Supplementary Notes for "Decentralized College Admissions"

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The notes consist of three sections. Section A presents the proof of Lemma 1 in the paper. Section B and Section C provide formal analyses of "restricted applications" and "information acquisition and evaluation costs," respectively.

A Proof of Lemma 1

The proof consist of three steps.

Step 1. *In equilibrium, $m_A(0) \le \kappa \le m_A(1)$ and $m_B(1) \le \kappa \le m_B(0)$.*

Proof. Consider college A. The analysis for B is analogous. Suppose $m_A(1) < \kappa$. Let A admit a mass $\kappa - m_A(1)$ of students in addition to those it currently admits. The resulting enrollment, denoted by $\widetilde{m}_A(s)$, satisfies that for any $s < 1$,

$$m_A(s) < m_A(s) + \mu_A(s)(\kappa - m_A(1)) \le \widetilde{m}_A(s) \le m_A(s) + (\kappa - m_A(1)) < \kappa,$$

where the the last inequality holds since $m_A(s) < m_A(1)$ for $s < 1$. Note that A benefits from the deviation since it admits more students without exceeding its capacity. Similarly, if $m_A(0) > \kappa$, then A can benefit by rejecting $m_A(0) - \kappa$ of students. Thus, we must have $m_A(0) \le \kappa \le m_A(1)$ in equilibrium. ■

Step 2. *In equilibrium, there is a unique $\hat{s}_i \in (0, 1)$, for $i = A, B$, such that $m_i(\hat{s}_i) = \kappa$.*

Proof. Suppose $\hat{s}_A = 1$. Then, $m_A(s) < m_A(1) = \kappa$ for all $s < 1$, so college A does not pay the capacity cost. Consider a deviating strategy such that A admits a small fraction of students, say $(v, e_A) \in (c, c + \delta) \times (e, e + \eta)$ for some $c, e < 1$ and small $\delta, \eta > 0$, who are not admitted currently. The resulting enrollment is

$$\widetilde{m}_A(s) = m_A(s) + \int_c^{c+\delta} \int_e^{e+\eta} (1 - \overline{\sigma}_B(v) + \mu_A(s)\overline{\sigma}_B(v)) de_A \, g(v) dv.$$

Let $\tilde{s}_A$ be such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A < \hat{s}_A = 1$ since $\widetilde{m}_A(s) > m_A(s)$. Next, choose δ and η such that $1 - \tilde{s}_A < \varepsilon$ for sufficiently small ε. From the deviation, A gains from admitting more students but incurs capacity costs. Hence, its net payoff is

$$\begin{aligned}
&\int_c^{c+\delta}\int_e^{e+\eta} U^A(v,e_A)(1-\overline{\sigma}_B(v)+\overline{\mu}_A\overline{\sigma}_B(v))de_A\,g(v)dv - \lambda\int_{\tilde{s}_A}^1(\widetilde{m}_A(s)-\kappa)ds \\
&= \int_c^{c+\delta}\int_e^{e+\eta} U^A(v,e_A)(1-\overline{\sigma}_B(v)+\overline{\mu}_A\overline{\sigma}_B(v))de_A\,g(v)dv \\
&\quad - \lambda\int_{\tilde{s}_A}^1\left(m_A(s)+\int_c^{c+\delta}\int_e^{e+\eta}(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\,g(v)dv-\kappa\right)ds \\
&= \int_c^{c+\delta}\int_e^{e+\eta} U^A(v,e_A)(1-\overline{\sigma}_B(v)+\overline{\mu}_A\overline{\sigma}_B(v))g(v)dv \\
&\quad - \lambda\int_{\tilde{s}_A}^1\int_c^{c+\delta}\int_e^{e+\eta}(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\,g(v)dv\,ds + \lambda\int_{\tilde{s}_A}^1(\kappa-m_A(s))ds \\
&> \int_0^{\tilde{s}_A}\int_c^{c+\delta}\int_e^{e+\eta} U^A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\,g(v)dv\,ds \\
&\quad - \int_{\tilde{s}_A}^1\int_c^{c+\delta}\int_e^{e+\eta}(\lambda-U^A(v,e_A))(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\,g(v)dv\,ds \\
&> 0,
\end{aligned}$$

where $\overline{\mu}_A = \mathbb{E}[\mu_A(s)]$, the penultimate inequality holds since $m_A(s) < \kappa$ for $s \in [\tilde{s}_A, 1)$, and the last inequality holds for sufficiently small ε. Similarly, if $\hat{s}_A = 0$, then A can benefit by rejecting a small fraction of students who are currently admitted. Hence, we must have $\hat{s}_A \in (0,1)$ in equilibrium. The proof for B is analogous. ■

Note that Step 1 and Step 2 imply that $m_A(0) < \kappa < m_A(1)$ and $m_B(1) < \kappa < m_B(0)$.

