in that example is implied by Theorem 2.

A stable matching may fail to exist even in the continuum economy unless all firms have continuous preferences, as the following example illustrates.

Example 2 (Role of continuity)**.** Consider the following economy, which is modified from Example 1. There are worker types θ and θ' (each with measure 1/2) and firms f_1 and f_2. Firm f_1 wants to hire only measure 1/2 of each of the two types, and would like to be unmatched otherwise; meanwhile, firm f_2's choice function is continuous: it exhibits "responsive" preferences preferring type-θ workers to type-θ' workers and in turn prefers the latter to leaving a position vacant, and faces a capacity of measure 1/2. Then, C_{f_1} violates continuity, while C_{f_2} does not. As before, we assume

$$\theta : f_1 \succ f_2;$$
$$\theta' : f_2 \succ f_1.$$

No stable matching exists in this environment. To see this, consider the following two cases:

1. Suppose f_1 hires measure 1/2 of each type of workers. For such a matching, none of the capacity of f_2 is filled. Thus such a matching is blocked by f_2 and type-θ' workers (note that every type-θ' worker is currently matched with f_1, so they are willing to participate in the block).

2. Suppose f_1 hires no worker. Then, the only candidate for a stable matching is one in which f_2 hires measure 1/2 of type-θ workers (or else, f_2 and unmatched workers of type θ would block the matching). Then, since f_1 is most preferred by all θ workers, and type-θ' workers prefer f_1 to ø, the matching is blocked by a coalition of measure 1/2 of type-θ workers, measure 1/2 of type-θ' workers, and f_1.

The continuity assumption is important for existence of a stable matching, as this example shows that nonexistence can happen even if only one firm has a discontinuous choice function. This example also suggests that non-existence can reemerge once some "lumpiness" is reintroduced into the continuum economy (i.e., one firm can only hire a certain minimum mass of workers). However, this kind of lumpiness may not be the most natural in a continuum economy in contrast to a finite economy, where lumpiness is a natural consequence of the fact that each worker is indivisible.

5 Approximate Stability in Finite Economies

As we have seen in the illustrative example of Section 2, no matter how large the economy is, as long as it is finite, there does not necessarily exist a stable matching. This motivates us to look for an approximately stable matching in a large finite economy. In this section, we build on the existence of a stable matching in the continuum economy to demonstrate that approximate stability can be achieved if the economy is finite but sufficiently large.

In order to analyze economies of finite sizes, we consider a sequence of economies $(\Gamma^q)_{q\in\mathbb{N}}$ indexed by a positive integer q. In each economy Γ^q, there is a set of n firms $f_1, \ldots, f_n$ which is fixed across all q. There are also q workers, each with a type in Θ. The worker distribution is normalized with the economy's size. Formally, let the (normalized) population G^q of workers in Γ^q be defined so that $G^q(E)$ represents the number of workers with types in E divided by q. A subpopulation X^q is **feasible** in economy Γ^q if $X^q \sqsubset G^q$ and it is a measure whose value for any E is a multiple of $1/q$. Let $\mathcal{X}^q$ denote the set of all feasible subpopulations. Note that G^q, and thus every $X^q \in \mathcal{X}^q$, belongs to $\overline{\mathcal{X}}$, though it does not have to be an element of $\mathcal{X}$, i.e. subpopulation of G. Let us say that a sequence of economies $(\Gamma^q)_{q\in\mathbb{N}}$ **converges to a continuum economy** Γ if the measure G^q of worker types converges weakly to the measure G of the continuum economy, that is, $G^q \xrightarrow{w^*} G$.

In order to formalize the approximate stability concept, we assume that in economy Γ^q, each firm f evaluates the set of workers it matches with using its preferences as in the continuum economy, but with the distribution of workers normalized by the economy's size. In particular, we first endow the firms with cardinal utility functions over distributions of workers. Let $u_f : \overline{\mathcal{X}} \to \mathbb{R}$ denote the continuous utility function of firm f, with $u_f(X)$ denoting the firm's utility from matching with a subpopulation of workers $X \in \overline{\mathcal{X}}$. This utility function rationalizes firm f's choice in both continuum and finite economies, in the sense that firm f chooses a subpopulation that is feasible and maximizes u_f in the respective economies.[20] We assume that the choice function C_f in the continuum economy is continuous and a unique maximizer of u_f: that is, for any $X \in \mathcal{X}$,

$$C_f(X) = \underset{X' \sqsubset X}{\arg\max}\, u_f(X'),$$

is a singleton.[21] That the same utility function applies to finite as well as continuum economies means that the preferences of the firms remain constant (or consistent) over a

[20] To guarantee existence of such a utility function, we may assume as in Remark 1 that each firm is endowed with a complete, continuous preference relation. Then, since the set of alternatives, $\overline{\mathcal{X}}$, is a compact metric space, such a preference can be represented by a continuous utility function according to the Debreu representation theorem (Debreu, 1954).

[21] The continuity of C_f follows from the continuity of u_f and the uniqueness of the maximizer.

sequence of economies $(\Gamma^q)_{q\in\mathbb{N}}$ and its limit Γ. In each finite economy Γ^q, for any subpopulation of available workers $X^q \in \mathcal{X}^q$, each firm f chooses a subpopulation of X^q in $\mathcal{X}^q$ that maximizes its utility u_f. The continuity of the utility function then implies that each firm's optimal choice in the finite economy (which we do not assume to be unique) converges to the optimal choice in the limit economy as $q \to \infty$.[22] An example is a sequence of replica economies in which each firm has a responsive preference that is constant throughout the sequence, but faces capacity of rk in the r-replica (where q increases proportionately to r as the latter increases).

A matching in finite economy Γ^q is $M^q = (M^q_f)_{f\in\tilde{F}}$ such that $M^q_f \in \mathcal{X}^q$ for all $f \in \tilde{F}$ and $\sum_{f\in\tilde{F}} M^q_f = G^q$. The measure of available workers for each firm f at matching $M^q \in (\mathcal{X}^q)^{n+1}$ is $D^{\preceq f}(M^q)$, where $D^{\preceq f}(\cdot)$ is defined as in (1).[23] Note that as M^q_f for each $f \in \tilde{F}$ is a multiple of $1/q$, so is $D^{\preceq f}(M^q)$, meaning that $D^{\preceq f}(M^q)$ is feasible in Γ^q. We now define ϵ-stability in finite economy Γ^q.

Definition 4. A matching $M^q \in (\mathcal{X}^q)^{n+1}$ in economy Γ^q is **ϵ-stable** if (i) it is individually rational, and (ii) $u_f(\tilde{M}^q) < u_f(M^q_f) + \epsilon$ for any $f \in F$ and $\tilde{M}^q \in \mathcal{X}^q$ with $\tilde{M}^q \sqsubset D^{\preceq f}(M^q)$.

Condition (i) of this definition is identical to the one for exact stability, so the only relaxation of ϵ-stability compared to stability is with respect to its condition (ii). Note ϵ-stability does not require the absence of a blocking coalition,[24] but in case a blocking coalition exists, it requires the utility gain for the blocking firm to be small, specifically less than ϵ.

Remark 4. For $\epsilon > 0$, we say that matching M is **ϵ-Pareto efficient** if there exists no matching $M' \neq M$ and firm $f \in F$ such that $M' \succeq_F M$, $M' \succeq_\Theta M$, and $u_f(M'_f) \geq u_f(M_f) + \epsilon$. By an argument analogous to Pareto efficiency of a stable matching presented in Section 3, it is easy to see that for sufficiently small $\epsilon' > 0$, any ϵ'-stable matching is ϵ-Pareto efficient.

Theorem 3. *Fix any $\epsilon > 0$ and a sequence of economies $(\Gamma^q)_{q\in\mathbb{N}}$ that converges to a continuum economy Γ. For any sufficiently large q, there exists an ϵ-stable matching M^q in Γ^q.*

Proof. See Appendix B. ▮

[22] This statement is formally proved in Lemma 7 of Appendix B.

[23] To be precise, $D^{\preceq f}(M^q)$ is given as in (1) with G and X being replaced by G^q and M^q, respectively.

[24] A blocking coalition is unavoidable if a stable matching does not exist, as is demonstrated in Section 2.

Remark 5. One could define approximate stability slightly differently. Say a matching M^q is **ϵ-distance stable** if (i) it is individually rational and (ii) $|\tilde{M}_f^q - M_f^q| < \epsilon$ for any coalition f and $\tilde{M}_f^q \in \mathcal{X}^q$ that blocks M^q in the sense that $\tilde{M}_f^q \sqsubset D^{\preceq_f}(M^q)$ and $u_f(\tilde{M}_f^q) > u_f(M_f^q)$.[25] In words, if a matching M^q is ϵ-distance stable, then any alternative matching a firm may propose for blocking must be within distance ϵ from the original matching. One advantage of this concept is that it is an ordinal concept, that is, we do not need to endow the firms with cardinal utility functions in order to formalize the notion. Note also that the notion requires the ϵ bound for *any* blocking coalition, not just the "optimal" blocking coalition in the sense defined in Definition 2-2, making the notion more robust than the standard one. Matching M^q constructed in the proof of Theorem 3 is ϵ-distance stable for any sufficiently large q, as shown in the proof.

Below we revisit the example in Section 2 to give a concrete example of an approximately stable matching.

Example 3. Recall the finite economy described in Section 2.[26] If the index q is odd, then a stable matching does not exist. Let us consider the following matching: firm f_1 matches with $\frac{q+1}{2}$ workers of each type and firm f_2 matches with all remaining workers, i.e. $\frac{q-1}{2}$ workers of each type. After normalization (i.e., dividing by $2q$), we obtain a matching

$$\begin{pmatrix} f_1 & f_2 \\ \frac{q+1}{4q}\theta + \frac{q+1}{4q}\theta' & \frac{q-1}{4q}\theta + \frac{q-1}{4q}\theta' \end{pmatrix},$$

which converges to the stable matching in the limit economy. Given this matching, it is straightforward to see that there is no blocking coalition involving firm f_1. Also, the only blocking coalition involving firm f_2 entails taking only a single worker of type θ' away from firm f_1. Thus, for any $\epsilon > 0$, if q is sufficiently large, then the above matching is ϵ-distance stable, as well as ϵ-stable (provided the firms' utility functions are continuous).

[25] The notion of distance $|\cdot|$ is Prokhorov metric that metrizes the weak-$*$ topology on a measure space. Specifically, for any $X, Y \in \mathcal{X}$, we let

$$|X - Y| := \inf \{\varepsilon > 0 \mid X(E) \leq Y(E^\varepsilon) + \varepsilon \text{ and } Y(E) \leq X(E^\varepsilon) + \varepsilon \text{ for all } E \in \Sigma\},$$

where $E^\varepsilon := \{\theta \in \Theta \mid \exists \theta' \in E,\ |\theta - \theta'| < \varepsilon\}$.

[26] With a slight abuse of notation, this example assumes that there are a total of $2q$ workers (q workers of θ and θ' each) rather than q. Of course, this is done for purely expositional purposes.

6 Indifferences, Fairness, and Strategy-Proofness

In this section, we generalize our continuum economy framework to allow for firms that are indifferent over different subpopulations of workers. There are at least three motivations for this generalization. First, there is a continuum of workers in our environment, and in such a situation it is natural to allow for a firm to be indifferent across some subpopulations and choose more than one subpopulation as the most preferred. Second, indifferences appear to be inherent in some applications. In school choice, for instance, schools are often required by law to regard many students to have the same priority,[27] in which case school preferences encoding the priority will exhibit indifferences over students. Last but not least, our continuum model can be applied to study "time share" or "probabilistic" matching in a finite market. In the time share model, a finite set of workers contract with a finite set of firms for fractional time share. As we describe more precisely later, we can use the continuous measures of workers in our large market model to represent "time units" a single worker may supply in this finite economy model. Hence, to study a firm (e.g., a school) indifferent over a set of workers (e.g., students in the same priority class) in the latter model, we must allow a firm to be indifferent across different types of workers in our large market model.

Formally, firms' indifferences can be accommodated by extending their choice functions to be multi-valued, namely by introducing choice correspondences for firms. The notion of stability as well as its characterization and existence must be extended for this new environment. We therefore begin with this generalization in Section 6.1. Our main focus will be, however, on the two new issues that become important in this new environment.

First, an important new question is how equitably the workers who are perceived as equivalent by a firm are treated in the matching. As mentioned in the introduction, stability involves a desirable fairness property, and in particular our stable matching (implicitly) treats the workers of the same type (in terms of their preferences as well as firms' preferences over them) identically. But our stability notion alone permits many different possible matchings, and some of them treat workers differently even if they are perceived as equivalent by the firms. As pointed out by Kesten and Ünver (2014), a stronger notion of fairness than implied by the standard notion of stability is not only attainable but may be important in settings such as school choice.

[27] In the public school choice program in Boston prior to 2005, for instance, a student's priority at a school is based only on coarse criteria such as the student's residence and whether her sibling is currently enrolled at that school. Consequently, at each school, many students are equipped with the same priority (Abdulkadiroğlu et al., 2005).

