Supplementary Appendix to
Weak Cartels and Collusion-Proof Auctions

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Proof of Lemma 3: Let us denote by I_T, the inequality (20) with $S = T \subset N$. We first show that any p satisfying (21) is an extreme point of $\mathcal{P}$. Note that the allocation probability p can be achieved in a feasible way by assigning the k-th order to each agent $\pi(k)$ and then allocating the object in this order as long as the agents have values no lower than their reserve prices, which implies that p is a reduced form. Thus Lemma 1 implies that p must satisfy (20) so $p \in \mathcal{P}_N$. Suppose for a contradiction that p is not an extreme point of $\mathcal{P}$. Then, we can write $p = \lambda p' + (1-\lambda)p''$ for some $p' \neq p$ and $p'' \neq p$. Since $p_{\pi(1)} = 1$ and $p'_{\pi(1)}, p''_{\pi(1)} \leq 1$ (by $I_{\{\pi(1)\}}$), we must have $p'_{\pi(1)} = p''_{\pi(1)} = 1$. Using this and applying $I_{\{\pi(1),\pi(2)\}}$ to p' and p'', we obtain $p'_{\pi(2)}, p''_{\pi(2)} \leq F_{\pi(1)}(r_{\pi(1)}) = p_{\pi(2)}$, which implies $p'_{\pi(2)} = p''_{\pi(2)} = F_{\pi(1)}(r_{\pi(1)}) = p_{\pi(2)}$. Proceeding in this fashion, one can easily verify that $p'_{\pi(k)} = p''_{\pi(k)} = \prod_{j<k} F_{\pi(j)}(r_{\pi(j)}) = p_{\pi(k)}$ for all $k \in N$, a contradiction. To show that any extreme point $p \in \mathcal{P}$ can be expressed as in (21) for some π, we first establish the following claim.

Claim 4. *If for any two sets $S, T \subset N$, I_S and I_T hold as equality, then either $S \subset T$ or $T \subset S$.*

Proof. Suppose for a contradiction that for some $S, T \subset N$, I_S and I_T hold as equality but $S \not\subseteq T$ and $T \not\subseteq S$. Then, we must have $S \cap T \subsetneq T$ and $S \backslash T \neq \emptyset$. To draw a contradiction, let us first show

$$\sum_{i \in T \backslash S} p_i(1 - F_i(r_i)) \leq \prod_{i \in S} F_i(r_i)\Big(1 - \prod_{i \in T \backslash S} F_i(r_i)\Big). \tag{S.1}$$

For this, note that by $I_{S \cup T}$ and the assumption,

$$\sum_{i \in S \cup T} p_i(1 - F_i(r_i)) \leq 1 - \prod_{i \in S \cup T} F_i(r_i) \tag{S.2}$$

$$\sum_{i \in S} p_i(1 - F_i(r_i)) = 1 - \prod_{i \in S} F_i(r_i). \tag{S.3}$$

It is straightforward to see that (S.1) results from subtracting (S.3) from (S.2) side by side. Also, by $I_{S \cap T}$, we have

$$\sum_{i \in S \cap T} p_i(1 - F_i(r_i)) \leq 1 - \prod_{i \in S \cap T} F_i(r_i).$$

Adding this inequality and (S.1) side by side yields

$$\sum_{i \in T} p_i(1 - F_i(r_i)) \leq 1 + \prod_{i \in S} F_i(r_i) - \prod_{i \in S \cap T} F_i(r_i) - \prod_{i \in S \cup T} F_i(r_i) < 1 - \prod_{i \in T} F_i(r_i), \tag{S.4}$$

where the strict inequality follows since $S \cap T \subsetneq T$ and $S \backslash T \neq \emptyset$, and thus

$$\prod_{i \in S \cap T} F_i(r_i) + \prod_{i \in S \cup T} F_i(r_i) - \prod_{i \in S} F_i(r_i) - \prod_{i \in T} F_i(r_i)$$
$$= \Big(\prod_{i \in S \cap T} F_i(r_i) - \prod_{i \in T} F_i(r_i) \Big) \Big(1 - \prod_{i \in S \backslash T} F_i(r_i) \Big) > 0.$$