Step 3. *In equilibrium, the measure of $\mathcal{U}_A \cup \mathcal{U}_B$ is less than 1.*

Proof. Suppose to the contrary that almost all students are admitted by either college. Without loss of generality, suppose $[0,c]\times[0,e]\times[0,1]\cap\mathcal{U}_A \neq \emptyset$ for some $c, e < 1$. Let $\hat{s}_A$ be the cutoff state. Now, let A reject a small fraction of students, say $(v, e_A) \in [0,\delta)\times[0,\eta)$ for small $\delta < c$ and $\eta < e$, who are currently admitted. The resulting enrollment is

$$\widetilde{m}_A(s) = m_A(s) - \int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\,g(v)dv, \tag{A.1}$$

and let $\tilde{s}_A$ be such that $\widetilde{m}_A(\tilde{s}_A) = \kappa$. Note that $\tilde{s}_A > \hat{s}_A$ since $\widetilde{m}_A(s) < m_A(s)$. Next, choose

δ and η such that $\lambda(1-\tilde{s}_A) > U^A(\delta,\eta)$. From the deviation, A loses its revenue by

$$
\begin{aligned}
(*) &= \int_0^\delta \int_0^\eta U^A(v,e_A)\sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\overline{\mu}_A\overline{\sigma}_B(v))de_A\, g(v)dv \\
&< U^A(\delta,\eta)\int_0^1\int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\, g(v)dv\, ds,
\end{aligned}
$$

where the inequality holds since U^A is strictly increasing in v and nondecreasing in e_A, but saves the capacity costs by

$$
\begin{aligned}
(**) &= \lambda\left(\int_{\hat{s}_A}^1 \big(m_A(s)-\kappa\big)ds - \int_{\tilde{s}_A}^1 \big(\widetilde{m}_A(s)-\kappa\big)ds\right) \\
&= \lambda\left(\int_{\hat{s}_A}^{\tilde{s}_A} \big(m_A(s)-\kappa\big)ds + \int_{\tilde{s}_A}^1 \big(m_A(s)-\kappa\big)ds - \int_{\tilde{s}_A}^1 \big(\widetilde{m}_A(s)-\kappa\big)ds\right) \\
&> \lambda\int_{\tilde{s}_A}^1\int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\, g(v)dv
\end{aligned}
$$

where the inequality follows from from (A.1) and the fact that $m_A(s) > \kappa$ for any $s \in (\hat{s}_A, \tilde{s})$. Hence, A's payoff increase from the deviation is

$$
\begin{aligned}
(**)-(*) &> (\lambda - U^A(\delta,\eta))\int_{\tilde{s}_A}^1\int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A, g(v)dv\, ds \\
&\quad - U^A(\delta,\eta)\int_0^{\tilde{s}_A}\int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(s)\overline{\sigma}_B(v))de_A\, g(v)dv\, ds \\
&> (\lambda - U^A(\delta,\eta))(1-\tilde{s}_A)\int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(\tilde{s}_A)\overline{\sigma}_B(v))de_A\, g(v)dv \\
&\quad - U^A(\delta,\eta)\,\tilde{s}_A\int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(\tilde{s}_A)\overline{\sigma}_B(v))de_A\, g(v)dv \\
&= \big(\lambda(1-\tilde{s}_A) - U^A(\delta,\eta)\big)\int_0^\delta\int_0^\eta \sigma_A(v,e_A)(1-\overline{\sigma}_B(v)+\mu_A(\tilde{s}_A)\overline{\sigma}_B(v))de_A\, g(v)dv \\
&> 0,
\end{aligned}
$$

where the first inequality holds since $\lambda > U^A(\delta,\eta)$ and the last equality holds since $\lambda(1-\tilde{s}_A) > U^A(\delta,\eta)$ by construction. ■

B Restricted Applications

A student's taste $y \in \mathcal{Y} \equiv [0,1]$ is drawn according to a distribution that depends on the underlying state. For a given s, let $K(y|s)$ be the distribution of y with density function

$k(y|s)$, which is continuous and obeys (strict) monotone likelihood ratio property: for any $y' > y$ and $s' > s$,

$$\frac{k(y'|s')}{k(y|s')} > \frac{k(y'|s)}{k(y|s)}, \tag{B.1}$$

meaning that a student's value of A is likely to be high when s is high. We further assume that there exists $\delta > 0$ such that $\left|\frac{k_y(y|s)}{k(y|s)}\right| < \delta$ for any $y \in [0,1]$ and $s \in [0,1]$, which indicates that students' tastes change moderately relative to the state. Each student with taste y forms a posterior belief about the state s, given by the following conditional density:

$$l(s|y) := \frac{k(y|s)}{\int_0^1 k(y|s)ds}.$$

Before proceeding, we make the following observations: First, for each student, applying to a college dominates not applying at all. Second, since a student does not know the score and the student's preference is independent of the score, the student's application depends solely on the preference. Third, since each student's preference depends on the state, the mass of students applying to each college varies across states. Let $n_i(s)$ be the mass of students who apply to college $i = A, B$ in state s.