Second, firms' indifferences make workers' incentives for truthful reporting nontrivial. In a large market setting, each worker is insignificant in her ability to influence the matching through her preference reporting; more precisely, a matching in our continuum economy framework is insensitive to a change in the measure zero types. This does not mean, however, that the incentive problem disappears on the worker side. A worker may still influence her (probabilistic) assignment by manipulating her preference report, if the workers with different preference but same productivity types—note they are perceived as equally desirable from the firms' perspectives—are treated differently. As it turns out, the incentive issue is closely related to the fairness issue. In Section 6.2, we propose a mechanism that will solve both issues and show that such a mechanism exists. Section 6.3 applies the result to a general time share model.

6.1 Stable Matching with Choice Correspondence

In order to accommodate firms' indifferences, we introduce choice correspondences for firms. Let $C_f : \mathcal{X} \rightrightarrows \mathcal{X}$ be a **choice correspondence**: i.e., for any $X \in \mathcal{X}$, $C_f(X) \subset \mathcal{X}_X$ is the set of subpopulations of X that are the most preferred by f among all subpopulations of X (recall $\mathcal{X}_X$ is the set of all subpopulations of X). By convention, we let $C_\varnothing(X) = \{X\}, \forall X \in \mathcal{X}$. Then, let $R_f(X) := X - C_f(X)$, or equivalently $R_f(X) = \{Y \in \mathcal{X} \,|\, Y = X - X'$ for some $X' \in C_f(X)\}$. We assume that for any $X \in \mathcal{X}$, $C_f(X)$ is nonempty. Assume further that $C_f(\cdot)$ satisfies the revealed preference property: For any $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, if $C_f(X) \cap \mathcal{X}_{X'} \neq \emptyset$, then $C_f(X') = C_f(X) \cap \mathcal{X}_{X'}$. Define a function $D^{\preceq f} : \mathcal{X}^{n+1} \to \mathcal{X}$ for each $f \in F$ in the same manner as in (1).

Definition 5. A matching M is **stable** if

1. For all $P \in \mathcal{P}$, we have $M_f(\Theta_P) = 0$ for any f satisfying $\varnothing \succ_P f$; and for each $f \in F$, $M_f \in C_f(M_f)$, and
2. There exist no $f \in F$ and $M'_f \in \mathcal{X}$ such that $M'_f \sqsubset D^{\preceq f}(M)$, $M'_f \in C_f(M'_f \vee M_f)$, and $M_f \notin C_f(M'_f \vee M_f)$.

Clearly, this definition is a generalization of Definition 2 to the case of choice correspondences. The existence of a stable matching then follows from existence of a fixed point for a correspondence operator defined analogously to the mapping T in Section 4. To this end, we extend the notion of continuous preferences as follows:

Definition 6. The firm f's choice correspondence C_f is **upper hemicontinuous** if, for any sequences $(X^k)_{k\in\mathbb{N}}$ and $(\tilde{X}^k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ such that $X^k \xrightarrow{w^*} X$, $\tilde{X}^k \xrightarrow{w^*} \tilde{X}$, and $\tilde{X}^k \in C_f(X^k), \forall k$, we have $\tilde{X} \in C_f(X)$.[28]

By Berge's maximum theorem (see, for instance, Ok (2011) for a textbook treatment), an upper hemicontinuous choice correspondence arises when a firm has a utility function $u : \mathcal{X} \to \mathbb{R}$ that is continuous in weak-$*$ topology.

Theorem 4. *Suppose that for each $f \in F$, C_f is convex-valued and upper hemicontinuous. Then, there exists a stable matching.*

Proof. See Appendix A. ∎

This result shows that our main result — that there exists a stable matching when there is a continuum of workers — does not hinge on the restrictive assumption that each firm's choice is unique. On the contrary, this result holds for a wide range of specifications that allow for indifferences and choice correspondences. As will be seen in the next section, this generalization turns out to be useful when analyzing fairness and incentive compatibility properties of matching mechanisms.

6.2 Fairness and Incentive Compatibility

In this section, we study the fairness and incentive issues that arise in a general model in which firms may be indifferent over a set of workers. For this purpose, it is convenient to describe the type of each worker by a pair $\theta = (a, P)$, where a denotes the worker's productivity or skill, and P is her preference for firms and the outside option, as before. We assume that a worker's preference does not affect firms' preferences and is private information, whereas the productivity type may affect firms' preferences and is observable to the firms (and also to the mechanism designer). Let A and $\mathcal{P}$ be the sets of productivity types and preference types, respectively, and $\Theta = A \times \mathcal{P}$. We assume that A is a finite set.[29] This implies that Θ is a finite set, so the population G of worker types is a discrete measure.

[28]This definition is often referred to as the "closed graph property," which implies (the standard definition of) upper hemicontinuity and closed-valuedness if the range space is compact, as is true in our case.

[29]Finiteness of A is necessitated by our use of weak-$*$ topology and the construction of strong stability and strategy-proof mechanisms below. To illustrate the difficulty, suppose A were a unit interval, say, and G has a well defined density. Our construction below would require that the density associated with firms' choice mappings satisfy a certain population-proportionality property. Convergence in our weak-$*$ topology does not preserve this restriction on density. As a consequence, the operator T may violate upper hemicontinuity, which may result in the failure of the nonempty-valuedness of our solution. It may

As before, a matching is described by a profile $M = (M_f)_{f \in \tilde{F}}$ of subpopulations of workers matched with alternative firms or the outside option (note that given discreteness of G, each matching M can be expressed as a profile of discrete distributions). We assume that all workers of the same (reported) type are treated ex ante identically. Hence, given matching M, a worker of type (a, P) in the support of G is matched to $f \in \tilde{F}$ with probability $\frac{M_f(a,P)}{G(a,P)}$. Note that $\sum_{f \in \tilde{F}} \frac{M_f(a,P)}{G(a,P)} = 1$ holds by construction, giving rise to a valid probability distribution over $\tilde{F}$. A **mechanism** is a function φ that maps any $G \in \overline{\mathcal{X}}$ to a matching.

We next describe firms' preferences. To accommodate indifferences, we let a partition $\mathbb{P}_f := \{\Theta_f^1, ..., \Theta_f^{K_f}\}$ of Θ denote the set of indifference classes for firm f. The partition means that firm f is indifferent over matching with different types of workers as long as the total measure of workers within each group Θ_f^k is unchanged. Let $I_f = \{1, ..., K_f\}$ be the associated index set. Note that the partitional structure may be arbitrary; in particular, it may differ across different firms.

We next assume that, for any given measure $X \sqsubset G$ of available workers, the firm has a unique optimal choice in terms of the measure of workers in each indifference class. Formally, for each $k \in I_f$, we let $\Lambda_f^k : \mathcal{X} \to \mathbb{R}_+$ denote firm f's unique choice of total measure of workers in the indifference class Θ_f^k. Specifically, we assume that for each $X \sqsubset G$, $\Lambda_f^k(X) \in [0, \sum_{\theta \in \Theta_f^k} X(\theta)]$ and $\Lambda_f^k(X') = \Lambda_f^k(X)$ whenever $\sum_{\theta \in \Theta_f^{k'}} X'(\theta) = \sum_{\theta \in \Theta_f^{k'}} X(\theta)$ for all $k' \in I_f$. We also assume that $\Lambda_f^k(X') = \Lambda_f^k(X)$ whenever $\Lambda_f^{k'}(X) \leq \sum_{\theta \in \Theta_f^{k'}} X'(\theta) \leq \sum_{\theta \in \Theta_f^{k'}} X(\theta)$ for all $k' \in I_f$, which captures the revealed preference property. Finally, we assume that $\Lambda_f^k(X)$ is continuous in $(\sum_{\theta \in \Theta_f^{k'}} X(\theta))_{k' \in I_f}$ (in the Euclidean topology).

The resulting economy described by $(G, F, \mathbb{P}_F, \Lambda_F)$, where $\mathbb{P}_F = (\mathbb{P}_f)_{f \in F}$, $\Lambda_f := (\Lambda_f^k)_{k \in I_f}$ for each $f \in F$ and $\Lambda_F = (\Lambda_f)_{f \in F}$, is called a **general indifferences model**. It is easy to see that a general indifference model admits a stable matching. Observe that the functions $\Lambda_f = (\Lambda_f^k)_{k \in I_f}$ induce a choice correspondence for firm f, given by

$$C_f(X) := \{Y \sqsubset X \mid \sum_{\theta \in \Theta_f^k} Y(\theta) = \Lambda_f^k(X), \forall k \in I_f\}, \tag{3}$$

for each $X \in \mathcal{X}$. One can see that C_f is convex-valued and upper hemicontinuous, and satisfies the revealed preference property (see Lemma 8 in Appendix C). It then follows by Theorem 4 that a stable matching exists in the general indifference model.

To motivate the fairness and incentive issues that arise with indifferences, we begin with an example.

be possible to address this issue by strengthening the topology, but whether the resulting space satisfies conditions that would guarantee the existence of a stable matching is an open question.

Example 4. There are two firms f_1 and f_2, and a unit mass of workers with two productivity types a and a'. Firm f_1 wishes to maximize $\min\{x, x'\}$, where x and x' are the measures of workers with productivity types a and a', respectively. Firm f_2 has a responsive preference with capacity 1/2 and prefers a-productivity type to a'-productivity type. There are three possible worker preferences:

$$
\begin{aligned}
P &: f_1 \succ f_2 \succ \emptyset;\\
P' &: f_2 \succ f_1 \succ \emptyset;\\
P'' &: f_2 \succ \emptyset \succ f_1.
\end{aligned}
$$

Letting $\theta = (a, P)$, $\theta' = (a', P')$, and $\theta'' = (a', P'')$, the type distribution is given as $G(\theta) = 1/2$ and $G(\theta') = 1/4 = G(\theta'')$. (Note that this example is the same as our leading example except mass 1/4 of type θ' workers now have a new preference P''.)

Consider first a mechanism that maps G to matching

$$
M = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{4}\theta + \frac{1}{4}\theta' & \frac{1}{4}\theta + \frac{1}{4}\theta'' \end{pmatrix}.
$$

This matching is stable, which can be seen by the fact that the firms are matched with the same measures of productivity types as in the stable matching in our leading example. Observe, however, that the matching treats the type-θ' and type-θ'' workers differently—the former workers match with f_1 and the latter workers match with f_2 (which they both prefer)—despite the fact that the firms perceive them as equivalent. This lack of "fairness" leads to an incentive problem: type-θ' workers have an incentive to (mis)report to be of type θ'' and thereby match with f_2 instead f_1.

These problems can be addressed by another mechanism that maps G to matching

$$
\bar{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{6}\theta + \frac{1}{6}\theta' & \frac{1}{3}\theta + \frac{1}{12}\theta' + \frac{1}{12}\theta'' \end{pmatrix}.
$$

This matching is stable just like M, but in addition firm f_2 treats type-θ' and type-θ'' workers identically in this matching. Further, neither type θ' nor type θ''-workers have incentive to misreport. (One can show that this is a unique matching that satisfies both stability and incentive compatibility.)

This example suggests that stability alone does not guarantee equal treatment of workers perceived as equally desirable by the firms and for that reason may also fail strategy-proofness. It thus motivates strengthening the notion of stability. In particular, we consider the following (stronger) notion of fairness, proposed by Kesten and Ünver (2014).

Definition 7. A matching M is **strongly stable** if (i) it is stable and (ii) for any $f \in F$, $k \in I_f$, and $\theta, \theta' \in \Theta_f^k$, if $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$, then $\sum_{f' \in \tilde{F}: f' \prec_\theta f} M_{f'}(\theta) = 0$.

In words, strong stability requires that if a worker type θ is assigned a firm f with strictly lower probability than another type θ' in the same indifference class for firm f, then the type-θ worker should never be assigned any firm f' the worker ranks below f. In that sense, the workers in the same priority class should not be discriminated against one another. This is an additional requirement that is not implied by stability (which requires fairness, or more precisely, elimination of justified envy, across workers that a firm is *not* indifferent among).

We next discuss incentive compatibility of a mechanism. To this end, we assume as before that the workers evaluate lotteries via the partial order given by first-order stochastic dominance, and define strategy-proofness as follows:

Definition 8. φ is **strategy-proof for workers** if, for each (reported) population $G \in \overline{\mathcal{X}}$, productivity type $a \in A$, preference types P and P' in $\mathcal{P}$ such that both (a, P) and (a, P') are in the support of G, and $f \in \tilde{F}$, we have

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} \geq \sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')}. \tag{4}$$

Some comments on our modeling assumptions are in order. First, a worker can misreport only her preference type, and not her productivity type (recall that a worker's productivity type determines firms' preferences on her): this assumption is the same as in the standard setting in the literature. Second, unlike in finite population models, the worker cannot alter the population G by unilaterally misreporting her preferences, because there are a continuum of workers. Third, we only impose the restriction (4) for types (a, P) and (a, P') that are in the support of G. For the true worker type (a, P), this is the same assumption as in the standard strategy-proofness concept for finite markets. We do not impose any condition for misreporting a measure-zero type, since if φ is a mechanism that is individually rational (which is the case for stable mechanisms), then the incentives for misreporting to be a measure-zero type can be eliminated by specifying the mechanism to assign a worker reporting such a type to the outside option with probability one. Finally, strategy-proofness is closely related to a standard fairness property. We say that a matching M is **envy-free** if, for each productivity type $a \in A$, as well as preference types P and P' in $\mathcal{P}$ such that both (a, P) and (a, P') are in the support of G, the matching for a worker of type (a, P) weakly first-order stochastically dominates the matching for type (a, P') at preference P. Clearly, this definition is "equivalent" to strategy-proofness in the sense that a mechanism φ is strategy-proof if and only if $\varphi(G)$ is envy-free for every worker population G. This connection between strategy-proofness and envy-freeness has been observed in the literature on mechanisms in large markets, by Che and Kojima (2010) for instance.