However, (S.4) contradicts the assumption that I_T holds as equality. ▌

Pick any extreme point $p \in \mathcal{P}$ satisfying (20). Let us denote by $S_1, \cdots, S_m$, all subsets of N for which (20) is satisfied as equality given p. Due to the above Claim, these subsets must have a nested structure, that is $S_1 \subset S_2 \subset \cdots \subset S_m$. Since N contains n elements, m cannot be greater than n. Suppose for a contradiction that $m < n$. Then, there must be some k and $h, j \in N$ with $h \neq j$ such that $h, j \notin S_\ell$ if $\ell < k$ and $h, j \in S_\ell$ if $\ell \geq k$. We now show that p can be obtained by linearly combining some p' and $p'' \in \mathcal{P}$, contradicting that p is an extreme point. To do so, we denote by e^i, n-dimensional vector with its i-the element being 1 and all others being zero. Let $p' = p - \epsilon e^h + \delta e^j$ and $p'' = p + \epsilon e^h - \delta e^j$, where ϵ and δ are sufficiently small positive real numbers satisfying

$$\epsilon(1 - F_h(r_h)) = \delta(1 - F_j(r_j)),$$

which implies that for all $\ell < k$, $\sum_{i \in S_\ell} p'_i(1 - F_i(r_i)) = \sum_{i \in S_\ell} p_i(1 - F_i(r_i))$ and for all $\ell \geq k$,

$$\sum_{i \in S_\ell} p'_i(1 - F_i(r_i)) = \sum_{i \in S_\ell} p_i(1 - F_i(r_i)) - \epsilon(1 - F_h(r_h)) + \delta(1 - F_j(r_j)) = \sum_{i \in S_\ell} p_i(1 - F_i(r_i))$$

and similarly for p''. From this, we see that whether p satisfies (20) as equality or strict inequality, the same is true for p' and p'', provided that ϵ and δ are sufficiently small, which means $p', p'' \in \mathcal{P}$. However, $p = \frac{1}{2}p' + \frac{1}{2}p''$, resulting in the desired contradiction. Thus, we must have $m = n$, which implies $|S_k \backslash S_{k-1}| = 1$ for all $k = 1, \cdots, n$ with $S_0 = \emptyset$.

To complete the proof, define the permutation function $\pi : N \to N$ such that $\pi(i) = k$ if $\{i\} = S_k \backslash S_{k-1}$. Then, by definition of S_1, I_{S_1} holds as equality or $p_i(1 - F_i(r_i)) = 1 - F_i(r_i)$ for $i = \pi^{-1}(1)$, which yields $p_i = 1$ for $i = \pi^{-1}(1)$. For an induction argument, suppose

$$p_i = \prod_{j: \pi(j) < \pi(i)} F_j(r_j) \text{ for } i = \pi^{-1}(k'), \forall k' = 1, \cdots, k-1. \tag{S.5}$$

Then, by definition of S_k and π, we must have for $i = \pi^{-1}(k)$

$$\sum_{j: \pi(j) \leq \pi(i) = k} p_j[1 - F_j(r_j)] = 1 - \prod_{j: \pi(j) \leq \pi(i) = k} F_j(r_j),$$

which after substituting (S.5) and canceling the terms, leads us to obtain $p_i = \prod_{j: \pi(j) < \pi(i)} F_j(r_j)$, as desired. ▌

Proof of Lemma 4: We let $Q = (Q_1, \cdots, Q_n)$ denote the optimum of $[P']$. First suppose to the contrary that $r_j < \hat{v}$ for some j. Let us construct an alternative rule $\overline{Q} = (\overline{Q}_1, \cdots, \overline{Q}_n)$ as

$$\overline{Q}_i(v_i) = \begin{cases} 0 & \text{if } i = j \text{ and } v_i < \hat{v} \\ Q_i(v_i) & \text{otherwise.} \end{cases}$$

Clearly, the value of objective function is higher with $\overline{Q}$ than with Q. Also, $\overline{Q}$ satisfies the constraints (M) and (CP). So it only remains to show that (B) is satisfied, for which it suffices to construct an ex-post rule generating $\overline{Q}$.[24]ãĚ■ To do so, let $(q_1, \cdots, q_n)$ denote the ex-post allocation rule for Q and we can construct the ex-post rule for $\overline{Q}$ as

$$\overline{q}_i(v) = \begin{cases} 0 & \text{if } i = j \text{ and } v_j < \hat{v} \\ q_i(v) & \text{otherwise.} \end{cases}$$