Next, consider colleges' admission strategies. Since colleges face no enrollment uncertainty, it is optimal to admit all students up to a cutoff: for $i = A, B$,

$$\hat{v}_i(s) := \inf\left\{v \in [0,1] \,|\, n_i(s)(1 - G(v)) \leq \kappa\right\}.$$

If $n_i(s) \geq \kappa$ in state s, then college i will set its cutoff so as to admit students up to its capacity. Otherwise, it will admit all applicants.

Consider students' application decisions. Fix any strategy $\sigma : \mathcal{Y} \to [0,1]$ which specifies a probability of applying to A for each $y \in \mathcal{Y}$. The mass of applicants to each college is

$$n_A(s|\sigma) := \int_0^1 \sigma(y)k(y|s)\,dy$$

and $n_B(s|\sigma) = 1 - n_A(s|\sigma)$. A student with taste y expects to be admitted by college $i = A, B$ with probability

$$P_i(y|\sigma) \equiv \mathbb{E}[1 - G(\hat{v}_i(s))\,|\,y, \sigma] = \int_0^1 q_i(s|\sigma)l(s|y)ds,$$

where $q_i(s|\sigma) := \min\left\{\kappa/n_i(s|\sigma), 1\right\}$. This probability depends on the student's preference y since it is correlated with the underlying state. Note that a student with taste y will apply to A if and only if $yP_A(y|\sigma) \geq (1-y)P_B(y|\sigma)$.

We now provide characterizations of students' application behavior. Lemma B1 shows that students follow a cutoff strategy in any equilibrium, given a moderate value of δ, and Theorem B1 shows that equilibrium involves strategic application by students if one school is more popular than the other.

Lemma B1. *Suppose $\delta \leq \frac{1}{2}$. In any equilibrium, there exists a cutoff $\hat{y}$ such that students with $y \geq \hat{y}$ apply to A and those with $y < \hat{y}$ apply to B. And such an equilibrium exists.*

Proof. Define $T(y|\sigma) := y\, P_A(y|\sigma) - (1-y)\, P_B(y|\sigma)$ and fix any σ. To prove the optimality of the cutoff strategy, we show that $T'(y|\sigma) > 0$ for any y. Note that

$$\begin{aligned} T'(y|\sigma) &= P_A(y|\sigma) + P_B(y|\sigma) + yP'_A(y|\sigma) - (1-y)P'_B(y|\sigma) \\ &\geq y\big(P_A(y|\sigma) + P'_A(y|\sigma)\big) + (1-y)\big(P_B(y|\sigma) - P'_B(y|\sigma)\big) \\ &= y\int_0^1 q_A(s|\sigma)\big(l(s|y) + l_y(s|y)\big)ds + (1-y)\int_0^1 q_B(s|\sigma)\big(l(s|y) - l_y(s|y)\big)ds. \end{aligned}$$

Observe that

$$l(s|y) + l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds}\left(1 + \frac{k_y(y|s)}{k(y|s)} - \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds}\right) > \frac{k(y|s)}{\int_0^1 k(y|s)ds}(1-2\delta),$$

where the inequality holds since

$$\frac{k_y(y|s)}{k(y|s)} > -\delta \quad \text{and} \quad \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds} = \frac{\int_0^1 \frac{k_y(y|s)}{k(y|s)}k(y|s)ds}{\int_0^1 k(y|s)ds} < \delta$$

because $\left|\frac{k_y(y|s)}{k(y|s)}\right| < \delta$. Similarly,

$$l(s|y) - l_y(s|y) = \frac{k(y|s)}{\int_0^1 k(y|s)ds}\left(1 - \frac{k_y(y|s)}{k(y|s)} + \frac{\int_0^1 k_y(y|s)ds}{\int_0^1 k(y|s)ds}\right) > \frac{k(y|s)}{\int_0^1 k(y|s)ds}(1-2\delta),$$

where the inequality holds since $\left|\frac{k_y(y|s)}{k(y|s)}\right| < \delta$. We thus have $T'(y|\sigma) > 0$ whenever $\delta \geq \frac{1}{2}$.

It remains to show that there exists an equilibrium in cutoff strategy. Let $\hat{y}$ be a cutoff. Then, $n_A(s|\hat{y}) = \int_{\hat{y}}^1 k(y|s)dy = 1 - K(\hat{y}|s)$. Hence,

$$P_A(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{1 - K(\hat{y}|s)}, 1\right\} l(s|y)ds \text{ and } P_B(y|\hat{y}) = \int_0^1 \min\left\{\frac{\kappa}{K(\hat{y}|s)}, 1\right\} l(s|y)ds.$$

Now, let $T(y|\hat{y}) := yP_A(y|\hat{y}) - (1-y)P_B(y|\hat{y})$. Note that

$$T(0|\hat{y}) = -P_B(0|\hat{y}) = -\int_0^1 \min\Big\{\frac{\kappa}{K(\hat{y}|s)}, 1\Big\} l(s|0)ds < 0,$$

where the inequality holds since $\min\big\{\frac{\kappa}{K(\hat{y}|s)}, 1\big\} > 0$ and $l(s|0) \geq 0$, with strict inequality for a positive measure of states. Similarly, $T(1|\hat{y}) > 0$. By the continuity of $T(\cdot|\hat{y})$, there is a $\tilde{y}$ such that $T(\tilde{y}|\hat{y}) = 0$. Moreover, such a $\tilde{y}$ is unique since $T'(y|\hat{y})\big|_{y=\tilde{y}} > 0$.