We are now ready to state our main result. Our approach is to show existence of a stable matching that satisfies an additional property. Say a matching M is **population-proportional** if, for each $f \in F$ and $k \in I_f$, there is some $\alpha_f^k \in [0,1]$ such that

$$M_f(\theta) = \min\{D^{\preceq f}(M)(\theta), \alpha_f^k G(\theta)\}, \forall \theta \in \Theta_f^k. \tag{5}$$

In words, the measure of workers hired by firm f from the indifference class Θ_f^k is given by the same proportion α_f^k of $G(\theta)$ for all $\theta \in \Theta_f^k$, unless the measure of worker types θ available to f is less than the proportion α_f^k of $G(\theta)$, in which case the entire available measure of that type is assigned to that firm. In short, a population-proportional matching seeks to match a firm with workers of different types in proportion to their population sizes at G whenever possible, if they belong to the same indifference class for the firm. The stability and population-proportionality of a mechanism translate into the desired fairness and incentive properties, as the following result shows.

Theorem 5. *(i) If a matching is stable and population-proportional, then it is strongly stable.*

(ii) If a mechanism φ implements a strongly stable matching for every measure in $\overline{\mathcal{X}}$, then the mechanism is strategy-proof for workers.

Proof. See Appendix C. ∎

The following result then guarantees existence of a stable and population-proportional matching.

Theorem 6. *For any population of workers, there exists a matching which is stable and population-proportional.*

Proof. See Appendix C. ∎

A key step in the proof of this result is to select from each firm's choice correspondence C_f an optimal choice $\tilde{C}_f \in C_f$ that satisfies population-proportionality. The selection $\tilde{C}_f$ is then shown to be continuous, so by Theorem 2, a "stable matching" exists in the "hypothetical" continuum economy in which firms have preferences represented by the choice functions $\tilde{C}_f$. The last step is to show that the stable matching of the hypothetical economy is in turn stable in the original economy and satisfies population-proportionality.

This result establishes the existence of a matching that satisfies strong stability and strategy-proofness for workers for the large economy environment. In contrast to the existing literature, our result holds under general preferences of firms that may involve both indifferences and complementarities.

Remark 6 (Non-strategy-proofness for firms). Even with a continuum of workers, no stable mechanism is strategy-proof for the firms. To show this fact, consider the following example.[30] Let $F = \{f_1, f_2\}$, $\Theta = \{\theta, \theta'\}$, and $G(\theta) = G(\theta') = 1/2$. Worker preferences are given as follows:

$$\theta : f_2 \succ f_1 \succ \emptyset,$$
$$\theta' : f_1 \succ f_2 \succ \emptyset.$$

Firm preferences are responsive; f_1 prefers θ to θ' to vacant positions, and wants to be matched with workers of up to measure 1; and f_2 prefers θ' to θ to vacant positions, and wants to be matched with workers of up to measure 1/2.

Let φ be any stable mechanism. Given the above input, the following matching is the unique stable matching:

$$M \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta' & \frac{1}{2}\theta \end{pmatrix}.$$

Matching M is clearly stable because it is individually rational and every worker is matched to her most preferred firm. To see the uniqueness, note first that in any stable matching, every worker has to be matched to a firm (if there is a positive measure of unmatched workers, then there is also a vacant position in firm f_1, and they block the matching). All workers of type θ' are matched with f_1, because otherwise f_1 and θ' workers who are not matched with f_1 block the matching (note that f_1 has vacant positions to fill with θ' workers). Given this, all workers of type θ are matched with f_2, because otherwise f_2 and θ workers who are not matched with f_2 block the matching (note that f_2 has vacant positions to fill with type θ workers).

Now, assume that f_1 misreports its preferences, declaring that θ is the only acceptable worker type, while it still has a responsive preferences and wants to be matched to the same measure as before, 1. And assume that preferences for other agents are unchanged.

[30] This example is a continuum-population variant of an example in Section 3 of Hatfield, Kojima and Narita (2014*a*). See also Azevedo (2014) who shows that stable mechanisms are manipulable via capacities even in markets with continuum of workers.

Then, it is easy to verify that the unique stable matching is

$$M' \equiv \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta & \frac{1}{2}\theta' \end{pmatrix}.$$

Therefore, firm f_1 prefers its outcome at M' to the one at M, proving that no stable mechanism is strategy-proof for the firms.

6.3 Applications to Time-Share/Probabilistic Matching Models

Our general indifference model introduced in Section 6.2 has a connection with time share and probabilistic matching models. In these models, a *finite* set of workers contract with a finite set of firms for time shares or they are matched probabilistically. Probabilistic matching is often used in allocation problems without money such as school choice, while time-share models have been proposed as a solution to labor matching markets in which part-time jobs are available (see Biró, Fleiner and Irving (2013) for instance).

Our general indifference model can be reinterpreted as a time-share model. Let Θ be the *finite* set of workers whose shares firms may contract for, *as opposed to the finite types of continuum of workers.* The measure $G(\theta)$ represents the total endowment of time or probability that a worker θ has available for matching. A matching M describes time or probability $M_f(\theta)$ that a worker θ and a firm f are matched.

The partition $\mathbb{P}_f$ then describes firm f's set of indifference classes, where each class describes a set of workers the firm finds equivalent. The function $\Lambda_f = (\Lambda_f^k)_{k \in I_f}$ describes the time shares firm f wishes to choose from available time shares in alternative indifference classes. The structure of indifferences as well as firms' preference we permit is very general. Recall that our indifferences model allows different firms to be indifferent across different sets of workers. On the worker side, for each worker $\theta \in \Theta_P$, the first-order stochastic dominance induced by P describes the preference of the worker in evaluating lotteries. With this reinterpretation, Definitions 5 and 7 provide appropriate notions of a stable matching and a strongly stable matching, respectively.[31]

[31] The notion of strong stability in Definition 7 requires the proportion of time spent with a firm out of total endowment to be equalized among workers that the firm considers equivalent. This notion is sensible in the context of time share model, particularly when $G(\theta)$ is the same across all workers, as with school choice (where each student has a unit demand). When $G(\theta)$ is different across θ's, however, one could consider an alternative notion, for instance one that equalizes the absolute amount of time (not divided by $G(\theta)$) a worker spends with a firm. Our analysis can be easily modified to prove the existence of matching that satisfies this alternative notion of strong stability.

We call a time share model induced in this way by our general indifference model a **general time share model**. This model clearly encompasses a broad class of existing time share models, both in terms of the structure of indifferences that it permits as well as firms' preferences, which include non-substitutable as well as substitutable preferences. The only significant assumption is the continuity of Λ_f. The following result is immediate:

Corollary 1. *A general time share model admits a strongly stable—and thus stable—matching.*[32]

This result generalizes the existence of a strongly stable matching in the school choice problem studied by Kesten and Ünver (2014), where schools may regard multiple students as having the same priority. In their model, there are a finite number of students and a matching specifies a profile of match probabilities (so each worker θ represents a student and $G(\theta) = 1$ for all θ). They define probabilistic matchings that satisfy our strong stability property (which they call strong ex ante stability), and show their existence under the assumption that schools have responsive preferences with ties. In their context, strong stability requires—in addition to stability—that two students with the same priority at a school be assigned to that school with the same probability, unless the student with the smaller probability is matched with each of her less preferred outcomes with zero probability. Our contribution here lies in establishing the existence of a stable matching with general preferences that may violate responsiveness or even substitutability. Our result could be useful for school choice environments in which the schools may need a balanced student body in terms of gender, ethnicity, income, or skill levels. For example, in New York City, the so-called Education Option (EdOpt) school programs are required to assign 16 percent, 68 percent, and another 16 percent of the seats to the top performers, middle performers, and the lower performers, respectively (Abdulkadiroğlu, Pathak and Roth, 2005). Strong stability will ensure ex ante fairness in the sense that it not only implements the desired mix of students but also ensures fairness in the treatment of students.

7 Conclusion

Complementarity is prevalent in matching markets, and it poses difficulty for designing desirable mechanisms. The present paper took a step toward solving this problem by studying stable matching in large economies. We demonstrated that if the firm preferences

[32] Unlike the continuum model, population proportionality does not guarantee strategy-proofness. As is shown by Kesten and Ünver (2014), strategy-proofness is generally impossible to attain in the time-share/probabilistic models with finitely many workers.

are continuous, then an economy with a continuum of workers always has a stable matching, even in the presence of complementarities. Building upon this result, we established that there exists an approximately stable matching in large finite economies. We extended our analysis to the case in which firms' choices may not be unique, and used this framework to show that there is a stable mechanism that is strategy-proof for workers and satisfies an additional fairness property, strong stability.

To our knowledge, this paper is the first to analyze a continuum matching model with the level of generality as presented here. As such, our paper may pose as many questions as it answers. One issue worth pursuing is the computation of a stable matching. Existence of a stable matching, as established in this paper, is clearly necessary to find a desired mechanism, but for practical implementation one needs an algorithm that is computable and fast. Our mapping T not only characterizes stable matchings but also is interpretable as a tâtonnement adjustment process for revising the "budget quotes" that has a potential to be used as an actual mechanism.[33] However, this process is not guaranteed to always converge to a stable matching.[34] This is not a surprise, given that similar algorithms such as a tâtonnement process in general equilibrium theory or a best response process in game theory also fail convergence in general. Studying the computationally efficient algorithms to find stable matchings would be an interesting and challenging future research topic.

Another research direction is to find structural properties of stable matchings. Section S.3 of the Supplementary Notes studies this issue for the case where firm preferences satisfy substitutability (but not necessarily continuity). We show that the set of stable matchings forms a (nonempty) complete lattice, which in particular implies the existence of side-optimal stable matchings. A version of the rural hospital theorem also holds given an appropriate version of the law of aggregate demand. While these results may be expected from the existing theory, we also provide a novel condition that generalizes the full support assumption of Azevedo and Leshno (2011) and guarantees the uniqueness of stable matching. Although our analysis in the Supplementary Notes has already established these

[33] In fact, in case of substitutable preferences, the map T entails a monotonic process that converges regardless of the initial state, and in particular corresponds to the celebrated and largely successful Gale and Shapley's deferred acceptance (DA) algorithms when the initial measures are set at either the largest (firm-proposing DA) or at the smallest measures (worker-proposing DA).

[34] Recall our leading example in Example 1. We can see that starting from any state $(\frac{1}{2}, x_1'; x_2, \frac{1}{2})$, T cycles back to itself:

$$(\tfrac{1}{2}, x_1'; x_2, \tfrac{1}{2}) \to (\tfrac{1}{2}, x_2; \tfrac{1}{2} - x_1', \tfrac{1}{2}) \to (\tfrac{1}{2}, \tfrac{1}{2} - x_1'; \tfrac{1}{2} - x_2, \tfrac{1}{2}) \to (\tfrac{1}{2}, \tfrac{1}{2} - x_2; x_1', \tfrac{1}{2}) \to (\tfrac{1}{2}, x_1'; x_2, \tfrac{1}{2}).$$

Hence, unless one starts with the stable matching $x_1 = x_2' = 1/2$, the map T never finds a stable matching.

key structural properties, further studies along this line seems promising.

While our model substantially generalizes previous models, further generalizations would be another interesting research topic. In Section S.4 of the Supplementary Notes, we generalize our continuum matching model to the setting of matching with contracts due to Hatfield and Milgrom (2005). We establish basic results, namely the existence and a fixed-point characterization of stable matchings, and it appears that many other basic properties should hold. Just as many papers have successfully generalized and applied the finite model of matching with contract, future research on matching with contracts in large economies may prove fruitful.

The approach of this paper was theoretical, and our theoretical findings suggest a number of complementary research directions using different methodologies. For instance, it would be important to see whether approximate stability is sufficient for keeping the given matching undisturbed in practice. Testing this hypothesis with real data is likely challenging and certainly outside the scope of this particular paper, but it may be an interesting future research topic. Given the difficulty with using real data, lab experiments or numerical methods may be promising. Theses are just a few examples of research topics, and we expect that many other questions can be asked in the framework presented in the current paper.

A Proofs of Theorems 1, 2 and 4

Since Theorems 1 and 2 follow as corollaries from their counterparts (Theorems 7 and 4) in the correspondence case, we only consider the case where C_f and thus R_f are correspondences.

