Allocating Resources to Wealth-Constrained Agents

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ABSTRACT: Governments have a profound impact on the allocation of many goods, services and rights. We study individuals who differ in their valuations and in their wealth. In particular, some individuals have binding wealth constraints, which inhibit voluntary exchange, so the initial assignment scheme matters. We show that non-market assignment schemes — even as simple as random rationing — may yield a more efficient allocation than a competitive market would, if initial assignees are allowed to resell. Need-based assignment schemes that favor the poor and merit-based schemes that favor those with high valuations are even more desirable. At the same time, non-market assignment invites speculation, which limits its benefit relative to the market. If speculation is prevalent, prohibiting transferability may be beneficial.

KEYWORDS: efficiency, non-market assignment, transferability, merit-based assignment rules, need-based assignment rules.

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1 Introduction

Governments have a profound impact on the allocation of many goods, services and rights. Many governments directly provide health care, housing and education. They also assign rights to access various public resources ranging from minerals, forests, fish and wildlife, and radio spectrum; to intangibles such as immigration and exemptions from military or jury duty. Government influence often extends beyond public resources to goods and services provided by private firms and individuals. They control prices of many markets (such as housing and health care) and sometimes determine who is served, and in what order (e.g., distribution of human organs).

Governments’ role has been especially pronounced in developing economies. For instance, the Korean industrialization process was marked by its industrial licensing policies, which targeted certain industries and firms for subsidies, loans, export quotas, trade protection and other privileges, as well as extensive regulation of markets (see Amsden (1989)). During an earlier period dubbed the “licence raj,” the Indian government controlled large areas of economic activity through the awarding of rights and “permissions” (see Esteban and Ray (2006)).

How resources are initially assigned need not matter, according to the Coase theorem: Efficient allocation will arise from voluntary trade if the initial assignee possesses the right to transfer the resource, transfer involves no transactions costs, and utility is (fully) transferable.1 These conditions are rarely met in practice, however. For example, utility would not be fully transferable if some individuals have limited wealth and/or limited access to capital markets. Wealth-constrained individuals may not be able to purchase goods from existing owners even when they value the goods more highly than those owners do. Hence, voluntary exchange may not result in efficient allocation, and initial assignment of resources will matter. How goods should be assigned and what role governments should play in such cases are important, particularly in light of capital market imperfections.

We investigate these questions by developing a model in which a scarce resource is assigned to agents who may differ in their willingness and abilities to pay. The resource

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1This follows from Coase (1960). A standard version of the Coase theorem states: “When rights are well defined and the costs of transacting are zero, resource allocation is efficient and independent of the pattern of ownership” (Barzel, 1997).
could be a productive asset such as a license to produce. In that case, an agent’s willingness
to pay will reflect the profit from utilizing that asset, and her ability to pay may depend
on her access to capital markets. (We focus on a simple case in which the external capital
market does not exist.) Alternatively, it could be a consumption good in which case a
buyer’s valuation reflects her utility of consuming the good, and her ability to pay will then
depend on her permanent income as well as the ability borrow against future income. Our
analysis will focus on three questions:

1. Should a market be used to assign initial property rights? Competitive markets
have been used to assign many production rights, and their use is expanding into other
areas where non-market methods such as hearings or lotteries had been used. Non-market
assignment is common, though. Immigration visas are allocated by eligibility criteria and
lotteries. Jury duty (and thus exemptions from it) is assigned by a lottery. A military draft
may select conscripts (and thus exemptions or deferments) by lottery as well. Various other
methods have been used to allocate unclaimed property or government property. For in-
stance, land was deeded on a first-come-first-served basis during the 1889 Oklahoma Land
Rush, and by lottery in 1901. In Korea and Singapore, a large fraction of new houses has
been provided at prices well below market levels, via lotteries and other eligibility rules. While much is known about the efficiency properties of markets, the case for their use as a method of assigning initial property right is less clear cut. In particular, the market assign-
ment of property rights will not be efficient in the presence of wealth-constrained agents,
for a market will favor the wealthy (who may have a low valuation) over the poor (who
may have a high valuation). An interesting question is whether non-market assignment —
for instance, random assignment — can do better than market assignment.

2. How should a (non-market) assignment rule treat agents with different character-
istics? The benefits of a non-market assignment rule depend in part on the information
possessed by the government or suppliers. Often, the government can do better than sim-
ples random assignment, given some information about agents’ willingness and abilities to

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2In Singapore, most citizens live in units sold by the government at prices that are well below the
market. Some 86 per cent of Singapore’s citizens live in such units and 92 per cent of those residents own
their units. (See “Building Homes, Shaping Communities,” at http://www.mnd.gov.sg/, accessed on May
28, 2005.) The price cap is as low as half of the price on the resale market (Tu and Wong (2002)). The
same multiple is given by Green, Malpezzi and Vandell (1994) for Korea. See also Kim (2002).

3In fact, the Coase theorem would not favor market over any other method, given the irrelevance.
pay. For housing or health care, an individual’s valuation may be estimated from her existing living arrangements or a doctor’s assessment of her medical condition. The value of a productive asset to an individual may be inferred from his previous job performance or test scores. Likewise, one’s wealth may be inferred from earnings and asset holdings. Given such information, a government may decide that a non-market assignment scheme should be used. If it does, what form should the scheme take? This question has a great deal of policy relevance, for many real world assignment rules treat individuals based on their merits (willingness-to-pay signal) and needs (ability-to-pay signal). While merit-based assignment (i.e., favoring those with the highest apparent valuation) would be clearly justifiable on allocative efficiency grounds, an efficiency justification for a need-based assignment (i.e., favoring the poor) is not so clear.

3. Should the owner of the good be allowed to resell it? A non-market assignment scheme typically does not eliminate all mutually beneficial trading opportunities. Allowing resale will enable such trades to occur, so it will be efficient from an \textit{ex post} perspective.\footnote{The set of participants may differ, with and without resale, so it is not trivial that resale helps from an \textit{ex ante} perspective. This point was made by Tobin (1952) in the context of coupon rationing.} Yet, transferability of many goods is limited. For instance, the sale of human organs is outlawed in most countries, and a student cannot normally sell his spot in an oversubscribed class.\footnote{This issue arose when a student at NYU Law School attempted to purchase a spot in two classes. Students there are permitted to trade spots in classes, but they may not offer cash or “cash substitutes” (Bitkower (2005)). The initial allocation was determined by a priority scheme based on submitted preferences. There was an appreciation by administrators and students alike that allowing the sale of spots would induce speculative demand for certain classes.} We will study the efficiency effects of allowing or prohibiting resale.

Our findings are as follows.

\textbf{Efficiency of non-market assignment}: Suppose that a government imposes a binding price cap and assigns the good randomly to those who demand it. We show that this scheme yields a more efficient allocation than the competitive market \textit{if and only if resale is allowed}. Random assignment has two effects relative to the competitive market: It shifts the allocation from individuals with higher wealth toward those with lower wealth, and from individuals with higher valuations toward those with lower valuations. The former shift has no effect on allocative efficiency here; the latter shift reduces it, all else equal. If resale is permitted, low-valuation individuals will resell the good to those with high
valuations. Since some low-wealth-high-valuation buyers who would not consume the good in the competitive market do so now, the overall effect is to raise efficiency relative to the market. This result suggests that government intervention in a competitive market may be justified if wealth constraints are important, as long as resale is permitted. It also implies that the case for using markets or auctions to assign various entitlements, although defensible in many circumstances, should not be taken for granted.

□ Desirability of need-based assignment:

The logic of the previous result suggests that need-based schemes that favor the poor will yield even more efficient allocation than a random assignment. Favoring the poor in the assignment shifts the burden of purchasing at a market price toward agents with substantial purchasing power and away from those without. Consequently, a more efficient allocation will arise. Of course, the effect is present only when resale is permitted; without transferability, need-based schemes will have no effect. This observation not only provides an efficiency rationale for having assignment schemes and social programs that favor the poor, but it also identifies transferability of benefits as necessary for such programs to achieve this desirable outcome.

□ Desirability of allowing resale:

As seen above, resale is crucial for non-market assignment — and thus government intervention into a competitive market — to be justified on allocative efficiency grounds. At the same time, allowing for resale creates windfall gains to those lucky enough to be assigned the good. The prospect of earning these gains attracts many agents who would not otherwise demand the good (at a competitive market price). The presence of such speculators reduces the odds of assigning the goods to more-deserving agents, and thus reduces the benefit of a non-market assignment. In fact, the benefit of non-market assignment relative to market vanishes as the size of potential speculators grows without bound. It may therefore be beneficial to restrict transferability. If the assignment technology is sufficiently effective at selecting high-valuation individuals, and if there are sufficiently many speculators, assignment of a non-transferable good may dominate assignment of a transferable good. Thus, in certain circumstances there is an efficiency argument for prohibiting transferability.
The remainder of this paper is organized as follows. Section 2 lays out the basic model and describes the efficient allocation and the competitive market allocation. Section 3 characterizes the outcome when the good is subject to a price cap and the available supply is assigned randomly. Section 4 analyzes general assignment schemes and derives conditions under which they outperform the market. We discuss various extensions of the basic model in Section 5. The related literature is discussed in Section 6, with Concluding remarks in Section 7.

2 The Model

2.1 Primitives

A good is available in fixed supply, \( S \in (0, 1) \). The good is indivisible, and it is supplied competitively at a constant marginal cost, \( c > 0 \), up to \( S \). Marginal cost is sufficiently low that there is a shortage of supply. Inelastic supply and shortages are characteristics of many markets in which instruments other than price are employed.

There exists a unit mass of buyers who each demand one unit of the good. Each buyer is characterized by two attributes: her wealth, \( w \), and her valuation, \( v \), which have a continuous distribution over \([0, 1]^2\). We refer to \((w, v)\) as the buyer’s type. The two components are independent, with \( w \) having the cumulative distribution function (cdf) \( G(w) \) and \( v \) having the cdf \( F(v) \). The corresponding densities are non-zero for almost every \( v \) and \( w \). Independence helps to isolate the role that each attribute plays and the effect of policy treatments that focus on just one.