Next, let $\tau : [0,1] \to [0,1]$ be the map from $\hat{y}$ to $\tilde{y}$, which is implicitly defined by $T(\tau(\hat{y})|\hat{y}) = 0$ according to the implicit function theorem (since $T'(y|\hat{y})\big|_{y=\tilde{y}} > 0$). Since $P_A(y|\cdot)$ is nondecreasing and $P_B(y|\cdot)$ is nonincreasing $\hat{y}$, $\tau(\cdot)$ is decreasing. Hence, there is a fixed point such that $\tau(\hat{y}) = \hat{y}$, and so there is $\hat{y}$ such that $T(\hat{y}|\hat{y}) = 0$. ∎

Theorem B1. *Suppose $\mu_A(s) > \frac{1}{2}$ $\big(\mu_A(s) = \frac{1}{2}\big)$ for almost all s. Then, $\hat{y} \in (\frac{1}{2}, 1)$ $\big(\hat{y} = \frac{1}{2}\big)$, where $\hat{y}$ is the equilibrium cutoff.*

Proof. We first show $\hat{y} < 1$. If $\hat{y} = 1$, then $n_A(s|1) = 1 - K(1|s) = 0$ and so $P_A(y|\hat{y}) = 1$ for any y. Hence, $T(1|1) = P_A(1|1) = 1$, which contradicts the fact that $T(\hat{y}|\hat{y}) = 0$. Next, we show that $\hat{y} > \frac{1}{2}$ whenever $\mu_A(s) > \frac{1}{2}$. Suppose to the contrary $\hat{y} \leq \frac{1}{2}$. Then, it holds that $\frac{1}{2} < \mu_A(s) = 1 - K(\frac{1}{2}|s) \leq 1 - K(\hat{y}|s)$, so $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. Therefore,

$$P_A(y|\hat{y}) - P_B(y|\hat{y}) = \int_0^1 \min\Big\{\frac{\kappa}{1-K(\hat{y}|s)}, 1\Big\} l(s|y)ds - \int_0^1 \min\Big\{\frac{\kappa}{K(\hat{y}|s)}, 1\Big\} l(s|y)ds \leq 0. \tag{B.2}$$

Hence, if $\hat{y} < \frac{1}{2}$, then

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1-\hat{y})P_B(\hat{y}|\hat{y}) < \tfrac{1}{2}\big(P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})\big) \leq 0, \tag{B.3}$$

where the first inequality holds since $\hat{y} < \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) < 0$, a contradiction. Suppose now $\hat{y} = \frac{1}{2}$. Notice that since $K(\hat{y}|s) < 1 - K(\hat{y}|s)$, we have $K(\frac{1}{2}|s) < \frac{1}{2} < 1 - \kappa$, where the the second inequality holds since $\kappa < \frac{1}{2}$. So, $\kappa/(1 - K(\frac{1}{2}|s)) < 1$. Therefore, the last inequality of (B.2) becomes strict, and hence $T(\hat{y}|\hat{y}) = \frac{1}{2}\big(P_A(\frac{1}{2}|\frac{1}{2}) - P_B(\frac{1}{2}|\frac{1}{2})\big) < 0$, a contradiction again.

Lastly, $\hat{y} = \frac{1}{2}$ whenever $\mu_A(s) = \frac{1}{2}$. If $\hat{y} < \frac{1}{2}$, then $\frac{1}{2} = \mu_A(s) = 1 - K(\frac{1}{2}|s) < 1 - K(\hat{y}|s)$, so we have $K(\hat{y}|s) < 1 - K(\hat{y}|s)$. By (B.2) and (B.3), we reach a contradiction. If $\hat{y} > \frac{1}{2}$, then $\frac{1}{2} = \mu_A(s) = 1 - K(\frac{1}{2}|s) > 1 - K(\hat{y}|s)$ and so $K(\hat{y}|s) > 1 - K(\hat{y}|s)$. We then have $P_A(y|\hat{y}) - P_B(y|\hat{y}) \geq 0$ and

$$T(\hat{y}|\hat{y}) = \hat{y}P_A(\hat{y}|\hat{y}) - (1-\hat{y})P_B(\hat{y}|\hat{y}) > \tfrac{1}{2}\big(P_A(\hat{y}|\hat{y}) - P_B(\hat{y}|\hat{y})\big) \geq 0,$$

where the first inequality holds since $\hat{y} > \frac{1}{2}$. Thus, $T(\hat{y}|\hat{y}) > 0$, a contradiction. ∎

The intuition behind Theorem B1 is clear. If A is more popular than B, then A becomes more difficult to get in than B, all else equal. Hence, students who prefer B ($y \leq \frac{1}{2}$) will definitely apply to B. But, even students who mildly prefer A (i.e., y is greater than but close to $\frac{1}{2}$) will apply to B instead of A.