To prove the second statement, we first show that any optimum must have $r_i \in [\hat{v}, v^*)$ for at least one agent i. Suppose not, i.e. $r_i \geq v^*, \forall i \in N$. We draw a contradiction by constructing an alternative interim allocation rule $\tilde{Q}$ which makes the value of objective function greater than Q does, and which differs from Q only in that for an arbitrarily chosen agent k, $\tilde{Q}_k(v_k) = \tilde{p}_k = F(v^*)^{n-1}$ for $v_k \in [\hat{v}, v^*]$. It is clear that the value of objective function is greater with $\tilde{Q}$ by as much as $F(v^*)^{n-1} \int_{\hat{v}}^{v^*} J(s) dF(s) > 0$. Since $Y(v)$ for all $v \geq v^*$ is unaffected by the change from Q to $\tilde{Q}$, it only remains to check (25). Letting $\tilde{r}_i = \inf\{v_i \in \mathcal{V}_i | \tilde{Q}_i(v_i) > 0\}$ and $\tilde{N}_* = \{i \in N | \tilde{r}_i < v^*\}$, we have $\tilde{N}_* = \{k\}$ so can focus on $M = \{k\}$. Then, (25) can be written as

$$F(v^*)^{n-1}[F(v^*) - F(\hat{v})] \leq Y(v^*) + F(v^*)^n - F(v^*)^{n-1}F(\hat{v}),$$

which clearly holds since $Y(v^*) \geq 0$.

Given a solution Q of $[P']$, the above argument means $N_* = \{i \in N | r_i < v^*\} \neq \emptyset$. Assuming that $N_* \neq N$, we construct an alternative solution $\tilde{Q}$ of $[P']$ such that $\tilde{r}_i = \inf\{v_i \in \mathcal{V}_i | \tilde{Q}_i(v_i) > 0\} < v^*, \forall i \in N$. Letting $N^* = N \backslash N_*$, select any agent $k \in N_*$ and then define $\tilde{Q}$ to be the same as Q, except that for each agent $i \in N^* \cup \{k\}$, $\tilde{Q}_i(v_i) = \tilde{p}_i = \frac{p_k}{|N^*|+1}$ for $v_i \in (r_k, v^*)$, where $p_k = Q_k(v_k) > 0$ for $v_k \in (r_k, v^*)$. It is clear that the value of objective function remains the same under $\tilde{Q}$. Since $Y(v)$ for all $v \geq v^*$ is unaffected, it only remains to check (25). We can focus on such M that $M \cap (N^* \cup \{k\}) \neq \emptyset$, since the allocation for each agent $i \notin N^* \cup \{k\}$ has not been changed. For any such M, note that

$$\sum_{i \in M} \tilde{p}_i [F(v^*) - F(\tilde{r}_i)] = \sum_{i \in M \cap (N_* \backslash \{k\})} p_i [F(v^*) - F(r_i)] + \sum_{i \in M \cap (N^* \cup \{k\})} \tfrac{p_k}{|N^*|+1} [F(v^*) - F(r_k)]$$

[24]Recall that the condition (B) is necessary and sufficient for there to be an ex-post allocation rule that generates Q as an associated interim allocation rule.

$$
\begin{aligned}
&\leq \sum_{i\in M\cap(N_*\setminus\{k\})} p_i[F(v^*)-F(r_i)]+p_k[F(v^*)-F(r_k)] \\
&= \sum_{i\in(M\cap N_*)\cup\{k\}} p_i[F(v^*)-F(r_i)]. \qquad \text{(S.6)}
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\sum_{i\in(M\cap N_*)\cup\{k\}} p_i[F(v^*)-F(r_i)] &\leq Y(v^*)+F(v^*)^n-F(v^*)^{n-|(M\cap N_*)\cup\{k\}|}\left(\prod_{i\in(M\cap N_*)\cup\{k\}} F(r_i)\right) \\
&\leq\ Y(v^*)+F(v^*)^n-F(v^*)^{n-|M|}\left(\prod_{i\in M} F(\tilde{r}_i)\right), \qquad \text{(S.7)}
\end{aligned}
$$

where the first inequality holds since Q satisfies (25) and the second due to the facts that $M\cap(N^*\cup\{k\})\neq\emptyset$ implies $|(M\cap N_*)\cup\{k\}|\leq|M|$ and that $\tilde{r}_i\leq\min\{r_i,v^*\},\forall i\in N$. Combining (S.6) and (S.7), we obtain

$$
\sum_{i\in M}\tilde{p}_i[F(v^*)-F(\tilde{r}_i)]\leq Y(v^*)+F(v^*)^n-F(v^*)^{n-|M|}\left(\prod_{i\in M} F(\tilde{r}_i)\right),
$$

so $\tilde{Q}$ satisfies (25). ∎