There is also a mass, \( m \), of buyers who participate solely for speculative reasons. They each have a valuation \( v = 0 \), while their wealth has the same cdf, \( G(w) \). We will sometimes refer to these buyers as pure speculators.

All buyers are risk-neutral, with quasilinear utility. In particular, a type-\((w, v)\) buyer gets utility \( v + w - p \) if she consumes the good and pays \( p \). A buyer with \( v > w \) is said to be wealth-constrained since she is not able to pay as much as she is willing to pay. This raises the possibility that an individual who values the good highly does not acquire it whereas another individual who values it less highly does acquire it.
We evaluate the implications of using different assignment mechanisms. Before describing the mechanisms, it is necessary to discuss our welfare criterion.

### 2.2 Welfare Criterion and Efficient Allocation

The welfare criterion we use is total realized value, which equals the sum of valuations for those who consume. This is the sensible criterion when the good is a resource or a license to produce, in which case \( v \) could represent output or the price-cost margin. For a consumption good, total realized value reflects the sum of utilities, given quasilinear utility. This is also a sensible criterion when utility is not transferable, since it reflects expected payoffs. Suppose that consumers and producers are drawn from a large pool, with each individual being equally likely to be selected for the consumer pool as the producer pool. Then, a second drawing selects active participants and their types. If the individuals were to select an assignment mechanism prior to realizing their types, they would vote for the one with the higher total realized value as they would put equal weight on consumer and producer surplus.\(^6\) Thus, we employ total realized value as the welfare measure here.

The efficient allocation maximizes total realized value by providing the good to the buyers with the highest valuations. Let \( v^* > 0 \) denote the critical valuation such that \( 1 - F(v^*) = S \). If all buyers with valuations of \( v^* \) and above acquire the good, the total realized value is

\[
V^* := \int_0^1 \int_{v^*}^1 v dF(v) dG(w) = \int_{v^*}^1 v dF(v) = S \phi(v^*),
\]

where

\[
\phi(z) := \int_z^1 \frac{v dF(v)}{1 - F(z)}
\]

is the expectation of a buyer’s valuation, conditional on exceeding \( z \). The aggregate quantity is \( S \), and the average valuation among those who acquire the good is equal to \( \phi(v^*) \), so the total realized value in the efficient allocation is \( S \phi(v^*) \). That allocation and the associated realized value, \( V^* \), will serve as benchmarks for the schemes that we consider.

\(^6\)Analogous arguments are provided in Vickrey (1945) and Harsanyi (1953). A formal justification of this utilitarian approach is in Harsanyi (1955).
2.3 Assignment Schemes

Throughout the paper we will compare the performance of three alternatives: (1) a competitive market, (2) an assignment scheme for a non-transferable good, and (3) an assignment scheme for a transferable good. The latter two schemes entail a binding price cap and an exogenous rule for assigning the good. Formally, an assignment rule is a (measurable) function, \( x : [0,1]^2 \rightarrow [0,1] \), which maps \((w,v)\) to a probability of receiving the good. If the good is transferable, there will be a resale market subsequently in which those who obtained the good may resell it.

The three regimes are either currently employed or could be employed in numerous settings, as the following examples show:

- **Fugitive Property and Government Resources:** Fugitive property—a good or resource whose ownership is not yet established—can be assigned in many different ways. It can be assigned to the individual who claims it first (the rule of first possession), or to the individual who owns property tied to that object (tied ownership). These methods correspond to assignment of a transferable good. Ownership can also be established by an auction, which corresponds to the market. The 1889 Oklahoma Land Rush and the 1901 Oklahoma Land Opening are examples of assignment with transferability. In 1906, government land in Oklahoma was sold at auction.

- **Education:** Public school enrollment is typically tied to one’s place of residence, so the housing market serves indirectly as a market for school enrollment. Suppose that public school students are assigned to two schools based solely on place of residence, but the number of students who prefer school A exceeds its capacity. The valuation, \( v \), now represents the premium that an individual is willing to pay for the right to attend A. Since the nominal price of attending the school is zero, the preference for A will be capitalized in housing prices. This scenario corresponds to the market regime. Now suppose that slots in school A are awarded by lottery. This is an assignment scheme with a non-transferable good. The final regime arises if a lottery awards transferable vouchers that confer the right to attend A.

\footnote{For instance, a landowner has the subsurface right to natural gas deposits underneath the land.}
- **Housing**: The three regimes arise naturally in housing markets. Consider a rental market that is subject to rent control. If leases are not transferable, the regime is that of assignment without transferability. If leases are transferable, we have assignment with transferability. If rent control is abolished, we have a competitive market.

- **Health care**: Health care is often provided via an assignment scheme without transferability. A specific example involves organ transplants, where patients are placed in a queue and cannot sell or swap their places in the queue.\(^8\) One could also employ a general assignment scheme with transferability in which patients waiting for a transplant may sell their places in the queue. Finally, a competitive market could assign organs to those willing and able to pay the market price.\(^9\)

- **Military recruitment**: An all-volunteer army corresponds to the competitive market. A draft lottery is effectively an assignment scheme for a non-transferable good — military deferments. A draft with tradable deferments represents an assignment scheme with a transferable good. An early example occurred in ancient Korean kingdoms when wealthy families could pay sharecroppers to enlist on their behalf. This practice also arose in the U.S. Civil War: conscripts could avoid service in the Union Army by hiring a substitute.

In the remainder of this section we focus on the competitive market.

### 2.4 A Competitive Market

A competitive market is characterized by the price at which supply is exhausted by the effective demand. When the market price is \( p > 0 \), the measure of buyers willing and able to pay \( p \) is

\[
D(p) := [1 - G(p)][1 - F(p)].
\]

If a competitive equilibrium exists, the equilibrium price, \( p^e \), satisfies

\[
D(p^e) = [1 - G(p^e)][1 - F(p^e)] = S.
\] \(^{(1)}\)

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\(^8\) In the U.S., a patient awaiting a kidney transplant can actually move to the front of the queue by locating a live donor. Roth, Sönmez, and Ünver (2004) discuss kidney exchanges wherein patients who have found incompatible live donors essentially trade donor kidneys.

\(^9\) Organs are available in certain countries at market prices.
Assuming that $p^e > c$, this is a competitive equilibrium.

A couple of remarks are in order. First, $1 - F(v^*) = S,$ so $[1 - G(v^*)][1 - F(v^*)] < S,$ which implies $p^e < v^*.$ This means that the market equilibrium does not yield the efficient allocation. In Figure 1, the efficient allocation gives the good to all buyers in the region $A + B$ while the market assigns to those in $B + C$. Relative to the efficient allocation, the market favors some high-wealth-low-valuation buyers (region $C$) over low-wealth-high-valuation buyers (region $A$).

The total realized value in the market equilibrium is:

$$V^e := \int_1^{p^e} \int_0^{p^e} vdF(v)dG(w) = [1 - G(p^e)] \int_0^{p^e} vdF(v) = S\phi(p^e) < S\phi(v^*) = V^*.$$  

The third equality holds since $[1 - G(p^e)][1 - F(p^e)] = S,$ by (1); and the inequality holds since $p^e < v^*$ and $\phi$ is a strictly increasing function. The inefficiency is entirely attributable to the binding wealth constraints. If no buyers were constrained, the market-clearing price would satisfy $p = v^*$, yielding an efficient allocation.

Second, even though the market allocation is inefficient, it would not trigger any resale. Individuals possessing the good have $v \geq p^e$, so they would only sell at prices exceeding $p^e$, but there are no additional buyers willing and able to pay that much. In other words, the inefficiency resulting from the market will not be resolved by opening another market. If there is a price cap, by contrast, resale will occur, thereby mitigating the inefficiency.

3 Analysis with Random Assignment

In this section, we consider a market with a binding price cap, $p \in [c, p^e)$. The good is then assigned completely randomly to participants. Random assignment is simple, as it does not require knowledge of buyers’ preferences or budgets. While a government may try to

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10If there were an active resale market, the resale market price would be the same as the price in the original equilibrium; otherwise, buyers would switch from one market to the other. Hence, the equilibrium price must be $p^e$, so the allocation is the same.
screen buyers, screening may be imperfect because types are private information. We will analyze general assignment schemes in Section 4, so details of the equilibrium analysis will be presented then.

3.1 Random Assignment of a Non-transferable Good

Many of the goods we have discussed are not transferable, either because the supplier mandates it or because there are legal restrictions. Enforcement of non-transferability is easy in many cases. For example, proof of identity may be required before services are rendered. Licenses to provide cellular telephone or cable television service can easily be made non-transferable since ownership or use can be monitored. We now analyze the case in which resale is not permitted.

Buyers whose valuation and wealth both exceed $\bar{p}$ will want to buy the good. Each of these buyers will receive it with probability

$$S \frac{1}{[1 - F(\bar{p})][1 - G(\bar{p})]}.$$ 

The expected valuation for such buyers is $\phi(\bar{p})$, and the aggregate quantity is $S$, so random assignment gives a total realized value of $S\phi(\bar{p})$. Since $\bar{p} < p^*$, we have $S\phi(\bar{p}) < S\phi(p^*)$, meaning that the allocation is less efficient than under the market.

The reason why a price cap is harmful differs from the standard explanation, which is that quantity falls, resulting in a deadweight loss. There is no quantity effect here; rather, the fixed quantity is simply allocated inefficiently. When the cap binds, the good is assigned to some buyers with lower valuations instead of others with higher valuations. Although some buyers with lower wealth now acquire the good, this factor has no aggregate effect since $v$ and $w$ are independent.

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11 We can assume that a buyer who participates must take delivery and pay for the good if successful. If the buyer does not fulfill these obligations, she will not get the good and she must pay a small penalty. This will mean that buyers only participate if they are able to pay the price.