Let us now consider fairness and welfare properties of the equilibrium outcome. First, the equilibrium is unfair. Justified envy arises in that (i) students who happen to have applied to a more popular college for a given state may be unassigned even though their scores are good enough for the other college; and (ii) students who prefer but avoid an ex ante more popular college get into an ex ante less popular college, but they could have gotten into the former when it becomes ex post less popular. Second, a college may be undersubscribed in equilibrium so that its capacity is not filled even though there are unassigned, acceptable students. By assigning those students to unfilled seats of that college, students and the college will be both better off. Thus, the equilibrium outcome is inefficient.

Theorem B2. *The outcome of the restricted applications is unfair. Suppose $K(\hat{y}|s) < \kappa$ for a positive measure of states. Then, college B suffers from under-subscription, and the outcome is Pareto inefficient.*

Proof. For the first part of the theorem, observe that for a given s, justified envy arises whenever $\hat{v}_A(s) \neq \hat{v}_B(s)$. Suppose to the contrary $\hat{v}_A(s) = \hat{v}_B(s)$ for almost all s. Recall that equilibrium admission cutoffs satisfy

$$G(\hat{v}_A(s)) = \max\left\{1 - \frac{\kappa}{1 - K(\hat{y}|s)}, 0\right\} \quad \text{and} \quad G(\hat{v}_B(s)) = \max\left\{1 - \frac{\kappa}{K(\hat{y}|s)}, 0\right\}.$$

Since $G(\cdot)$ is strictly increasing, if $\hat{v}_A(s) = \hat{v}_B(s)$, then we must have either $n_i(s) < \kappa$ for all $i = A, B$ (so that $\hat{v}_A(s) = \hat{v}_B(s) = 0$) or $n_A(s) = n_B(s) \geq \kappa$. Note, however, that we cannot have $n_i(s) < \kappa$ for all i in equilibrium, since this means that all applicants are admitted by either college, which contradicts the fact that $2\kappa < 1$. Next, suppose $n_A(s) = n_B(s) \geq \kappa$. This implies that $K(\hat{y}|s) = \frac{1}{2}$ for all s (recall that $n_A(s) = 1 - K(\hat{y}|s)$ and $n_B(s) = K(\hat{y}|s)$). However, by (B.1), we have $K(\hat{y}|s') < K(\hat{y}|s)$ for all $s' > s$. Therefore, we reach a contradiction again.

For the second part of the theorem, recall that for given $\hat{y}$ in equilibrium, the mass of students applying to B is $K(\hat{y}|s)$. Thus, if there is a positive measure of states in which $K(\hat{y}|s) < \kappa$, then college B faces under-subscription in such states, implying that the equilibrium outcome is inefficient. ■

C DA with Costly Learning of Preferences by Students

Consider DA with only common measure. In order to study the information acquisition behavior of students, we need to introduce cardinal utilities of the students: in state s, a fraction $\mu_i(s)$ of students gets utility u from college i and $u'(< u)$ from j, where $i, j = A, B$ and $i \neq j$. Students do not know their preferences but can learn about them at small cost $c > 0$. We assume that the cost is sufficiently small so that if a student is certain of gaining admissions from both colleges, then he will incur the cost. (As in the paper, students know their own v).

Assume $\mathbb{E}[\mu_A(s)] > \frac{1}{2}$ without loss of generality (the case in which $\mathbb{E}[\mu_A(s)] = \frac{1}{2}$ is similar but requires separate analysis). We look for an equilibrium in which students with v incur the cost to learn their preferences if and only if $v \geq \hat{v}$ for some cutoff $\hat{v}$; we later show such an equilibrium exists. In this equilibrium with cutoff $\hat{v}$, the strategyproofness of DA means that those who incur the cost rank the colleges according to their actual realized preferences and those who do not incur the cost rank them according to their prior, which means they rank A ahead of B. It then follows that a fraction $\mu_A(s)(1 - G(\hat{v})) + G(\hat{v})$ of students ranks A above B, and the remaining fraction $(1 - \mu_A(s))(1 - G(\hat{v}))$ of students ranks B above A.