Let us extend the mapping T to the case of correspondences as follows: For any $X \in \mathcal{X}^{n+1}$,

$$T(X) = \{\tilde{X} \in \mathcal{X}^{n+1} \,|\, \tilde{X}_f(\cdot) = \sum_{P \in \mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot), \forall f \in \tilde{F},$$
$$\text{for some } (Y_f)_{f \in \tilde{F}} \text{ such that } Y_f \in R_f(X_f), \forall f \in \tilde{F}\},$$

where $Y_{f_-^P}(\Theta_P \cap \cdot) = G(\Theta_P \cap \cdot)$ if f is the top-ranked firm at P.

First, we obtain the characterization of stable matchings as fixed points of T, from which Theorem 1 follows as a corollary since the above T mapping coincides with that in (2), as can be easily verified.

Theorem 7. *A matching $M \in \mathcal{X}^{n+1}$ is stable if and only if there is a fixed point $X \in T(X)$ such that $M_f \in C_f(X_f), \forall f \in \tilde{F}$. Also, any such X and M satisfy $X_f = D^{\preceq f}(M), \forall f \in \tilde{F}$.*

Proof. **("Only if" part):** Suppose M is a stable matching in $\mathcal{X}^{n+1}$. Define $X = (X_f)_{f \in \tilde{F}}$ as

$$X_f(E) = D^{\preceq f}(M)(E) = \sum_{P \in \mathcal{P}} \sum_{f' \in \tilde{F} : f' \preceq_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

We prove that X is a fixed point of T. Let us first show that for each $f \in \tilde{F}$, $X_f \in \mathcal{X}$. It is clear that as each $M_{f'}(\Theta_P \cap \cdot)$ is nonnegative and countably additive, so is $X_f(\cdot)$. It is also clear that since $(M_f)_{f \in \tilde{F}}$ is a matching, $X_f \sqsubset G$. Thus, we have $X_f \in \mathcal{X}$.

We next claim that $M_f \in C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \varnothing$ since $M_\varnothing \sqsubset X_\varnothing = C_\varnothing(X_\varnothing)$. To prove the claim for $f \neq \varnothing$, suppose for a contradiction that $M_f \notin C_f(X_f)$, which means that there is some $M'_f \in C_f(X_f)$ such that $M'_f \neq M_f$. Note that $M_f \sqsubset X_f$ and thus $(M'_f \vee M_f) \sqsubset X_f$. Then, by revealed preference, we have $M_f \notin C_f(M'_f \vee M_f)$ and $M'_f \in C_f(M'_f \vee M_f)$ or $M'_f \succ_f M_f$, which means that M is unstable since $M'_f \sqsubset X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X \in T(X)$. The fact that $M_f \in C_f(X_f), \forall f \in \tilde{F}$ means that $X_f - M_f \in R_f(X_f), \forall f \in \tilde{F}$. We set $Y_f = X_f - M_f$ for each $f \in \tilde{F}$ and obtain for any $E \in \Sigma$

$$\sum_{P \in \mathcal{P}} Y_{f_-^P}(\Theta_P \cap E) = \sum_{P \in \mathcal{P}} \Big(X_{f_-^P}(\Theta_P \cap E) - M_{f_-^P}(\Theta_P \cap E) \Big)$$

$$= \sum_{P\in\mathcal{P}} \left(\sum_{f'\in\tilde{F}:f'\preceq_P f_-^P} M_{f'}(\Theta_P\cap E) - M_{f_-^P}(\Theta_P\cap E) \right)$$

$$= \sum_{P\in\mathcal{P}} \sum_{f'\in\tilde{F}:f'\preceq_P f} M_{f'}(\Theta_P\cap E) = X_f(E),$$

where the second and fourth equalities follow from the definition of $X_{f_-^P}$ and X_f, respectively, while the third from the fact that f_-^P is an immediate predecessor of f. The above equation holds for every firm $f\in\tilde{F}$, so we conclude that $X\in T(X)$, i.e. X is a fixed point of T.

("If" part): Let us first introduce some notations. Let f_+^P denote an **immediate successor** of $f\in\tilde{F}$ at $P\in\mathcal{P}$: that is, $f_+^P \prec_P f$, and for any $f'\prec_P f$, $f'\preceq_P f_+^P$. Also, let $X_{f_+^P}(\Theta_P\cap\cdot)\equiv 0$ for the firm f that is ranked last at P. Note that for any $f,\tilde{f}\in\tilde{F}$, $f=\tilde{f}_-^P$ if and only if $\tilde{f}=f_+^P$.

Suppose now that $X\in\mathcal{X}^{n+1}$ is a fixed point of T. For each firm $f\in\tilde{F}$ and $E\in\Sigma$, define

$$M_f(E) = X_f(E) - \sum_{P\in\mathcal{P}} X_{f_+^P}(\Theta_P\cap E). \tag{6}$$

We first verify that for each $f\in\tilde{F}$, $M_f\in\mathcal{X}$. First, it is clear that for each $f\in\tilde{F}$, as both $X_f(\cdot)$ and $X_{f_+^P}(\Theta_P\cap\cdot)$ are countably additive, so is M_f. It is also clear that for each $f\in\tilde{F}$, $M_f\sqsubset X_f$.

Let us next show that for all $f\in\tilde{F}$, $P\in\mathcal{P}$, and $E\in\Sigma$,

$$X_f(\Theta_P\cap E) = \sum_{f'\in\tilde{F}:f'\preceq_P f} M_{f'}(\Theta_P\cap E), \tag{7}$$

which means that $X_f = D^{\preceq f}(M)$. To do so, consider first a firm f that is ranked last at P. By (6) and the fact that $X_{f_+^P}(\Theta_P\cap\cdot)\equiv 0$, we have $M_f(\Theta_P\cap E) = X_f(\Theta_P\cap E)$. Hence, (7) holds for such f. Consider now any $f\in\tilde{F}$ which is not ranked last, and assume for an inductive argument that (7) holds true for f_+^P, so $X_{f_+^P}(\Theta_P\cap E) = \sum_{f'\in\tilde{F}:f'\preceq_P f_+^P} M_{f'}(\Theta_P\cap E)$. Then, by (6), we have

$$X_f(\Theta_P\cap E) = M_f(\Theta_P\cap E) + X_{f_+^P}(\Theta_P\cap E) = M_f(\Theta_P\cap E) + \sum_{f'\in\tilde{F}:f'\preceq_P f_+^P} M_{f'}(\Theta_P\cap E)$$

$$= \sum_{f'\in\tilde{F}:f'\preceq_P f} M_{f'}(\Theta_P\cap E),$$

as desired.

To show that $M = (M_f)_{f\in\tilde{F}}$ is a matching, let f be a top-ranked firm at P and note that by definition of T, if $\tilde{X} \in T(X)$, then $\tilde{X}_f(\Theta_P \cap \cdot) = G(\Theta_P \cap \cdot)$. Since $X \in T(X)$, we have for any $E \in \Sigma$

$$G(\Theta_P \cap E) = X_f(\Theta_P \cap E) = \sum_{f'\in\tilde{F}: f' \preceq_P f} M_{f'}(\Theta_P \cap E) = \sum_{f'\in\tilde{F}} M_{f'}(\Theta_P \cap E),$$

where the second equality follows from (7). Since the above equation holds for every $P \in \mathcal{P}$, M is a matching.

Let us now fix any $f \in \tilde{F}$ and show that $M_f \in C_f(X_f)$, which is equivalent to showing $X_f - M_f \in R_f(X_f)$. Recall that $X_{f_+^P}(\Theta_P \cap E) = Y_f(\Theta_P \cap E)$ for all $P \in \mathcal{P}$ and $E \in \Sigma$. Then, (6) implies

$$X_f(\cdot) - M_f(\cdot) = \sum_{P\in\mathcal{P}} X_{f_+^P}(\Theta_P \cap \cdot) = \sum_{P\in\mathcal{P}} Y_f(\Theta_P \cap \cdot) = Y_f(\cdot) \in R_f(X_f),$$

as desired, where the last inclusion relationship follows from the definition of T.

We now prove that $(M_f)_{f\in\tilde{F}}$ is stable. To prove the first part of Condition 1 of Definition 5, note first that $M_\emptyset \in C_\emptyset(X_\emptyset) = \{X_\emptyset\}$. Then, for every $P \in \mathcal{P}$ and $E \in \Sigma$,

$$\sum_{f: f \prec_P \emptyset} M_f(\Theta_P) = \sum_{f: f \preceq_P \emptyset} M_f(\Theta_P) - M_\emptyset(\Theta_P) = X_\emptyset(\Theta_P) - M_\emptyset(\Theta_P) = 0,$$

where the middle equality follows from (7). The above equation means that for each $f \prec_P \emptyset$, we have $M_f(\Theta_P) = 0$, as desired. The second part of Condition 1 of Definition 5 (i.e. $M_f \in C_f(M_f)$) follows from revealed preference since $M_f \sqsubset X_f$ by (7) and $M_f \in C_f(X_f)$.

It only remains to check Condition 2 of Definition 5. Suppose for a contradiction that it fails. Then, there exist f and M'_f such that

$$M'_f \sqsubset D^{\preceq f}(M),\ M'_f \in C_f(M'_f \vee M_f), \text{ and } M_f \notin C_f(M'_f \vee M_f). \tag{8}$$

So $M'_f \sqsubset D^{\preceq f}(M) = X_f$. Since then $M_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$ and $M_f \in C_f(X_f)$, the revealed preference property implies $M_f \in C_f(M'_f \vee M_f)$, contradicting (8). We have thus proven that M is stable. ▌

We now prove Theorem 4, from which Theorem 2 follows as a corollary since, if T is a single-valued mapping, then the convex- and closed-valuedness hold trivially while the upper hemicontinuity is equivalent to continuity. To prove existence, by Theorem 7, it suffices to show that the mapping T has a fixed point. To this end, we establish a series of Lemmas.

We now establish the compactness of $\mathcal{X}$ and the upper hemicontinuity of T. Recall that $\mathcal{X}$ is endowed with weak-$*$ topology. The notion of convergence in this topology, i.e. weak convergence, can be stated as follows: A sequence of measures $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ weakly converges to a measure $X \in \mathcal{X}$, written as $X_k \xrightarrow{w^*} X$, if and only if $\int_\Theta h dX_k \to \int_\Theta h dX$ for all $h \in C(\Theta)$, where $C(\Theta)$ is the space of all continuous functions defined on Θ. The next result provides some conditions that are equivalent to weak convergence.

Theorem 8. *Let X and $(X_k)_{k\in\mathbb{N}}$ be finite measures on Σ. The following conditions are equivalent:*[35]

(a) $X_k \xrightarrow{w^*} X$*;*

(b) $\int_\Theta h dX_k \to \int_\Theta h dX$ *for all* $h \in C_u(\Theta)$*, where* $C_u(\Theta)$ *is the space of all uniformly continuous functions defined on* Θ*;*

(c) $\liminf_k X_k(A) \geq X(A)$ *for every open set* $A \subset \Theta$*, and* $X_k(\Theta) \to X(\Theta)$*;*

(d) $\limsup_k X_k(A) \leq X(A)$ *for every closed set* $A \subset \Theta$*, and* $X_k(\Theta) \to X(\Theta)$*;*

(e) $X_k(A) \to X(A)$ *for every set* $A \in \Sigma$ *such that* $X(\partial A) = 0$ *(*∂A *denotes the boundary of* A*).*

Lemma 2. *The space $\mathcal{X}$ is convex and compact. Also, for any $X \in \mathcal{X}$, $\mathcal{X}_X$ is compact.*

Proof. Convexity of $\mathcal{X}$ follows trivially.

To prove the compactness of $\mathcal{X}$, let $C(\Theta)^*$ denote the dual (Banach) space of $C(\Theta)$ and note that $C(\Theta)^*$ is the space of all (signed) measures on (Θ, Σ), given Θ is a compact metric space.[36] Then, by Alaoglu's Theorem, the closed unit ball of $C(\Theta)^*$, denoted U^*, is weak-$*$ compact.[37] Clearly, $\mathcal{X}$ is a subspace of U^* since for any $X \in \mathcal{X}$, $\|X\| = X(\Theta) \leq G(\Theta) = 1$.

[35] This theorem is a modified version of "Portmanteau Theorem" that is modified to deal with any finite (i.e. not necessarily probability) measures. See Theorem 4.5.1 of Ash (1977) for this result, for instance.

[36] More precisely, $C(\Theta)^*$ is isometrically isomorphic to the space of all signed measures on (Θ, Σ) according to the Riesz Representation Theorem (see Royden and Fitzpatrick (2010) for instance).