12 The same point is made by Glaeser and Luttmer (2003) in the context of rent control. Conversely, the standard argument is that an increase in aggregate quantity is necessary for price discrimination to raise welfare. Once again, that will not be the case here, as welfare may rise even though there is no quantity effect.
3.2 Random Assignment of a Transferable Good

It may be costly or impossible to enforce a non-transferability provision. Even if not, the ex post benefits from resale may make it beneficial to allow transferability. Consequently, we now assume that the good is costlessly transferable. Since it will be profitable to acquire multiple units and resell some, we assume that each individual may only attempt to purchase one unit of the good initially.

There will be a competitive resale market after the initial assignment. The equilibrium resale price, \( r(\bar{p}) \), will exceed \( \bar{p} \). (If not, \( r(\bar{p}) < \bar{p}^c \), so there would be excess demand on the resale market.) A successful speculator can pocket \( r(\bar{p}) - \bar{p} > 0 \) by reselling so all \([1 + m][1 - G(\bar{p})]\) buyers who can afford to pay \( \bar{p} \) will participate in the initial assignment. Each participant will receive the good with probability

\[
\rho(\bar{p}; m) := \frac{S}{[1 + m][1 - G(\bar{p})]}
\]

Resale demand at a price of \( r \) comprises the buyers willing and able to pay \( r \) who did not get the good initially:

\[
RD(r) := [1 - F(r)][1 - G(r)][1 - \rho(\bar{p}; m)].
\] (2)

Now consider resale supply. If a buyer with valuation \( v \) gets the good and keeps it, she will receive a net surplus of \( v - \bar{p} \). Reselling nets \( r - \bar{p} \), so she will resell if \( v < r \). Resale supply comprises all buyers with \( v < r \) who got the good initially. Their measure is \( \rho(\bar{p}; m)[F(r) + m][1 - G(\bar{p})] \), so

\[
RS(r) := S \left( \frac{F(r) + m}{1 + m} \right).
\] (3)

Equating resale demand and supply, we have

\[
[1 - F(r)][1 - G(r)][1 - \rho(\bar{p}; m)] = S \left( \frac{F(r) + m}{1 + m} \right)
\]

\[\Rightarrow D(r) = D(r)\rho(\bar{p}; m) + S \left( \frac{F(r) + m}{1 + m} \right)\]

\[\Rightarrow D(r) = S - \rho(\bar{p}; m)[1 - F(r)][G(r) - G(\bar{p})].\]

The product on the last line is the measure of buyers who receive the good and retain it, but would be unable to purchase on the resale market. For any \( r \leq (\leq) \bar{p}^c \), we have
\(D(r) \geq (>) S\), so \(RD(r) > RS(r)\). Hence, the resale equilibrium must have \(r(\bar{p}) > p^e\). This can be seen intuitively through Figure 2.

[PLACE FIGURE 2 ABOUT HERE.]

Suppose that the resale price were \(r = p^e\). The buyers who are willing and able to pay \(r = p^e\) (area \(B\)) all end up with the good, just as in the competitive market equilibrium. (They either acquire it initially and keep it, or they acquire it in the resale market.) In addition, some buyers with \((w, v) \in [\bar{p}, r] \times [r, 1]\) obtain the good initially and keep it. This group accounts for the area \(A\). Since they keep the good, there must be excess demand on the resale market when \(r = p^e\). It follows that the resale price must exceed \(p^e\).

The total realized value is now \(S\phi(r(\bar{p})) > S\phi(p^e) = V^e\).\(^{13}\) This tells us that capping price and randomly allocating a transferable good produces a strictly more efficient allocation than either randomly allocating a non-transferable good or operating a competitive market.\(^{14}\) Despite the speculation that it engenders, permitting transferability is socially beneficial. Indeed, there is no efficiency rationale for capping price absent transferability.

The post-resale allocation becomes more efficient when the initial allocation is shifted away from those with the wealth to acquire the good and toward those without. Random assignment effectively reallocates the good from the wealthy to the poor; those with high valuations keep it while those with low valuations resell. In particular, some high-valuation-low-wealth buyers now get the good. The same logic applies to reductions in \(\bar{p}\). As the cap drops, more of the good is assigned to the poor. The ultimate allocation becomes more efficient since \(r(\bar{p})\) increases as \(\bar{p}\) falls, so efficiency is highest when the cap is at the lowest level at which supply is available, \(c\). The formal results are now given.

**Proposition 1.** Random assignment of a non-transferable good is less efficient than the competitive market, and it becomes progressively less so as the price cap, \(\bar{p} > c\), falls. Random assignment of a transferable good is more efficient than the competitive market, and it becomes strictly more so as the price cap falls.

\(^{13}\)Independence of \(v\) and \(w\) implies that, for any given \(w \geq \bar{p}\), the expected value of \(v\), conditional on exceeding \(r(\bar{p})\), is \(\phi(r(\bar{p}))\). Since quantity equals \(S\), the total realized value is \(S\phi(r(\bar{p}))\).

\(^{14}\)High-valuation buyers with wealth \(w < \bar{p}\) cannot get the good, so \(r(\bar{p}) < v^*\). Since \(S\phi(r(\bar{p})) < S\phi(v^*) = V^*\), there is not full efficiency.
A final point is that speculation reduces the benefits from capping price by lowering the quantity assigned to low-wealth buyers. In 1982, Congress authorized the FCC to use a lottery to assign cellular telephone licenses. Since the price to a buyer was low (the application fee was zero initially, and only $230 in 1993), nearly 400,000 applications were received. The volume of applications caused shelves to break at the FCC’s processing center (Kwerel and Williams (1993)).

To see the effect of speculation here, let
\[
\psi(m, r) := D(r) - S + \rho(\bar{p}; m)[1 - F(r)][G(r) - G(\bar{p})]
\]
denote the excess demand on the resale market when \( p = r \). Along a level set of \( \psi \), the slope is \( -\frac{\psi_m}{\psi_r} < 0 \) since excess demand declines with \( r \) and the probability of receiving the good declines with \( m \). Fixing \( \bar{p} \), a rise in \( m \) lowers the total realized value, \( S\phi(r(\bar{p})) \). As \( m \) increases without bound, \( \psi(m, r) \) approaches \( D(r) - S \) so the equilibrium resale price approaches \( p^e \). Speculators acquire the entire supply in the limit so the resale market mimics the original competitive market. The next section shows the robustness of these results.

4 Analysis with a General Assignment Scheme

The previous section considered random assignment of a good. In practice, the government or suppliers may have information about buyers’ wealths or valuations that allows them to implement alternative schemes. For instance, one patient may be given priority for an organ transplant based on age and medical urgency, which implicitly says that the one patient’s valuation is higher than the others’.

The chosen assignment scheme may also depend on the government’s or suppliers’ objectives. Some schemes aim to help the poor while others aim to select those deemed to have the greatest merit. For example, need-based scholarships target students with low wealth while merit-based scholarships can be seen as targeting those with high valuations. Need-based assignment can also be found in the preferences for “designated entities” (i.e., small businesses) in Federal Communications Commission license auctions. Even a first-come-first-served scheme will assign the good in a manner that depends on the correlation
between cost of time and bidders’ types. While merit-based assignment is clearly justifiable on allocative efficiency grounds, an efficiency justification for need-based assignment (i.e., favoring the poor) is not obvious.

In this section we will consider a family of general assignment schemes. We do not model seller objectives or the information available; instead, we take the assignment rule as given. An assignment rule, \( x \), gives the probability that a buyer gets the good initially. The assignment rule is feasible if

\[
\int_0^1 \int_0^1 x(w, v) dF(v) dG(w) + m \int_0^1 x(w, 0) dG(w) = S.
\]

A feasible assignment rule is separable if \( x(w, v) = \alpha_x(w) \beta_x(v) \) for functions \( \alpha_x : [0, 1] \rightarrow \mathbb{R}_+ \) and \( \beta_x : [0, 1] \rightarrow \mathbb{R}_+ \). Let \( \mathcal{X} \) denote the set of feasible, separable assignment rules. We henceforth consider rules in this family. Separable assignment schemes allow us to evaluate policies targeting a single attribute in the most transparent way.

### 4.1 Characterization of Assignment Rules

Let

\[
a_x(w) := \frac{\alpha_x(w)}{\int_0^1 \alpha_x(\tilde{w}) dG(\tilde{w})}
\]

and

\[
b_x(v) := \frac{\beta_x(v)}{\int_0^1 \beta_x(\tilde{v}) dF(\tilde{v}) + m \beta_x(0)}.
\]

The probability of assignment then satisfies

\[
x(w, v) = S \cdot a_x(w) b_x(v),
\]

since feasibility implies

\[
\int_0^1 \int_0^1 \alpha_x(\tilde{w}) \beta_x(\tilde{v}) dF(\tilde{v}) dG(\tilde{w}) + m \int_0^1 \alpha_x(\tilde{w}) \beta_x(0) dG(\tilde{w})
\]

\[
= \left( \int_0^1 \alpha_x(\tilde{w}) dG(\tilde{w}) \right) \left( \int_0^1 \beta_x(\tilde{v}) dF(\tilde{v}) + m \beta_x(0) \right) = S.
\]

Then,

\[
A_x(w) := \int_0^w a_x(\tilde{w}) dG(\tilde{w})
\]
and
\[ B_x(v) := \int_0^v b_x(\tilde{v})dF(\tilde{v}) + \frac{m\beta_x(0)}{\int_0^\infty \beta_x(\tilde{v})dF(\tilde{v}) + m\beta_x(0)} \]
are cumulative distribution functions for quantity across different wealths and valuations, respectively. It will prove useful for efficiency comparisons to characterize assignment rules in terms of these functions.

We begin with rules that favor buyers with higher valuations. The assignment rule \( x \in X \) merit-dominates \( y \in X \) if \( B_x \) first-order stochastically dominates (FOSD) \( B_y \): \( \forall v, B_x(v) \leq B_y(v) \).\(^\text{15}\) In words, \( x \) puts more probability on high-valuation buyers than \( y \) does. Rules \( x \) and \( y \) are merit-equivalent if \( b_x(\cdot) = b_y(\cdot) \). (Note that this implies \( B_x(0) = B_y(0) \).)