Following Azevedo and Leshno (2012), a DA allocation in each state s is characterized by the cutoffs $(\hat{v}_A(s; \hat{v}), \hat{v}_B(s; \hat{v}))$ for the colleges such that each student is assigned the most preferred college (according to his ranking, described above) among the colleges whose cutoff is below his score. Clearly, the outcome maintains the feature that only top 2κ in v are assigned either college, which implies that $\min\{\hat{v}_A(s; \hat{v}), \hat{v}_B(s; \hat{v})\} = \overline{v}$, where $\overline{v}$ satisfies $1 - G(\overline{v}) = 2\kappa$. Obviously, students with $v < \overline{v}$ will not be assigned anywhere. This in turn implies that no student with $v \leq \overline{v}$ has any incentive to incur the learning cost. Hence, from now on, we restrict attention to $\hat{v} > \overline{v}$.

To be precise, $(\hat{v}_A(s; \hat{v}), \hat{v}_B(s; \hat{v}))$ satisfies $D_i(\hat{v}_i(s; \hat{v}); \hat{v}_j(s; \hat{v})) = \kappa$, for $i, j = A, B$ and $i \neq j$, where D_i is the "demand" for college i:

$$D_A(\tilde{v}; \hat{v}_B) := \begin{cases} \mu_A(s)(1 - G(\tilde{v})) & \text{if } \tilde{v} \geq \hat{v}, \\ \mu_A(s)(1 - G(\hat{v})) + \max\{\mu_B(s)(G(\hat{v}_B(s)) - G(\hat{v})), 0\} + G(\hat{v}) - G(\tilde{v}) & \text{if } \tilde{v} < \hat{v}, \end{cases}$$

and

$$D_B(\tilde{v}; \hat{v}_A) := \begin{cases} \mu_B(s)(1 - G(\tilde{v})) & \text{if } \tilde{v} \geq \hat{v}, \\ \mu_B(s)(1 - G(\hat{v})) + \max\{\mu_A(s)(G(\hat{v}_A(s) - G(\hat{v})), 0\} & \\ \quad + \min\{G(\hat{v}_A(s; \hat{v})), G(\hat{v})\} - G(\tilde{v}) & \text{if } \tilde{v} < \hat{v}. \end{cases}$$

To see this, consider D_A. (D_B can be understood similarly.) The mass of students with $v \geq \hat{v}$

and ranking A above B is $\mu_A(s)(1-G(\hat{v}))$. If $\mu_A(s)(1-G(\hat{v})) \geq \kappa$, then only the top κ students among them are assigned to A, so $\mu_A(s)(1-G(\hat{v}_A(s;\hat{v}))) = \kappa$. If $\mu_A(s)(1-G(\hat{v})) < \kappa$, then all of such students are assigned to A, and the students with $v \geq \hat{v}$ and ranking B above A are also assigned to A whenever they are rejected by B (if exist). And then, the remaining seats are filled by the top students below $\hat{v}$. Hence, $\hat{v}_A(s;\hat{v})$ satisfies

$$\mu_A(s)(1-G(\hat{v})) + \max\{\mu_B(s)(G(\hat{v}_B(s)) - G(\hat{v})), 0\} + G(\hat{v}) - G(\hat{v}_A(s;\hat{v})) = \kappa.$$

Consider now a student with v. Suppose the student incurs the cost and finds himself preferring i. Then, he will be assigned to i and get u if $v \geq \hat{v}_i(s;\hat{v})$ or will be assigned to j and get u' if $\hat{v}_i(s;\hat{v}) > v \geq \overline{v}$. Denote the student's payoff by $\hat{u}(v;\hat{v}) - c$, where

$$\hat{u}(v;\hat{v}) := \int_0^1 \Big(\mu_A(s)\big(\mathbb{1}_{\{v\geq\hat{v}_A(s;\hat{v})\}}u + \mathbb{1}_{\{\hat{v}_A(s)>v\geq\overline{v}\}}u'\big) + \mu_B(s)\big(\mathbb{1}_{\{v\geq\hat{v}_B(s;\hat{v})\}}u + \mathbb{1}_{\{\hat{v}_B(s)>v\geq\overline{v}\}}u'\big)\Big)ds.$$

Suppose the student does not incur the cost. Since the student ranks A above B, he will be assigned to A and get $\mu_A(s)u + \mu_B(s)u'$ if $v \geq \hat{v}_A(s)$, or will be assigned to B and get $\mu_A(s)u' + \mu_B(s)u$ if $\hat{v}_A(s) > v \geq \overline{v}$. Thus, the student's payoff is

$$\tilde{u}(v;\hat{v}) := \int_0^1 \Big(\mathbb{1}_{\{v\geq\hat{v}_A(s;\hat{v})\}}(\mu_A(s)u + \mu_B(s)u') + \mathbb{1}_{\{\hat{v}_A(s)>v\geq\overline{v}\}}(\mu_A(s)u' + \mu_B(s)u)\Big)ds$$