[37] The closed unit ball is defined as $U^* := \{X \in C^*(\Theta) : \|X\| \leq 1\}$, where $\|X\|$ is the dual norm, i.e.,

$$\|X\| = \sup\{\left|\int_\Theta h dX\right| : h \in C(\Theta) \text{ and } \max_{\theta\in\Theta} |h(\theta)| \leq 1\}.$$

If X is a nonnegative measure, then the supremum is achieved by taking $h \equiv 1$, and thus $\|X\| = X(\Theta)$. It is well known (see Royden and Fitzpatrick (2010) for instance) that if $C(\Theta)^*$ is infinite dimensional, then U^* is not compact under the norm topology (i.e., the topology induced by the dual norm). On the other hand, U^* is compact under the weak-$*$ topology, which follows from Alaoglu's Theorem (see Royden and Fitzpatrick (2010) for instance).

The compactness of $\mathcal{X}$ will thus follow if $\mathcal{X}$ is shown to be a closed set. To prove this, we prove that for any sequence $(X_k)_{k\in\mathbb{N}}$ in $\mathcal{X}$ and $X \in C(\Theta)^*$ such that $X_k \xrightarrow{w^*} X$, we must have $X \in \mathcal{X}$, which will be shown if we prove that $0 \le X(E) \le G(E)$ for any $E \in \Sigma$. Let us first make the following observation: every finite (Borel) measure X on the metric space Θ is normal,[38] which means that for any set $E \in \Sigma$,

$$X(E) = \inf\{X(O) : E \subset O \text{ and } O \in \Sigma \text{ is open}\} \tag{9}$$

$$= \sup\{X(F) : F \subset E \text{ and } F \in \Sigma \text{ is closed}\}. \tag{10}$$

To show first that $X(E) \le G(E)$, consider any open set $O \in \Sigma$ such that $E \subset O$. Then, since $X_k \in \mathcal{X}$ for every k, we must have $X_k(O) \le G(O)$ for every k, which, combined with (c) of Theorem 8 above, implies that $X(O) \le \liminf_k X_k(O) \le G(O)$. Given (9), this means that $X(E) \le G(E)$.

To show next that $X(E) \ge 0$, consider any closed set $F \in \Sigma$ such that $F \in \Sigma$. Since $X_k \in \mathcal{X}$ for every k, we must have $X_k(F) \ge 0$, which, combined with (d) of Theorem 8 above, implies that $X(F) \ge \limsup_k X_k(F) \ge 0$. Given (10), this means $X(E) \ge 0$.

The proof for the compactness of $\mathcal{X}_X$ is analogous and hence omitted. ▌

Lemma 3. *The map T is a correspondence from $\mathcal{X}^{n+1}$ to itself. Also, it is convex-valued and upper hemicontinuous.*

Proof. To show that T maps from $\mathcal{X}^{n+1}$ to itself, observe that for any $X \in \mathcal{X}^{n+1}$ and $\tilde{X} \in T_f(X)$, there is $Y_f \in R_f(X_f)$ for each $f \in \tilde{F}$ such that for all $E \in \Sigma$,

$$\tilde{X}(E) = \sum_{P\in\mathcal{P}} Y_{\underline{f}^P}(\Theta_P \cap E) \le \sum_{P\in\mathcal{P}} X_{\underline{f}^P}(\Theta_P \cap E) \le \sum_{P\in\mathcal{P}} G(\Theta_P \cap E) = G(E),$$

which means that $\tilde{X} \in \mathcal{X}$, as desired.

To prove that T is convex-valued, it suffices to show that for each $f \in \tilde{F}$, R_f is convex-valued. Consider any $X \in \mathcal{X}$ and $Y', Y'' \in R_f(X)$. There are some $X', X'' \in C_f(X)$ satisfying $Y' = X - X'$ and $Y'' = X - X''$. Then, the convexity of $C_f(X)$ implies that for any $\lambda \in [0, 1]$, $\lambda X' + (1-\lambda)X'' \in C_f(X)$ so $\lambda Y' + (1-\lambda)Y'' = X - (\lambda X' + (1-\lambda)X'') \in R_f(X)$.

To establish the upper hemicontinuity of T, we first establish the following claim:

Claim 1. *For any sequence $(X_k)_{k\in\mathbb{N}} \subset \mathcal{X}$ that weakly converges to $X \in \mathcal{X}$, a sequence $(X_k(\Theta_P \cap \cdot))_{k\in\mathbb{N}}$ also weakly converges to $X(\Theta_P \cap \cdot)$ for all $P \in \mathcal{P}$.*

[38] See Theorem 12.5 of Aliprantis and Border (2006).

Proof. Let X^P and X_k^P denote $X(\Theta_P \cap \cdot)$ and $X_k(\Theta_P \cap \cdot)$, respectively. Note first that for any $X \in \mathcal{X}$, we have $X^P \in \mathcal{X}$ for all $P \in \mathcal{P}$. Due to Theorem 8, it suffices to show that X^P and $(X_k^P)_{k\in\mathbb{N}}$ satisfy the condition (c) of Theorem 8. To do so, consider any open set $O \subset \Theta$. Then, letting Θ_P° denote the interior of Θ_P,

$$\begin{aligned}\liminf_k X_k^P(O) &= \liminf_k X_k(\Theta_P^\circ \cap O) + X_k(\partial\Theta_P \cap O) \\ &= \liminf_k X_k(\Theta_P^\circ \cap O) \geq X(\Theta_P^\circ \cap O) = X^P(O),\end{aligned}$$

where the second equality follows from the fact that $X_k(\partial\Theta_P \cap O) \leq X_k(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$, the lone inequality from $X_k \xrightarrow{w^*} X$, (c) of Theorem 8, and the fact that $\partial\Theta_P^\circ \cap O$ is an open set, and the last equality from repeating the first two equalities with X instead of X_k. Also, we have

$$X_k^P(\Theta) = X_k(\Theta_P) \to X(\Theta_P) = X^P(\Theta),$$

where the convergence is due to $X_k \xrightarrow{w^*} X$, (e) of Theorem 8, and the fact that $X(\partial\Theta_P) \leq G(\partial\Theta_P) = 0$. Thus, the two requirements in condition (c) of Theorem 8 are satisfied, so $X_k^P \xrightarrow{w^*} X^P$, as desired. ∎

It is also straightforward to observe that if C_f is upper hemicontinuous, then R_f is also upper hemi-continuous.

We now prove the upper hemicontinuity of T by considering any sequences $(X_k)_{k\in\mathbb{N}}$ and $(\tilde{X}_k)_{k\in\mathbb{N}}$ in $\mathcal{X}^{n+1}$ weakly converging to some X and $\tilde{X}$ in $\mathcal{X}^{n+1}$, respectively, such that $\tilde{X}_k \in T_f(X_k)$ for each k. To show that $\tilde{X} \in T(X)$, let $X_{k,f}$ and $\tilde{X}_{k,f}$ denote the components of X_k and $\tilde{X}_k$, respectively, that correspond to $f \in \tilde{F}$. Then, we can find $Y_{k,f} \in R_f(X_{k,f})$ for each k and f such that $\tilde{X}_{k,f}(\cdot) = \sum_{P\in\mathcal{P}} Y_{k,f_-^P}(\Theta_P \cap \cdot)$, which implies that for all $f \in \tilde{F}$ and $P \in \mathcal{P}$, $\tilde{X}_{k,f_+^P}(\Theta_P \cap \cdot) = Y_{k,f}(\Theta_P \cap \cdot)$ since f is the immediate predecessor of f_+^P at P. As $\tilde{X}_{k,f} \xrightarrow{w^*} \tilde{X}_f, \forall f \in \tilde{F}$, by assumption, we have $\tilde{X}_{k,f_+^P}(\Theta_P \cap \cdot) \xrightarrow{w^*} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$ by Claim 1, which means that $Y_{k,f}(\Theta_P \cap \cdot) \xrightarrow{w^*} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$ for all $f \in \tilde{F}$. This in turn implies that $Y_{k,f}(\cdot) = \sum_{P\in\mathcal{P}} Y_{k,f}(\Theta_P \cap \cdot) \xrightarrow{w^*} \sum_{P\in\mathcal{P}} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$. Now let $Y_f(\cdot) = \sum_{P\in\mathcal{P}} \tilde{X}_{f_+^P}(\Theta_P \cap \cdot)$. We then have $\tilde{X}_f(\Theta_P \cap \cdot) = Y_{f_-^P}(\Theta_P \cap \cdot)$ and thus $\tilde{X}_f(\cdot) = \sum_{P\in\mathcal{P}} Y_{f_-^P}(\Theta_P \cap \cdot)$. Also, since $X_{k,f} \xrightarrow{w^*} X_f$ and $Y_{k,f} \xrightarrow{w^*} Y_f$, we must have $Y_f \in R_f(X_f)$ by the upper hemicontinuity of R_f, which means $\tilde{X} \in T(X)$, as desired. ∎

Proof of Theorem 4. Lemmas 2 and 3 show that T is convex-valued and upper hemi-continuous while it is a mapping from convex, compact space $\mathcal{X}^{n+1}$ into itself, which implies that T is also closed-valued. Thus, we can invoke Kakutani-Fan-Glicksberg's fixed point theorem to conclude that the mapping T has a nonempty set of fixed points.[39] ∎

B Proof of Theorem 3

Since the utility function of each firm is continuous, it suffices to show that there exists an ϵ-distance stable matching as defined in Remark 5. The rest of this proof constructs an ϵ-distance stable matching.

Let Γ be the limit continuum economy which the sequence $(\Gamma^q)_{q\in\mathbb{N}}$ converges to. For any population G, fix a sequence $(G^q)_{q\in\mathbb{N}}$ of finite-economy populations such that $G^q \xrightarrow{w^*} G$. Let $\Theta^q = \{\theta_1^q, \theta_2^q, \ldots, \theta_{\bar{q}}^q\} \subset \Theta$ be the support for G^q.[40] For each firm $f \in \tilde{F}$, define Θ_f to be the set of types that find firm f acceptable, i.e., $\Theta_f := \cup_{P\in\mathcal{P}: f\succ_P \varnothing}\Theta_P$ (let $\Theta_\varnothing = \Theta$ by convention). Let $\overline{\Theta}_f$ denote the closure of Θ_f. We first prove a few preliminary results.

Lemma 4. *For any $r > 0$, there is a finite number of open balls, $B_1, \ldots, B_L$, in Θ that have radius smaller than r with a boundary of zero measure (i.e. $G(\partial B_\ell) = 0, \forall \ell$) and cover $\overline{\Theta}_f$ for each $f \in F$.*

Proof. Let $B(\theta, r) = \{\theta' \in \Theta \,|\, \|\theta' - \theta\| < r\}$ and $S(\theta, r) = \{\theta' \in \Theta \,|\, \|\theta' - \theta\| = r\}$, where $\|\cdot\|$ is a metric for the space Θ. For all $\theta \in \overline{\Theta}_f$ and $r > 0$, there must be some $r_\theta \in (0, r)$ such that $G(S(\theta, r_\theta)) = 0$.[41] This means that $\partial B(\theta, r_\theta) = S(\theta, r_\theta)$ has a zero measure. Consider now a collection $\{B(\theta, r_\theta) \,|\, \theta \in \Theta\}$ of open balls that covers $\overline{\Theta}_f$. Since $\overline{\Theta}_f$ is a closed subset of the compact set Θ, it is compact and thus has a finite cover. ∎

Lemma 5. *Consider any $X, Y \in \mathcal{X}$ and any sequence $(Y^q)_{q\in\mathbb{N}}$ such that $Y^q \in \mathcal{X}^q$, $Y^q \xrightarrow{w^*} Y$, $X \sqsubset Y$, and $X(\Theta\backslash\Theta_f) = 0$ for some $f \in \tilde{F}$.*[42] *Then, there exists a sequence $(X^q)_{q\in\mathbb{N}}$ such that $X^q \in \mathcal{X}^q$, $X^q \xrightarrow{w^*} X$, $X^q \sqsubset Y^q$, and $X^q(\Theta\backslash\Theta_f) = 0$ for all q.*

Proof. Consider a decreasing sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of real numbers converging to 0. Fix any k. Then, by Lemma 4, we can find a finite cover $\{B_\ell^k\}_{\ell=1,\ldots,L_k}$ of $\overline{\Theta}_f$ for each k such that for each ℓ, B_ℓ^k has a radius smaller than ϵ_k and $G(\partial B_\ell^k) = 0$. Define $A_1^k = B_1^k \cap \Theta_f$ and

[39] For Kakutani-Fan-Glicksberg's fixed point theorem, refer to Theorem 16.12 and Corollary 16.51 in Aliprantis and Border (2006).

[40] Note that we allow for possibility that there are more than one worker of the same type even in finite economies, so $\bar{q}$ may be strictly smaller than q.

[41] To see this, note first that $B(\theta, r) = \cup_{\tilde{r}\in[0,r)}S(\theta, \tilde{r})$ and $G(B(\theta, r)) < \infty$. Then, $G(S(\theta, \tilde{r})) > 0$ for at most countably many $\tilde{r}$'s, since otherwise the set $R_n \equiv \{\tilde{r} \in [0, r) \,|\, G(S(\theta, \tilde{r})) \geq 1/n\}$ has to be infinite for at least one n, which yields $G(B(\theta, r)) \geq G(\cup_{\tilde{r}\in R_n}S(\theta, \tilde{r})) \geq \frac{\infty}{n}$, a contradiction.