The assignment rule is merit-blind if \( b_x(v) \) is constant for all \( v \geq \bar{p} \). Random assignment is an obvious example of a merit-blind rule as it awards the good with the same probability to all buyers whose valuations exceed the price cap.

There are analogous conditions for cases in which lower wealth is favored. We say that \( x \) need-dominates \( y \) if \( A_y \) FOSD \( A_x \): \( \forall w, A_x(w) \geq A_y(w) \). In words, \( x \) is more likely to assign the good to low-wealth buyers than \( y \) is. Rules \( x \) and \( y \) are said to be need-equivalent if \( a_x(\cdot) = a_y(\cdot) \). Finally, \( x \) is need-blind if \( a_x(w) \) is constant for all \( w \geq \bar{p} \), in which case the probability of receipt is independent of wealth.

The outcomes with and without transferability can now be characterized.

### 4.2 Assigning a Non-transferable Good

Given a price cap, \( \bar{p} < p^* \), only buyers with \((w, v) \geq (\bar{p}, \bar{p})\) will participate if the good is not transferable. The assignment rule will then have \( x(w, v) = 0 \) if \( w < \bar{p} \) or \( v < \bar{p} \). The total realized value is
\[
V_x := \int_{\bar{p}}^1 \int_{\bar{p}}^1 vx(w, v)dF(v)dG(w).
\]

The following proposition provides a ranking based on merit-dominance.

**Proposition 2.** If \( x \in X \) [strictly] merit-dominates \( y \in X \), then \( V_x \geq [>]V_y \).

\(^{15}\)The merit-dominance is strict if the inequality is strict for a positive measure of valuations. The analogous condition makes subsequent dominance definitions strict as well.
Proof: Rewrite the total realized value as

\[ V_x = S \int_0^1 \int_0^1 va_x(w)b_x(v)dF(v)dG(w) \]
\[ = S \int_0^1 vb_x(v)dF(v) \]
\[ = S \int_0^1 vdB_x(v). \]

Since \( x \) [strictly] merit-dominates \( y \in X \), \( B_x \) [strictly] FOSD \( B_y \), so the result follows.

This result says that an assignment rule that puts relatively more weight on high valuations yields greater efficiency. A couple of implications can be drawn immediately. We first note that it does not matter how an assignment rule treats buyers with different wealth levels here: All that matters for efficiency is how it screens based on valuations.

Corollary 1. (Irrelevance of need-based screening) If \( x \in X \) and \( y \in X \) are merit-equivalent, then \( V_x = V_y \).

Any assignment rule that is merit-dominated by a merit-blind rule is less efficient than the merit-blind rule, which is welfare-equivalent to random allocation, by Corollary 1. The previous section established that random assignment is strictly less efficient than the competitive market, so the following result is also immediate.

Corollary 2. (Drawback of assignment without transferability) Any assignment rule in \( X \) that is merit-dominated by the random assignment rule (given the same binding price cap) yields a strictly less efficient allocation than the market does.

It follows that a merit-blind assignment rule (which is merit-equivalent to random allocation) is strictly less efficient than the competitive market. That is, any assignment rule associated with purely need-based screening is less efficient than the market. It follows that favoring the poor cannot be justified from an efficiency perspective if the good is not transferable.

The strict dominance of the market over merit-blind rules implies that any rule that is modestly merit-superior to random assignment also falls short from an efficiency perspective. In order for the assignment of a non-transferable good to improve efficiency,
substantial merit-based screening must be feasible. In the extreme case in which buyers’ valuations are observable, the efficient allocation can be achieved. In practice, however, such precise information will rarely be available.

4.3 Assigning a Transferable Good

We now consider the use of general assignment schemes when the good is transferable. Suppose that the price cap is \( p < p^e \), and the subsequent resale market price is \( r > p \). A buyer would participate in the initial assignment if and only if \((w, v) \geq (p, 0)\), so the assignment rule has \( x(w, v) = 0 \) for any \( w < p \).

A buyer who fails to get the good initially will demand a unit on the resale market if she is willing and able to pay \( r \), so resale demand is

\[
RD(r) := \int_r^1 \int_r^1 [1 - x(w, v)] dF(v) dG(w).
\]

Buyers who get the good initially will keep it if \( v \geq r \). The quantity supplied on the resale market at \( r \) is therefore

\[
RS(r) := S - \int_{\overline{p}}^r \int_r^1 x(w, v) dF(v) dG(w).
\]

\( RD(\cdot) \) is nonincreasing and \( RS(\cdot) \) is nondecreasing, and both are continuous functions. Further, \( RD(1) = 0 < S = RS(1) \), and \( RD(0) = 1 - S > 0 = RS(0) \). Hence, there exists a resale price, \( r_x \), that clears the market: \( RD(r_x) = RS(r_x) \).

Rewrite the market-clearing condition as:

\[
RD(r) - RS(r) = D(r) - S + K_x(r) = 0,
\]

where

\[
K_x(r) := \int_{\overline{p}}^r \int_r^1 x(w, v) dF(v) dG(w) = S \cdot A_x(r)[1 - B_x(r)]
\]

is the measure of buyers with wealth \( w \in [\overline{p}, r] \) who get the good initially and keep it. Note that if \( K_x(p^e) > 0 \), then \( RD(p^e) - RS(p^e) = K_x(p^e) > 0 \). It follows that \( RD(r) > RS(r) \) for any \( r \leq p^e \), implying that \( r_x > p^e \), which confirms that the resale price exceeds \( \overline{p} \).

The total realized value is now

\[
\hat{V}_x := \int_{\overline{p}}^{r_x} \int_{r_x}^1 vx(w, v) dF(v) dG(w) + \int_{r_x}^1 \int_{r_x}^1 vdF(v) dG(w).
\]

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The first term pertains to high-valuation-low-wealth buyers who keep the good, while the second covers those with high valuations and high wealth; some of them get the good initially, while the others purchase it on the resale market.

The subsequent characterization refers to a new property. We say that the assignment rule \( x \) relatively merit-dominates the rule \( y \) if \( B_x(0) \leq B_y(0) \) and, for every \( v' \in (v, 1) \),

\[
\frac{b_x(v')}{b_x(v)} \geq \frac{b_y(v')}{b_y(v)}.
\]

This notion implies merit-dominance as it requires the latter to hold for all subsets of valuations. We say that \( x \) is meritorious if it relatively merit-dominates a merit-blind rule, in which case \( b_x(v') \geq b_x(v) \) for all \( v' > v \). Likewise, \( x \) is demeritorious if it is relatively merit-dominated by a merit-blind rule. Meritorious (demeritorious) assignment schemes dominate (are dominated by) merit-blind rules in a stronger sense than merit-dominance.

Let \( \mathcal{X}^+ \) and \( \mathcal{X}^- \) denote, respectively, the set of all meritorious and demeritorious rules in \( \mathcal{X} \). Note that both sets include merit-blind rules.

**Proposition 3.** If a meritorious rule, \( x \in \mathcal{X}^+ \), relatively merit-dominates \( y \in \mathcal{X} \), and if \( r_x \geq [r_y] \), then \( \hat{V}_x \geq [\hat{V}_y] \).

**Proof:** See the Appendix.

As was the case with random assignment, Proposition 3 shows that a higher resale price indicates a more efficient allocation. Several important implications can then be drawn. The first is that we can provide a sufficient condition for assignment schemes to improve upon the competitive market equilibrium.

**Corollary 3.** (Superiority of meritorious assignment schemes for a transferable good) Any meritorious assignment rule, \( x \in \mathcal{X}^+ \), with \( \overline{p} < p^e \) and \( A_x(p^e) > 0 \), produces a strictly more efficient allocation than the competitive market.

**Proof:** Fix \( x \in \mathcal{X}^+ \) with \( \overline{p}_x < p^e \). The total realized value in the competitive market is equal to \( \hat{V}_y \) for any \( y \in \mathcal{X} \) satisfying \( \overline{p}_y = r_y = p^e \). (Then, only buyers with \( w \geq p^e \) participate.) In particular, we can choose \( y \) to be relatively merit-dominated by \( x \). Since \( x \) is meritorious, we have \( B_x(p^e) \leq p^e < 1 \); together with \( A_x(p^e) > 0 \), this means \( K_x(p^e) > 0 \).
It follows that \( r_x > p^e \). In sum, \( x \in X^+ \) relatively merit-dominates \( y \), and \( r_x > p^e = r_y \). Proposition 3 then implies \( \hat{V}_x > \hat{V}_y \).

The condition \( A_x(p^e) > 0 \) simply means that the scheme awards the good to some buyers who are willing but unable to pay the competitive market price. These buyers would not resell the good if \( r = p^e \), so the resale price must exceed the competitive market price. Except for this condition, the result does not require much in terms of how the assignment depends on wealth levels. In other words, a weakly meritorious assignment rule does strictly better than the competitive market, largely independent of how it treats different wealths. Also worth noting is that even a merit-blind assignment rule strictly dominates the market. This means that some demeritorious assignment schemes could do better than the market when the good is transferable.

We next examine the effect of a change in the price cap. This analysis involves a delicate issue: When the cap changes, it alters the set of buyers who participate, so the assignment rule itself changes. One must therefore specify how the rule changes when the price cap changes. We make the following assumptions.

**Condition (RC):** Suppose that \( x \) and \( x' \) are assignment rules induced by a given assignment technology, with price caps \( \bar{p} < p^e \) and \( \bar{p}' < \bar{p} \), respectively. Then, (i) \( x \) and \( x' \) are merit-equivalent (i.e., \( b_{x'}(\cdot) = b_x(\cdot) \)), and (ii) \( a_{x'}(w) < a_x(w) \) for \( w \in [\bar{p}, 1] \).