Next, let $\Delta_u(v;\hat{v}) := \hat{u}(v;\hat{v}) - \tilde{u}(v;\hat{v})$ and observe that

$$\begin{aligned}\Delta_u(v;\hat{v}) &= \int_0^1 \mu_B(s)\Big(u\big(\mathbb{1}_{\{v\geq\hat{v}_B(s;\hat{v})\}} - \mathbb{1}_{\{\hat{v}_A(s;\hat{v})>v\geq\overline{v}\}}\big) - u'\big(\mathbb{1}_{\{v\geq\hat{v}_A(s;\hat{v})\}} - \mathbb{1}_{\{\hat{v}_B(s;\hat{v})>v\geq\overline{v}\}}\big)\Big)ds \\ &= \int_0^1 \mu_B(s)\,(u-u')\,\big(\mathbb{1}_{\{v\geq(\hat{v}_A(s;\hat{v})\vee\hat{v}_B(s;\hat{v}))\}} - \mathbb{1}_{\{(\hat{v}_A(s;\hat{v})\wedge\hat{v}_B(s;\hat{v}))>v\geq\overline{v}\}}\big)\,ds \\ &= \int_0^1 \mu_B(s)\,(u-u')\,\mathbb{1}_{\{v\geq(\hat{v}_A(s;\hat{v})\vee\hat{v}_B(s;\hat{v}))\}}\,ds\end{aligned}$$

where the last equality holds since $\hat{v}_A(s;\hat{v}) \wedge \hat{v}_B(s;\hat{v}) = \overline{v}$. Note that

$$\begin{aligned}\Delta_u(\hat{v};\hat{v}) &= \int_0^1 \mu_B(s)\,(u-u')\,\mathbb{1}_{\{\hat{v}\geq(\hat{v}_A(s;\hat{v})\vee\hat{v}_B(s;\hat{v}))\}}\,ds \\ &= \int_0^1 \mu_B(s)\,(u-u')\,\mathbb{1}_{\left\{1-\frac{\kappa}{1-G(\hat{v})}<\mu_A(s)<\frac{\kappa}{1-G(\hat{v})}\right\}}\,ds,\end{aligned}$$

where the last equality holds since $\mu_i(s)(1-G(\hat{v})) < \kappa$ for $i = A, B$ if and only if $\hat{v} \geq \hat{v}_i(s;\hat{v})$ for $i = A, B$. Note that $\Delta_u(\overline{v}, \hat{v}) = 0$ and $\Delta_u(\hat{v};\hat{v})$ is strictly increasing in $\hat{v}$. Hence, there exists $\hat{v}^*(> \overline{v})$ such that $\Delta_u(\hat{v}^*;\hat{v}^*) = c$. We observe that $\hat{v}^* < \max_s \hat{v}_A(s;\hat{v}^*)$, or else

$\mathbb{1}_{\{1-\frac{\kappa}{1-G(\hat{v}^*)}<\mu_A(s)<\frac{\kappa}{1-G(\hat{v}^*)}\}} = 0$ for almost all s, violating the equilibrium condition.

Since $\Delta_u(v;\hat{v})$ is nondecreasing in v, students with $v \geq \hat{v}^*$ will incur the cost. For students with $v < \hat{v}^*$,

$$\begin{aligned}\Delta_u(v;\hat{v}^*) &= \int_0^1 \mu_B(s)\,(u-u')\,\mathbb{1}_{\{\hat{v}^*>v\geq(\hat{v}_A(s;\hat{v}^*)\vee\hat{v}_B(s;\hat{v}^*))\}}\,ds \\ &= \int_0^1 \mu_B(s)\,(u-u')\,\mathbb{1}_{\left\{1-\frac{\kappa}{1-G(\hat{v}^*)}<\mu_A(s)<\frac{\kappa-(G(\hat{v}^*)-G(v))}{1-G(\hat{v}^*)}\right\}}\,ds \\ &< c,\end{aligned}$$

showing that those students do not incur the cost. Hence, the students' cutoff strategy with cutoff $\hat{v}^*$ forms an equilibrium indeed.

We highlight several properties of DA allocation in this case:

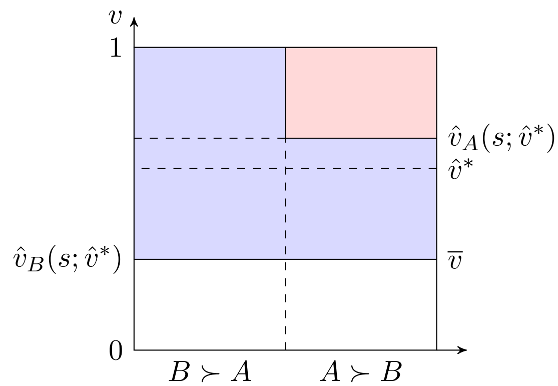
1. As with the baseline case, neither college over-enrolls or under-enrolls, and the outcome is jointly optimal for the colleges in the sense that the top 2κ students are assigned a colleges.