[42] Note that if $f = \varnothing$, then $\Theta\backslash\Theta_f = \emptyset$. Thus, the restriction that $X(\Theta\backslash\Theta_f) = 0$ becomes vacuous.

$A_\ell^k = \left(B_\ell^k \setminus (\cup_{\ell'=1}^{\ell-1} B_{\ell'}^k)\right) \cap \Theta_f$ for each $\ell \geq 2$. Then, $\{A_\ell^k\}_{\ell=1,\dots,L_k}$ constitutes a partition of Θ_f. It is straightforward to see that $G(\partial A_\ell^k) = 0, \forall \ell$, since $G(\partial B_\ell^k) = 0, \forall \ell$, and that $G(\partial \Theta_f) = 0$.[43] This implies that $Y(\partial A_\ell^k) = 0, \forall \ell$. Given this and the assumption that $Y^q \xrightarrow{w^*} Y$, condition (e) of Theorem 8 implies that there exists sufficiently large q, denoted q_k, such that for all $q \geq q_k$

$$\frac{1}{q} < \frac{\epsilon_k}{L_k} \text{ and } |Y(A_\ell^k) - Y^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}, \forall \ell = 1, \dots, L_k. \tag{11}$$

Let us choose $(q_k)_{k \in \mathbb{N}}$ to be a sequence that strictly increases with k.

We are prepared to construct X^q as follows: (i) $X^q(\theta) \leq Y^q(\theta), \forall \theta \in \Theta^q$; (ii) for each $q \in \{q_k, \dots, q_{k+1} - 1\}$,

$$X^q(A_\ell^k) = \max\left\{ \frac{m}{q} \;\middle|\; m \in \mathbb{N} \cup \{0\} \text{ and } \frac{m}{q} \leq \min\{X(A_\ell^k), Y^q(A_\ell^k)\} \right\} \text{ for each } \ell = 1, \dots, L_k.$$

It is straightforward to check the existence of X^q that satisfies both (i) and (ii). Note that (i) ensures that $X^q \sqsubset Y^q$ and $X^q(\Theta \backslash \Theta_f) \leq Y^q(\Theta \backslash \Theta_f) = 0$.

We show that for all $q \in \{q_k, \dots, q_{k+1} - 1\}$, we have

$$|X(A_\ell^k) - X^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}. \tag{12}$$

To see this, consider first the case where $X(A_\ell^k) < Y^q(A_\ell^k)$. Then, by definition of X^q and (11), we have $0 \leq X(A_\ell^k) - X^q(A_\ell^k) < \frac{1}{q} < \frac{\epsilon_k}{L_k}$. In the case where $X(A_\ell^k) \geq Y^q(A_\ell^k)$, we have $X^q(A_\ell^k) = Y^q(A_\ell^k) \leq X(A_\ell^k) \leq Y(A_\ell^k)$, which implies by (11)

$$|X(A_\ell^k) - X^q(A_\ell^k)| \leq |Y(A_\ell^k) - Y^q(A_\ell^k)| < \frac{\epsilon_k}{L_k}.$$

Let us now prove that $X^q \xrightarrow{w^*} X$. We do so by invoking (b) of Theorem 8, according to which $X^q \xrightarrow{w^*} X$ if and only if $|\int h dX^q - \int h dX| \to 0$ as $q \to \infty$, for any uniformly continuous function $h \in C_u(\Theta)$.

Hence, to begin, fix any $h \in C_u(\Theta)$, and fix any $\epsilon > 0$. Next we define for each k and $q \in \{q_k, \dots, q_{k+1} - 1\}$

$$\bar{h}_\ell^{q,k} \equiv \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta) h(\theta)}{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta)} = \frac{\sum_{\theta \in \Theta^q \cap A_\ell^k} X^q(\theta) h(\theta)}{X^q(A_\ell^k)}$$

[43] The latter fact holds since $\Theta_f = \cup_{P \in \mathcal{P}: f \succ \emptyset} \Theta_P$ and thus $\partial \Theta_f \subset \cup_{P \in \mathcal{P}: f \succ_P \emptyset} \partial \Theta_P$, which implies

$$G(\partial \Theta_f) \leq G(\cup_{P \in \mathcal{P}: f \succ_P \emptyset} \partial \Theta_P) \leq \sum_{P \in \mathcal{P}: f \succ_P \emptyset} G(\partial \Theta_P) = 0.$$

if $X^q(A_\ell^k) > 0$, and if $X^q(A_\ell^k) = 0$, then define $\bar{h}_\ell^{q,k} \equiv h(\theta)$ for some arbitrarily chosen $\theta \in A_\ell^k$.

Note that $C_u(\Theta)$ is endowed with the sup norm $\|\cdot\|_\infty$ and $\|h\|_\infty$ is finite for any $h \in C_u(\Theta)$. Thus, there exists sufficiently large $K \in \mathbb{N}$ that for all $k > K$ and $q \in \{q_k, \ldots, q_{k+1} - 1\}$,

$$\|h\|_\infty \epsilon_k < \frac{\epsilon}{2} \text{ and } \sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| X(A_\ell^k) < \frac{\epsilon}{2}, \tag{13}$$

which is possible since h is uniformly continuous, $A_\ell^k \subset B_\ell^k$, and B_ℓ^k has a radius smaller than ϵ_k with ϵ_k converging to 0 as $k \to \infty$.

Then, for any $q > Q := q_K$, there exists $k > K$ satisfying $q \in \{q_k, \ldots, q_{k+1} - 1\}$ such that

$$\begin{aligned}
&\left| \int h dX^q - \int h dX \right| \\
=&\left| \int_{\theta \in \Theta_f} h dX^q - \int_{\theta \in \Theta_f} h dX \right| \\
=&\left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X^q(A_\ell^k) - \int_{\theta \in \Theta_f} h dX \right| \\
\leq&\left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} (X^q(A_\ell^k) - X(A_\ell^k)) \right| + \left| \sum_{\ell=1}^{L_k} \bar{h}_\ell^{q,k} X(A_\ell^k) - \int_{\theta \in \Theta_f} h dX \right| \\
\leq&\sum_{\ell=1}^{L_k} \|h\|_\infty |X^q(A_\ell^k) - X(A_\ell^k)| + \left| \sum_{\ell=1}^{L_k} \int_{\theta \in \Theta_f} \bar{h}_\ell^{q,k} \mathbb{1}_{A_\ell^k} dX - \sum_{\ell=1}^{L_k} \int_{\theta \in \Theta_f} h \mathbb{1}_{A_\ell^k} dX \right| \\
\leq&\|h\|_\infty \epsilon_k + \sum_{\ell=1}^{L_k} \sup_{\theta \in A_\ell^k} |\bar{h}_\ell^{q,k} - h(\theta)| X(A_\ell^k) \\
\leq&\frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

where the first equality holds since $X(\Theta \backslash \Theta_f) = X^q(\Theta \backslash \Theta_f) = 0$ while the third and fourth inequalities follow from (12) and (13), respectively. ∎

Lemma 6. *For any two sequences $(X^q)_{q \in \mathbb{N}}$ and $(Y^q)_{q \in \mathbb{N}}$ such that $X^q, Y^q \in \mathcal{X}^q$, $X^q \sqsubset Y^q, \forall q$, $X^q \xrightarrow{w^*} X$, and $Y^q \xrightarrow{w^*} Y$, we have $X \sqsubset Y$.*

Proof. Letting $Z^q = Y^q - X^q$ and $Z = Y - X$, we have $Z^q \xrightarrow{w^*} Z$ because of the fact that for any $h \in C_u(\Theta)$,

$$\int_\Theta h dZ^q = \int_\Theta h dY^q - \int_\Theta h dX^q \to \int_\Theta h dY - \int_\Theta h dX = \int_\Theta h dZ$$

and (b) of Theorem 8. Since $Z^q = Y^q - X^q \in \mathcal{X}$, $Z^q \xrightarrow{w^*} Z$, and $\mathcal{X}$ is compact, we have $Z \in \mathcal{X}$, which implies that $Z(E) = Y(E) - X(E) \geq 0$ for all $E \in \Sigma$, as desired. ▌

Given the utility function u_f, we define each firm f's optimal choice(s) in the economy Γ^q as follows: for any $X \in \mathcal{X}^q$,

$$C_f^q(X) = \underset{X' \sqsubset X, X' \in \mathcal{X}^q}{\arg\max}\; u_f(X').$$

Note that we allow for multiple optimal choices.

Lemma 7. *Consider any $X \in \mathcal{X}$ and sequence $(X^q)_{q\in\mathbb{N}}$ such that $X^q \in \mathcal{X}^q$ and $X^q \xrightarrow{w^*} X$. Then, for any sequence $(M^q)_{q\in\mathbb{N}}$ such that $M^q \in C_f^q(X^q)$ for some (fixed) $f \in F$, we must have $M^q \xrightarrow{w^*} C_f(X)$.*

Proof. Suppose to the contrary that M^q does not converge to $C_f(X)$. Then, there exists some subsequence of M^q, denoted $(M^{q_k})_{k\in\mathbb{N}}$, that converges to some $M' \neq C_f(X)$.[44] Since $M^{q_k} \sqsubset X^{q_k}$ and $X^{q_k} \xrightarrow{w^*} X$, we have $M' \sqsubset X$ by Lemma 6. Since both $C_f(X)$ and M' are subpopulations of X, the assumption that $C_f(X)$ chooses a uniquely utility-maximizing subpopulation implies that $u_f(C_f(X)) - \epsilon' > u_f(M') + \epsilon'$ for some $\epsilon' > 0$. To draw a contradiction, we apply Lemma 5 to construct a sequence $(\hat{M}^k)_{k\in\mathbb{N}}$ such that $\hat{M}^k \in \mathcal{X}^{q_k}$, $\hat{M}^k \sqsubset X^{q_k}$, and $\hat{M}^k \xrightarrow{w^*} C_f(X)$. We can then find sufficiently large k such that $u_f(\hat{M}^k) > u_f(C_f(X)) - \epsilon' > u_f(M') + \epsilon' > u_f(M^{q_k})$, which contradicts the optimality of M^{q_k} given X^{q_k}, since both $\hat{M}^k$ and M^{q_k} are feasible subpopulations of X^{q_k}. ▌

We are now ready to prove Theorem 3. First of all, by Theorem 2, there exists a stable matching M in the continuum economy. We use this matching to construct a matching $\bar{M}^q = (\bar{M}_f^q)_{f\in\tilde{F}}$ in the finite economy Γ^q as follows: order the firms in F by $f_1, \ldots, f_n$, and

1. define $\bar{M}_{f_1}^q$ as X^q in Lemma 5 with $X = M_{f_1}$, $Y = G$, and $Y^q = G^q$[45];

2. define $\bar{M}_{f_2}^q$ as X^q in Lemma 5 with $X = M_{f_2}$, $Y = G - M_{f_1}$, and $Y^q = G^q - \bar{M}_{f_1}^q$ (this is possible since $G^q - \bar{M}_{f_1}^q \xrightarrow{w^*} G - M_{f_1}$);

3. in general, for each $f_k \in \tilde{F}$, define inductively $\bar{M}_{f_k}^q$ as X^q in Lemma 5 with $X = M_{f_k}$, $Y = G - \sum_{k'<k} M_{f_{k'}}$, and $Y^q = G^q - \sum_{k'<k} \bar{M}_{f_{k'}}^q$;

[44]This existence is guaranteed by the compactness of $\mathcal{X}$, which is equivalent to the sequential compactness in a metric space, as is the case in our model.

[45]Note that $M_f(\Theta \backslash \Theta_f) = 0$ for all $f \in F$ since M is individually rational, so Lemma 5 can be applied.

and define $\bar{M}^q_\varnothing = G^q - \sum_{f \in \tilde{F}} \bar{M}^q_f$.

By Lemma 5, $\bar{M}^q$ is feasible in Γ^q and individually rational for workers while $\bar{M}^q \xrightarrow{w^*} M$. To ensure the individual rationality for firms, we define another matching $M^q = (M^q_f)_{f \in \tilde{F}}$ as follows: for each $f \in F$, select any $M^q_f \in C^q_f(\bar{M}^q_f)$, and then set $M^q_\varnothing = G^q - \sum_{f \in F} M^q_f$. By revealed preference, we have $M^q_f \in C^q_f(M^q_f)$ and thus M^q is individually rational for firms. Also, the individual rationality of M^q for workers follows immediately from the individual rationality of $\bar{M}^q$ and the fact that $M^q_f \sqsubset \bar{M}^q_f$ for all $f \in F$. Furthermore, by Lemma 7, $M^q_f \xrightarrow{w^*} M_f$ for all $f \in F$ since $M^q_f \in C^q_f(\bar{M}^q_f)$, $\bar{M}^q_f \xrightarrow{w^*} M_f$, and $M_f = C_f(M_f)$. This implies that $M^q_\varnothing = G^q - \sum_{f \in F} M^q_f \xrightarrow{w^*} G - \sum_{f \in F} M_f = M_\varnothing$.