Property (i) is appealing since a change in the price cap affects the set of participants along the wealth dimension only. It is then reasonable that the merit aspect of the assignment does not change. Property (ii) reflects the equally plausible requirement that adding buyers with lower wealth to the pool reduces the assignment probability for all of the original participants. A special case of Condition (RC) arises if the relative probabilities among the original participants are unchanged; i.e., \( x'(w, v) = \lambda x(w, v) \) for \( w \in [\bar{p}, 1] \), for some \( \lambda < 1 \).

**Corollary 4. (Benefit of lowering price caps)** Lowering the price cap increases efficiency, given a meritorious assignment technology satisfying Condition (RC).

**Proof:** Let \( x \) and \( x' \) be meritorious assignment rules induced by a given assignment technology, with price caps \( \bar{p} < p^e \) and \( \bar{p}' < \bar{p} \), respectively. Then, Condition (RC) means
that \( x \) and \( x' \) are also merit-equivalent. Hence, \( x' \) relatively merit-dominates \( x \), and \( B_{x'}(\cdot) = B_x(\cdot) \).

We next prove that \( A_{x'}(w) > A_x(w) \) for all \( w \in [\bar{p}, 1) \). Observe that, \( \forall w \geq \bar{p} \), we have

\[
A_x(w) = \int_{\bar{p}}^w a_x(\hat{w})dG(\hat{w}),
\]

and

\[
A_{x'}(w) = \int_{\bar{p}}^w a_{x'}(\hat{w})dG(\hat{w}).
\]

Hence, for \( w \geq \bar{p} \),

\[
\frac{dA_{x'}(w)}{dw} = a_{x'}(w)g(w) < a_x(w)g(w) = \frac{dA_x(w)}{dw},
\]

where the inequality follows from Condition (RC). This, together with \( A_{x'}(1) = A_x(1) \), implies that \( A_{x'}(w) > A_x(w) \) for all \( w \in [\bar{p}, 1) \).

Combining these facts, we have

\[
K_{x'}(r) = A_{x'}(r)[1 - B_{x'}(r)] > A_x(r)[1 - B_x(r)] = K_x(r)
\]

for any \( r > \bar{p} \), so \( r_{x'} > r_x \). Consequently, Proposition 4 implies that \( \hat{V}_{x'} > \hat{V}_x \). \( \blacksquare \)

This result shows that the lowest feasible cap is optimal. The proposition also allows us to investigate which assignment schemes promote efficiency.

**Corollary 5. (Benefit of need-based assignment schemes)** If \( x \in \mathcal{X} \) relatively merit-dominates and need-dominates \( y \in \mathcal{X} \), then \( \hat{V}_x \geq \hat{V}_y \). If either dominance is strict, then \( \hat{V}_x > \hat{V}_y \).

**Proof:** Given Proposition 3, it suffices to show that \( r_x \geq r_y \), with a strict inequality for a strict ranking. Relative merit-dominance by \( x \) over \( y \) implies \( 1 - B_x(r) \geq 1 - B_y(r) \) for all \( r \), whereas need-dominance implies \( A_x(r) \geq A_y(r) \). Hence, \( K_x(r) \geq K_y(r) \), for all \( r \), which implies \( r_x \geq r_y \). If either dominance is strict, then \( K_x(r) > K_y(r) \) when \( r = r_y \), so \( r_x > r_y \). \( \blacksquare \)

Need-based assignment schemes can yield greater efficiency than the market does. While it is not surprising that merit-based rules can improve efficiency, it is striking that need-based rules can have the same effect. If \( x \) and \( y \) are merit-equivalent, but the former
need-dominates the latter, then $x$ produces a more efficient allocation. Wealthy buyers can buy the good from a reseller if they do not get it initially, whereas the poor lack the means to do so. As a consequence, an initial assignment that gives the good disproportionately to the poor enables the resale market to produce a more efficient allocation.

Corollary 5 also provides a strong result if the assignment rule favors low-wealth buyers completely. Consider a merit-blind rule with $x^*(w, v) = 1$ if $w \leq w^*$, and $x^*(w, v) = 0$ otherwise, where $(1 + m)G(w^*) = S$. Let $\bar{p} = 0$. That is, $x^*$ assigns the good to all buyers with wealth $w^*$ or below (the region $A + B$ in Figure 3). This rule will achieve full efficiency if $v^* \leq w^*$, where $1 - F(v^*) = S$. To see this, suppose that $v^* \leq w^*$. When the resale price is $r$, supply will be $RS(r) = S(F(r) + m).$ Meanwhile, those not assigned the good are willing to buy if $v \geq r$, so resale demand is $RD(r) = [1 - F(r)][1 - G(\max\{r, w^*\})].$ Since $RD(r) \geq RS(r)$ if $r \geq v^*$, the equilibrium resale price is $v^*$. Consequently, the buyers with $v \geq v^*$ will end up with the good, which is fully efficient. This group comprises the region $A + C$ in Figure 3.

When the supply is sufficiently large, the infimum wealth among buyers who do not get the good initially is high. These buyers are then able to purchase on the resale market. An important observation is that this benefit from need-based screening does not depend on the independence of $v$ and $w$.

A final point concerns the impact of the pure speculators. Consider a more general (i.e., not necessarily separable) assignment technology. As $m$ rises, the assignment rule itself must change to remain feasible. We note this dependence by writing the assignment probability as $x(w, v; m)$. Let the set of participants be $\Omega^*$ and let $\mu(\cdot)$ denote the measure of a set. Then, an assignment rule is called non-concentrating if, for all $\Omega \subset \Omega^*$, we have

$$\int_{\Omega^+} x(w, v; m)dF(v)dG(w) + m \int_{\Omega^0} x(w, 0; m)dG(w) \leq \left(\frac{N\mu(\Omega)}{\mu(\Omega^*)}\right) S$$

for some fixed $N > 1$, where $\Omega^+ := \{(w, v) \in \Omega | v > 0\}$ and $\Omega^0 := \{(w, v) \in \Omega | v = 0\}$. The condition says that any set of types receives a quantity that is not too large relative to the

\[16\]This condition is satisfied if $S$ is sufficiently large. Letting $t(z) := [1 - F(z)] - (1 + m)G(z)$, we just need to find a root of the equation $t(z) = 0$. Since $t(0) = 1$, $t(1) = -(1 + m)$, and $t(\cdot)$ is continuous and strictly decreasing, there is a unique root, $z^*$. Full efficiency requires $S \geq (1 + m)G(z^*) = 1 - F(z^*)$. 

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measure of the set. This leads to an asymptotic version of the Coase Theorem; specifically, the initial allocation does not matter much when there is substantial participation by pure speculators.

**Proposition 4.** Given a non-concentrating assignment rule, as \( m \) rises without bound, the equilibrium resale price converges to the competitive market price.

**Proof:** The current analog to the market-clearing condition in (4) is:

\[
RD(r) - RS(r) = D(r) - S + \tilde{K}_x(r) = 0,
\]

where

\[
\tilde{K}_x(r) := \int_p^r \int_r^1 x(w, v; m) dF(v) dG(w)
\]

is the measure of buyers with wealth \( w \in [p, r] \) who get the good initially and keep it. Since the assignment rule is non-concentrating, we have

\[
\int_p^r \int_r^1 x(w, v; m) dF(v) dG(w) \leq \left( \frac{N \int_p^r \int_r^1 dF(v) dG(w)}{[1 + m][1 - G(p)]} \right) S \leq \frac{NS}{1 + m}.
\]

Thus, \( \tilde{K}_x(p^e) \) must converge to zero as \( m \) rises without bound, so \( RD(p^e) - RS(p^e) \) also converges to zero, meaning that \( r_x \) converges to \( p^e \).

It follows that prohibiting transferability may be welfare-enhancing when speculation is a major concern. With transferability, the allocation approaches the original competitive market allocation as \( m \) rises without bound. Without transferability, if the corresponding assignment rule is sufficiently meritorious, the ultimate outcome will be more efficient than the market outcome (and the outcome of the assignment scheme with transferability).\(^{17}\)

**Corollary 6.** Consider a non-concentrating assignment technology that merit-dominates the competitive market allocation when the good is not transferable. Then, prohibiting transferability raises total realized value if \( m > M \), for some fixed \( M > 0 \).

When the assignment technology allows sufficient merit-based screening, it is preferable to rely on that technology instead of the resale market since speculation blunts the benefits

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\(^{17}\)Andolfatto (2002) makes a very different case for restricting transferability. His result relies on financial markets that work “too well” in the sense that they allow individuals to borrow against future income, leaving them in need of largess in the future.
of resale. At the same time, one can often screen out speculators, making this result less relevant. For instance, it is possible to verify whether people seeking an organ transplant actually need the operation, whether people seeking a military deferment are eligible to be drafted, or whether people seeking transferable education vouchers have children who are eligible to attend the school.\textsuperscript{18}

5 Discussion

In this section we explore the impact of several natural extensions. We will see that the basic qualitative results continue to hold in a range of circumstances. For ease of exposition, we consider the case of $m = 0$ here so there are no pure speculators.

5.1 A Dual System

We have so far assumed that the entire supply was distributed according to a single assignment rule. Now suppose that part of the supply is assigned according to one assignment rule, while the remainder is sold at a higher price on a black market. There is a long history of dual markets. During World War II, black markets for beef, chicken, gasoline, and tires operated in the U.S. For instance, some farmers sold part of their crop on the market at the price cap, and the remainder at the farm gate at higher prices (Rockoff (1998)). Underground markets for a multitude of goods thrived for decades in planned economies. More recently, gasoline was selling for 2,000 dinars on the black market in Baghdad while the price cap was 80 dinars.\textsuperscript{19} Finally, despite restrictions on the sale of human organs, there have been sales on black markets.\textsuperscript{20}

Dual market systems may also arise by design. Consider school choice. One existing scheme guarantees admission to students living in the vicinity of a public school and uses a

\textsuperscript{18}It may be difficult to discern whether applicants for immigration visas or subsidized housing are speculators, however.