2. The fact that $\hat{v}^* \in (\overline{v}, \max_s \hat{v}_A(s;\hat{v}^*))$ means that there are states in which $\hat{v}^* < \max\{\hat{v}_A(s;\hat{v}^*), \hat{v}_B(s;\hat{v}^*)\}$ (depicted in Case 1 and 2-2 below) so that students with $v \in (\hat{v}^*, \hat{v}_A(s;\hat{v}^*))$ have no choice between the two colleges and should not have incurred the learning cost but they do; and there may arise states in which $\hat{v}^* > \max\{\hat{v}_A(s;\hat{v}^*), \hat{v}_B(s;\hat{v}^*)\}$ (depicted in Case 2-2 below) so that students with $v \in (\max\{\hat{v}_A(s;\hat{v}^*), \hat{v}_B(s;\hat{v}^*)\}, \hat{v}^*)$ do have a choice between the two colleges yet do not incur the learning cost. In short, the information acquisition behavior of students is inefficient.

3. The informational inefficiency in turn implies that justified envy and allocational inefficiency may arise. It remains true, however, that no student who is unassigned has justified envy.

□ **Allocations.**

1. Suppose $\mu_A(s)(1 - G(\hat{v}^*)) \geq \kappa$ in equilibrium.

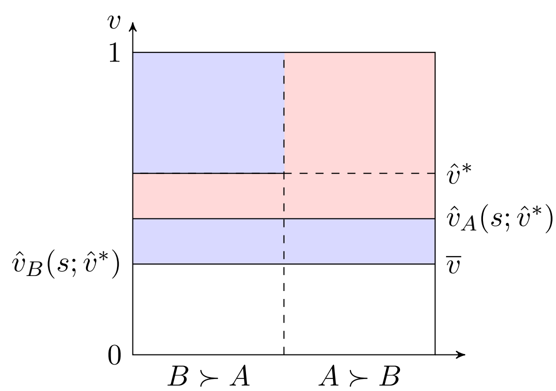


College A admits the top κ students among those with $v \geq \hat{v}^*$ and ranking it above B, so $\hat{v}_A(s;\hat{v}^*)$ satisfies $\mu_A(s)(1 - G(\hat{v}_A(s;\hat{v}^*))) = \kappa$. College B admits students from the top except those who are admitted by A, so $\hat{v}_B(s;\hat{v}^*) = \overline{v}$ satisfies

$$(1 - \mu_A(s))(1 - G(\hat{v}^*)) + \mu_A(s)(G(\hat{v}_A(s;\hat{v}^*)) - G(\hat{v}^*)) + G(\hat{v}^*) - G(\overline{v}) = \kappa.$$

2. Suppose $\mu_A(s)(1 - G(\hat{v}^*)) < \kappa$ in equilibrium. There are two cases as follows:

2-1. $(1 - \mu_A(s))(1 - G(\hat{v}^*)) < \kappa$.



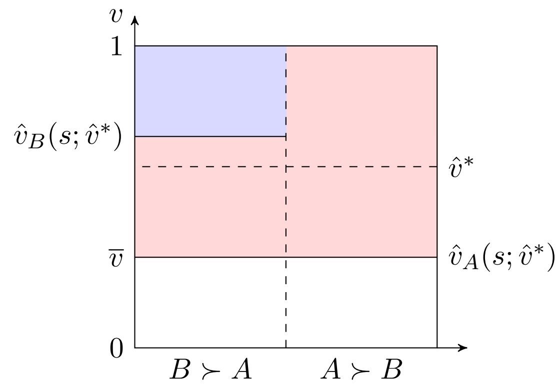
College A admits all students with $v \geq \hat{v}^*$ and raking it above B, and it also admits those with $v < \hat{v}^*$ to fill the remaining seats. Hence, $\hat{v}_A(s;\hat{v}^*)$ satisfies

$$\mu_A(s)(1 - G(\hat{v}^*)) + G(\hat{v}^*) - G(\hat{v}_A(s;\hat{v}^*)) = \kappa.$$

College B admits all students with $v \geq \hat{v}^*$ and raking it above A. It fills the remaining seats with students scoring below $\hat{v}^*$ except those who are admitted by A. Hence, $\hat{v}_B(s; \hat{v}^*) = \overline{v}$ satisfies

$$(1 - \mu_A(s))(1 - G(\hat{v}^*)) + G(\hat{v}_A(s; \hat{v}^*)) - G(\overline{v}) = \kappa.$$

2-2. $(1 - \mu_A(s))(1 - G(\hat{v}^*)) \geq \kappa$.



College B fills its capacity with the students scoring above $\hat{v}^*$ and raking it above A, so its cutoff $\hat{v}_B(s; \hat{v}^*)$ satisfies $(1 - \mu_A(s))(1 - G(\hat{v}_B(s; \hat{v}^*))) = \kappa$. College A admits students from the top except those who are admitted by B. Hence, $\hat{v}_A(s; \hat{v}^*) = \overline{v}$ satisfies

$$\mu_A(s)(1 - G(\hat{v}^*)) + (1 - \mu_A(s))(G(\hat{v}_B(s; \hat{v}^*)) - G(\hat{v}^*)) + G(\hat{v}^*) - G(\overline{v}) = \kappa.$$

References

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