Let us now denote the set of all blocking coalitions involving f under M^q as $\mathcal{B}^q_f$: that is, $\mathcal{B}^q_f = \{\tilde{M} \in \mathcal{X}^q \,|\, \tilde{M} \sqsubset D^{\preceq f}(M^q_f) \text{ and } u_f(\tilde{M}) > u_f(M^q)\}$. Since $\mathcal{B}^q_f$ is finite for each q, the set $\mathcal{B}_f := \cup_{q \in \mathbb{N}} \mathcal{B}^q_f$ is countable. One can index the blocking coalitions in $\mathcal{B}_f$ to form a sequence $(\tilde{M}^k)_{k \in \mathbb{N}}$ such that for any $\tilde{M}^k \in \mathcal{B}^q_f$ and $\tilde{M}^{k'} \in \mathcal{B}^{q'}_f$ with $q < q'$, we have $k' > k$. Define $q(k)$ to be such that $\tilde{M}^k \in \mathcal{B}^{q(k)}_f$. We show that $\tilde{M}^k \xrightarrow{w^*} M_f$. If not, there must be a subsequence $(\tilde{M}^{k_m})_{m \in \mathbb{N}}$ that converges to some $M' \in \mathcal{X}$ with $M' \neq M_f$. To draw a contradiction, note first that since $D^{\preceq f}(\cdot)$ is continuous and $M^q \xrightarrow{w^*} M$, we have $D^{\preceq f}(M^q) \xrightarrow{w^*} D^{\preceq f}(M)$.[46] Combining this with the fact that $\tilde{M}^{k_m} \xrightarrow{w^*} M'$ and $\tilde{M}^{k_m} \sqsubset D^{\preceq f}(M^{q(k_m)})$, and invoking Lemma 6, we obtain $M' \sqsubset D^{\preceq f}(M)$, which implies that $u_f(M_f) - \epsilon' > u_f(M') + \epsilon'$ for some $\epsilon' > 0$, since C_f chooses a uniquely utility-maximizing subpopulation. Since $\tilde{M}^{k_m} \xrightarrow{w^*} M'$ and $M^q \xrightarrow{w^*} M_f$, we can find sufficiently large m such that $u_f(M^{q(k_m)}_f) > u_f(M_f) - \epsilon' > u_f(M') + \epsilon' > u_f(\tilde{M}^{q(k_m)})$, which contradicts with the fact that $\tilde{M}^{q(k_m)} \in \mathcal{B}^{q(k_m)}_f$. This establishes that $\tilde{M}^k \xrightarrow{w^*} M_f$. Using this and the fact that $M^q_f \xrightarrow{w^*} M_f$, one can choose sufficiently large K such that for all $k > K$, we have $|\tilde{M}^k - M_f| < \frac{\epsilon}{2}$ and $|M_f - M^{q(k)}_f| < \frac{\epsilon}{2}$, which implies that $|\tilde{M}^k - M^{q(k)}_f| < |\tilde{M}^k - M_f| + |M_f - M^{q(k)}_f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This means that for all $q > q(K)$ and $\tilde{M} \in \mathcal{B}^q_f$, we have $|\tilde{M} - M^q_f| < \epsilon$, showing that M^q is an ϵ-distance stable matching. Because u_f is continuous for all $f \in F$, this completes the proof of Theorem 3.

C Proofs for Section 6.2

Proof of Theorem 5. To prove (i), suppose a matching M is stable and population-proportional. We shall show that M satisfies the property (ii) of Definition 7. The population-proportionality of M, equivalently equality (5), implies that, if $\frac{M_f(\theta)}{G(\theta)} < \frac{M_f(\theta')}{G(\theta')}$

[46] The continuity of $D^{\preceq f}(\cdot)$ can be shown using an argument similar to that which we have used to establish the continuity of T in the proof of Lemma 3.

for any $\theta, \theta' \in \Theta_f^k$, then we must have $M_f(\theta) = D^{\preceq f}(M)(\theta)$, or else $\frac{M_f(\theta)}{G(\theta)} = \alpha_f^k$, but in that case, we have a contradiction since $\alpha_f^k \geq \frac{M_f(\theta')}{G(\theta')}$. Then, by definition of $D^{\preceq f}$,

$$M_f(\theta) = D^{\preceq f}(M)(\theta) = \sum_{f' \in \tilde{F}: f' \preceq f} M_{f'}(\theta) = M_f(\theta) + \sum_{f' \in \tilde{F}: f' \prec f} M_{f'}(\theta),$$

so $\sum_{f' \in \tilde{F}: f' \prec f} M_{f'}(\theta) = 0$. We have thus proven that M is strongly stable.

To prove (ii), fix any mechanism φ that implements a strongly stable matching for any measure. Suppose for contradiction that inequality (4) fails for some measure $G \in \overline{\mathcal{X}}$, for some a, P, P', with (a, P) and (a, P') in the support of G, and for some f. Then, let f be the most preferred firm (or the outside option) at P among those for which inequality (4) fails. Then,

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} < \sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')}, \tag{14}$$

while

$$\sum_{f': f' \succeq_P f_-^P} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} \geq \sum_{f': f' \succeq_P f_-^P} \frac{\varphi_{f'}(G)(a, P')}{G(a, P')},$$

so it follows that

$$\frac{\varphi_f(G)(a, P)}{G(a, P)} < \frac{\varphi_f(G)(a, P')}{G(a, P')}. \tag{15}$$

By the strong stability of $\varphi(G)$ and the fact that (a, P) and (a, P') are in the same indifference class for firm f by assumption, inequality (15) holds only if $\sum_{f': f \succ_P f'} \varphi_{f'}(G)(a, P) = 0$. Thus, because $\sum_{f' \in \tilde{F}} \frac{\varphi_{f'}(G)(a,P)}{G(a,P)} = 1$ as $\varphi(G)$ is a matching, we obtain

$$\sum_{f': f' \succeq_P f} \frac{\varphi_{f'}(G)(a, P)}{G(a, P)} = 1.$$

This equality contradicts inequality (14) because the right hand side of inequality (14) cannot be strictly larger than 1 as $\varphi(G)$ is a matching, which completes the proof. ▌

Proof of Theorem 6 requires several lemmas.

Lemma 8. *The correspondence defined in (3) is convex-valued and upper hemicontinuous, and satisfies the revealed preference property.*

Proof. To first show that C_f is convex-valued, for any given X, consider any $X', X'' \in C_f(X)$. Note first that $X', X'' \sqsubseteq X$ implies $\lambda X' + (1-\lambda)X'' \sqsubseteq X$. Also, for any $\lambda \in [0,1]$ and $k \in I_f$,

$$\sum_{\theta \in \Theta_f^k} (\lambda X' + (1-\lambda)X'')(\theta) = \lambda \sum_{\theta \in \Theta_f^k} X'(\theta) + (1-\lambda) \sum_{\theta \in \Theta_f^k} X''(\theta) = \Lambda_f^k(X),$$

where the second equality holds since the assumption that $X', X'' \in C_f(X)$ implies $\Lambda_f^k(X) = \sum_{\theta \in \Theta_f^k} X'(\theta) = \sum_{\theta \in \Theta_f^k} X''(\theta)$. Thus, $\lambda X' + (1-\lambda)X'' \in C_f(X)$.

To next show the upper hemicontinuity, consider two sequences $(X^\ell)_{\ell \in \mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell \in \mathbb{N}}$ converging to some X and $\tilde{X}$, respectively, such that for each ℓ, $\tilde{X}^\ell \in C_f(X^\ell)$, i.e., $\tilde{X}^\ell \sqsubseteq X^\ell$ and $\Lambda_f^k(X^\ell) = \sum_{\theta \in \Theta_f^k} \tilde{X}^\ell(\theta), \forall k \in I_f$. Since Λ_f is continuous, we have $\Lambda_f^k(X) = \lim_{\ell \to \infty} \Lambda_f^k(X^\ell) = \lim_{\ell \to \infty} \sum_{\theta \in \Theta_f^k} \tilde{X}^\ell(\theta) = \sum_{\theta \in \Theta_f^k} \tilde{X}(\theta)$, which, together with the fact that $\tilde{X} \sqsubseteq X$, means that $\tilde{X} \in C_f(X)$, establishing the upper hemicontinuity of C_f.[47] To show the revealed preference property, let $X, X' \in \mathcal{X}$ with $X' \sqsubset X$, and suppose $C_f(X) \cap \mathcal{X}_{X'} \neq \emptyset$. Consider any $Y \in C_f(X)$ such that $Y(\theta) \leq X'(\theta)$ for all θ. Then, $\Lambda_f^k(X) = \sum_{\theta \in \Theta_f^k} Y(\theta) \leq \sum_{\theta \in \Theta_f^k} X'(\theta)$ for all $k \in I_f$. By the revealed preference property of Λ_f, it follows that $\Lambda_f(X') = \Lambda_f(X)$. Therefore, Y satisfies $\sum_{\theta \in \Theta_f^k} Y(\theta) = \Lambda_f^k(X) = \Lambda_f^k(X')$ for all $k \in I_f$, which implies that $Y \in C_f(X')$ and thus $C_f(X) \cap \mathcal{X}_{X'} \subseteq C_f(X')$. To show $C_f(X) \cap \mathcal{X}_{X'} \supseteq C_f(X')$, consider any $Y \in C_f(X')$ and $\tilde{X} \in \mathcal{X}_{X'}$ such that $\tilde{X} \in C_f(X)$. By the previous argument, we have $\tilde{X} \in C_f(X')$, which implies that for each $f \in F$ and $k \in I_f$, $\sum_{\theta \in \Theta_f^k} Y(\theta) = \sum_{\theta \in \Theta_f^k} \tilde{X}(\theta)$. Since $\tilde{X} \in C_f(X)$, this means that $Y \in C_f(X)$ and thus $Y \in C_f(X) \cap \mathcal{X}_{X'}$. Therefore, we conclude that $C_f(X') = C_f(X) \cap \mathcal{X}_{X'}$ as desired. ▮

From now, we establish a couple of lemmas (Lemmas 9 and 10) and use them to prove Theorem 6. To do so, define a correspondence B_f from $\mathcal{X}$ to itself as follows:

$$B_f(X) := \{X' \sqsubset X \mid \text{ for each } k \in I_f, \text{ there is some } \alpha^k \in [0,1] \text{ such that } X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\} \text{ for all } \theta \in \Theta_f^k\}. \tag{16}$$

We then modify the choice correspondence C_f in (3) to

$$\tilde{C}_f(X) = C_f(X) \cap B_f(X), \tag{17}$$

for every $f \in F$ while we let $\tilde{C}_\varnothing = C_\varnothing$.

[47]The argument for $\tilde{X} \sqsubseteq X$ is that for each $\theta \in \Theta$, $\tilde{X}^\ell(\theta) \leq X^\ell(\theta)$, so taking the limit with respect to ℓ yields $\tilde{X}(\theta) \leq X(\theta)$.

Lemma 9. *For any $X \sqsubset G$, $\tilde{C}_f(X)$ is nonempty and a singleton set (i.e., $\tilde{C}_f$ is a function). Also, $\tilde{C}_f$ satisfies the revealed preference property.*

Proof. We first establish that for X, $\tilde{C}_f(X)$ is a singleton set. To do so, for any $X \in \mathcal{X}$, $f \in F$, $k \in I_f$, and $\alpha^k \in [0,1]$, define $\zeta_f^k(\alpha^k) := \sum_{\theta \in \Theta_f^k} \min\{X(\theta), \alpha^k G(\theta)\}$. From now on, we assume $C_f(X) \neq \{X\}$ since, if $C_f(X) = \{X\}$, then we have $\tilde{C}_f(X) = \{X\}$, a singleton set as desired. We show that there exists a unique $\hat{\alpha}^k$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Lambda_f^k(X)$, which means that $\tilde{C}_f(X)$ is a singleton set. First, we must have $\hat{\alpha}^k < \max_{\theta \in \Theta_f^k} X(\theta)$ since otherwise $\zeta_f^k(\hat{\alpha}^k) = \sum_{\theta \in \Theta_f^k} X(\theta) > \Lambda_f^k(X)$ (which follows from the assumption that $C_f(z) \neq \{X\}$ and thus, for any $X' \in C_f(X)$, $X' \sqsubset X$ and $X' \neq X$). Next, observe that $\zeta_f^k(\cdot)$ is strictly increasing in the range $[0, \max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)})$. Then, the continuity of ζ_f^k, along with the fact that $\zeta_f^k(0) = 0$ and $\zeta_f^k(\max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)}) > \Lambda_f^k(X)$, implies that there is a unique $\hat{\alpha}^k \in [0, \max_{\theta \in \Theta_f^k} \frac{X(\theta)}{G(\theta)})$ satisfying $\zeta_f^k(\hat{\alpha}^k) = \Lambda_f^k(X)$.