\textsuperscript{20}A middle ground is occupied by online services that match live donors and recipients. In jurisdictions where it is legal for recipients to reimburse donors for costs, this opens one avenue for compensation through inflated costs.
lottery to assign the remaining slots (Abdulkadiroğlu and Sönmez (2003)). Since one can acquire a slot by purchasing a house in the immediate vicinity of the school, there is a high price to get a slot with probability one, and a low (zero) price to get it with a probability less than one through the lottery.

An analogous situation occurs when a high school student applies to a college or university under the “early decision” program. All else equal, the probability of being accepted is higher when one applies for an early decision (Avery, Fairbanks, and Zeckhauser (2003)). The price is also higher since financial aid offers may be significantly lower.\footnote{21See “The Benefits and Drawbacks of Early Decision and Early Action Plans,” at http://www.collegeboard.com/prof/counselors/apply/5.html, accessed on June, 8, 2005.}

In other words, students who are accepted under the early decision program tend to pay more, all else equal.

Our model can be altered to accommodate dual-market systems. Suppose that a fraction \( \gamma \in (0, 1) \) of the supply is sold in a competitive market, while the remaining \( 1 - \gamma \) is subject to a price cap, \( \bar{p} < p^* \), and random allocation. Assume that the good is not transferable.

In equilibrium, buyers with \((w, v) \geq (\bar{p}, \bar{p})\) participate in the assignment scheme. There is also a market price, \( p^* > p^e \), such that unsuccessful buyers with \((w, v) \geq (p^*, p^*)\) will purchase on the market.\footnote{22The organizer may be able to monitor participation in the dual system so that one individual may get access to the assignment scheme or the market, but not both. The result is qualitatively the same.} Let \( L(z) := \{(w, v)|(w, v) \geq (\bar{p}, z), \min\{w, v\} < p^*\} \) be the L-shaped set containing types with valuations exceeding \( z \) who participate in the assignment scheme but who will not purchase at \( p^* \); and let \( \ell(z) := \Pr\{(w, v) \in L(z)\} \) and \( \xi(z) := \mathbb{E}[v|(w, v) \in L(z)\} \). The total realized value from the dual system is

\[
V_D := \sigma \ell(\bar{p}) \xi(\bar{p}) + D(p^*) \phi(p^*),
\]

where \( \sigma \) is the probability of assignment in the lottery, which satisfies the market-clearing condition, \( \sigma \ell(\bar{p}) + D(p^*) = S \).

The efficiency comparison between the dual system and the unregulated market is actually ambiguous.\footnote{23When \( \gamma = 1 \), the dual market outcome is the same as a competitive market. Lowering \( \gamma \) slightly raises \( p^* \). Moreover, buyers in \( L(\bar{p}) \) now get the good with some small probability. These changes may raise or lower the total realized value, depending on the density in this region.} At the same time, incorporating random assignment of a transferable good would enhance the dual system, as we now show. Suppose that the quantity \( \gamma S \) is as-
signed via transferable vouchers and the remaining \((1 - \gamma)S\) is assigned via non-transferable vouchers, both at the price of \(\bar{p}\). Consistent with the earlier approach, assume that each individual may only enter the pool for a single type of voucher. Those who are unsuccessful can purchase a transferable voucher in the resale market, which is again assumed to be competitive.

We look for an equilibrium with two properties: (1) There exists \(\hat{v} \in [0, p^*]\) such that buyers with \(v \geq \hat{v}\) enter the pool for non-transferable vouchers, and those with \(v < \hat{v}\) enter the pool for transferable ones; and (2) the resale price is \(p^*\). Sufficient conditions for such an equilibrium to exist are derived as follows. First, the critical type, \(\hat{v}\), must be indifferent between the two voucher types:

\[
\frac{(1 - \gamma)S}{[1 - F(\hat{v})][1 - G(\bar{p})]} [\hat{v} - \bar{p}] = \frac{\gamma S}{F(\hat{v})[1 - G(\bar{p})]} [p^* - \bar{p}].
\]

(6)

Given (6), any buyer with \(v > \hat{v}\) will strictly prefer non-transferable [transferable] vouchers. Next, the resale market will clear at \(p^*\) if

\[
\left(1 - \frac{(1 - \gamma)S}{[1 - F(\hat{v})][1 - G(\bar{p})]}\right) D(p^*) = \gamma S.
\]

(7)

If there exists a pair, \((\hat{v}, \hat{\gamma})\), satisfying (6) and (7), then such an equilibrium exists. The aggregate value realized in that equilibrium is

\[
\hat{V}_D = \hat{\sigma} \ell(\hat{v}) \xi(\hat{v}) + D(p^*) \phi(p^*),
\]

where \(\hat{\sigma} \ell(\hat{v}) + D(p^*) = S\). Since \(\hat{v} > \bar{p}\), we conclude that \(\hat{V}_D > V_D\).

**Proposition 5.** For any dual-market system, there exists an assignment scheme with vouchers (some transferable) that is more efficient.

**Proof:** The proof is in the Appendix.

### 5.2 Regulation of Resale

When a good is transferable, speculation by low-valuation buyers reduces the probability that any given buyer gets the good initially, which lowers ultimate efficiency. One possible response is to tax resales. Suppose that the government caps the price of the good at
\( \bar{p} < p^* \), assigns it randomly, and imposes a lump-sum tax of \( \tau \geq 0 \) on resellers. All buyers with \( w \geq \bar{p} \) will seek to acquire the good if the resale price, \( \tilde{r}(\tau) \), satisfies \( \tilde{r}(\tau) \geq \bar{p} + \tau \), which we will assume. This condition means that it is still profitable to engage in resale.\(^{24}\) The equilibrium resale price increases with the tax in this region, and the net proceeds from resale fall.

To see the effect of a higher tax, it is useful to distinguish the two groups that get the good ultimately. First, some buyers with \( (w, v) \geq (\bar{p}, \tilde{r}(\tau) - \tau) \) get it initially and do not resell. Second, some buyers with \( (w, v) \geq (\tilde{r}(\tau), \tilde{r}(\tau)) \) purchase on the resale market. When the tax rate rises from \( \tau \) to \( \tau' \), the former group expands and the latter group contracts. In particular, some buyers with \( v \geq \tilde{r}(\tau) \) now get the good with a probability below one. Conversely, buyers with \( w \geq \bar{p} \) and \( \tilde{r}(\tau') - \tau < v < \tilde{r}(\tau) - \tau \) now get it with strictly positive probability, rather than zero. Since the probability of receipt is unchanged in the other regions, the total realized value is lower with the higher tax, \( \tau' \). It follows that a tax on resales is counterproductive.\(^{25}\)

5.3 Pre-payment Resale

Since resale has been shown to be beneficial, we now ask whether one should encourage even more resale. This can be done by allowing a buyer to resell the good before taking delivery. We consider such cases by allowing a buyer who acquires the good to resell it after paying a deposit, \( \delta \leq \bar{p} \).\(^{26}\) This allows the buyer to engage in speculation even if his wealth is insufficient to pay the price.

Assume that the good is randomly assigned again. Fixing \( \bar{p} \), speculation declines as the required deposit rises, so more high-valuation buyers with lower wealth get the good initially. The resale price must therefore rise, so a higher deposit means improved allocative efficiency. Fixing the price cap, it is optimal to set the deposit equal to the cap. Doing so excludes buyers with \( w < \bar{p} \), but they would not be able to keep the good anyway, so there

\(^{24}\)If the inequality does not hold, speculation is unprofitable. Only those with \( (w, v) \geq (\bar{p}, \bar{p}) \) will participate in the initial allocation, and the resale market will be inactive.

\(^{25}\)One can also interpret \( \tau \) as a measure of resale transaction costs, unrelated to government regulation. Then, \( \tau \) is a social cost as well.

\(^{26}\)Requiring the deposit has essentially the same effect as raising the price cap to \( \delta \), so there is no loss of generality in assuming that \( \delta \leq \bar{p} \).
is no efficiency loss from excluding them. We conclude that a deposit equal to the cap is optimal, meaning that there is no benefit from allowing pre-payment resale.

5.4 Direct Subsidy vs. Random Allocation

Another option for improving the allocation is to alleviate wealth constraints directly with a subsidy, and then let the market operate. Direct subsidies have two shortcomings, however. First, the financing of the subsidy may itself be distortionary. Second, if buyers’ valuations and wealth are not readily observable, it is difficult to target the subsidy to the wealth-constrained individuals who value the good highly.

Random assignment of a transferable good subsidizes the poor while being budget-balanced as it does not entail any outlay from the government. The scheme is also targeted in that not all of the wealthy are subsidized. (In fact, the wealthy may be worse-off, on average, since there is probability \((1 - S)\) that they pay more with the price cap and resale than without the cap.) The only unintended beneficiaries are the resellers, who have low valuations. Enriching this group appears to be necessary to subsidize buyers in a budget-balanced fashion.

We now provide a comparison of two schemes. Consider a direct subsidy that is financed in a budget-balanced way, i.e., through the sale of the good itself. We will show that such a scheme produces a less efficient outcome than the random assignment scheme if \(G(\cdot)\) is (weakly) concave. (This condition means, roughly speaking, that the poor are relatively more numerous than the wealthy.) We suppose that neither \(v\) nor \(w\) is observable so that only a uniform subsidy is feasible.

Let the government offer each consumer a lump-sum subsidy, \(\sigma \geq 0\), and charge a price, \(p\), such that the budget balances and the market clears. Budget-balance means that \((p - c)S = \sigma\). Given \(p\) and \(\sigma\), a buyer with \((w, v)\) will demand the good if and only if \(v \geq p\) and \(w + \sigma \geq p\), so aggregate demand is \((1 - F(p))(1 - G(p - \sigma))\). Using the budget-balance condition, market-clearing requires

\[
ED^*(p) := (1 - F(p))(1 - G(p - (p - c)S)) - S = 0.
\]

It is straightforward to check that a unique equilibrium price, \(p^*\), exists. As in Section 3, the value realized is \(S\phi(p^*)\).
Proposition 6. Suppose that $G(\cdot)$ is (weakly) concave. Then, setting a price cap, $\bar{p} = c$, and randomly allocating a transferable good yields higher total realized value than the budget-balanced uniform subsidy.

Proof: It suffices to show that the equilibrium resale price under the random assignment scheme, $r(c)$, exceeds the market-clearing price under the subsidy regime. Let $ED^r(p) := RD(p) - RS(p)$ denote the excess demand under the former regime, where $RD(p) - RS(p)$ follows from (2) and (3) with $\bar{p} = c$. The difference between the excess demands is:

$$\Delta(p) := ED^r(p) - ED^s(p) = \left(\frac{1 - F(p)}{1 - G(c)}\right) \delta(p, S),$$

where

$$\delta(p, S) := (G(p - (p - c)S) - G(c))(1 - G(c)) - (G(p) - G(c))(1 - G(c) - S).$$

To prove $r(c) > p^s$, it suffices to show that $\delta(p, S) > 0$ for all $p$ and that $S \in (0, 1 - G(c))$. Note first that $\delta(p, 0) = 0$. Next, we show that $\partial \delta(p, S)/\partial S > 0$ whenever $\delta(p, S) = 0$:

$$\frac{\partial \delta(p, S)}{\partial S} \bigg|_{\delta(p, S) = 0} = \{ -g(p - (p - c)S)(p - c)(1 - G(c)) + G(p) - G(c) \} \delta(p, S) = 0$$

$$= \left( \frac{1 - G(c)}{1 - G(c) - S} \right) \left[ G(p - (p - c)S) - G(c) - g(p - (p - c)S)(p - c)(1 - G(c) - S) \right]$$

$$> \left( \frac{1 - G(c)}{1 - G(c) - S} \right) \left[ G(p - (p - c)S) - G(c) - g(p - (p - c)S)(p - c)(1 - S) \right]$$

$$\geq 0.$$

The last inequality holds by concavity of $G$: Letting $z := p - (p - c)S$, the expression in square brackets equals $G(z) - G(c) - (z - c)g(z) \leq 0$.

Since $\delta(p, 0) = 0$, $\partial \delta(p, S)/\partial S > 0$ whenever $\delta(p, S) = 0$, and $\delta(p, \cdot)$ is continuous; we conclude that $\delta(p, S) > 0$ for all $p \in (c, 1)$ and $S \in (0, 1 - G(c))$. The result then follows.

This analysis presumed that a government sold the good and used the revenue to finance the subsidy. If the sellers are private firms, a direct subsidy may require financing through a distortionary tax, another source of inefficiency. In addition, a subsidy based on wealth may cause individuals to conceal their wealth. Basing a subsidy on individuals’ choices
is not foolproof either. Suppose that the government offers a subsidy only to those who purchase the good. This will raise each buyer’s willingness-to-pay by exactly the amount of the subsidy, so wealth constraints are not alleviated at all. The same point would hold with uniform bidding credits. A final point is that when there are pure speculators (i.e., \(m > 0\)), targeting a subsidy becomes more difficult.

### 5.5 Social Cost of Speculation

Speculation produces an indirect welfare cost here by reducing the probability of assignment for those with low wealth but a high valuation. Speculation may also entail a direct welfare cost if it involves real transaction costs or opportunity costs. Indeed, the price caps on new housing in Korea have been criticized for encouraging speculation and diverting resources away from other productive investment activities. We now illustrate this effect and show that it need not overturn the desirability of the aforementioned schemes.

Suppose that there exists a project requiring a capital investment of \(k \in [0, 1)\) that yields a net return of \(R \in (0, 1)\). An individual who participates in the assignment process must forego that project. (Assume that \(k + c > 1\) so no individual can both purchase the good and pursue the project.) It is socially desirable for individuals with \(w \geq k\) and \(v < R + c\) to invest in the project. This will occur in the competitive equilibrium since \(p^e \geq c\), which implies \(v - p^e < R\) if \(v < R + c\).

Now suppose that the price is capped at \(\bar{p} < p^e\) such that

\[
\rho(\bar{p})[r(\bar{p}) - \bar{p}] > R,
\]

where

\[
\rho(\bar{p}) := \frac{S(\bar{p})}{1 - G(\bar{p})}
\]

is the assignment probability for all participants. Speculation then offers a higher expected profit than the project, so no one will invest in the project. In this case, our scheme for allocating a transferable good has a direct welfare cost — the unrealized project return for buyers with \(w \geq k\) and \(v < R + c\). Even if this case arises, the welfare cost need not overturn the desirability of the scheme, although moderation in capping the price may be warranted. The improvement in efficiency that the cap brings may still outweigh the cost.
But if not, there exists $\bar{p}$ sufficiently close to $p^*$ such that $r(\bar{p}) - \bar{p} < R$. It follows that such a cap would not create the kind of speculation that entails inefficient project choices.\(^{27}\)

### 5.6 Elastic Supply

Our analysis has assumed that supply is perfectly inelastic, which may be a reasonable assumption for health care, school choice, and even housing. Supply may be responsive to a price cap in situations where capacity is not an issue, however. Moreover, even if capacity is limited in the short run, the ability to invest means that supply may be elastic in the long run.\(^{28}\) It is therefore important to examine the robustness of our results to the presence of elastic supply.

Suppose that the good is supplied competitively according to a twice-differentiable, increasing, aggregate cost function, $c(\cdot)$. The supply at price $p$ is then given by $S(p) \in \arg\max_{q \geq 0} pq - c(q)$, or implicitly by $c'(S(p)) = p$; $S(\cdot)$ is increasing and differentiable. The competitive equilibrium is characterized by a price, $p^e$, satisfying $D(p^e) = S(p^e)$.

When the price is capped at $\bar{p} < p^e$, there is excess demand, which will be resolved by allocating the good randomly. Assume that the good is transferable, so all individuals with $(w, v) \geq (\bar{p}, 0)$ will participate. A resale equilibrium exists and is characterized by the price $r(\bar{p})$ satisfying

$$D(r(\bar{p})) = S(\bar{p}) - \rho(\bar{p})[1 - F(r(\bar{p}))][G(r(\bar{p})) - G(\bar{p})].$$

The total realized value is now

$$\hat{V}(\bar{p}) = S(\bar{p})\phi(r(\bar{p})) - c(S(\bar{p})).$$

Capping the price and randomly allocating the good presents a tradeoff. On the positive side, it again reallocates the good from some low-valuation-high-wealth buyers to some

\(^{27}\)Essentially the same conclusion would arise with a continuum of projects with returns ranging from 0 to $R > 0$. In this case, random assignment of a transferable good will crowd out some low-return projects. Yet, lowering the cap slightly below $p^c$ will entail only a negligible welfare cost, since the projects that are foregone in pursuit of speculation would have generated only small social returns, whereas the gain from improved efficiency would be nonnegligible.

\(^{28}\)For example, new apartments can be built in the long run. Conversely, apartments can be converted to offices while rent control is in place.
others with high valuations and low wealth. On the negative side, supply falls, so some current buyers lose access to the good. This latter effect is nonnegligible, even if the cap is just below the market price, since the buyers losing access may have high valuations. (Some current buyers are wealth constrained, with wealth just above $p^c$ but valuations well above $p^c$.) The net effect is positive if supply is not too elastic.

**Proposition 7.** If \( \frac{S(p^c)}{S(p^c)} < \frac{f(p^c)}{F(p^c)} \), there exists a price cap, \( p < p^c \), such that random assignment of a transferable good yields greater total realized value than the competitive market does.

**Proof:** The proof is in the Appendix.

The applicability of Proposition 7 is clear in the two polar cases. If supply is almost perfectly inelastic, the proposition applies. The previous results continue to hold and random assignment outperforms the competitive market. Conversely, if supply is almost perfectly elastic, the proposition does not apply. In that case, the drop in quantity is so great that this effect swamps the improved mix of buyers.

An obvious point is that if the speculation problem could be addressed directly, the random assignment scheme would be desirable regardless of the supply elasticity. That is, if \( x(w, v) = 0 \) for all \( v < \bar{p} \) (instead of random allocation), the result of Proposition 7 always holds. In addition, the same result would hold if the buyers’ price could be lowered without altering the price that the sellers receive, although such a policy would violate the budget-balancing property of the assignment scheme.

6 Related Literature

The current paper fits in a line of research showing that binding wealth constraints may lead to inefficient market allocations. Che and Gale (1998) show this in the context of standard auctions when bidders differ in their valuations and wealths. A high-valuation-low-wealth bidder may lose to a wealthier bidder with a lower valuation. Gali and Fernandez (1999) study the matching of workers to inputs when workers differ in ability and wealth. They compare the market and a tournament, and find that both regimes provide efficient matching with perfect capital markets. Inefficiencies arise with imperfect capital markets, however. Then, matching efficiency is greater with the tournament, and aggregate consumption
may be higher as well. Finally, Esteban and Ray (2006) consider a government awarding licenses to produce; their analysis corresponds to the market regime in our context. The government assigns licenses based on lobbying expenditures since more-productive sectors have a greater incentive to lobby; however, wealthier sectors find it less costly to lobby, which jams the productivity signal. Esteban and Ray find conditions under which a reduction in inequality necessarily improves efficiency, which is analogous to our result that a need-dominant assignment scheme yields a more efficient allocation, all else equal.

The current paper finds the same signal-jamming effect of binding wealth constraints, which renders the market allocation inefficient. It differs from the preceding papers in several respects, however. Most notable is the analysis of resale, which is an indispensable element of many efficiency-enhancing assignment schemes. Conversely, the current paper also provides a novel rationale for the prohibition of resale based on the volume of pure speculation.

A second, related literature considers assignment of a good in settings where markets are not permitted to operate. Abdulkadiroğlu and Sönmez (1999) and (2003), along with Roth, Sönmez, and Unver (2004) have proposed ingenious algorithms for improving allocative efficiency—without using transfers—when binding wealth constraints may be important. The algorithms may not deliver full efficiency, however. The current paper suggests that introducing transfers in the form of resale may produce a more efficient allocation, so our approach is complementary.