To show the revealed preference property, consider any $X, X', X'' \in \mathcal{X}$ such that $\tilde{C}_f(X) = \{X'\}$ and $X' \sqsubset X'' \sqsubset X$. Since we already know that $C_f(\cdot)$ satisfies the revealed preference property, we have $X' \in C_f(X'')$. It suffices to show that $X' \in B_f(X'')$, since it means $\tilde{C}_f(X'') = \{X'\}$, from which the revealed preference property follows. To do so, note that $X' \in B_f(X)$ means that $X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}$ for each k and $\theta \in \Theta_f^k$. Then, since $X(\theta) \geq X''(\theta) \geq X'(\theta)$ and $\alpha^k G(\theta) \geq X'(\theta)$, we have

$$X'(\theta) = \min\{X(\theta), \alpha^k G(\theta)\} \geq \min\{X''(\theta), \alpha^k G(\theta)\} \geq X'(\theta),$$

so $X'(\theta) = \min\{X''(\theta), \alpha^k G(\theta)\}$ as desired. ∎

Lemma 10. *Any stable matching in the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ is stable and population-proportional in the economy $(G, F, \mathcal{P}_\Theta, C_F)$.*

Proof. Consider a stable matching $M = (M_f)_{f \in \tilde{F}}$ in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$ and let $X_f = D^{\preceq f}(M)$ for each $f \in \tilde{F}$. We first show that M is stable in $(G, F, \mathcal{P}_\Theta, C_F)$. It is straightforward, thus omitted, to check the individual rationality. To check the condition of no blocking coalition, suppose to the contrary that there is a blocking pair f and M'_f, which means that $M'_f \sqsubset X_f$, $M'_f \in C_f(M'_f \vee M_f)$, and $M_f \notin C_f(M'_f \vee M_f)$. Given this, by Lemma 9, there exists $\tilde{M}_f$ such that $\tilde{C}_f(M'_f \vee M_f) = \{\tilde{M}_f\}$. First, by the revealed preference property of $\tilde{C}_f$ and the fact that $\tilde{M}_f \sqsubset (\tilde{M}_f \vee M_f) \sqsubset (M'_f \vee M_f)$, we have $\tilde{M}_f \in \tilde{C}_f(\tilde{M}_f \vee M_f)$ and $M_f \notin \tilde{C}_f(\tilde{M}_f \vee M_f)$. Second, since $M_f \sqsubset X_f$ and $M'_f \sqsubset X_f$, we have $\tilde{M}_f \sqsubset (M'_f \vee M_f) \sqsubset X_f$. In sum, f and $\tilde{M}_f$ form a blocking pair in $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, which is a contradiction.

To show the population-proportionality of M, observe that since M is stable in the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$, we have $M_f = \tilde{C}_f(D^{\preceq f}(M)) = C_f(D^{\preceq f}(M)) \cap B_f(D^{\preceq f}(M))$ for each $f \in F$. Thus, $M_f \in B_f(D^{\preceq f}(M))$, that is, there is some α^k for each $k \in I_f$ such that (5) holds. ▮

Proof of Theorem 6. Given Lemma 10, it suffices to establish the existence of stable matching in the economy $(G, F, \mathcal{P}_\Theta, \tilde{C}_F)$. For doing so, we prove the continuity of $\tilde{C}_f$ and invoke Theorem 2. The continuity of $\tilde{C}_f = C_f \cap B_f$ follows if both C_f and B_f are shown to be upper hemicontinuous, since the intersection of a family of closed-valued upper hemicontinuous correspondences, one of which is also compact-valued, is upper hemicontinuous (see 16.25 Theorem of Aliprantis and Border (2006) for instance), implying that $\tilde{C}_f$, which is a single-valued correspondence by Lemma 9, is continuous.

Since C_f is upper hemicontinuous by Lemma 8, it remains to show that B_f is upper hemicontinuous. Consider sequences $(X^\ell)_{\ell\in\mathbb{N}}$ and $(\tilde{X}^\ell)_{\ell\in\mathbb{N}}$ with $\tilde{X}^\ell \in B_f(X^\ell), \forall \ell$, converging weakly to X and $\tilde{X}$, respectively. So, for each $k \in I_f$, there is a sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ such that $\tilde{X}^\ell(\theta) = \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}, \forall \theta \in \Theta^k_f$. For each k, let α^k be a limit to which a subsequence of the sequence $(\alpha^k_\ell)_{\ell\in\mathbb{N}}$ converges. We claim that $\tilde{X}(\theta) = \min\{X(\theta), \alpha^k G(\theta)\}, \forall \theta \in \Theta^k_f$. If $\tilde{X}(\theta) > \min\{X(\theta), \alpha^k G(\theta)\}$, then one can find sufficiently large ℓ to make $\tilde{X}^\ell(\theta)$, $X^\ell(\theta)$, and α^k_ℓ close to $\tilde{X}(\theta)$, $X(\theta)$, and α^k, respectively, so that $\tilde{X}^\ell(\theta) > \min\{X^\ell(\theta), \alpha^k_\ell G(\theta)\}$, which is a contradiction. The same argument applies to the case with $\tilde{X}(\theta) < \min\{X(\theta), \alpha^k G(\theta)\}$. ▮

References

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 2003. "School Choice: A Mechanism Design Approach." *American Economic Review*, 93: 729–747.

Abdulkadiroğlu, Atila, Parag A. Pathak, Alvin E. Roth, and Tayfun Sönmez. 2005. "The Boston Public School Match." *American Economic Review Papers and Proceedings*, 95: 368–372.

Abdulkadiroğlu, Atila, Parag A. Pathak, and Alvin E. Roth. 2005. "The New York City High School Match." *American Economic Review Papers and Proceedings*, 95: 364–367.

Adachi, Hiroyuki. 2000. "On a Characterization of Stable Matchings." *Economics Letters*, 68(1): 43–49.

Aliprantis, Charalambos D., and Kim C. Border. 2006. *Infinite Dimensional Analysis: A Hitchhiker's Guide.* Berlin:Springer.

Alkan, Ahmet, and David Gale. 2003. "Stable Schedule Matching Under Revealed Preferences." *Journal of Economic Theory*, 112: 289–306.

Ashlagi, Itai, Mark Braverman, and Avinatan Hassidim. 2014. "Stability in Large Matching Markets with Complementarities." *Operations Research*, 62(4): 713–732.

Ashlagi, Itai, Yash Kanoria, and Jacob Leshno. 2014. "Unbalanced Random Matching Markets: The Stark Effect of Competition." Unpublished mimeo.

Ash, Robert B. 1977. *Real Analysis and Probability.* San Diego:Academic Press.

Aygün, Orhan, and Tayfun Sönmez. 2013. "Matching with Contracts: Comment." *American Economic Review*, 103(5): 2050–2051.

Azevedo, Eduardo M. 2014. "Imperfect Competition in Two-Sided Matching Markets." *Games and Economic Behavior*, 83: 207–223.

Azevedo, Eduardo M., and Jacob D. Leshno. 2011. "A Supply and Demand Framework for Two-Sided Matching Markets." *Unpublished mimeo, Harvard Business School.*

Azevedo, Eduardo M., and John William Hatfield. 2012. "Complementarity and Multidimensional Heterogeneity in Matching Markets." Unpublished mimeo.

Azevedo, Eduardo M., E. Glen Weyl, and Alexander White. 2013. "Walrasian Equilibrium in Large, Quasilinear Markets." *Theoretical Economics*, 8(2): 281–290.

Biró, Péter, Tamás Fleiner, and Rob Irving. 2013. "Matching Couples with Scarf's Algorithm." *Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and its Applications*, 55–64.

Blair, Charles. 1984. "Every Finite Distributive Lattice Is a Set of Stable Matchings." *Journal of Combinatorial Theory*, 37: 353–356.

Che, Yeon-Koo, and Fuhito Kojima. 2010. "Asymptotic Equivalence of Probabilistic Serial and Random Priority Mechanisms." *Econometrica*, 78(5): 1625–1672.

Che, Yeon-Koo, and Olivier Tercieux. 2015*a*. "Efficiency and Stability in Large Matching Markets." Unpublished mimeo.

Che, Yeon-Koo, and Olivier Tercieux. 2015*b*. "Payoff Equivalence of Efficient Mechanisms in Large Matching Markets." Unpublished mimeo.

Che, Yeon-Koo, and Youngwoo Koh. 2013. "Decentralized College Admissions." Unpublished mimeo.

Debreu, Gerald. 1954. "Representation of a Preference Ordering by a Numerical Function." *Decision Processes*, 159–165.

Echenique, Federico, and Jorge Oviedo. 2004. "Core Many-to-One Matchings by Fixed Point Methods." *Journal of Economic Theory*, 115: 358–376.

Echenique, Federico, and Jorge Oviedo. 2006. "A theory of stability in many-to-many matching." *Theoretical Economics*, 1: 233–273.

Echenique, Federico, and M. Bumin Yenmez. 2007. "A Solution to Matching with Preferences over Colleagues." *Games and Economic Behavior*, 59(1): 46–71.

Fleiner, Tamás. 2003. "A Fixed-Point Approach to Stable Matchings and Some Applications." *Mathematics of Operations Research*, 28: 103–126.

Gale, David, and Lloyd S. Shapley. 1962. "College Admissions and the Stability of Marriage." *American Mathematical Monthly*, 69: 9–15.

Hatfield, John William, and Fuhito Kojima. 2008. "Matching with Contracts: Comment." *American Economic Review*, 98: 1189–1194.

Hatfield, John William, and Paul Milgrom. 2005. "Matching with Contracts." *American Economic Review*, 95: 913–935.

Hatfield, John William, and Scott Duke Kominers. 2014. "Contract Design and Stability in Many-to-Many Matching." Unpublished mimeo.

Hatfield, John William, Fuhito Kojima, and Yusuke Narita. 2014*a*. "Many-to-Many Matching with Max–Min Preferences." *Discrete Applied Mathematics*, 179: 235–240.

Hatfield, John William, Fuhito Kojima, and Yusuke Narita. 2014*b*. "Promoting School Competition Through School Choice: A Market Design Approach." Unpublished mimeo.

Immorlica, Nicole, and Mohammad Mahdian. 2005. "Marriage, Honesty, and Stability." *SODA 2005*, 53–62.

Kagel, John H., and Alvin E. Roth. 2000. “The Dynamics of Reorganization in Matching Markets: A Laboratory Experiment Motivated by a Natural Experiment.” *The Quarterly Journal of Economics*, 115(1): 201–235.

Kelso, Alexander, and Vincent Crawford. 1982. “Job matching, coalition formation, and gross substitutes.” *Econometrica*, 50: 1483–1504.

Kesten, Onur, and U. Ünver. 2014. “A Theory of School Choice Lotteries.” forthcoming, *Theoretical Economics.*

Klaus, Bettina, and Flip Klijn. 2005. “Stable Matchings and Preferences of Couples.” *Journal of Economic Theory*, 121(1): 75–106.

Kojima, Fuhito, and Mihai Manea. 2010. “Incentives in the Probabilistic Serial Mechanism.” *Journal of Economic Theory*, 145(1): 106–123.

Kojima, Fuhito, and Parag A. Pathak. 2009. “Incentives and Stability in Large Two-Sided Matching Markets.” *American Economic Review*, 608–627.

Kojima, Fuhito, Parag A. Pathak, and Alvin E. Roth. 2013. “Matching with Couples: Stability and Incentives in Large Markets.” *Quarterly Journal of Economics*, 128: 1585–1632.

Lee, SangMok. 2012. “Incentive Compatibility of Large Centralized Matching Markets.” Unpublished mimeo.

Liu, Qingmin, and Marek Pycia. 2013. “Ordinal Efficiency, Fairness, and Incentives in Large Markets.” Unpublished mimeo.

Manea, Mihai. 2009. “Asymptotic Ordinal Inefficiency of Random Serial Dictatorship.” *Theoretical Economics*, 4(2): 165–197.

Ok, Efe. 2011. *Real Analysis with Economic Applications.* Princeton:Princeton University Press.

Ostrovsky, Michael. 2008. “Stability in Supply Chain Networks.” *American Economic Review*, 897–923.

Pycia, Marek. 2012. “Stability and Preference Alignment in Matching and Coalition Formation.” *Econometrica*, 80(1): 323–362.

Roth, Alvin E. 1984. "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory." *Journal of Political Economy*, 92: 991–1016.

Roth, Alvin E. 1985. "The college admission problem is not equivalent to the marriage problem." *Journal of Economic Theory*, 36: 277–288.

Roth, Alvin E. 1991. "A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom." *American Economic Review*, 415–440.

Roth, Alvin E. 2002. "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics." *Econometrica*, 70: 1341–1378.

Roth, Alvin E., and Elliott Peranson. 1999. "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design." *American Economic Review*, 89: 748–780.

Royden, Halsey, and Patrick Fitzpatrick. 2010. *Real Analysis.* New Jersey:Pearson.

Sönmez, Tayfun, and Utku Ünver. 2010. "Course Bidding at Business Schools." *International Economic Review*, 51(1): 99–123.

Sotomayor, Marilda. 1999. "Three Remarks on the Many-to-Many Stable Matching Problem." *Mathematical Social Sciences*, 38(1): 55–70.