A final, relevant literature rationalizes market intervention based on criteria that differ from utilitarian efficiency. Weitzman (1977) took as a benchmark the allocation of goods that would prevail if all consumers had the mean income. He then showed that an even allocation of goods may be closer to the benchmark than the market allocation. Sah (1987) compared different regimes from the perspective of the (homogeneous) members of the poorest group. They preferred quantity controls (maximum purchases) with resale permitted to quantity controls without resale, which they in turn preferred to the competitive market. Wijkander (1988) showed that capping price and allocating the good randomly favors certain income groups. Given a social welfare function that puts different weights on different consumers’ utility, capping price may therefore raise welfare. The current paper

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29 Closeness to the benchmark was determined using a mean-square-error criterion. This criterion does not give an exact measure of the welfare cost of misallocation, however.
differs from this strand of the literature by maintaining an efficiency criterion throughout.

7 Concluding Remarks

This paper has asked how a good should be assigned in the presence of buyers’ limited wealth. We have shown that a range of assignment schemes may perform better than the unregulated market when wealth constraints are important. Schemes that place goods directly in the hands of high-valuation buyers obviously work well. It is striking, however, that placing goods in the hands of low-wealth buyers is may also be beneficial. Even more striking is the possibility that full efficiency may be realized by allocating the entire stock to the poor. Transferability was critical to these results.

The results here make a case for a paradigm shift in how particular goods are assigned. Certain goods tend to be assigned by a pure market mechanism whereas others are assigned using instruments besides price. The current paper suggests a blurring of this line when wealth constraints are important: one should consider alternative assignment schemes in certain unregulated markets, and one should consider the removal of prohibitions on transferability in markets that are currently subject to regulation.

The results also suggest how existing assignment schemes can be improved. U.S. immigration policy provides an illustrative example. The U.S. assigns 50,000 immigration visas per year by lottery. Becker (1987) proposed selling visas to a pool of qualified applicants. A simple alternative is to retain the current lottery system but permit recipients to resell their visas to other qualified applicants. Our results suggest that this change would yield greater efficiency than both the current mechanism and the Becker proposal.\textsuperscript{30} There are many other possible applications. For instance, using a lottery to assign transferable educational vouchers may be preferable to a system of local attendance zones or to the use of a lottery to assign non-transferable vouchers. Keeping the price of health care below competitive market levels and allowing patients to sell their spot in the queue could improve efficiency. It may likewise be beneficial to employ a draft with tradable deferments for

\textsuperscript{30}An implicit assumption is that immigrants create rents and those who value visas most highly create the most rents. Other factors may enter a policymaker’s welfare function. For instance, visa sales would generate revenue for the Treasury.
military recruitment. In each of these cases there may be other objectives or institutional details that loom large, but the results here argue for consideration of different assignment schemes and transferability.

References


Appendix: Proofs

Proof of Proposition 3: Let $\psi_x(v) := \frac{\int_1^{\psi(x(v)}}{1-B_x(v)}$ denote the expected value conditional on exceeding $v > 0$, given the ultimate allocation. The total realized value can be expressed as:

$$
\hat{V}_x = \int_{r_x}^{\psi_x(v)} \int_{r_x}^{1} \nu x(w,v) dF(v) dG(w) + \int_{r_x}^{1} \int_{r_x}^{1} v dF(v) dG(w)
$$

$$
= S \int_{r_x}^{\psi_x(v)} \int_{r_x}^{1} v a_x(w) b_x(v) dF(v) dG(w) + (1 - G(r_x)) \int_{r_x}^{1} v dF(v)
$$

$$
= S \cdot A_x(r_x)[1 - B_x(r_x)] \left( \int_{r_x}^{1} v dB_x(v) \right) \frac{1}{1 - B_x(r_x)} + D(r_x) \left( \int_{r_x}^{1} v dF(v) \right) \frac{1}{1 - F(r_x)}
$$

$$
= K_x(r_x) \psi_x(r_x) + D(r_x) \phi(r_x)
$$

$$
= S \left[ \left( 1 - \frac{D(r_x)}{S} \right) \psi_x(r_x) + \left( \frac{D(r_x)}{S} \right) \phi(r_x) \right];
$$

where the first equality follows by definition; the second and third follow by substituting for $x(w,v)$ and integrating; and the last one follows from (4). If $x$ is meritorious, then $\psi_x(r_x) \geq \phi(r_x)$. 


Suppose, further, that $x$ merit-dominates $y \in X$ and $r_x \geq [>]r_y$. Then,

$$
\hat{V}_x = S \left[ \left( 1 - \frac{D(r_x)}{S} \right) \psi_x(r_x) + \left( \frac{D(r_x)}{S} \right) \phi(r_x) \right] \\
\geq S \left[ \left( 1 - \frac{D(r_y)}{S} \right) \psi_x(r_x) + \left( \frac{D(r_y)}{S} \right) \phi(r_x) \right] \\
\geq S \left[ \left( 1 - \frac{D(r_y)}{S} \right) \psi_y(r_x) + \left( \frac{D(r_y)}{S} \right) \phi(r_x) \right] \\
\geq [>] S \left[ \left( 1 - \frac{D(r_y)}{S} \right) \psi_y(r_y) + \left( \frac{D(r_y)}{S} \right) \phi(r_y) \right] \\
= \hat{V}_y;
$$

where the first inequality follows from $r_x \geq r_y$ (which implies $D(r_x) \leq D(r_y)$) and from $\psi_x(r_x) \geq \phi(r_x)$; the second follows from the relative merit-dominance of $x$ over $y$; and the third one follows from the fact that the conditional expectations, $\psi_y(\cdot)$ and $\phi(\cdot)$, are strictly increasing in the relevant region.

**Proof of Proposition 5:** It suffices to show that there exists a pair $(\hat{v}, \hat{\gamma})$ satisfying the two nonlinear equations (6) and (7). Observe first that for each $\hat{\gamma} = t_1(\hat{v})$ satisfying (6). Further, $t_1(\cdot)$ is increasing and satisfies $t_1(\bar{p}) = 0$. Next, let $\hat{\phi} > p^\circ$ be such that $[1 - F(\hat{\phi})][1 - G(\bar{p})] = S$. Then, for each $\hat{v} \in [\bar{p}, \hat{\phi}]$, there exists $\hat{\gamma} = t_2(\hat{v})$ satisfying (7). Further, $t_2(\cdot)$ is decreasing, and it satisfies $t_2(\bar{p}) > 0$ and $t_2(\hat{\phi}) = 0$. Since both $t_1(\hat{v})$ and $t_2(\hat{v})$ are continuous, there exists a unique $\hat{v}^* \in (\bar{p}, \hat{\phi})$ such that $t_1(\hat{v}^*) = t_2(\hat{v}^*) =: \hat{\gamma}^*$. Obviously, $(\hat{v}^*, \hat{\gamma}^*)$ satisfies (6) and (7). Since $\hat{v}^* > \bar{p}$ and $\xi(\cdot)$ is increasing, we have $\hat{V}_D > V_D$.

**Proof of Proposition 7:** It suffices to show that $\hat{V}'(p^\circ) < 0$ here, which means that lowering the cap increases the total realized value. To that end, for $\bar{p} \leq p^\circ$, rewrite

$$
\hat{V}(\bar{p}) = \{ \rho(\bar{p})[G(r(\bar{p})) - G(\bar{p})] + [1 - G(r(\bar{p}))] \} \int_{r(\bar{p})}^1 v dF(v) - c(S(\bar{p})).
$$

Since $r(p^\circ) = p^\circ$, we have

$$
\hat{V}'(p^\circ) = -(1 - G(p^\circ))f(p^\circ)r'(p^\circ)p^\circ - (1 - \rho(p^\circ))g(p^\circ)r'(p^\circ) \int_{p^\circ}^1 v dF(v) \\
- \rho(p^\circ)g(p^\circ) \int_{p^\circ}^1 v dF(v) - c'(S(p^\circ))S'(p^\circ)
$$

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\[ -(1 - G(p^e))f(p^e)r'(p^e)p^e - F(p^e)[1 - F(p^e)]g(p^e)r'(p^e)\phi(p^e) \]
\[ -[1 - F(p^e)]^2 g(p^e)\phi(p^e) - S'(p^e)p^e, \]  

(10)

where the second equality holds since \( \rho(p^e) = \frac{S(p^e)}{1 - G(p^e)} = \frac{D(p^e)}{1 - G(p^e)} = 1 - F(p^e) \), \( \phi(z) = \int_z^1 \frac{v dF(v)}{[1 - F(z)]} \), and \( c'(S(p^e)) = p^e \).

Meanwhile, totally differentiating both sides of (8) and using \( \rho(p^e) = 1 - F(p^e) \) yields
\[ r'(p^e) = -\frac{S'(p^e) + g(p^e)[1 - F(p^e)]^2}{g(p^e)(1 - F(p^e))F(p^e) + f(p^e)(1 - G(p^e))}. \]

Substituting this into (10) and collecting terms, we get
\[ \hat{V}'(p^e) = -\frac{[\phi(p^e) - p^e](1 - F(p^e))g(p^e)[D(p^e)f(p^e) - S'(p^e)F(p^e)]}{g(p^e)(1 - F(p^e))F(p^e) + f(p^e)(1 - G(p^e))} \]
\[ = -\frac{[\phi(p^e) - p^e](1 - F(p^e))g(p^e)[S(p^e)f(p^e) - S'(p^e)F(p^e)]}{g(p^e)(1 - F(p^e))F(p^e) + f(p^e)(1 - G(p^e))}. \]

Hence, \( \hat{V}'(p^e) < 0 \) if and only if \( \frac{S(p^e)}{S(p^e)} < \frac{f(p^e)}{F(p^e)} \). 

\[ \blacksquare \]
Figure 1: Benchmark allocations

$B+C$: Market
$A+B$: Efficient allocation

Figure 2: Rationing with resale

$A$: Probability $= \rho < 1$
$B$: Probability $= 1$
Figure 3: Full efficiency of need-based screening

\[ A + B \]: Need-based allocation

\[ A + C \]: Efficient allocation